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一般化されたT-双対性の量子論

Quantum aspects of generalized T-duality

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Introduction

- String duality is a key to grasp whole picture of string theory

  string vacua, non-perturbative aspects, ···

- T(target space)-duality is well understood both from target-space and world-sheet point of view

  [Kikkawa-Yamasaki '84, Sakai-Senda '86, ···]

- – natural to try to extend T-duality indeed, such development is still ongoing

Introduction

- Analyses are mainly classical
- In spite of progress for 30 years
  their quantum aspects remain unclear

In this talk,

- – discuss quantum aspects of generalized T–duality
  (Poisson–Lie T–duality) [Klimcik–Severa '95]
- – give the first example of quantum PL T–duality
  valid to all orders in α' , at any genus

Plan

1. Introduction
2. (Abelian) T-duality
3. Generalization of T-duality
   non-Abelian T-duality, Poisson-Lie (PL) T-duality
4. PL T-duality of WZNW model (classical)
5. PL T-duality of WZNW model (quantum)
6. Summary and discussion

(Abelian) T-duality

Abelian T-duality (classical)

- Let us consider 2 dim. sigma model for strings

$$S = \frac{1}{2\pi}\int d^2z\,(g_{ij}+b_{ij})\,\partial x^i \bar{\partial} x^j$$

g_{ij}: metric, b_{ij} : anti-symm. tensor

- Suppose g_{ij} , b_{ij} indep. of x^1 (isometry)
- Then, consider (gauging isometry)

$$\hat{S} = \frac{1}{2\pi}\int d^2z\,\left[g_{11}A\bar{A} + E_{1a}A\bar{\partial}x^1 + E_{a1}\partial x^1\bar{A} + E_{ab}\partial x^a\bar{\partial}x^b\right]$$

$$+\frac{1}{2\pi}\int d\tilde{\theta}\wedge\mathcal{A}$$

$\{x^i\} = \{x^1, x^a\}$, $E_{ij} = g_{ij} + b_{ij}$

$\mathcal{A} = A_\alpha dz^\alpha$

Abelian T-duality (classical)

- Integrating out $\tilde{\theta}$

$$\rightarrow \quad d\mathscr{A} = 0 \quad \rightarrow \quad A = \partial\theta, \;\; \bar{A} = \bar{\partial}\theta \quad \text{(locally)}$$

$$\rightarrow \quad \widehat{S} \Rightarrow S \,, \quad x^1 = \theta \quad \text{(original model)}$$

- Integrating out $\mathscr{A}$

$$\widehat{S} \quad \Rightarrow \quad \tilde{S} = \frac{1}{2\pi}\int d^2z\,(\tilde{g}_{ij} + \tilde{b}_{ij})\,\partial\tilde{x}^i\,\bar{\partial}\tilde{x}^j \qquad \text{(dual model)}$$

$$\{\tilde{x}^i\} = \{\tilde{x}^1 = \tilde{\theta},\, x^a\}\ , \quad \tilde{g}_{11} = \frac{1}{g_{11}}, \quad \tilde{g}_{1a} = \frac{b_{1a}}{g_{11}}, \quad \tilde{b}_{1a} = \frac{g_{1a}}{g_{11}}$$

$$\tilde{g}_{ab} = g_{ab} - \frac{g_{1a}g_{1b} - b_{1a}b_{1b}}{g_{11}}, \quad \tilde{b}_{ab} = b_{ab} + \frac{g_{1a}b_{1b} - b_{1a}g_{1b}}{g_{11}}$$

"Buscher rules"

Abelian T-duality (classical)

- Two descriptions by (g_{ij}, b_{ij}) or $(\tilde{g}_{ij}, \tilde{b}_{ij})$ for same $\widehat{S}$
  $\longrightarrow$ duality

- Map/symmetry of e.o.m. , sugra

So far, discussion is local, classical

Abelian T-duality (quantum)

- Measure (Jacobian for integration of $\mathscr{A}$) :

  dilaton $\phi \to \phi - \frac{1}{2}\log g_{11}$ [Buscher '88]

- Global issue :

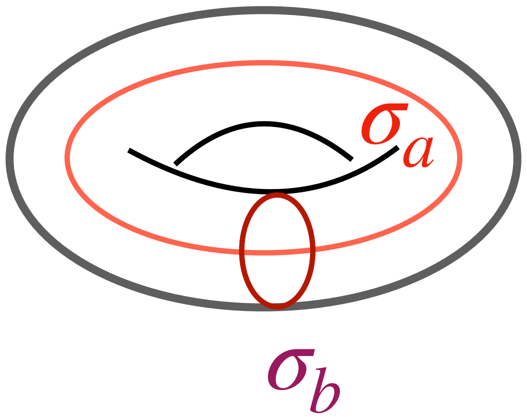
  when $\tilde{\theta} = \tilde{x}^1$ has period 2π along world-sheet torus

  $\to \tilde{\theta} = n_a \sigma_a + n_b \sigma_b + \cdots$ (winding modes)

  $\to \hat{S}$ contains $\hat{S}_{\text{wind}} \sim n_a \oint_b \mathscr{A} + n_b \oint_a \mathscr{A}$

  $\to$ summing over n_a, n_b gives $\oint_{a,b} \mathscr{A} = 2\pi\mathbb{Z}$

  $\to \theta = x^1$ $(\mathscr{A} = d\theta)$ has period 2π [Rocek-Verlinde '91]



Exact T-duality as isomorphism of CFT

- Let $g_{11} = R^2$ (constant) , $g_{1a} = 0$, $b_{ij} = 0$
  for circle compactification w/radius R

- $$\sqrt{g_{11}}\,x^1 = Rx^1 = n_w R\sigma + \frac{n_m}{R}\tau + \cdots$$
  $$= \frac{1}{2}\left(\frac{n_m}{R} + n_w R\right)(\tau+\sigma) + \frac{1}{2}\left(\frac{n_m}{R} - n_w R\right)(\tau-\sigma) + \cdots$$
  $$=: X(\tau+\sigma) + \bar{X}(\tau-\sigma)$$

- $$\sqrt{\tilde{g}_{11}}\,\tilde{x}^1 = R^{-1}\tilde{x}^1 = \left(R \leftrightarrow R^{-1}\right) \text{ in the above}$$
  $$= X(\tau+\sigma) - \bar{X}(\tau-\sigma)$$

- T-duality : $(X, \bar{X}) \to (X, -\bar{X})$

Exact T–duality as isomorphism of CFT

- This is isomorphism of CFT (conformal field theory) valid to all orders in α', at any genus

- For compactification on d–dim. torus (with const. b_{ij})
T–duality forms

  $\mathrm{O}(d,d,\mathbb{R})$ for e.o.m., sugra (classical)

  $\mathrm{O}(d,d,\mathbb{Z})$ as exact symmetry of string

- Isomorphism of CFT is a way to explicitly show fully quantum duality (others may be difficult…)

Non-Abelian T-duality

Non-Abelian T-duality

- Procedure using $\mathscr{A}$ and $\tilde{\theta}$ can be generalized to non-Abelian isometry [de la Ossa-Quevedo '92; Giveon-Rocek '93; Alvarez et al '94]

$$\tilde{\theta}\,(\partial\bar{A} - \bar{\partial}A) \quad \rightarrow \quad \mathrm{tr}\left(\,\tilde{\theta}\,F(\mathscr{A})\,\right)$$

$$\tilde{\theta}\;,\mathscr{A} \;\in \mathscr{G}\ \text{(algebra)}\;,\; F(\mathscr{A}) = \partial\bar{A} - \bar{\partial}A + [A,\bar{A}]$$

- Symmetry of dual model is not local
  $\longrightarrow$ dual of dual ? (not "duality")
- Holonomy $Pe^{\oint \mathscr{A}}$ is non-local
  Winding, periodicity of $\tilde{\theta}$ etc ? $\Rightarrow$ global issues ?
  Dilaton ?
- – may not be promoted to quantum duality

Poisson–Lie T–duality

Poisson-Lie (PT) T-duality

- Another generalization is Poisson-Lie T-duality

[Klimcik and Severa '95]

- It is based on Drinfeld double

- Applications to deformation of sigma models

[Yoshida-san, Sakamoto-san's talks]

- Formulation using Doubled Formalism

[Sakatani-san's talk]

Here, let us follow the original formulation

Drinfeld double D

- D is a Lie Group
- Its Lie algebra $\mathscr{D}$ can be decomposed into a pair subalgebras $\mathscr{G}, \tilde{\mathscr{G}}$

s.t. $\mathscr{D} = \mathscr{G} + \tilde{\mathscr{G}}$

$$\langle T^a, \tilde{T}_b \rangle = \delta^a_b \; ; \qquad \langle T^a, T^b \rangle = \langle \tilde{T}_a, \tilde{T}_b \rangle = 0$$

$T^a \in \mathscr{G}\,,\ \tilde{T}_a \in \tilde{\mathscr{G}}\ ;\quad \langle \cdot\,,\cdot \rangle$: ad–inv. bilinear form

- Such D is called Drinfeld double

  $(\mathscr{D}; \mathscr{G}, \tilde{\mathscr{G}})$ is called Manin triple

Poisson–Lie T-duality

- Suppose $\mathscr{D}$ is decomposed

  also into orthogonal subspaces as $\mathscr{D} = \mathscr{E}^+ + \mathscr{E}^-$

- For $l \in D$, consider equations

$$\left\langle \partial_\pm l \cdot l^{-1}, \mathscr{E}^\mp \right\rangle = 0 \qquad (*)$$

- Substituting decomposition

$$l(\sigma^+, \sigma^-) = g(\sigma^+, \sigma^-)\, \tilde{h}(\sigma^+, \sigma^-)\ ;\ \ g \in G\,,\ \tilde{h} \in \tilde{G}$$

$$\rightarrow \quad A^a_\pm(g) := E^{ab}(g)(g^{-1}\partial_\pm g)^b = -\,(\partial_\pm \tilde{h}\tilde{h}^{-1})^a$$

$$\rightarrow \quad \partial_+ A^a_-(g) - \partial_- A^a_+(g) \ = \ \tilde{c}_{bc}{}^a A^b_-(g) A^c_+(g)$$

"Poisson–Lie structure"

$\tilde{c}_{bc}{}^a$: structure const. of $\tilde{\mathscr{G}}$, E^{ab} : see below

Poisson–Lie T-duality

- This is e.o.m. from

$$L = E^{ab}(g)(g^{-1}\partial_- g)_a (g^{-1}\partial_+ g)_b$$

$$g^{-1}\mathscr{E}^+ g = \mathrm{Span}\left(T^a + E^{ab}(g)\tilde{T}_b\right)$$

- Exchanging roles of G and $\tilde{G}$, (∗) gives e.o.m. from

$$\tilde{L} = \tilde{E}_{ab}(\tilde{g})(\tilde{g}^{-1}\partial_- \tilde{g})^a (\tilde{g}^{-1}\partial_+ \tilde{g})^b$$

$$\tilde{g}^{-1}\mathscr{E}^+ \tilde{g} = \mathrm{Span}\left(\tilde{T}_a + \tilde{E}_{ab}(\tilde{g})T^b\right)$$

Two expressions/backgrounds for same (∗)

→ Poisson–Lie T-duality

Poisson–Lie T–duality

- "Duality" is just $G \leftrightarrow \tilde{G}$
- When another Manin triple exists $\mathscr{D} = \mathscr{G}' + \tilde{\mathscr{G}}'$
  $\longrightarrow$ "PL T–plurality"
- At origins $g = e$, $\tilde{g} = \tilde{e}$, one finds $E(e)\tilde{E}(\tilde{e}) = \mathrm{id}$.

  generalization of $R \leftrightarrow 1/R$ duality
- If G and $\tilde{G}$ are Abelian, reduces to Abelian T–duality
- So far, local, classical

Global issues? Quantum aspects?

PL T-duality of WZNW model

- classical -

WZNW models

- WZNW (Wess–Zumino–Novikov–Witten) model describes string propagation on group manifold G
- Action takes

$$S = \frac{k}{4\pi}\int_{\Sigma} d^2z \ \mathrm{tr}\left(g^{-1}\partial g\right)^2 + \frac{ik}{12\pi}\int_{B} \mathrm{tr}\left(g^{-1}dg\right)^3$$

$$g(z,\bar{z}) \in \mathrm{G}, \quad \partial B = \Sigma, \quad k : \mathrm{level}$$

- – has left and right affine-Lie (current) algebra symmetry $\widehat{\mathcal{G}}_k$

WZNW models

- For left-mover

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c J^c(w)}{z-w}$$

- In terms of modes $J^a_n = \oint dz\, z^n J^a(z)$

$$[J^a_n, J^b_m] = k\delta^{ab} + i f^{ab}{}_c J^c_{n+m}$$

- Similarly for right-mover $\bar{J}_a(\bar{z})$

Classical self-duality of WZNW model

- Toward quantum analysis,
  simplest case (not reduced to Abelian duality on tori) may be WZNW models w/ 3 dim. target space; SU(2), SL(2)

  ⬆ 6 dim. Drinfeld doubles

- 6 dim. Drinfeld doubles are classified
  ∃ 22 classes [Snobl and Hlavaty ’02]

- Search over 6 dim. Drinfeld doubles
  → found a self-dual pair of SU(2) WZNW model

SU(2) WZNW from Drinfeld double

- We start from Drinfeld double given by

$$[T_a, T_b] = f_{ab}{}^c T_c\,, \qquad [\tilde{T}^a, \tilde{T}^b] = f^{ab}{}_c \tilde{T}^c$$

$$\rightarrow [T_a, \tilde{T}^b] = f_a{}^{bc} T_c - f_{ac}{}^b \tilde{T}^c = -[\tilde{T}^b, T_a]$$

$$f_{ab}{}^c = -f_{ba}{}^c\,, \;\; f_a{}^{bc} = -f_a{}^{cb}\,, \;\; f_{abc} = f_{[abc]}$$

compatibility w/ $\langle \cdot , \cdot \rangle$

$$f_{13}{}^1 = f_{23}{}^2 = -\omega\,, \;\; f_{13}{}^2 = -f_{23}{}^1 = 1$$

$$f_1{}^{13} = f_2{}^{23} = -\omega\,, \;\; f_2{}^{13} = -f_1{}^{23} = \omega^2$$

$$\omega > 0\,, \;\; (1+\omega^2)^2 \sim k^{-1}$$

- Duality transformations are non-linear, non-local
  → need find good coordinates to see explicit relations

Non-local automorphism of $\widehat{su}(2)_k$

- After some algebra/efforts,
  indeed found self-dual SU(2) WZNW (local, classical)

- WZNW models are often self-dual classically
  [Klimcik-Severa '02]

- Duality here is summarized
  in terms of currents of affine Lie algebra $\widehat{su}(2)_k$ as
  Integer

$$J^a(z) \to J'^a(z) = J^a(z) \quad (a = 3, \pm)$$

$$\bar{J}^3(\bar{z}) \to \bar{J}'^3(\bar{z}) = -\bar{J}^3(\bar{z})$$

$$\bar{J}^\pm(\bar{z}) \to \bar{J}'^\pm(\bar{z}) = e^{\mp \frac{4i}{k} \int \bar{J}^3(\bar{z})} \cdot \bar{J}^\pm(\bar{z})$$

non-local, involving infinitely many modes

Non–local automorphism of $\widehat{su}(2)_k$

- PL T–duality induces canonical transformations (classical) [Klimcik–Severa ’95; Sfetsos ’97]

- Above duality trans. should be automorphism of $\widehat{su}(2)_k$
  → indeed can be confirmed

- This automorphism is not on list in math. literature
  (inner, outer, anti–involution, ···)

What is our PL T–duality ??

PL T-duality of WZNW model

– quantum –

Parafermionic realization

- $\widehat{su}(2)_k$ is realized by parafermions as $\dfrac{\widehat{su}(2)_k}{\hat{u}(1)_k} \times \hat{u}(1)_k$

free boson

$$J^3(z) = i\sqrt{k/2}\,\partial\varphi(z)$$

$$J^+(z) = \sqrt{k}\,\psi(z)\,e^{i\sqrt{2/k}\,\varphi(z)}, \quad J^-(z) = \sqrt{k}\,\psi^\dagger(z)\,e^{-i\sqrt{2/k}\,\varphi(z)}$$

parafermion

- Primary fields of $\widehat{su}(2)_k$ are

$$G^{l,\bar{l}}_{m,\bar{m}}(z,\bar{z}) = \Phi^{l,\bar{l}}_{m,\bar{m}}(z,\bar{z})\,e^{i\left(m\varphi(z)+\bar{m}\bar{\varphi}(\bar{z})\right)/\sqrt{2k}}$$

  - $\Phi^{l,\bar{l}}_{m,\bar{m}}$: primary of parafermionic CFT
  - $j = l/2$: su(2) spin; $m/2$: eigenvalue of J^3_0

Quantum PL T-duality

- Automorphism from PL T-duality is found to be

$$(\varphi,\ \bar{\varphi}) \quad \to \quad (\varphi,\ -\bar{\varphi})$$

same as Abelian T-duality

or

$$\Phi^{l,\bar{l}}_{m,\bar{m}} \;\to\; \Phi^{l,\bar{l}}_{m,-\bar{m}}$$

- These are isomorphisms of CFT
  Latter is order/disorder duality of parafermionic CFT
- Non-locality etc are absorbed in def. of φ , ψ , $\psi^\dagger$
- Global issue : SU(2) → SU(2)/$\mathbb{Z}_k$ ≅ SU(2) (for WZNW)
- Dilaton decouples

Quantum PL T–duality

- Promoting classical trans. above,

First example of quantum equivalence under PL T–duality
to all orders in α', at any genus

Analog of mirror symmetry

- At CFT (Genper) point, strings on Calabi–Yau manifold is described by Gepner models
  $\sim$ tensor product of $\mathcal{N}=2$ S(uper)CFT

- $\mathcal{N}=2$ SCFT is realized as $\dfrac{\widehat{su}(2)_k}{\hat{u}(1)_{k+2}} \times \hat{u}(1)_2$

- Mirror symmetry is represented also by [Greene–Plesser ’90]

$$(\varphi,\ \bar{\varphi}) \quad \to \quad (\varphi,\ -\bar{\varphi})$$

Family of exact duality

- – exists RG flow connecting $\mathcal{N} = 2$ SCFT and $\widehat{su}(2)_k$

[Yang '88]

- For $k = 1$, our PL T–duality reduces to Abelian T–duality of $\varphi(z, \bar{z})$

- Abelian T–duality, mirror symmetry, our PL T–duality are of same family : chiral sign flip of free boson

$$(\varphi, \bar{\varphi}) \quad \rightarrow \quad (\varphi, -\bar{\varphi})$$

- As isomorphism of CFT describing strings duality/symmetry holds to all order in α', at any genus

* – turns out Abelian T–duality of WZNW gives same trans. as ours (w/ appropriate corrections to literature)

Summary

Summary

- – found a PL T-duality, which induces non-trivial isomorphism of SU(2) WZNW model

  → first example of quantum equivalence of dual pair

- Abelian T-duality, mirror symmetry, our PL T-duality all are chiral sign flip of $u(1)$ current (free boson) , of same "family"

Discussion

- Classical canonical property
  → similar isomorphism is expected to hold
  for higher dim. cases
- Dealing with higher dim. Drinfeld doubles is not easy
  → need new insights
- Duals of non-compact cases involve non-geometric b.g.
- More exact duality of strings than we know now
  beyond e.o.m., sugra ?
- World-sheet description of strings (CFT) is indeed
  valuable in considering quantum aspects of strings