Replica evolution of classical field in 4+1 dimensional spacetime as a simulator of quantum field evolution

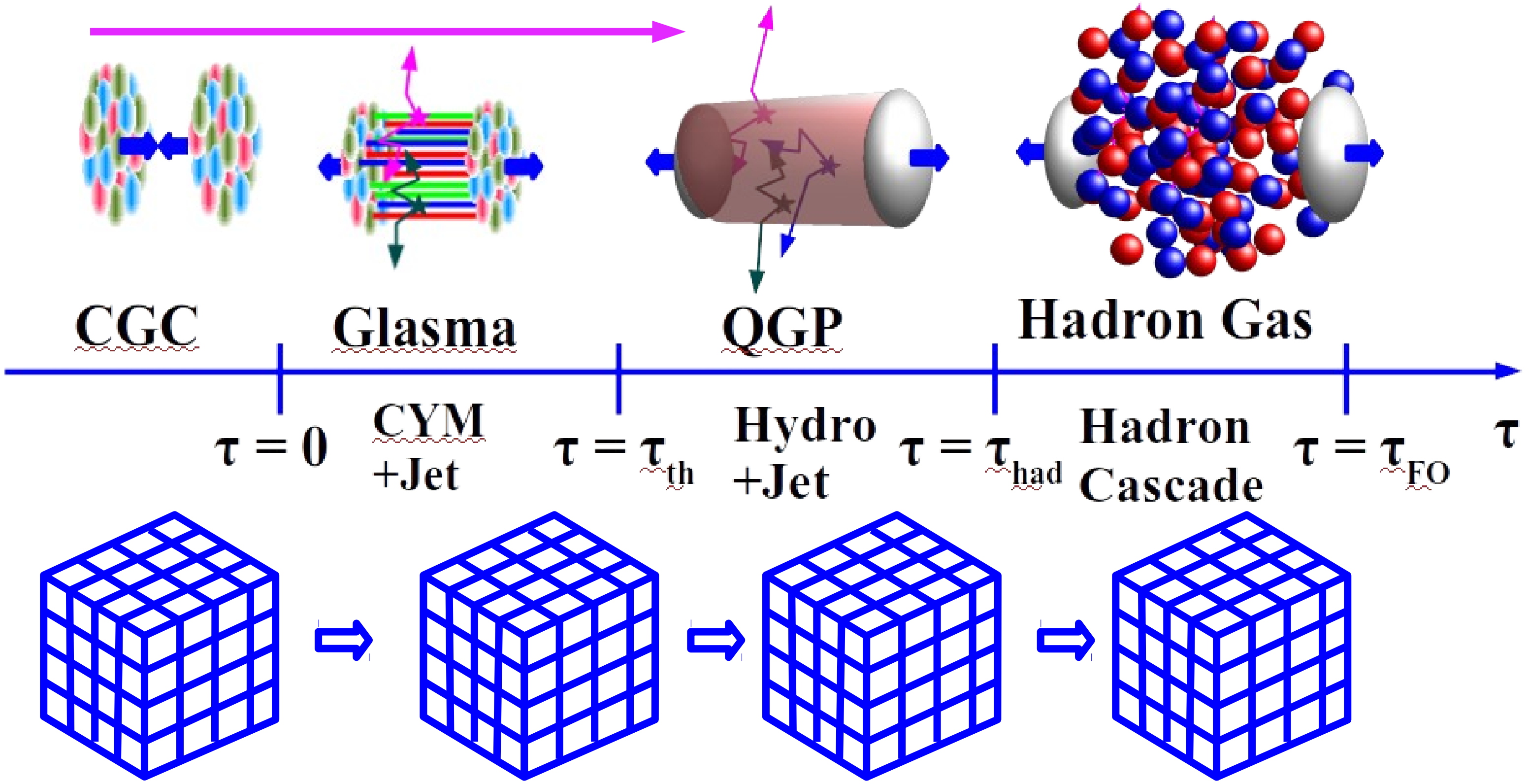
Akira Ohnishi[1], H. Matsuda[2], T. Kunihiro[1], T. T. Takahashi[3]
1. YITP, Kyoto U., 2. Dept. Phys., Kyoto U., 3. NCT, Gunma
Prog. Theor. Exp. Phys. 2021 (2021), 023B09 [arXiv:2008.09556]

Heidelberg/Stavanger Lattice & ML seminar, April 21, 2021.

- **Introduction**
- **Replica Evolution**
  - **Basic Idea**
  - **Application to Scalar Field Theory**
  - **Momentum dist. and Rayleigh-Jeans divergence**
  - **Time-corr. func. and thermal mass**
- **Summary**

Heavy-Ion Collisions on the Lattice ?

- **Classical field (glasma) thermalizes and forms QGP**



- **In the initial stages, we need to solve quantum field evolution under inhomogeneous & non-equilibrium background field.**

- **HIC on the lattice ?**
  → But there is a strong sign problem in rea

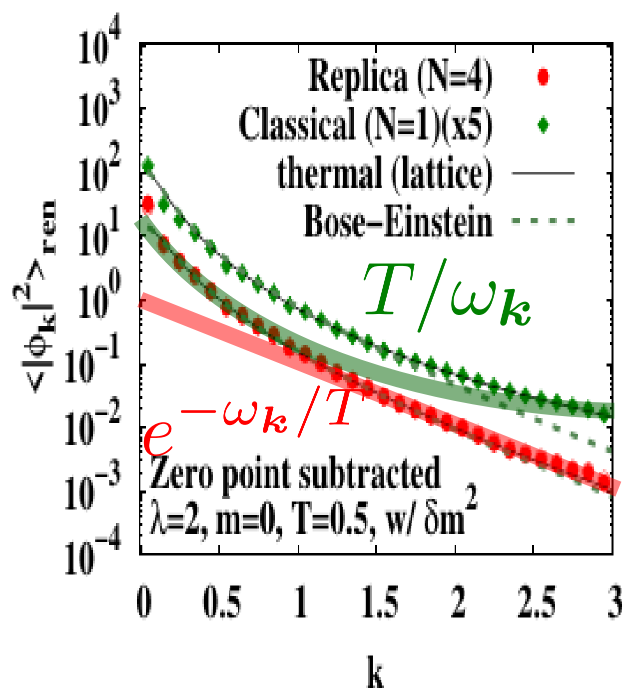
Unreachable Dream ?

Toward Real-Time Dynamics of Quantum Field

- **Classical field dynamics**
  - **Amplitude=exp(iS), Classical EOM δS=0 → No Sign problem**
  - **Equilibrium is classical, and energy density diverges in the continuum limit (Rayleigh-Jeans Divergence)**

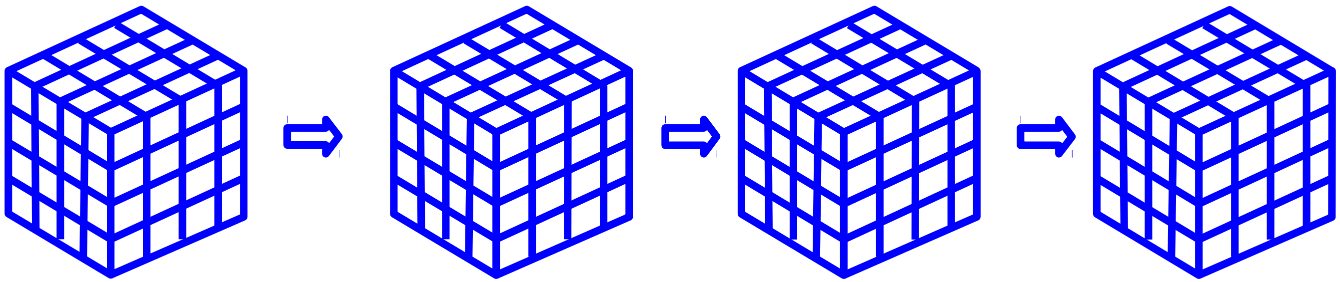
$$n_{\boldsymbol{k}} = T/\omega_{\boldsymbol{k}} (\text{Classical}), \quad n_{\boldsymbol{k}} = [\exp(\omega_{\boldsymbol{k}}/T) \mp 1]^{-1} \,(\text{Quantum})$$

- **Closed Time Path+ 2PI effective action (Schwinger-Keldysh-Kadanoff-Baym Eq.)**
  Aarts, Berges ('02), Hatta, Nishiyama ('12)
  → CPU time is huge when background field is inhomogeneous



- **High-p DOFs are integrated or implemented by particles**
  Bodeker, McLerran, Smilga ('95), Greiner, B.Muller ('97), Dumitru, Nara, (Strickland) ('05,'07) …
  → Classical field part still obeys classical statistics in equilibrium

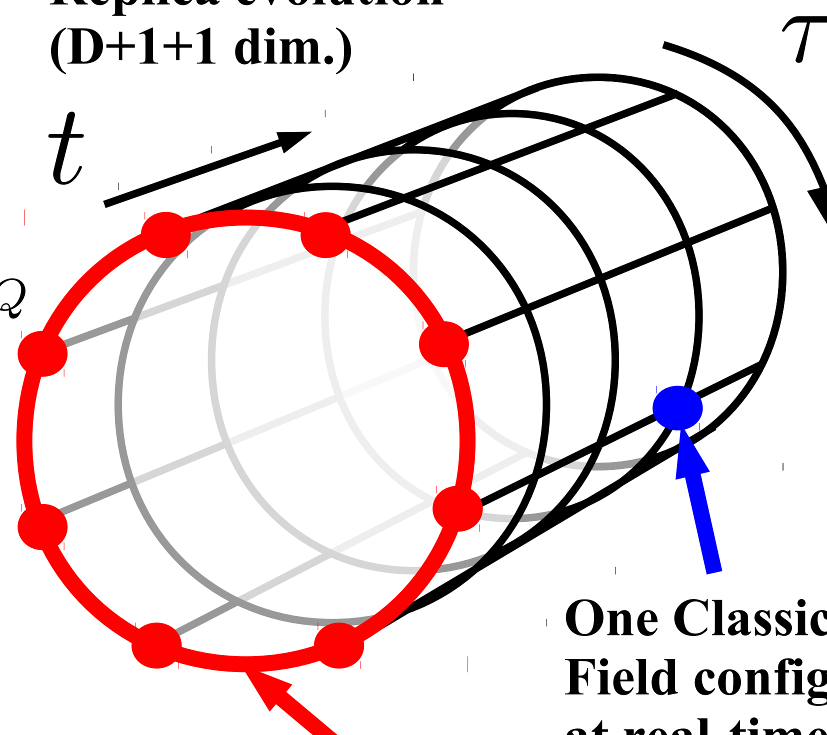
Replica Evolution



$$\mathcal{H} = \sum_{\tau \boldsymbol{x}} \frac{1}{2}\pi_{\tau \boldsymbol{x}}^2 + \xi S[\phi]$$

$$\mathcal{Z}_R = \int \mathcal{D}\phi\, \mathcal{D}\pi\, e^{-\mathcal{H}/\xi} \propto Z_Q$$

$$\mathcal{Z}_Q = \int \mathcal{D}\phi\, e^{-S[\phi]}$$



Replica Evolution
=Classical Dynamics
with Quantum Statistics

Replica evolution

Replica Evolution (Quantum Mechanics)

- **Replica Hamiltonian**

$$\mathcal{H} = \sum_{\tau=1}^{N} \left[\frac{p_\tau^2}{2} + U(x_\tau) + \frac{\xi^2}{2}(x_{\tau+1} - x_\tau)^2 \right]$$

$$\simeq \xi \int_0^{1/T} d\bar{\tau} \left\{ \frac{p^2(\bar{\tau})}{2} + U(x(\bar{\tau})) + \frac{1}{2}\left[\frac{\partial x(\bar{\tau})}{\partial \bar{\tau}}\right]^2 \right\} \quad (\xi = NT, \bar{\tau} = \tau/NT)$$

$H_{\rm cl}(x_\tau, p_\tau)$

τ-derivative term

$\xi S[x]$

- **Quantum Statistical Equilibrium from Replica Evolution**
  - **Replicas = Set of N classical fields (Imag. Time formalism)**
  - **Gauss integral over p of Boltzmann weight [exp(-H/ξ)] leads to quantum statistical partition function.**

$$\mathcal{Z}_R(\xi) = \int \frac{\mathcal{D}x\mathcal{D}p}{2\pi} \exp(-\mathcal{H}/\xi) = \mathcal{N}\,(2\pi\xi)^{NL^3/2} \int \mathcal{D}x \exp(-S[x])$$

$\mathcal{Z}_Q(T)$

"Classical" part. fn. of replicas ∝ "Quantum" part. fn.

Replica Evolution (Quantum Mechanics)

- **Replica Equation of Motion**

$$\frac{dx_\tau}{dt} = \frac{\partial \mathcal{H}}{\partial p_\tau} = p_\tau , \quad \frac{dp_\tau}{dt} = -\frac{\partial \mathcal{H}}{\partial x_\tau} = -\frac{\partial U(x_\tau)}{\partial x_\tau} + \xi^2 (x_{\tau+1} + x_{\tau-1} - 2x_\tau)$$

Force from τ-derivative term

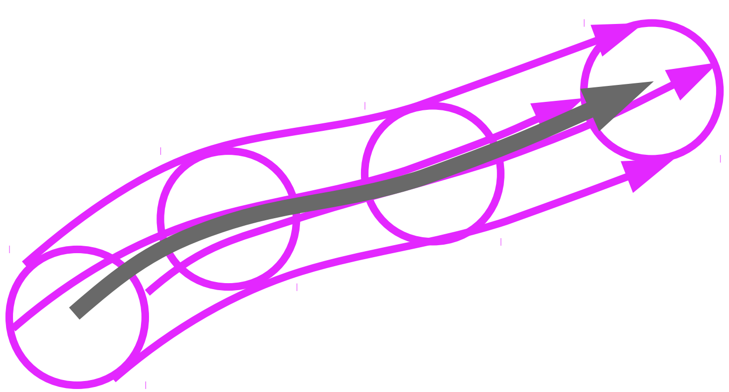
- **Replica index average**

$$\widetilde{x} \equiv \frac{1}{N}\sum_\tau x_\tau , \quad \widetilde{p} \equiv \frac{1}{N}\sum_\tau p_\tau$$

$$\frac{d\widetilde{x}}{dt} = \widetilde{p}, \qquad \frac{d\widetilde{p}}{dt} = -\frac{1}{N}\sum_\tau \frac{\partial U(x_\tau)}{\partial x_\tau} + 0 = -\frac{\partial U(\widetilde{x})}{\partial \widetilde{x}} + \mathcal{O}((\delta x)^2)$$

Ehrenfest's theorem

Replica index average obeys classical EOM (when flucuations are small).



Replica Evolution in Scalar Field Theory

- **Replica evolution in field theory**
  - **Replace variables** $(x_\tau, p_\tau) \to (\phi_{\tau\boldsymbol{x}}, \pi_{\tau\boldsymbol{x}})$
  - **Mass renormalization & Subtracting zero point contribution**
- **Example: Φ^4 theory** **τ-deriv. term**

$$\mathcal{H} = \sum_{\tau,\boldsymbol{x}} \left[\frac{\pi_{\tau\boldsymbol{x}}^2}{2} + \frac{1}{2}(\boldsymbol{\nabla}\phi_{\tau\boldsymbol{x}})^2 + \frac{m_0^2}{2}\phi_{\tau\boldsymbol{x}}^2 + \frac{\lambda}{24}\phi_{\tau\boldsymbol{x}}^4 + \frac{\xi^2}{2}(\partial_\tau\phi_{\tau\boldsymbol{x}})^2 \right]$$

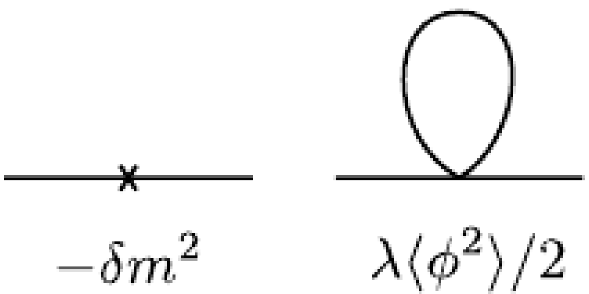
$H(\phi_{\tau\boldsymbol{x}}, \pi_{\tau\boldsymbol{x}})$ $\xi S[\phi]$

$$m_0^2 = m^2 - \delta m^2, \partial_\tau\phi_{\tau\boldsymbol{x}} \equiv \phi_{\tau+1,\boldsymbol{x}} - \phi_{\tau\boldsymbol{x}}$$

- **Mass Counterterm (one loop)** *Aarts, Smit ('97), Kapusta, Gale (textbook)*

$$\delta m^2 = \frac{\lambda}{2}\langle\phi^2\rangle_{\mathrm{div}}$$

$$\langle\phi^2\rangle_{\mathrm{div}} = \frac{1}{L^3}\sum_{\boldsymbol{k}} \frac{1}{2\omega_{\boldsymbol{k}}\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}}$$



Numerical Calculation Setup

- **Lattice size = 32^3 x 4 (L=32, N=4)**
- **T=0.5 (ξ=NT=2); m=0, 0.5; λ=0.5, 1, 2, 4, 6, 8, 10.**
- **One loop renormalization of mass, no counterterm for λ.**
- **Initial conditions are obtained by solving the Langevin equation.**
- **Solve replica EOM until t=500 with the time step of Δt=0.025.**
- **Number of replica configurations = 1000**
  → 3-6 hours on one core of core i7 PC for a given (m, λ)

Momentum Distribution

- **Momentum distribution in replica = zero point + Bose-Einstein**

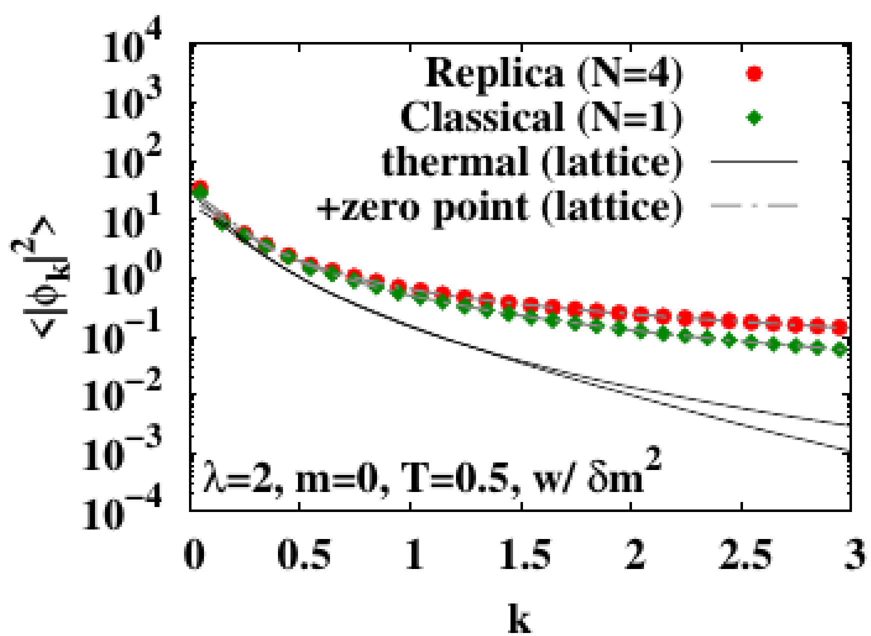
$$\langle|\phi_{\boldsymbol{k}}|^2\rangle = \frac{1}{N}\sum_n \langle\phi_{n\boldsymbol{k}}\phi^*_{n\boldsymbol{k}}\rangle = \frac{1}{\omega_{\boldsymbol{k}}\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}}\left[\frac{1}{2}+\frac{1}{e^{\Omega_{\boldsymbol{k}}/T}-1}\right]$$

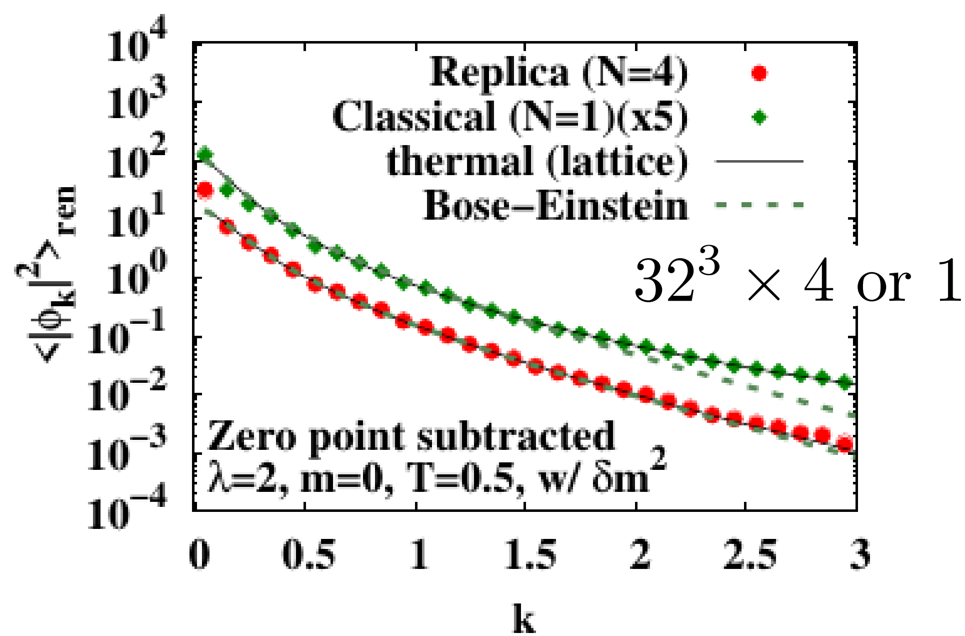
Free field, Matsubara sum

Zero point

Thermal → Bose-Einstein

- **By subtracting the zero point part, we can avoid equipartition & Rayleigh-Jeans divergence.**





Rayleigh-Jeans Divergence

- **With N >= 2, free field energy converges in the replica method.**

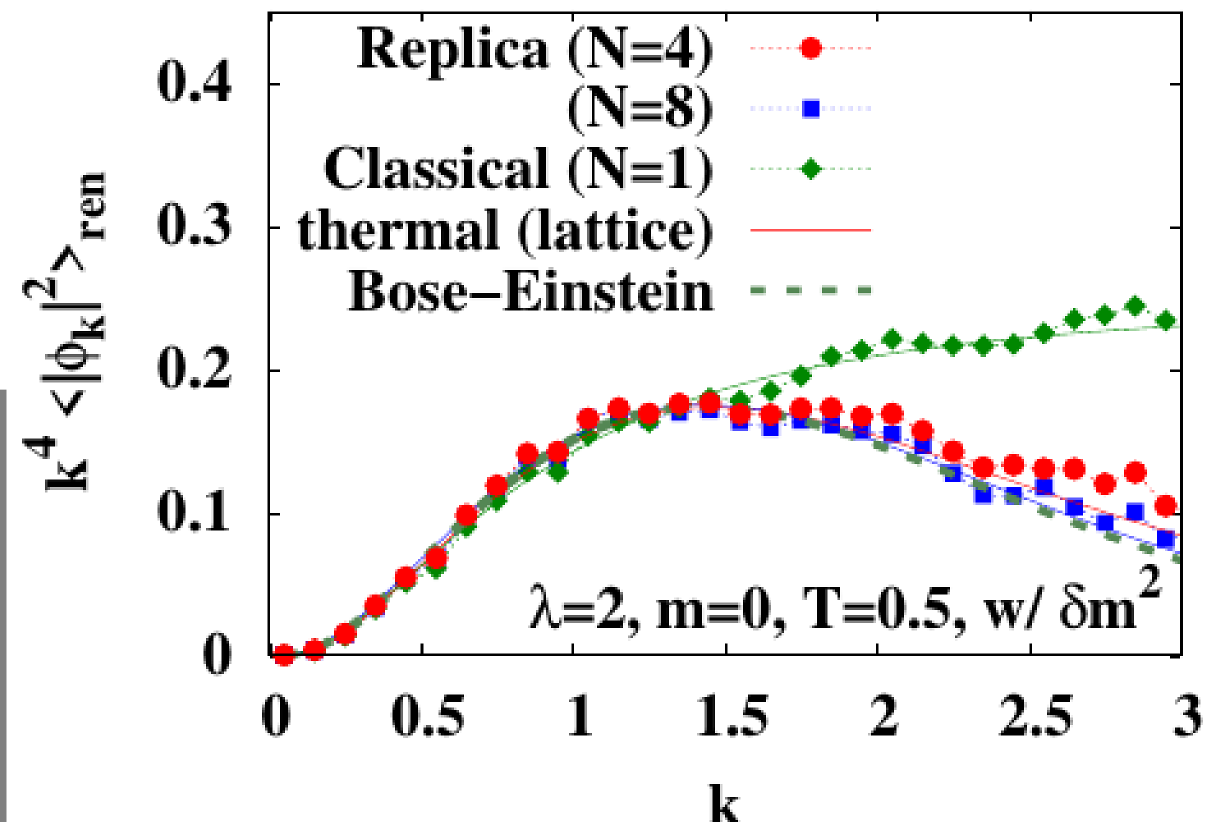
$$\Omega = 2NT\,\mathrm{arcsinh}\,(\omega/2NT) \underset{\omega\gg NT}{\longrightarrow} 2NT\log(\omega/NT)$$

$$\langle|\phi_{\boldsymbol{k}}|^2\rangle_{\rm ren} \simeq \frac{2NT}{k^2}\exp(-\Omega_{\boldsymbol{k}}/T) \to 2(NT)^{2N+1}k^{-2(N+1)}$$

$$k^4\,\langle|\phi_{\boldsymbol{k}}|^2\rangle_{\rm ren} \to 2(NT)^{2N+1}k^{-2(N-1)} \quad (\text{K.E.} \propto \int dk k^4\,\langle|\phi_{\boldsymbol{k}}|^2\rangle_{\rm ren})$$

  - **Convergence cond.**
    2(N-1) > 1 → N > 1.5

We can remove divergence of energy in the replica method (N>=2) with mass counterterm.



Time-correlation function

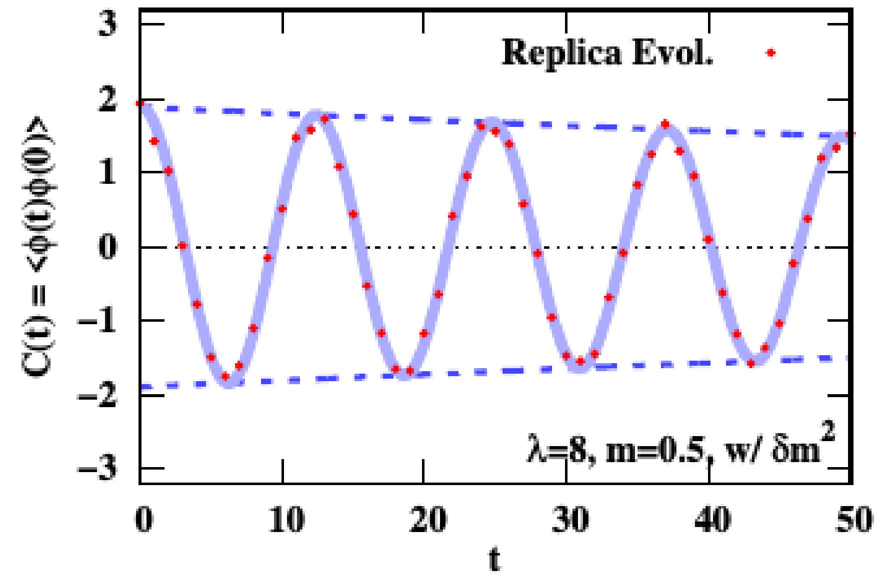
- **Time-correlation function of free field (zero momentum)**

$$C(t) = \frac{1}{L^3}\sum_{\boldsymbol{x},\boldsymbol{y}} \langle \phi_{\boldsymbol{x}}(t)\phi_{\boldsymbol{y}}(0)\rangle$$

$$= \frac{1}{NL^3}\sum_{\tau,\boldsymbol{x},\boldsymbol{y}} \langle \phi_{\tau\boldsymbol{x}}(t)\phi_{\tau\boldsymbol{y}}(0)\rangle$$

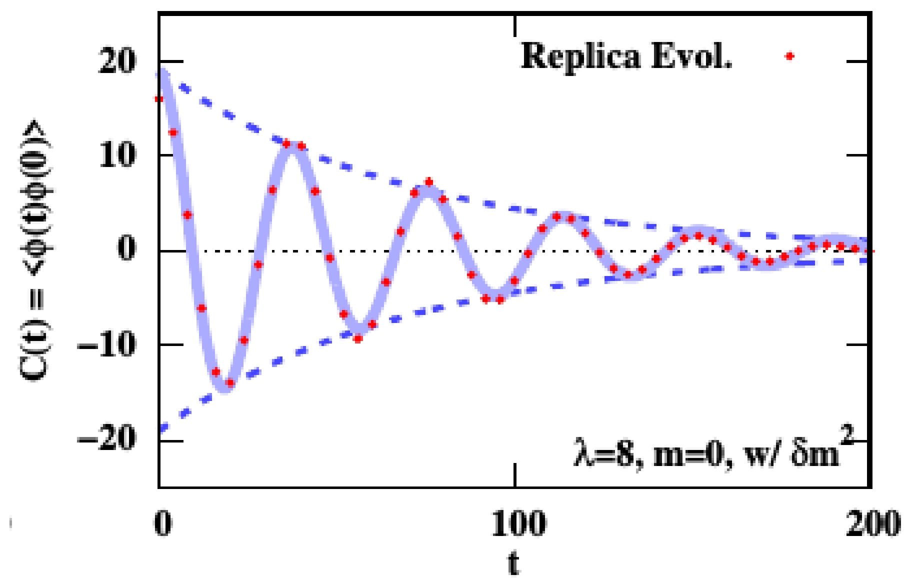
$$= \sum_n \frac{T}{M_n^2}\cos M_n t$$

$$(M_n^2 = \omega^2 + 4\xi^2 \sin^2(n\pi/N))$$



From the dominant frequency of C(t), we obtain thermal mass.

- **TCF of interacting field**
  - **Interaction induces thermal mass**
  - **Coupling of different momentum modes induces damping.**

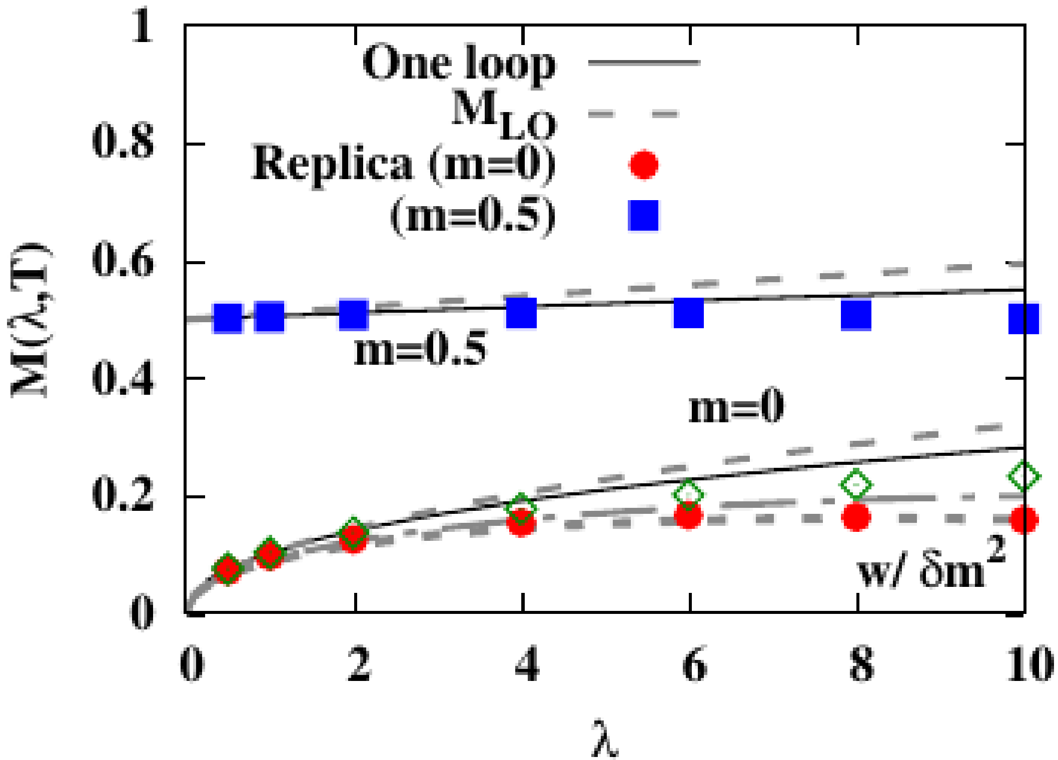


Thermal Mass

- **Thermal Mass**
  - **Leading Order (one-loop)** $M^2_{\rm LO} = m^2 + \lambda T^2/24.$
  - **Resummed One-Loop** $M^2_{\rm resum} = \frac{\lambda T^2}{24}\left[1 - \frac{3}{\pi}\sqrt{\frac{\lambda}{24}}\right]$
  - **Two-Loop**

$$M^2_{2-\rm loop} = \frac{\lambda T^2}{24}\left\{1 - \frac{3}{\pi}\sqrt{\frac{\lambda}{24}} + \frac{\lambda}{(4\pi)^2}\left[\frac{3}{2}\log\left(\frac{T^2}{4\pi\mu^2}\right) + 2\log\left(\frac{\lambda}{24}\right) + \alpha\right]\right\},$$

Kapusta, Gale (textbook)
Parwani ('92, '93)



Thermal mass in Replica Evolution ~ Two-loop results

Summary

- **Replica evolution is classical field dynamics, which reaches quantum statistical equilibrium and approximately describes quantum evolution.**
  AO, H.Matsuda, T.Kunihiro, T.T.Takahashi, PTEP 2021('21), 023B09 [2008.09556]
  - **Configurations of N classical fields (replicas) interacting via τ-derivative terms (kinetic E. in imag. time formalism) at temperature ξ=NT reach quantum statistical distribution at temperature T. (Chaoticity is assumed.)**
  - **Replica-index (~ imag. time) average obeys classical field EOM, when fluctuations are small.**
    Unequal-time two-point function gives thermal mass close to 2-loop results in QFT.
    → Real-time evolution of quantum field is approximately described.
  - **Subtracting zero point motion part from $\langle\Phi^2\rangle$**
    → mass renormalization and removing Rayleigh-Jeans divergence
- **Similar ideas are found in the molecular dynamics part of HMC, and in the Path Integral Molecular Dynamics (PIMD) in quantum chemistry.**

To do list

- **There are many subjects to be investigated**
  - **Comparison with previously proposed frameworks.**
    Hard mode effects [Bodeker, McLerran, Smilga ('95), Greiner, B.Muller ('97)],
    Field-particle sim. [Dumitru, Nara ('05), Dumitru, Nara, Strickland ('07)],
    2PI [Aarts, Berges ('02), Hatta, Nishiyama ('12)], …
  - **Formal discussions, e.g. relation to Boltzmann Eq.,**
    A.Muller, Son ('04).
  - **Shear viscosity, Thermalization of classical field, Entropy production, …**
    - **EM tensor needs to be renormalized. Gradient flow ?**

Thank you for your attention !

AO

Hidefumi Matsuda

Teiji Kunihiro

Toru T. Takahashi

"Replica evolution of classical field in 4+1 dimensional spacetime toward real time dynamics of quantum field",
A. Ohnishi, H. Matsuda, T. Kunihiro, T. T. Takahashi,
PTEP 2021 (2021), 023B09 [arXiv:2008.09556 [hep-lat]].

Harmonic Oscillator

- **Replica Hamiltonian = N free HO Hamiltonian**

$$\mathcal{H} = \sum_\tau \left[\frac{p_\tau^2}{2} + \frac{\omega^2 x_\tau^2}{2} + \underline{\frac{\xi^2}{2}(x_{\tau+1} - x_\tau)^2}\right] = \sum_n \left[\frac{\bar{p}_n^2}{2} + \frac{M_n^2 \bar{x}_n^2}{2}\right]$$

τ-deriv. term

Fourier transf.

$$M_n^2 = \omega^2 + 4\xi^2 \sin^2(\pi n/N)$$

- **Expectation value of x^2 in Replica**

exp$(-\mathcal{H}/\xi)$

Matsubara freq. sum

$$\langle x^2 \rangle = \frac{1}{N}\sum_\tau \langle x_\tau^2 \rangle = \frac{1}{N}\sum_n \langle \bar{x}_n^2 \rangle = \frac{1}{N}\sum_n \frac{\xi}{M_n^2} = \frac{\coth(\Omega/2T)}{2\omega\sqrt{1+\omega^2/4\xi^2}}$$

$$\Omega = 2\xi \mathrm{arcsinh}(\omega/2\xi)$$

$$\frac{T}{\omega^2} (N = 1, \mathrm{Classical})$$

zero point

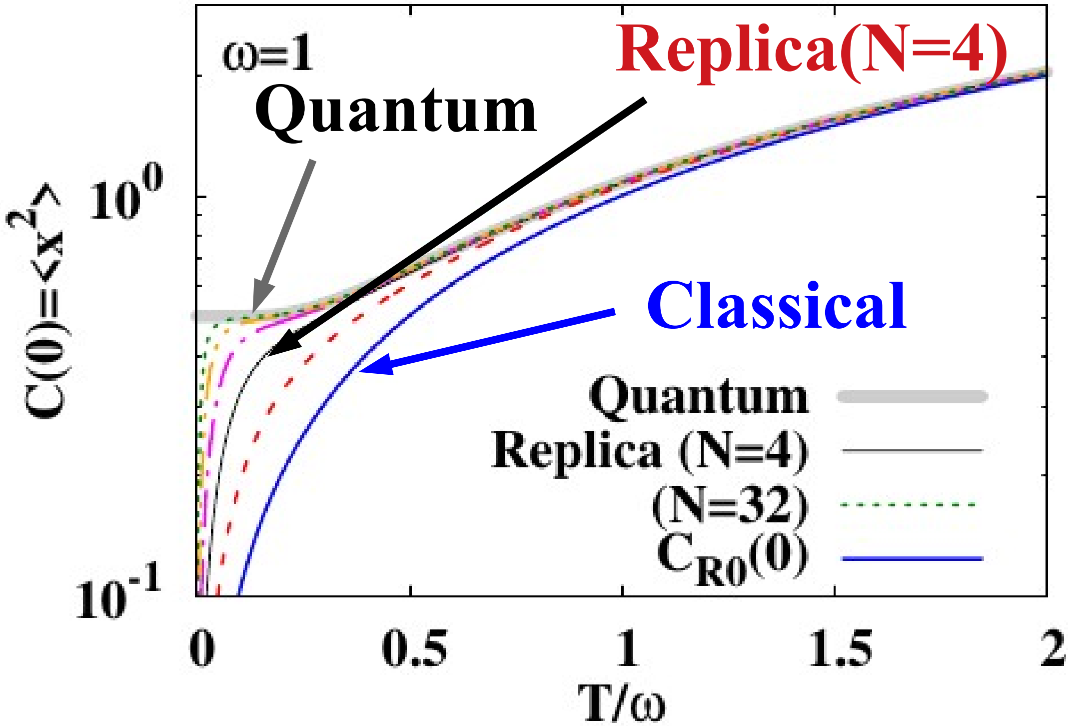
thermal

$$\to \frac{\coth(\omega/2T)}{2\omega} = \frac{1}{\omega}\left[\frac{1}{2} + \frac{1}{e^{\omega/T} - 1}\right] (N \to \infty, \mathrm{Quantum})$$

Equal time observables of x are reproduced at $N \to \infty$

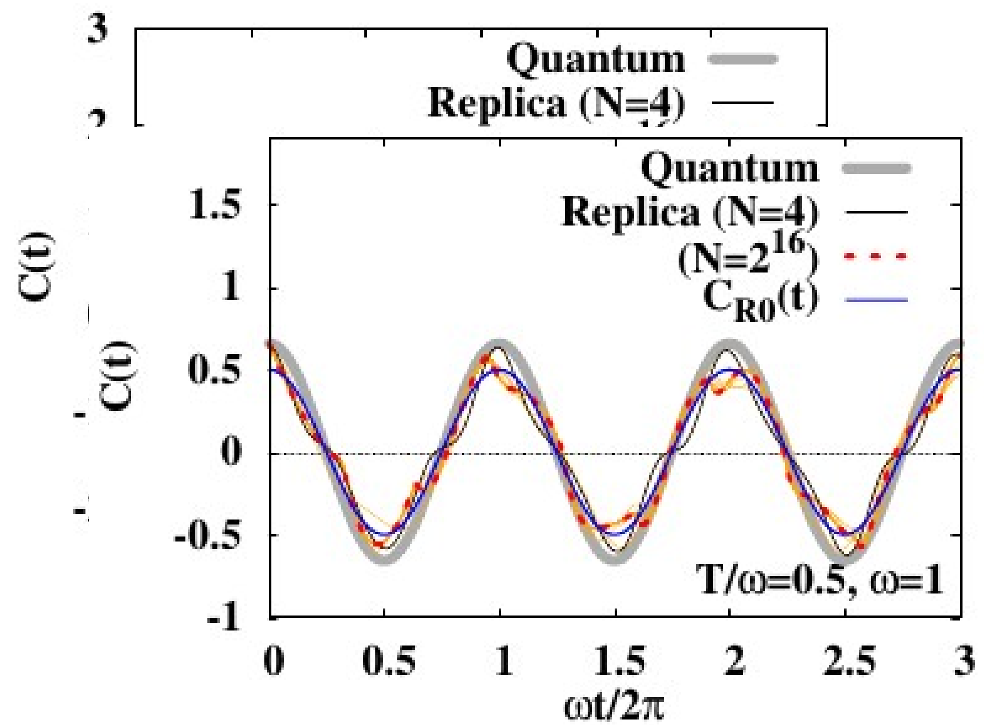
Time Correlation Function in HO

$$C(t) = \langle x(t)x(0) \rangle$$



Equal time observables
→ Exact at N → ∞

Unequal time corr. fn.
→ Not exact,
but good for T/ω > 0.5



Sounds nice. How about field theory ?

Approaches using 2PI effective action

VOLUME 88, NUMBER 4 PHYSICAL REVIEW LETTERS 28 JANUARY 2002

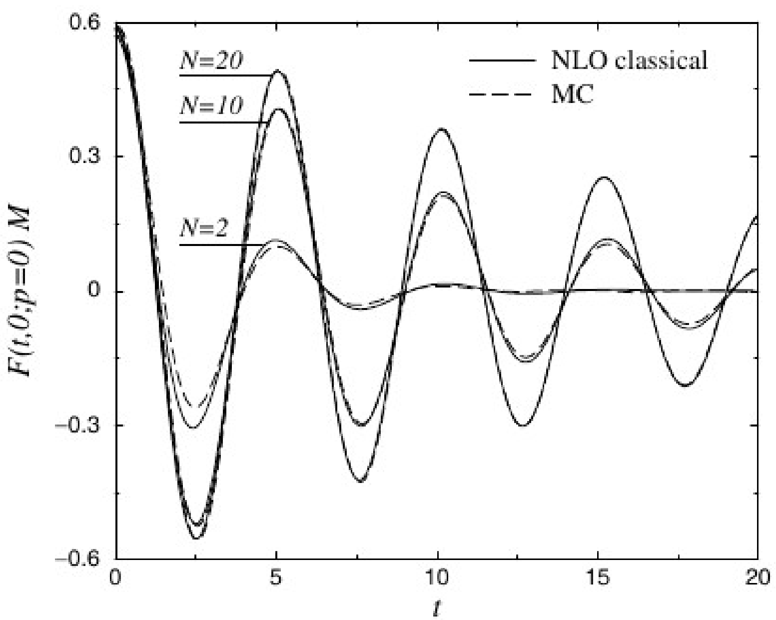
Classical Aspects of Quantum Fields Far from Equilibrium

Gert Aarts and Jürgen Berges

Institut für Theoretische Physik, Philosophenweg 16, 69120 Heidelberg, Germany

(Received 16 July 2001; published 15 January 2002)

We consider the time evolution of nonequilibrium quantum scalar fields in the $O(N)$ model, using the next-to-leading order $1/N$ expansion of the two-particle irreducible effective action. A comparison with exact numerical simulations in 1 + 1 dimensions in the classical limit shows that the $1/N$ expansion gives quantitatively precise results already for moderate values of N. For sufficiently high initial occupation numbers the time evolution of quantum fields is shown to be accurately described by classical physics. Eventually the correspondence breaks down due to the difference between classical and quantum thermal equilibrium.



Time-correlation function is reasonably described by classical field, but statistics in equilibrium is problematic.