Comments on Scaling Limits of 4d $\mathcal{N} = 2$ theories

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based on [arXiv:1011.4568]

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Introduction & Summary

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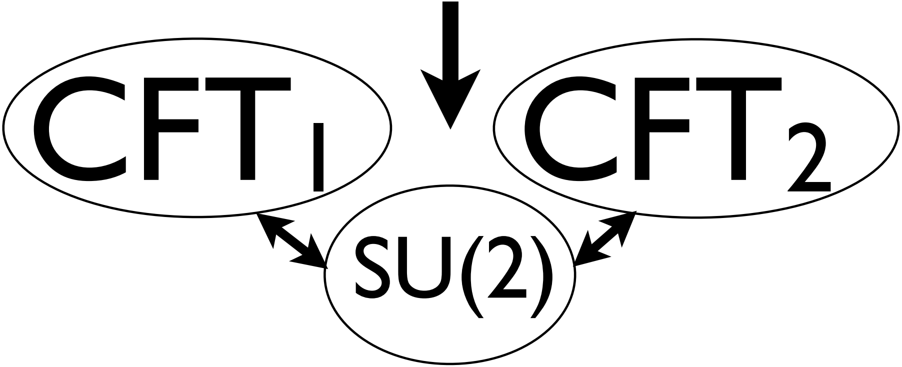
Introduction & Summary

- The day before yesterday, I mainly talked about the behind-the-scene story concerning my personal relation to the a-theorem.
- I don't have time to talk about a today; the method to calculate a of an $\mathcal{N} = 2$ SCFT is a talk in itself.
- Instead I'd like to talk in detail about the structure of the low energy limit of $\mathcal{N} = 2$ $\mathbf{SU}(N)$ with $N_f = 2n$ flavors.
- I guess it's not so bad to recall the Seiberg-Witten theory.

$\mathcal{N}$=2 SUSY SU(N) with N_f flavors



It's a strange structure, but not that strange.

- Consider $\mathbf{SU(2)}$ with a number of **doublets** and a number of **triplets**.

  Doublets, taken alone, comprise a **free** CFT with $\mathbf{SU(2)}_{\boldsymbol{F}}$

  Triplets, taken alone, comprise a **free** CFT with $\mathbf{SU(2)}_{\boldsymbol{F}}$

- So the theory is: CFT_1 and CFT_2 coupled via $\mathbf{SU(2)}$ gauge bosons

The only difference here is that both CFT_1 and CFT_2 are **non-free**.

You can call them unparticle sectors if you like.

- We habitually think of field theory as gauge groups + **matter fields** ...
- but **matter fields** might not be free.

- In this case we had a conventional UV description which was a SUSY QCD,
- but there's no guarantee there is a conventional UV description either.
- **Just studying conventional field theories might not be enough.**

- I'm very sorry I didn't have time to put figures into the slides.
- I'll use the small whiteboard there to draw them.

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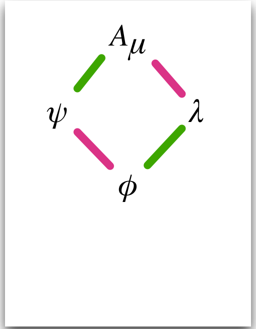

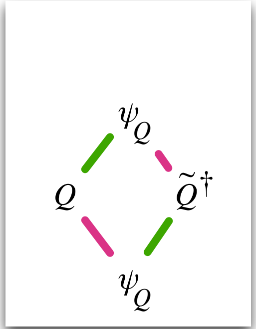

vector multiplet

adjoint of G

traceless $N \times N$ matrix

hypermultiplet

some rep of G

N-dim column vector

- $\langle Q \rangle = \langle \tilde{Q} \rangle = 0$ for simplicity.
- $V(\phi) = \mathbf{tr}[\phi, \phi^\dagger]^2$
- Classically, $\phi = \begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_N \end{pmatrix}$ → $V(\phi) = 0$

  (n.b. $\sum a_i = 0$)
- $(N-1)$-dimensional **moduli space of vacua**.

$$\phi = \begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_N \end{pmatrix}$$

Classically,

- Masses of W-bosons : $|a_i - a_j|$

$$\mathcal{L} \supset [A_\mu, \langle\phi\rangle][A^\mu, \langle\phi\rangle]$$

- $U(1)^{N-1}$ remains unbroken and massless
- Masses of the quarks : $|a_i + m|$

$$W = \tilde{Q}\langle\phi\rangle Q + m\tilde{Q}Q = \sum_{i=1}^{N}(a_i + m)\tilde{Q}^i Q_i$$

$$\phi = \begin{pmatrix} a_1 & & & & \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_N \end{pmatrix}$$

Classically,

- Masses of W-bosons : $|a_i - a_j|$
- $\langle \mathrm{tr}\, \phi^k \rangle = \sum a_i^k$, or equivalently
- $\langle \det(x - \phi) \rangle = x^N + u_2 x^{N-2} + u_3 x^{N-3} + \cdots + u_N$

$$\text{where} \qquad u_k = a_1 a_2 \cdots a_k + \text{permutations.}$$

Quantum effect modifies this relation. But how?

Quantum mechanically,

- Writing $\phi = \mathbf{diag}(a_1, \ldots, a_N)$ **doesn't make much sense** because they are gauge dependent.

- W-boson masses are **physical**. →
  Define a_i so that W-boson masses are $|a_i - a_j|$.

- Vevs of operators are **physical**. →
  Define u_k so that
  $\langle \mathbf{det}(x - \phi) \rangle = x^N + u_2 x^{N-2} + u_3 x^{N-3} + \cdots + u_N$

- $u_k = a_1 a_2 \cdots a_k$ + permutations + **quantum corrections**

What are these quantum corrections?

ANSWER: Take

$$\Sigma : y^2 = P(x)^2 - \Lambda^{2N-N_f} \prod_{k=1}^{N_f} (x + m_k)$$

where

$$P(x) = \langle \det(x - \phi) \rangle = x^N + u_2 x^{N-2} + u_3 x^{N-3} + \cdots + u_N$$

and the differential on it

$$\lambda = \frac{x}{2\pi i} d \log \frac{P(x) + y}{P(x) - y}$$

Then

$$a_i = \int_{A_i} \lambda.$$

This is **exact**.

(n.b. λ needs slight modification when $N_f = 2N$)

- **SU(2)** by [Seiberg-Witten] 1994
- **SU**(N) by [Argyres-Shapere], [Hanany-Oz] 1995
- done by combining
  correct guesses at a few singular points + holomorphy
- (Re)derived by performing the path integral by [Nekrasov] 2003.

Another interpretation [Witten] 1997, [Gaiotto] 2009:

- On a single M5 lives a **6d** theory.
  M2s ending on the M5 give **strings** on it.
- Put the theory on 2d Σ. Gives a 4d theory.
- The string tension is position-dependent, $|\lambda|$.
- A string wrapped on C has the mass

$$\int_C |\lambda| \geq \left| \int_C \lambda \right|$$

It's not just that $a_i = \int_{A_i} \lambda$.   E.g.

- Choice of C $\longrightarrow$ W-bosons, quarks, monopoles, dyons ...
- Which choice of C really gives rise to particles
- Whether that particle is a vector or a hyper or one with higher-spin ...

[Shapere-Vafa] 1999, [Gaiotto-Moore-Neitzke] 2008∼

- When an electric particle goes around a magnetic particle, it gets the phase

$$\mathbf{exp}[2\pi i q_e q'_m]$$

- If the first particle has the charge (q_e, q_m) and the second (q'_e, q'_m) then

$$\mathbf{exp}[2\pi i (q_e q'_m - q'_e q_m)]$$

- The number $q_e q'_m - q'_e q_m$: **Dirac quantization pairing**
- If the first particle comes from C_1 and the second C_2,

$$q_e q'_m - q'_e q_m = \#(C_1 \cap C_2)$$

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$\boldsymbol{\Sigma}$: ([Klemm-Lerche-Yankielowicz-Theisen], [Argyres-Faraggi], 1994)

$$\begin{aligned} y^2 &= P(x)^2 - \Lambda^{2N} \\ &= (x^N + u_2 x^{N-2} + \cdots + u_N + \Lambda^N) \times \\ &\quad (x^N + u_2 x^{N-2} + \cdots + u_N - \Lambda^N) \end{aligned}$$

Let

$$x^N + u_2 x^{N-2} + \cdots + u_N = (x - \underline{a}_1)(x - \underline{a}_2) \cdots (x - \underline{a}_N)$$

$\Lambda = 0$: Doubles zeros at $x = \underline{a}_i$ ⟶

Small Λ : Cuts between $x = \underline{a}_i^{\pm} = \underline{a}_i \pm O(\Lambda^N)$

The differential was

$$\boldsymbol{\lambda = xd\,\mathbf{log}(P+y)/(P-y)}$$

Close to $\boldsymbol{x = \underline{a}_i}$ it is

$$\boldsymbol{\lambda \sim \underline{a}_i dx/x}$$

Therefore

$$\boldsymbol{a_i = \frac{1}{2\pi i}\int_{A_i}\lambda \sim \underline{a}_i}$$

$$\boldsymbol{\frac{1}{2\pi i}\int_{B_{ij}}\lambda \sim \frac{2N}{2\pi i}(\underline{a}_i - \underline{a}_j)\,\mathbf{log}\,\Lambda}$$

W-bosons:

$$A_i - A_j : \qquad a_i - a_j$$

$\mathbf{SU(2)}$ 't Hooft-Polyakov monopoles embedded using the (i,j)-th entry

$$B_{ij} : \qquad (a_i - a_j)\frac{2N}{2\pi i}\log\Lambda.$$

Classically, the mass of the monopole is

$$\tau(a_i - a_j) \qquad \text{where} \qquad \tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$$

$$\longrightarrow \qquad \Lambda\frac{\partial}{\partial\Lambda}\tau = \frac{2N}{2\pi i}$$

This correctly reproduces one-loop running from the vector multiplet,

$$b_0 = 2N.$$

What happens when $\mathbf{\Lambda}$ is very big?

$$\boldsymbol{y^2 = (x^N + u_2 x^{N-2} + \cdots + u_N + \Lambda^N) \times}$$
$$\boldsymbol{(x^N + u_2 x^{N-2} + \cdots + u_N - \Lambda^N)}$$

Set

$$\begin{aligned} \boldsymbol{u_2} &= \boldsymbol{\epsilon^2 \hat{u}_2,} \\ &\vdots \\ \boldsymbol{u_{N-1}} &= \boldsymbol{\epsilon^{N-1} \hat{u}_{N-1},} \\ \boldsymbol{u_N} &= \boldsymbol{\Lambda^N + \epsilon^N \hat{u}_N} \end{aligned}$$

⟶ $\boldsymbol{y^2 \sim 2\Lambda^N x^N}$ + small deformation.

Electric & magnetic particles are both light.

[Argyres-Douglas] 1995, [Eguchi-Hori-Ito-Yang] 1996

$$y^2 \sim 2\Lambda^N x^N + \text{small deformation.}$$

- The differential is $\lambda \sim x dy$ when $x \sim 0$.
- Cuts are at $x \sim \epsilon$ → $y \sim \epsilon^{N/2}$ → $\int \lambda \sim \epsilon^{1+N/2}$.
- $\int \lambda$ determines the physical mass. → $[\epsilon] = \dfrac{2}{N+2}$.
- The scaling dimension is then $[\hat{u}_k] = \dfrac{2k}{N+2}$.
- Originally, $u_k \sim \mathbf{tr}\, \phi^k$ and $[u_k] \sim k$.
- Dimensions got reduced by $\dfrac{2}{N+2}$!

Contents

$$\Sigma : y^2 = P(x)^2 - \Lambda^{2N-N_f} \prod_k (x - m_k)$$

where

$$\begin{aligned} P(x) &= \langle \mathbf{det}(x - \phi) \rangle \\ &= x^N + u_2 x^{N-2} + \cdots + u_N \\ &= (x - \underline{a}_1)(x - \underline{a}_2) \cdots (x - \underline{a}_N) \end{aligned}$$

with the differential

$$\lambda = x d \log \frac{P + y}{P - y}$$

λ has poles at $x = m_i$ with residue m_i.

$$\Sigma : y^2 = P(x)^2 - \Lambda^{2N-N_f} \prod_k (x - m_k)$$

where

$$P(x) = (x - \underline{a}_1)(x - \underline{a}_2) \cdots (x - \underline{a}_N)$$

When $\Lambda \ll \underline{a}_i \ll m_k$, there are two regions of the curve

- around $x \sim m$
- around $x \sim \underline{a}$ ⟶ almost the same as the pure case with

$$\Lambda'^{2N} = \Lambda^{2N-N_f} \prod_k m_k.$$

There are cycles C_{ik} with

$$\frac{1}{2\pi i} \int_{C_{ik}} \lambda = a_i + m_k \quad : \quad \text{hypers}$$

What happens when Λ is very big? Let $N_f = 2n$, set $m = 0$.

$$\begin{aligned} y^2 &= P(x)^2 - \Lambda^{2N-2n} x^{2n} \\ &= (x^N + \cdots + u_{N-n} x^n + \cdots + u_N + \Lambda^{N-n} x^n) \times \\ &\quad (x^N + \cdots + u_{N-n} x^n + \cdots + u_N - \Lambda^{N-n} x^n) \end{aligned}$$

Set

$$u_{N-n} = \Lambda^{N-n}, \qquad u_k = 0 \quad \text{otherwise}.$$

$$\longrightarrow y^2 = (x^N + 2\Lambda^{N-n} x^n) x^N$$

- You can proceed as in $N_f = 0$...
  [Argyres-Plesser-Seiberg-Witten] 1994 [Eguchi-Hori-Ito-Yang] 1996
- But extra care is necessary when $N_f \geq 4$. ($N_f = 2$ is OK).

- So, let's study the easiest case with $N_f = 4$, i.e. $\mathbf{SU}(2)$ with $N_f = 4$.
- Some aspects can be easily generalized to $\mathbf{SU}(N)$ with $N_f = 2N$.
- Note that the one-loop beta function vanishes. Known to be vanishing even non-perturbatively. τ is exactly marginal.

$$y^2 = (x^N + u_2 x^{N-2} + \cdots + u_N)^2 - f(\tau) \prod_{k=1}^{2N} (x - g(\tau)\mu - \mu_i)$$

- $f(\tau) = 1 - g(\tau)^2$ is a certain function of τ.
- The mass of the i-th hyper is $m_i = \mu + \mu_i$; $\sum \mu_i = 0$.
- $f \sim 0$ when the theory is weakly-coupled.
- $f \sim 1$ when the theory is very, very strongly-coupled.

So, let's study what happens when $f = 1 - g^2 \sim 1$.

$$y^2 = (x^N + u_2 x^{N-2} + \cdots + u_N)^2 - f(\tau) \prod_{k=1}^{2N} (x - g(\tau)\mu - \mu_i)$$

$$= (\tilde{x}^N + Ng\mu\tilde{x}^{N-1} + \cdots + \tilde{u}_N)^2 - f(\tau) \prod_{k=1}^{2N} (\tilde{x} - \mu_i)$$

$$\sim (\frac{g^2}{2}\tilde{x}^N + Ng\mu\tilde{x}^{N-1} + \tilde{u}_2 x^{N-2} + \cdots + \tilde{u}_N)\times$$

$$(2\tilde{x}^N + Ng\mu\tilde{x}^{N-1} + \cdots + \tilde{u}_N) + \sum_{k=2}^{2N} M_k \tilde{x}^{N-k}$$

- Two zeros around $x \sim 1/g \gg 0$
- $2N - 2$ zeros around $x \sim O(1)$
- $\lambda \sim dx/x$ in the middle.

Let $\lambda \sim a dx/x$ in the middle tube.

Middle tubular region

- particles of mass $\pm 2a$ → W-boson of "magnetic" $\mathbf{SU(2)}$

Region at $x \sim 1/g$

- particles of mass $\pm a \pm N\mu$
  → a doublet hyper charged with magnetic $\mathbf{SU(2)}$ with mass $N\mu$.

Region at $x \sim O(1)$

- ???

If we **originally have SU(2) with $N_f = 4$**, it can be better understood.

Middle tubular region

- particles of mass $\pm 2a$ → W-boson of "magnetic" $\mathbf{SU(2)}$

Region at $x \sim 1/g$

- particles of mass $\pm a \pm N\mu$
  → a doublet hyper charged with magnetic $\mathbf{SU(2)}$ with mass $N\mu$.

Region at $x \sim O(1)$

- particles of mass $\pm a + \mu_i - \mu_j$
- μ_i was in the 4-dim. rep of $\mathbf{SU(4)}_F$
- $\mu_i - \mu_j$ are for the anti-sym. rep of $\mathbf{SU(4)}_F$,
  i.e. the vector of $\mathbf{SO(6)}_F$.

Originally:

$\mathbf{SU(2)}$ with four doublets,
transforming as $\mathbf{4_{+1}} \oplus \overline{\mathbf{4}}_{\mathbf{-1}}$ under $\mathbf{U(1)} \times \mathbf{SU(4)}$

Strong-coupling limit:

$\mathbf{SU(2)}$ with one doublets + three doublets,
transforming as $\mathbf{1_{+2}} \oplus \mathbf{1_{-2}} \oplus \mathbf{6_0}$ under $\mathbf{SO(2)} \times \mathbf{SO(6)}$

[Seiberg-Witten], 1994

Originally:

$\mathbf{SU}(2)$ with four doublets,
transforming as 8_V under $\mathbf{SO}(8)$

Strong-coupling limit:

$\mathbf{SU}(2)$ with four doublets
transforming as 8_S under $\mathbf{SO}(8)$

[Seiberg-Witten], 1994

$\mathbf{SU}(N)$ with $\mathbf{2}N$ flavors in the strongly-coupled limit.

Middle tubular region

- particles of mass $\pm 2a$ → W-boson of "magnetic" $\mathbf{SU}(2)$

Region at $x \sim 1/g$

- particles of mass $\pm a \pm N\mu$
  → a doublet hyper charged with magnetic $\mathbf{SU}(2)$ with mass $N\mu$.

Region at $x \sim O(1)$

- Some strange theory with $\mathbf{SU}(2) \times \mathbf{SU}(2N)$ symmetry. Call it R_N.

Originally:

$\mathbf{SU}(N)$ with $\mathbf{2}N$ doublets,
transforming as $\mathbf{2N_{+1}} \oplus \overline{\mathbf{2N}}_{\mathbf{-1}}$ under $\mathbf{U(1)} \times \mathbf{SU}(\mathbf{2}N)$

Strong-coupling limit:

$\mathbf{SU(2)}$ with one doublets of charge N under $\mathbf{U(1)}$, plus the strange theory $\boldsymbol{R_N}$ with $\mathbf{SU(2)} \times \mathbf{SU}(\mathbf{2}N)$ symmetry.

[Argyres-Seiberg] 2007	did $N = \mathbf{3}$
[Gaiotto] 2009	gave the general direction
[Distler-Chacaltana] 2010	did this particular case

- R_2 is just three doublets. Has $\mathbf{SU(2)} \times \mathbf{SO(6)} \sim \mathbf{SU(2)} \times \mathbf{SU(4)}$ symmetry.
- R_3 is the E_6 theory of [Minahan-Nemeschansky], 1996.
  Note that $E_6 \supset \mathbf{SU(2)} \times \mathbf{SU(6)}$.
- R_N for $N \geq 4$ is, well, R_N.

Originally:

$\mathbf{SU}(N)$ with $\mathbf{2}N$ doublets,
transforming as $\mathbf{2N_{+1}} \oplus \overline{\mathbf{2N}}_{\mathbf{-1}}$ under $\mathbf{U(1)} \times \mathbf{SU}(\mathbf{2}N)$

Strong-coupling limit:

$\mathbf{SU(2)}$ with one doublets of charge N under $\mathbf{U(1)}$, plus the strange theory $\boldsymbol{R_N}$ with $\mathbf{SU(2)} \times \mathbf{SU}(\mathbf{2}N)$ symmetry.

The dual is also conformal.

$$\boldsymbol{b_0} = \mathbf{4} - \mathbf{1} - \boldsymbol{R_N}\text{'s contribution} = \mathbf{0}$$

→ $\boldsymbol{R_N}$'s contribution to $\boldsymbol{b_0}$ = 3.

Let's come back to $\mathbf{SU}(N)$ with $N_f = 2n < 2N$.
(This is the new thing; everything so far was a review!)

$$\begin{aligned} y^2 &= (x^N + u_2 x^{N-2} + \cdots + u_N)^2 - \Lambda^{2N-2n} \prod_k (x - m_k) \\ &= (\tilde{x}^N + \tilde{u}_1 \tilde{x}^{N-1} + \cdots + \tilde{u}_N)^2 - \Lambda^{2N-2n} \tilde{x}^N - \sum_{k=2}^{2n} M_k x^k \\ &= (\tilde{x}^N + \cdots + \hat{u}_{N-n} x^n + \cdots + \tilde{u}_N) \times \\ &\qquad (\tilde{x}^N + \cdots + (2\Lambda^{N-n} + \hat{u}_{N-n}) x^n + \cdots + \tilde{u}_N) - \sum_{k=2}^{2n} M_k x^k \end{aligned}$$

$\lambda = y d\tilde{x}/\tilde{x}^n$ when $\tilde{x} \ll \Lambda$. Let $\tilde{y} = y/\tilde{x}^{n-1}$ so that $\lambda = \tilde{y} d\tilde{x}/\tilde{x}$.

For simplicity I'll drop all the hats and the tildes.

$$y^2 = \Big[x^{N-n+2} + \cdots + u_{N-n+1}x + u_{N-n+2} + \frac{u_{N-n+3}}{x} + \cdots + \frac{u_N}{x^{n-2}}\Big] \times \Big[x^{N-n} + \cdots + (2\Lambda^{N-n} + u_{N-n}) + \frac{u_{N-n+1}}{x} + \cdots + \frac{u_N}{x^n}\Big] - \sum_{k=2}^{2n} \frac{M_k}{x^{k-2}} \qquad \text{with} \quad \lambda = y\frac{dx}{x}.$$

We choose to scale as

$$u_1 \propto \epsilon, u_2 \propto \epsilon^2, \cdots, u_{N-n+2} \propto \epsilon^{N-n+2}$$

and

$$u_{N-n+2} \propto \delta^2, u_{N-n+3} \propto \delta^3, \cdots u_N \propto \delta^n; \qquad M_k \propto \delta^k.$$

Therefore $\epsilon^{N-n+2} = \delta^2$, and $\delta \ll \epsilon$.

Around $x \sim \delta$, the curve is

$$y^2 = \left[u_{N-n+2} + \frac{u_{N-n+3}}{x} + \cdots + \frac{u_N}{x^{n-2}}\right] \times \left[2 + \frac{u_{N-n+2}}{x^2} + \cdots + \frac{u_N}{x^n}\right] - \sum_{k=2}^{2n} \frac{M_k}{x^{k-2}} \quad \text{with} \quad \lambda = y\frac{dx}{x}.$$

Around $x \sim \delta$, the curve is

$$y^2 = \left[\check{u}_2 + \frac{\check{u}_3}{x} + \cdots + \frac{\check{u}_n}{x^{n-2}}\right] \times \left[2 + \frac{\check{u}_2}{x^2} + \cdots + \frac{\check{u}_n}{x^n}\right] - \sum_{k=2}^{2n} \frac{M_k}{x^{k-2}} \qquad \text{with} \quad \lambda = y\frac{dx}{x}.$$

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$$\left[2 + \frac{\check{u}_2}{x^2} + \cdots + \frac{\check{u}_n}{x^n}\right] - \sum_{k=2}^{2n} \frac{M_k}{x^{k-2}} \qquad \text{with} \quad \lambda = y\frac{dx}{x}.$$

Depends only on n.
Can be understood with $\mathbf{SU}(n)$ with $2n$ flavors...

Around $x \sim \delta$, the curve is

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Depends only on n.
Can be understood with $\mathbf{SU}(n)$ with $2n$ flavors...

In fact this is the R_n. $[\delta] = 1$.

Around $x \sim \epsilon$, the curve is

$$y^2 = x^{N-n+2} + \cdots + u_{N-n+1}x + u_{N-n+2} \quad \text{with} \quad \lambda = y\frac{dx}{x}.$$

Depends only $N - n$.

Around $x \sim \epsilon$, the curve is

$$y^2 = x^{N-n+2} + \cdots + u_{N-n+1}x + u_{N-n+2} \quad \text{with} \quad \lambda = y\frac{dx}{x}.$$

Depends only $N - n$.

In fact, it's just $\mathbf{SU}(N - n + 1)$ with $N_f = 2$ flavors studied by Eguchi-Hori-Ito-Yang.

(Note that $N' = N - n + 1$, $n' = 1$ and therefore $N' - n' = N - n$.)

Call it S_{N-n+1}. $\epsilon^{N-n+2} = \delta^2$. $[\epsilon] = \dfrac{2}{N-n+2}$.

And there is the tube in $\delta \ll x \ll \epsilon$.

$$\text{W-boson with mass} \quad 2a$$
$$\text{Monopole with mass} \quad \frac{1}{2\pi i} a \log \frac{\epsilon}{\delta}$$

Recall $\epsilon^{N-n+2} = \delta^2$. Then

$$\frac{1}{2\pi i} a \log \frac{\epsilon}{\delta} = \frac{1}{2\pi i} a \frac{N-n}{N-n+2} \log \delta$$

$$\longrightarrow \qquad b_0 = -\frac{N-n}{N-n+2}.$$

$$b_0 = -\frac{N-n}{N-n+2} = 4 - 3 - \frac{2(N-n+1)}{N-n+2}$$

from $\mathbf{SU}(2)$ vector	$: +4$
from R_n	$: -3$
from S_{N-n+1}	$: -\frac{2(N-n+1)}{N-n+2}$

I explained how you get -3 from R_n.

The last one was known in [Shapere-YT], 2007.

Summary

$\mathcal{N} = 2$ $\mathbf{SU}(N)$ $N_f = 2n$ flavors at a very special choice of $\langle \mathbf{tr}\, \phi^k \rangle$:

→

- $\mathbf{SU}(2)$ coupled to
- R_n: a part of the strong coupling dual of $\mathbf{SU}(2n)$ with $2n$ flavors
- S_{N-n+1} : the low energy limit of $\mathbf{SU}(N-n+1)$ with 2 flavors

Exercises:

- $\mathbf{SU}(N)$ with $2n+1$ flavors ???
- $\mathbf{SO}(N)$? $\mathbf{Sp}(N)$?
- I believe the structure is very generic