Supergravity 入門

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1 $D = 11$ Supergravity

Guess: on-shell mode number counting

$$
\begin{aligned}
e^m{}_\mu &: \quad 11^2 - {}_{11}C_2(\mathrm{LL}) - 2 \times 11(\mathrm{GC\ ghosts}) = 44 \\
\psi_\mu &: \quad (11 \times 32 + 32(B_\alpha)) \times \tfrac{1}{2} - 2 \times 32(\mathrm{Super\ ghosts}) = 128
\end{aligned} \tag{1}
$$

requires

$$
128 - 44 = 84 = 9{\cdot}8{\cdot}7/3! = {}_9C_3 \text{ bosons} \quad \rightarrow \quad \text{antisymmetric gauge field} \quad A_{\mu\nu\rho}\ ! \tag{2}
$$

Lagrangian

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{2\kappa^2} e R(e,\omega) - \frac{1}{2} e \bar{\psi}_\mu \Gamma^{\mu\rho\sigma} D_\rho\Big(\frac{\omega+\hat{\omega}}{2}\Big)\psi_\sigma - \frac{1}{2\cdot 4!} e F^2_{\mu\nu\rho\sigma} \\
&\quad - \frac{\sqrt{2}\kappa}{16\cdot 4!} e(\bar{\psi}_\mu \Gamma^{\mu\alpha\beta\gamma\delta\nu}\psi_\nu + 12\bar{\psi}^\alpha\Gamma^{\beta\gamma}\psi^\delta)(F_{\alpha\beta\gamma\delta} + \hat{F}_{\alpha\beta\gamma\delta}) \\
&\quad + \frac{\sqrt{2}\kappa}{6\cdot(4!)^2\cdot 3!} \varepsilon^{\mu_1\cdots\mu_{11}} F_{\mu_1\cdots\mu_4} F_{\mu_5\cdots\mu_8} A_{\mu_9\mu_{10}\mu_{11}}
\end{aligned} \tag{3}
$$

where

$$
\begin{aligned}
F &= dA, \qquad F \equiv \frac{1}{4!} F_{\mu\nu\rho\sigma} dx^\mu dx^\nu dx^\rho dx^\sigma, \quad A \equiv \frac{1}{3!} A_{\mu\nu\rho} dx^\mu dx^\nu dx^\rho \\
\hat{\omega}_{\mu\, mn} &= \omega_{\mu\, mn}(e) + \tfrac{1}{4}(\bar{\psi}_\mu \Gamma_m \psi_n - \bar{\psi}_\mu \Gamma_n \psi_m + \bar{\psi}_m \Gamma_\mu \psi_n) \quad \leftarrow \text{supercov'tion of } \omega(e) \\
\omega_{\mu\, mn} &= \hat{\omega}_{\mu\, mn} - \tfrac{1}{8}(\bar{\psi}^\alpha \Gamma_{\alpha\mu mn\beta}\psi^\beta) \quad \leftarrow \delta S/\delta\omega = 0 \text{ の解} \\
\hat{F}_{\mu\nu\rho\sigma} &= F_{\mu\nu\rho\sigma} + \tfrac{1}{8}\bar{\psi}_{[\mu}\Gamma_{\nu\rho}\psi_{\sigma]} \quad \leftarrow (4\times 3 \text{ 通りの反対称和}) \\
D_\mu(\omega)\psi &= (\partial_\mu - \tfrac{1}{4}\omega_\mu{}^{mn})\Gamma_{mn}\psi
\end{aligned} \tag{4}
$$

This is invariant under the following (local) SUSY transformation:

$$
\begin{aligned}
\delta e^m{}_\mu &= \frac{\kappa}{2}\bar{\varepsilon}\Gamma^m\psi_\mu, \\
\delta\psi_\mu &= \frac{1}{\kappa} D_\mu(\hat{\omega})\varepsilon + \frac{\sqrt{2}}{12\cdot 4!}(\Gamma_\mu{}^{\alpha\beta\gamma\delta} - 8\delta^\alpha_\mu \Gamma^{\beta\gamma\delta})\varepsilon\, \hat{F}_{\alpha\beta\gamma\delta}, \\
\delta A_{\mu\nu\rho} &= -\frac{\sqrt{2}}{4}\bar{\varepsilon}\Gamma_{[\mu\nu}\psi_{\rho]}, \quad \leftarrow 3 \text{ 項の反対称和}
\end{aligned} \tag{5}
$$

SUSY algebra:

$$\begin{aligned}
[\delta_Q(\varepsilon_2),\, \delta_Q(\varepsilon_1)] &= \delta_{\rm GC}(\xi^\mu) + \delta_Q(-\xi^\nu\psi_\nu) + \delta_M(\lambda^{mn}) + \delta_{\rm gauge}(\Lambda_{\mu\nu}) \\
\text{where}\ \ \xi^\mu &= \frac{1}{2}\bar\varepsilon_1\Gamma^m\varepsilon_2\, e_m{}^\mu \\
\lambda^{mn} &= \xi^\mu\hat\omega_\mu{}^{mn} + \bar\varepsilon_1(\Gamma^{mn\alpha\beta\gamma\delta} - 24e^{m\alpha}e^{n\beta}\Gamma^{\gamma\delta})\hat F_{\alpha\beta\gamma\delta} \\
\Lambda_{\mu\nu} &= -\frac{1}{2}\bar\varepsilon_1\Gamma_{\mu\nu}\varepsilon_2 - \xi^\rho A_{\rho\mu\nu} \qquad \text{with}\ \ \delta_{\rm gauge}A = d\Lambda.
\end{aligned} \tag{6}$$

2 Reduction to $D = 4$ $N = 8$ SUGRA

Simple Reduction:

$$\begin{aligned}
&\Phi(x^M{=}(x^\mu, x^i)) \quad \rightarrow \quad \Phi(x^\mu) \\
&e_M^A = \begin{pmatrix} e^\alpha{}_\mu & e^a{}_\mu \\ e^\alpha{}_i \equiv 0 & e^a{}_i \end{pmatrix} \quad \begin{array}{ll} \text{curved index:} & M = (\mu(=0,1,2,3)\ , i(=4,5,\cdots,10)), \\ \text{flat index:} & A = (\alpha(=0,1,2,3)\ , a(=4,5,\cdots,10)). \end{array}
\end{aligned} \tag{7}$$

Now each 11 dimensional index split into a 4 dimensional part and an external part, and the 32 component Majorana spinor in 11 dimensions is decomposed into 8 four-component Majorana spinors in 4 dimensions corresponding to the following tensor product decomposition of 11 γ-matrices Γ^A:

$$\left\{ \begin{array}{ll} \Gamma^\alpha & = \gamma^\alpha \otimes (\mathbf{1})_A^B, \\ \Gamma^a & = \gamma_5 \otimes (\Gamma^a_{(7)})_A^B, \end{array} \right. \qquad A, B = 1, 2, \cdots, 2^{[7/2]} = 8.$$

Thus this dimensional reduction gives rise to the following field constant in 4 dimensions:

$$\begin{aligned}
e_M^A(g_{MN}) \quad &\rightarrow \quad \left\{ \begin{array}{ll} e_\mu^\alpha(g_{\mu\nu}) & 1\ {\scriptstyle (J=2)} \\ e_\mu^a(g_{\mu i}) \sim B_\mu^i & 7 \\ e_i^a(g_{ij}) \sim \phi_{ij} & 28 \end{array} \right. \\
A_{MNP} \quad &\rightarrow \quad \left\{ \begin{array}{llll} A_{\mu\nu\rho} \sim \text{auxiliary} & 28\ {\scriptstyle (J=1)} & 35\ {\scriptstyle (J^P=0^+)} & \\ A_{\mu\nu i} \sim \phi_i & 7 & & 70\ {\scriptstyle (J=0)} \\ A_{\mu ij} \sim B_\mu^{ij} & 21 & & \\ A_{ijk} & {}_7C_3 = 35\ {\scriptstyle (J^P=0^-)} & & \end{array} \right. \\
\psi_M \quad &\rightarrow \quad \left\{ \begin{array}{ll} \psi_\mu^A & 8\ {\scriptstyle (J=3/2)} \\ \psi_i^A \sim \lambda_{ABC} \propto (\Gamma^i_{(7)})_{[AB}\psi_{iC]} & {}_8C_3 = 56\ {\scriptstyle (J=1/2)} \end{array} \right.
\end{aligned} \tag{8}$$

Now the general coordinate transformation $GC(11)$ and the local Lorentz invariance $SO(1,10)_{\rm LL}$ in 11 dimensions reduce to

$$GC(11) \quad \rightarrow \quad \left\{ \begin{array}{l} GC(4) \\ GL(7,R)_{\rm global} \\ \text{Abelian gauge } [U(1)]^7 \end{array} \right.$$

$$SO(1,10)_{\rm LL} \;\rightarrow\; \begin{cases} SO(1,3)_{\rm LL} \\ SO(7)_{\rm local}, \end{cases} \tag{9}$$

respectively, in the resultant 4-dimensional theory. Here we should note that the general coordinate transformations in the extra dimensions which survive the dimensional reduction are only those with the transformation parameter $\xi^i(x^M)$ of the form $\xi^i(x^\mu, x^i) = \Lambda^i_j x^j + \xi^i(x^\mu)$. Λ^i_j corresponds to the global (x^μ-independent) general linear transformation $GL(7,R)_{\rm global}$ and $\xi^i(x^\mu)$ to the abelian gauge transformation $[U(1)]^7$.

So $GL(7,R)_{\rm global} \times SO(7)_{\rm local}$ is a manifest internal symmetry of this 4-dimensional theory, $N=8$ supergravity. Cremmer and Julia, however, noted that these 7-dimensional symmetries are in fact easily enlarged to the 8-dimensional ones:

$$\begin{aligned} GL(7,R)_{\rm global} &\rightarrow SL(8,R)_{\rm global}, \\ SO(7)_{\rm local} &\rightarrow SO(8)_{\rm local}. \end{aligned} \tag{10}$$

The symptoms of these 8-dimensional symmetries appear in the following facts:

i) The γ-matrix generators Γ^{ab} of $SO(7)$ are combined with $\Gamma^a \quad (a = 4, \cdots, 10)$ to form $SO(8)$ generators.

ii) The vector fields appearing in the above, seven B^i_μ and twenty one B^{ij}_μ, are combined to yield **28** representation of $SL(8,R)$ (or $SO(8)$).

iii) The scalar fields, twenty eight ϕ_{ij} and seven ϕ_i, are also combined to yield **35** representation of $SL(8,R)$.

Cremmer and Julia have shown at this stage that the lagrangian for the 35 scalar($J^P = 0^+$) sector can be written in the form of nonlinear sigma model based on $SL(8,R)/SO(8)$ $[\dim(SL(8,R)) - \dim(SO(8)) = 63 - 28 = 35]$.

Proceeding further, they have found that the total scalar field sector, the 35 sector plus 35 pseudo-scalar fields, is just described by the nonlinear sigma model on the coset [1]

$$E_{7(+7)}/SU(8) \qquad (\dim(E_{7(+7)}) - \dim(SU(8)) = 133 - 63 = 70),$$

and that the symmetries of $N=8$ supergravity are in fact larger ones than (10):

$$\begin{aligned} SL(8,R)_{\rm global} &\rightarrow E_{7(+7)\ \rm global}, \\ SO(8)_{\rm local} &\rightarrow SU(8)_{\rm local}. \end{aligned} \tag{11}$$

All the fermionic fields are inert under $E_{7(+7)\ \rm global}$ and $J = 1,2$ bosonic fields are inert under $SU(8)_{\rm local}$. Since the fermionic fields $\psi_{\mu A}$ and λ_{ABC} are Majorana spinors, the $SU(8)_{\rm local}$ transformation $\Lambda'^B_A + i\Lambda''^B_A$ are understood to be $\Lambda'^B_A + i\gamma_5\Lambda''^B_A$ on them. [Since the smallest

[1] $E_{7(+7)}$ denotes a non-compact form of E_7, which has $SU(8)$ as maximum compact subgroup and $+7 = 70 - 63$ is the signature of the non-compact group (implying the number of negative metric generators minus positive ones.)

non-trivial representation of E_7 is 56 dimensional, the 28 vector fields do not fit in the E_7 representation and hence the $E_{7(+7)}$ cannot be a lagrangian symmetry of the vector field sector. However, the 28 equations of motion for their field strength and the 28 Bianchi-identities turn out to fall into **56** representation of $E_{7(+7)}$, and the $E_{7(+7)}$ is therefore a symmetry on-the-mass shell in the vector sector.]

It will be instructive to see explicitly how the **70** scalar ($J = 0$) fields appear in the $N = 8$ supergravity lagrangian. The 133 generators of $E_{7(+7)}$ group are given by 63 generators $T_A^B \quad (A, B = 1, 2, \cdots, 8)$ of its maximal subgroup $SU(8)$ plus 70 generators X_{ABCD} which is totally anti-symmetric and self-dual i.e.,

$$\bar{X}^{ABCD} = (X_{ABCD})^* = \frac{1}{4!}\epsilon^{ABCDEFGH} X_{EFGH}.$$

The infinitesimal transformation

$$\delta = i\Lambda_A^B T_B^A + \frac{1}{4!}\left(\bar{\Sigma}_{ABCD}\bar{X}^{ABCD} + \bar{\Sigma}^{ABCD} X_{ABCD}\right)$$

with parameters Λ_A^B and Σ_{ABCD},

$$\bar{\Sigma}^{ABCD} = (\Sigma_{ABCD})^* = -\frac{1}{4!}\epsilon^{ABCDEFGH}\Sigma_{EFGH}$$

is expressed as follows in the case of fundamental representation **56** whose basis vector is given by $(Z_{AB},\ \bar{Z}^{AB})$ with $Z_{AB} = -Z_{BA},\ \bar{Z}^{AB} = (Z_{AB})^*$:

$$\begin{aligned} \delta\, Z_{AB} &= \Lambda_A^C Z_{CB} + \Lambda_B^C Z_{AC} + \Sigma_{ABCD}\bar{Z}^{CD}, \\ \delta\, \bar{Z}^{AB} &= \bar{\Lambda}_C^A \bar{Z}^{CB} + \bar{\Lambda}_C^B \bar{Z}^{AC} + \bar{\Sigma}^{ABCD} Z_{CD}. \end{aligned} \tag{12}$$

The basic variable ξ in the scalar field sector is the $E_{7(+7)}$ group element

$$\xi(x) = e^{i\phi_A^B(x)T_B^A} \cdot e^{\frac{1}{4!}\left(\phi_{ABCD}(x)\bar{X}^{ABCD} + \bar{\phi}^{ABCD}(x)X_{ABCD}\right)},$$

which is a 56×56 matrix by taking the matrix representation of T_A^B and X_{ABCD} in the above fundamental representation **56**, and transforms under $g \in E_{7(+7)\mathrm{global}}$ and $h(x) \in SU(8)_{\mathrm{local}}$ as

$$\xi(x) \to \xi'(x) = h(x)\xi(x)g^\dagger.$$

Then the Maurer-Cartan 1-form becomes

$$\alpha_\mu \equiv \frac{1}{i}\partial_\mu\xi \cdot \xi^\dagger = \begin{pmatrix} 2Q_{\mu[A}^{[C}\delta_{B]}^{D]} & P_{\mu ABCD} \\ \bar{P}_\mu^{ABCD} & 2\bar{Q}_{\mu[C}^{[A}\delta_{D]}^{B]} \end{pmatrix}$$

where

$$Q_{\mu[A}^{[C}\delta_{B]}^{D]} = \frac{1}{2}(Q_{\mu A}^C\delta_B^D - Q_{\mu A}^D\delta_B^C - Q_{\mu B}^C\delta_A^D + Q_{\mu B}^D\delta_A^C).$$

The Q_μ part corresponds to $\alpha_{\mu||}$, proportional to the unbroken" generators T_A^B of $H = SU(8)$, and the P_μ part, to $\alpha_{\mu\perp}$, proportional to the broken" generators X_{ABCD} and $\bar{X}^{ABCD}$ in the terminology of section 4.1. Therefore the scalar field lagrangian invariant under $E_{7(+7)\text{global}}$ and $SU(8)_{\text{local}}$ is given by

$$\mathcal{L}_{\text{scalar}} \sim \text{tr } (\alpha_{\mu\perp})^2 \sim P_{\mu ABCD}\bar{P}^{\mu ABCD}.$$

This is exactly the lagrangian of the scalar field sector in $N = 8$ supergravity found by Cremmer and Julia. The original 70 scalar plus pseudo-scalar fields correspond to ϕ_{ABCD} and $\bar{\phi}^{ABCD}$ which survive the $SU(8)_{\text{local}}$ gauge fixing $\phi_A^B = 0$.

It is also interesting to see that the kinetic term of 8 Rarita-Schwinger fields $\psi_{\mu A}$ take the following form in the $N = 8$ supergravity lagrangian:

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu A}\gamma_\sigma\gamma_5(\delta_A^B D_\nu - Q_{\nu A}^B)\psi_{\rho B}.$$

where D_μ is local Lorentz covariant derivative. This form is understandable since $\alpha_{\mu||} \sim Q_{\mu A}^B$ transforms inhomogeneously just like a hidden local $SU(8)$ gauge field and hence $\delta_A^B D_\nu - Q_{\nu A}^B$ is an $SU(8)_{\text{local}}$-covariantized derivative on $\psi_{\mu A}$, transforming as $\underline{8}$ of $SU(8)_{\text{local}}$.

Table 1: Supergravity σ-model symmetries.

D	G	H
9	$\mathrm{GL}(2,\mathbb{R})$	$\mathrm{SO}(2)$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$
6	$\mathrm{SO}(5,5)$	$\mathrm{SO}(5) \times \mathrm{SO}(5)$
5	$\mathrm{E}_{6(+6)}$	$\mathrm{USP}(8)$
4	$\mathrm{E}_{7(+7)}$	$\mathrm{SU}(8)$
3	$\mathrm{E}_{8(+8)}$	$\mathrm{SO}(16)$

$E_{7(+7)}$ is an on-shell symmetry:
Vector $B_\mu^{ij} \in \mathbf{28}$. The representation **56** of $E_{7(+7)}$ is given by

$$\begin{pmatrix} F_{\mu\nu}^{ij} \\ G_{\mu\nu}^{ij} \end{pmatrix} \qquad \begin{array}{ll} F_{\mu\nu}^{ij}: & \text{field strength of } B_\mu^{ij} \\ \tilde{G}_{\mu\nu}^{ij} \equiv & \dfrac{4}{e}\dfrac{\delta S}{\delta F_{\mu\nu}^{ij}} \end{array} \tag{13}$$

so that

$$\text{Binachi: } \partial_\mu(e\,\tilde{F}^{\mu\nu\,ij}) = 0 \quad \leftrightarrow \text{eq. of motion: } \partial_\mu(e\,\tilde{G}^{\mu\nu\,ij}) = 0. \tag{14}$$

$\rightarrow$ $E_{7(+7)}$ is a (Cremmer-Julia) *duality symmetry.*

3 Superconformal group $SU(2,2|1)$

4D Conformal Group: $SO(4,2) \cong SU(2,2)$

$$\begin{aligned}
&[M_{\mu\nu},\, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho}),\\
&[P_\rho,\, M_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu),\\
&[K_\rho,\, M_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu),\\
&[D,\, M_{\mu\nu}] = [P_\mu,\, P_\nu] = [K_\mu,\, K_\nu] = 0,\\
&[P_\mu,\, D] = iP_\mu, \quad [K_\mu,\, D] = -iK_\mu,\\
&[P_\mu,\, K_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu}).
\end{aligned} \tag{15}$$

This 4D conformal group is in fact identical with the extended Lorentz group $SO(4,2)$ in 6 dimensions with metric

$$\eta_{ab} = \begin{pmatrix} \eta_{\mu\nu} & & \\ & -1 & \\ & & +1 \end{pmatrix}, \tag{16}$$

for which the generators $M_{ab} = -M_{ba}$ $(a, b = 0, 1, \cdots, 5)$ satisfy

$$[M_{ab},\, M_{cd}] = -i(\eta_{ac}M_{bd} - \eta_{bc}M_{ad} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac}). \tag{17}$$

and

$$M_{\mu 4} \equiv \frac{1}{2}(P_\mu - K_\mu), \quad M_{\mu 5} \equiv \frac{1}{2}(P_\mu + K_\mu), \quad M_{54} \equiv D. \tag{18}$$

By considering the (Weyl) spinor representation, this algebra is also seen to be isomorphic with $SU(2,2)$. The generators Γ^a of the Clifford algebra for $SO(4,2)$, can be represented, for instance, by the following 8×8 matrices:

$$\begin{aligned}
\Gamma^\mu = \gamma^\mu \otimes \sigma^1 &= \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix},\\
\Gamma^4 = i\gamma_5 \otimes \sigma^1 &= \begin{pmatrix} 0 & i\gamma_5 \\ i\gamma_5 & 0 \end{pmatrix},\\
\Gamma^5 = 1_4 \otimes (-\sigma^2) &= \begin{pmatrix} 0 & i1_4 \\ -i1_4 & 0 \end{pmatrix}.
\end{aligned} \tag{19}$$

The Lorentz generators M_{ab} of $SO(4,2)$ are then represented by

$$\begin{aligned}
&M_{ab} = \frac{i}{4}[\Gamma_a,\, \Gamma_b] = \frac{1}{2}\begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \bar{\sigma}_{ab} \end{pmatrix},\\
&\sigma_{a=\mu,b=\nu} = \sigma_{\mu\nu}, \quad \sigma_{\mu 4} = \gamma_\mu\gamma_5, \quad \sigma_{\mu 5} = \gamma_\mu, \quad \sigma_{54} = i\gamma_5\\
&\bar{\sigma}_{a=\mu,b=\nu} = \sigma_{\mu\nu}, \quad \bar{\sigma}_{\mu 4} = \gamma_\mu\gamma_5, \quad \bar{\sigma}_{\mu 5} = -\gamma_\mu, \quad \bar{\sigma}_{54} = -i\gamma_5.
\end{aligned} \tag{20}$$

Clearly the 4-component Weyl spinor gives an irreducible representation of the Lorentz group $SO(4,2)$, for which the Lorentz group element $\Lambda = \exp(\frac{i}{2}\varepsilon^{ab}M_{ab})$ is represented by

$$\exp(\tfrac{i}{4}\varepsilon^{ab}\sigma_{ab}). \tag{21}$$

These 4×4 matrices belong to $SU(2,2)$ since σ_{ab} are traceless and hermitian under the metric $a \equiv \gamma^0$ (which has two $+1$ and two -1 eigenvalues). Moreover, Since $6\times 5/2 = 15$ σ_{ab} exist and give a complete set for such traceless and hermitian 4×4 matrices, any $SU(2,2)$ matrix is expressed in the form Eq. (21) (at least in the neighborhood of the identity) and so we have the isomorphism of the algebra $SO(4,2) \simeq SU(2,2)$.

We have seen that the 4D conformal algebra $SO(4,2) \simeq SU(2,2)$ can be represented by traceless 4×4 matrices $M_{ab} = \frac{1}{2}\sigma_{ab}$ acting on a 4-component spinor ψ. Then it is clear that it can be extended to the superconformal algebra $SU(2,2|1)$ acting on a (4+1)-component super-spinor (ψ,φ) by adding another single component φ (which should have opposite statistics to the original component ψ). $SU(2,2|1)$ is defined to be a supergroup consisting of 5×5 matrices (of unimodular superdeterminant) which leave the innerproduct

$$\psi_1^\dagger\gamma_0\psi_2 + \varphi_1^\dagger\varphi_2 \tag{22}$$

invariant. Clearly, there are 24 independent generators as a whole, which we can take, for instance,

$$\begin{aligned} M_{ab} &= \tfrac{1}{2}\begin{pmatrix} \sigma_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \qquad & A &= -\tfrac{1}{4}\begin{pmatrix} 1_4 & 0 \\ 0 & 4 \end{pmatrix}, \\ \Sigma_\alpha &= 2\begin{pmatrix} 0_4 & 0 \\ \delta_\alpha^j & 0 \end{pmatrix}, \qquad & \bar\Sigma^\alpha &= 2\begin{pmatrix} 0_4 & \delta_i^\alpha \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{23}$$

Note that a diagonal (supertraceless) matrix A appears here. This gives the defining representation of $SU(2,2|1)$ algebra. From this we can easily find the following albebra written in 6 dimensinal notation:

$$\begin{aligned} &[\Sigma,\, M_{ab}] = \tfrac{1}{2}\sigma_{ab}\Sigma, \qquad [\bar\Sigma,\, M_{ab}] = -\tfrac{1}{2}\bar\Sigma\sigma^{ab} \\ &[\Sigma,\, A] = +\tfrac{3}{4}\Sigma, \qquad [\bar\Sigma,\, A] = -\tfrac{3}{4}\bar\Sigma, \qquad [M_{ab},\, A] = 0, \\ &\{\Sigma,\, \Sigma\} = \{\bar\Sigma,\, \bar\Sigma\} = 0, \qquad \{\Sigma,\, \bar\Sigma\} = \sigma^{ab}M_{ab} - 4A. \end{aligned} \tag{24}$$

This shows that Σ is an $SU(2,2) \simeq SO(4,2)$ spinor generator and $\bar\Sigma$ charge is its conjugate, so that they can be decomposed into two 2-component Weyl spinors in 4-dimension as follows:

$$\Sigma = \begin{pmatrix} Q_\alpha \\ \bar S^{\dot\alpha} \end{pmatrix}, \qquad \bar\Sigma = \Sigma^\dagger\gamma_0 = (S^\alpha,\ \bar Q_{\dot\alpha}). \tag{25}$$

Clearly, these $15+4+4+1=24$ matrices again span a complete set of 5×5 (supertraceless) matrices and give the whole generators of $SU(2,2|1)$ superconformal algebra. The $SU(2,2|1)$ group acts on the 5 component super spinor as

$$\exp i(\tfrac{1}{2}\theta^{ab}M_{ab}+\theta A+\bar{\varepsilon}\Sigma+\bar{\Sigma}\varepsilon)\begin{pmatrix}\psi\\ \varphi\end{pmatrix} \tag{26}$$

(where ψ is an $SU(2,2)$ spinor field and φ a single component field and they should be fermion and boson (or vice versa), respectively, since the spinor transformation parameter ε is Grassmann odd) with $\bar{\varepsilon}=\varepsilon^\dagger a$, which leaves invariant the innerproduct with metric

$$\alpha\equiv\begin{pmatrix}\gamma^0 & \\ & 1\end{pmatrix}. \tag{27}$$

Rewriting Eq. (24) into 4 dimensional notation, we find the following algebra in addition to the $SO(4,2)\simeq SU(2,2)$ subalgebra:

$$\begin{array}{l}
[\begin{pmatrix}Q\\ S\end{pmatrix},\, M_{\mu\nu}]=\tfrac{1}{2}\sigma_{\mu\nu}\begin{pmatrix}Q\\ S\end{pmatrix}, \qquad [\begin{pmatrix}Q\\ S\end{pmatrix},\, A]=\tfrac{3}{4}\gamma_5\begin{pmatrix}Q\\ -S\end{pmatrix},\\
\begin{array}{ll}[Q,\,P_\mu]=0, & [Q,\,K_\mu]=\gamma_\mu S,\\ [S,\,P_\mu]=\gamma_\mu Q, & [S,\,K_\mu]=0,\end{array}, \qquad [\begin{pmatrix}Q\\ S\end{pmatrix},\, D]=i\tfrac{1}{2}\begin{pmatrix}Q\\ -S\end{pmatrix},\\
[A,\,M_{\mu\nu}]=[A,\,P_\mu]=[A,\,K_\mu]=[A,\,D]=0,\\
\{Q,\,\bar{Q}\}=2\gamma^\mu P_\mu, \qquad \{S,\,\bar{S}\}=2\gamma^\mu K_\mu,\\
\{S,\,\bar{Q}\}=2iD+\sigma^{\mu\nu}M_{\mu\nu}+4\gamma_5 A, \quad (\{Q,\,\bar{S}\}=-2iD+\sigma^{\mu\nu}M_{\mu\nu}-4\gamma_5 A).
\end{array} \tag{28}$$

4 Yang-Mills theory for superalgebra

Consider a superalgebra whose generators (devided by i), $X_A=T_A/i$ satisfying

$$[X_A,\,X_B\}=f_{AB}{}^C X_C. \tag{29}$$

For definiteness, we here mean by X_A always a certain *matrix* representation acting a supermultiplet matter field Φ, for which the infinitesimal transformation is given by

$$\delta(\varepsilon)\Phi=\varepsilon\Phi, \qquad \varepsilon\equiv\varepsilon^A X_A, \tag{30}$$

where ε^A are the transformation parameters. Introduce the gauge field by

$$h_\mu=h_\mu^A X_A. \tag{31}$$

The covariant derivative

$$D_\mu\Phi \equiv (\partial_\mu - h_\mu)\Phi \tag{32}$$

is defined by a property

$$\delta(\varepsilon)(D_\mu\Phi) = \varepsilon^A\, D_\mu(X_A\Phi), \tag{33}$$

from which follows

$$\delta(\varepsilon)h_\mu = \partial_\mu\varepsilon + [\varepsilon, h_\mu], \quad \to \quad \delta(\varepsilon)h^A_\mu = \partial_\mu\varepsilon^A + \varepsilon^B h^C_\mu f_{BC}{}^A. \tag{34}$$

The curvature tensor (field strength) is defined by

$$\begin{aligned} R_{\mu\nu} &\equiv [D_\mu,\, D_\nu] = \partial_\nu h_\mu - \partial_\mu h_\nu - [h_\nu,\, h_\mu], \\ &\to \quad R^A_{\mu\nu} = \partial_\nu h^A_\mu - \partial_\mu h^A_\nu - h^B_\nu h^C_\mu f_{BC}{}^A. \end{aligned} \tag{35}$$

The curvature tensor is covariant as usual:

$$\delta(\varepsilon)R_{\mu\nu} = [\varepsilon, R_{\mu\nu}], \quad \to \quad \delta(\varepsilon)R^A_{\mu\nu} = \varepsilon^B R^C_{\mu\nu} f_{BC}{}^A. \tag{36}$$

$$\begin{aligned} X_A &= i^{-1}(P_m,\, Q,\, M_{mn},\, D,\, A,\, S,\, K_m) \\ &\equiv (\boldsymbol{P}_m,\, \boldsymbol{Q},\, \boldsymbol{M}_{mn},\, \boldsymbol{D},\, \boldsymbol{A},\, \boldsymbol{S},\, \boldsymbol{K}_m), \\ \varepsilon^A X_A &= \xi^m \boldsymbol{P}_m + \bar{\varepsilon}\boldsymbol{Q} + \tfrac{1}{2}\lambda^{mn}\boldsymbol{M}_{mn} + \rho\boldsymbol{D} + \theta\boldsymbol{A} + \bar{\zeta}\boldsymbol{S} + \xi^m_K \boldsymbol{K}_m, \\ h^A_\mu X_A &= e^m{}_\mu \boldsymbol{P}_m + \bar{\psi}_\mu\boldsymbol{Q} + \tfrac{1}{2}\omega_\mu{}^{mn}\boldsymbol{M}_{mn} + b_\mu\boldsymbol{D} + A_\mu\boldsymbol{A} + \bar{\varphi}_\mu\boldsymbol{S} + f^m{}_\mu\boldsymbol{K}_m. \end{aligned} \tag{37}$$

Curvatures:

$$\begin{aligned} &R_{\mu\nu}{}^m(P) = 2\partial_\nu e^m{}_\mu - 2\omega_\nu{}^{mn} e_{n\mu} + 2b_\nu e^m{}_\mu + 2i\bar{\psi}_\nu\gamma^m\psi_\mu, \\ &R_{\mu\nu}{}^{mn}(M) = 2\partial_\nu\omega_\mu{}^{mn} - 2\omega_\nu{}^{mc}\omega_{\mu c}{}^n + 4(f^m{}_\nu e^n{}_\mu - f^n{}_\nu e^m{}_\mu) + 4i\bar{\psi}_\nu\sigma^{mn}\varphi_\mu, \\ &R_{\mu\nu}(D) = 2\partial_\nu b_\mu + 4f^n{}_\nu e_{n\mu} + 4\bar{\psi}_\nu\varphi_\mu \\ &R_{\mu\nu}(A) = 2\partial_\nu A_\mu - 8i\bar{\psi}_\nu\gamma_5\varphi_\mu \\ &R_{\mu\nu}{}^m(K) = 2\partial_\nu f^m_\mu - 2\omega_\nu{}^{mn} f_{n\mu} - 2b_\nu f^m{}_\mu + 2i\bar{\varphi}_\nu\gamma^m\varphi_\mu, \\ &R_{\mu\nu}(Q) = 2D^\omega_\nu\psi_\mu + b_\nu\psi_\mu - \tfrac{3}{2}iA_\nu\gamma_5\psi_\mu - 2i\gamma_m\varphi_\nu e^m{}_\mu, \\ &R_{\mu\nu}(S) = 2D^\omega_\nu\varphi_\mu - b_\nu\varphi_\mu + \tfrac{3}{2}iA_\nu\gamma_5\varphi_\mu - 2i\gamma_m\psi_\nu f^m{}_\mu, \end{aligned} \tag{38}$$

with

$$D^\omega_\nu\psi_\mu \equiv \partial_\nu\psi_\mu + \tfrac{i}{4}\omega_\nu{}^{mn}\sigma_{mn}\psi_\mu, \tag{39}$$

(and the same for φ_μ,)

変換則:

$$\delta e^m_{\ \mu} = \partial_\mu \xi^m + \lambda^{ml} e_{l\mu} - \omega_\mu^{\ mn}\xi_n - \rho e^m_{\ \mu} + b_\mu \xi^m - 2i\bar{\varepsilon}\gamma^m \psi_\mu,$$
$$\delta\omega_\mu^{\ mn} = \partial_\mu \lambda^{mn} + 2\lambda^{ml}\omega_{\mu l}^{\ \ n} - 2(\xi^m_K e^n_{\ \mu} - \xi^n_K e^m_{\ \mu}) + 2(f^m_{\ \mu}\xi^n - f^n_{\ \mu}\xi^m)$$
$$-2i\bar{\varepsilon}\sigma^{mn}\varphi_\mu - 2i\bar{\psi}_\mu \sigma^{mn}\zeta,$$
$$\delta b_\mu = \partial_\mu \rho - 2\xi^n_K e_{n\mu} + 2f^n_{\ \mu}\xi_n - 2\bar{\varepsilon}\varphi_\mu + 2\bar{\psi}_\mu \zeta$$
$$\delta A_\mu = \partial_\mu \theta + 4i\bar{\varepsilon}\gamma_5 \varphi_\mu - 4i\bar{\psi}_\mu \gamma_5 \zeta$$
$$\delta f_{\mu\nu}^{\ \ \ m} = \partial_\mu \xi^m_K + \lambda^{\ mn} f_{n\mu} - \omega_\mu^{\ mn}\xi_{K\,n} + \rho f^m_{\ \mu} - b_\mu \xi^m_K - 2i\bar{\zeta}\gamma^m \varphi_\mu,$$
$$\delta\psi_\mu = D^\omega_\mu \varepsilon - \frac{i}{4}\lambda^{mn}\sigma_{mn}\psi_\mu - \frac{1}{2}\rho\psi_\mu + \frac{1}{2}b_\mu \varepsilon + \frac{3}{4}i\theta\gamma_5\psi_\mu - \frac{3}{4}iA_\mu \gamma_5 \varepsilon$$
$$+ie^m_{\ \mu}\gamma_m \zeta - i\xi^m \gamma_m \varphi_\mu,$$
$$\delta\varphi_\mu = D^\omega_\mu \zeta - \frac{i}{4}\lambda^{mn}\sigma_{mn}\varphi_\mu + \frac{1}{2}\rho\varphi_\mu - \frac{1}{2}b_\mu \zeta - \frac{3}{4}i\theta\varphi_\mu\gamma_5 + \frac{3}{4}iA_\mu \zeta\gamma_5$$
$$+if^m_{\ \mu}\gamma_m \varepsilon - i\xi^m_K \gamma_m \psi_\mu, \tag{40}$$

For inverse vierbein,

$$\begin{aligned} \delta e_m^{\ \mu} &= -e_n^{\ \mu} e_m^{\ \nu}(\delta e^n_{\ \nu}) \\ &= -e_n^{\ \mu}\partial_m \xi^n - e_l^{\ \mu}\lambda^l_{\ m} + \omega_m^{\ \mu l}\xi_l + \rho e_m^{\ \mu} - b_m \xi^\mu + 2i\bar{\varepsilon}\gamma^\mu \psi_m. \end{aligned} \tag{41}$$

Curvature の group 変換則 $\delta R^A_{\mu\nu}$ は、上の gauge 場の変換則 δh^A_μ で、$\partial_\mu \varepsilon^A$ を捨て、全ての h^B_μ を $R^B_{\mu\nu}$ に置き換えれば良い。

5 Deformation of the the $SU(2,2|1)$ algebra

$$\begin{aligned} \delta_{\rm GC}(\xi^\lambda)h^A_\mu &= \partial_\mu \xi^\lambda \cdot h^A_\lambda + \xi^\lambda \partial_\lambda h^A_\mu \\ &= D_\mu(\xi^\lambda \cdot h^A_\lambda) + \xi^\lambda(\partial_\lambda h^A_\mu - D_\mu h^A_\lambda) \\ &= [D_\mu(\xi\cdot h)]^A + \xi^\lambda R^A_{\mu\nu} \\ &= \delta(\xi\cdot h)\, h^A_\mu + \xi^\lambda R^A_{\mu\nu}, \end{aligned} \tag{42}$$

The last equality is because

$$\begin{aligned} \delta(\varepsilon)h^A_\mu &= (D_\mu \varepsilon)^A \\ \delta(\xi\cdot h)h^A_\mu &= \partial_\mu(\xi\cdot h^A) + (\xi\cdot h)^B h^C_\mu f_{BC}^{\ \ \ A}. \end{aligned} \tag{43}$$

Note that

$$\begin{aligned} \delta(\xi\cdot h) &= \delta_P(\xi^m) + \sum_{A'(\neq P)} \delta_{A'}(\xi\cdot h^{A'}), \\ \xi^m &= \xi^\lambda e^m_{\ \lambda}, \\ \xi\cdot h^{A'} &= \xi^\lambda h^{A'}_\lambda = \xi^m h^{A'}_m \end{aligned} \tag{44}$$

Therefore, we have *a key relation*:

$$\delta_P(\xi^m)h_\mu^A \;=\; \underbrace{\delta_{\rm GC}(\xi^\lambda)h_\mu^A - \sum_{B'}\delta_{B'}(\xi\cdot h^{B'})h_\mu^A}_{\equiv\,\delta_{\widetilde{P}}(\xi^m)} - \xi^\lambda R_{\mu\lambda}^A. \tag{45}$$

Now, we deform the $SU(2,2|1)$ algebra by making a replacement

$$\delta_P(\xi^m) \;\to\; \delta_{\widetilde{P}}(\xi^m) = \delta_{\rm GC}(\xi^\lambda) - \sum_{B'}\delta_{B'}(\xi\cdot h^{B'}). \tag{46}$$

First we note that, among the commutators $[\delta_{A'},\,\delta_{B'}]$ for $A',\ B' \neq P$, the only one yielding δ_P in the RHS is $[\delta_Q(\varepsilon_2),\,\delta_Q(\varepsilon_1)] = \delta_P(-2i\bar\varepsilon_1\gamma^m\varepsilon_2)$. So we require first that

$$[\delta_Q(\varepsilon_2),\,\delta_Q(\varepsilon_1)] = \delta_{\widetilde{P}}(\xi^m), \quad \text{with} \quad \xi^m \equiv -2i\bar\varepsilon_1\gamma^m\varepsilon_2, \tag{47}$$

holds on *any independent gauge fields*, and find constraints necessary for that.

5.1 On $e^m_{\ \mu}$

On $e^m_{\ \mu}$, we originary have

$$\begin{aligned}[\delta_Q(\varepsilon_2),\,\delta_Q(\varepsilon_1)]e^m_{\ \mu} &= \delta_P(\xi^m)e^m_{\ \mu}, \\ &= \delta_{\widetilde{P}}(\xi^m)e^m_{\ \mu} - \xi^\lambda R_{\mu\lambda}{}^m(P).\end{aligned} \tag{48}$$

So it is necessary and sufficient to impose the constraint:

$$\boxed{0 = R_{\mu\nu}{}^m(P)} = 2\partial_\nu e^m_{\ \mu} - 2\omega_\nu{}^{mn}e_{n\mu} + 2b_\nu e^m_{\ \mu} + 2i\bar\psi_\nu\gamma^m\psi_\mu \tag{49}$$

This can be solved by the M gauge field $\omega_\mu{}^{mn}$ and yields

$$\omega_\mu{}^{mn} = \omega_\mu{}^{mn}(e,\psi,b), \tag{50}$$

so that $\omega_\mu{}^{mn}$ is no longer an *independent* gauge field. However, since the constraint $R_{\mu\nu}{}^m(P) = 0$ is invariant under M_{mn}, D, A, S, K_m, $\omega_\mu{}^{mn}$ still keeps the same transformation law as the original group transformation under M_{mn}, D, A, S, K_m transformations. On the other hand, the constraint $R_{\mu\nu}{}^m(P) = 0$ is *not* invariant under Q transformation, the Q transformation of $\omega_\mu{}^{mn}$ becomes different from the original group transformation law:

$$\delta_Q(\varepsilon)\omega_\mu{}^{mn}(e.\psi,b) = \delta_Q^{\rm group}(\varepsilon)\omega_\mu{}^{mn} + \delta'_Q(\varepsilon)\omega_\mu{}^{mn}. \tag{51}$$

The difference can be easily found by noting that the constraint $R_{\mu\nu}{}^m(P) = 0$ is of course an identity and Q-invariant if $\omega_\mu{}^{mn}$ there is replaced by $\omega_\mu{}^{mn}(e, \psi, b)$, so that we have

$$\begin{aligned} 0 &= \delta_Q^{\rm group}(\varepsilon) R_{\mu\nu}{}^m(P) + \delta'_Q(\varepsilon)\omega_\mu{}^m{}_\nu - \delta'_Q(\varepsilon)\omega_\nu{}^m{}_\mu \\ &= -2i\bar\varepsilon\gamma^m R_{\mu\nu}(Q) + \delta'_Q(\varepsilon)\omega_\mu{}^m{}_\nu - \delta'_Q(\varepsilon)\omega_\nu{}^m{}_\mu. \end{aligned} \tag{52}$$

(Note that we are anticipating that $e^m_{\ \mu}, \psi_\mu, b_\mu$ will remain to be independent gauge fields and receive no changes in the Q-transformation laws.) Solving this (in a similar way to solve Christoffel symbol in terms of $g_{\mu\nu}$), we find

$$\delta'_Q(\varepsilon)\omega_{\mu\,mn} = i\bar\varepsilon(\gamma_\mu R_{mn}(Q) + \gamma_m R_{\mu n}(Q) - \gamma_n R_{\mu m}(Q)) \equiv i\bar\varepsilon\mathcal{R}_{\mu mn}(Q). \tag{53}$$

5.2 On ψ_μ

Noting

$$\delta_Q(\varepsilon)\psi_\mu = (\partial_\mu + \tfrac{i}{4}\omega_\mu{}^{mn}\sigma_{mn} + \tfrac{1}{2}b_\mu - \tfrac{3}{4}i\gamma_5 A_\mu)\varepsilon \tag{54}$$

and that $\omega_\mu{}^{mn}$ now receives an extra Q transformation $\delta'_Q(\varepsilon)$ in addition to the original group transformation $\delta_Q^{\rm group}(\varepsilon)$, we find that the $[\delta_Q,\, \delta_Q]$ commutator on ψ_μ now reads

$$\begin{aligned} [\delta_Q(\varepsilon_2),\, \delta_Q(\varepsilon_1)]\psi_\mu &= [\delta_Q^{\rm group}(\varepsilon_2),\, \delta_Q^{\rm group}(\varepsilon_1)]\psi_\mu + \tfrac{i}{4}(\delta'_Q(\varepsilon_2)\omega_\mu\cdot\sigma\varepsilon_1 - (1\leftrightarrow 2)) \\ &= \delta_{\tilde P}(\xi)\psi_\mu - \xi^m R_{\mu m}(Q) + \tfrac{i}{4}(\delta'_Q(\varepsilon_2)\omega_\mu\cdot\sigma\varepsilon_1 - (1\leftrightarrow 2)). \end{aligned} \tag{55}$$

So we see that the condition

$$\tfrac{i}{4}((i\bar\varepsilon_2\mathcal{R}_{\mu\,mn}(Q))\,\sigma^{mn}\varepsilon_1 - (1\leftrightarrow 2)) = -2i(\bar\varepsilon_1\gamma^m\varepsilon_2)R_{\mu m}(Q) \tag{56}$$

should hold. From this, after some some calculations like Fierzing, we find a constraint

$$\boxed{\gamma^\rho R_{\mu\rho}(Q) = 0.} \tag{57}$$

is the necessary and sufficient condition for the $[\delta_Q,\, \delta_Q]$ algebra Eq. (47) hold on ψ_μ. The extra Q transformation for $\omega_\mu{}^{mn}$ now takes a simple form:

$$\boxed{\delta'_Q(\varepsilon)\omega_{\mu\,mn} = 2i\bar\varepsilon\gamma_\mu R_{mn}(Q)}\,(= -2i\bar R_{mn}(Q)\gamma_\mu\varepsilon). \tag{58}$$

The constraint (57), $\gamma^\mu R_{\mu\nu}(Q) = 0$, is solved by the S-gauge field φ_μ:

$$\begin{aligned} 0 &= \gamma^\mu R_{\mu\nu}(Q) = \gamma^\mu[(\partial_\nu + \tfrac{i}{4}\omega_\nu\cdot\sigma + \tfrac{1}{2}b_\nu - \tfrac{3}{4}i\gamma_5 A_\nu)\psi_\mu - (1\leftrightarrow 2)] - i\gamma^\mu(\gamma_\mu\varphi_\nu - \gamma_\nu\varphi_\mu) \\ &\Rightarrow \quad \varphi_\mu = \varphi_\mu(e, \psi, b, A). \end{aligned} \tag{59}$$

So φ_μ now become *dependent* gauge field. Since the constraint $\gamma^\mu R_{\mu\nu}(Q) = 0$ is M_{mn}, D, A, S, K_m invariant but not invariant under Q, the Q-transformation of φ_μ is modified:

$$
\begin{aligned}
0 &= \delta_Q(\varepsilon)(\gamma^\mu R_{\mu\nu}(Q)) = \gamma^\mu \delta_Q^{\rm group}(\varepsilon) R_{\mu\nu}(Q) + (\delta_Q(\varepsilon) e_m{}^\mu)\gamma^m R_{\mu\nu}(Q) \\
&\quad + \tfrac{i}{4}\gamma^\mu [(\delta'_Q(\varepsilon)\omega_\nu{}^{mn})\sigma_{mn}\psi_\mu - (\mu \leftrightarrow \nu)] - i(4\delta^\mu_\nu - \gamma^\mu\gamma_\nu)\delta'_Q(\varepsilon)\varphi_\mu
\end{aligned} \tag{60}
$$

where

$$
\begin{aligned}
\delta_Q(\varepsilon) e_m{}^\mu &= 2i\bar\varepsilon\gamma^\mu\psi_m \\
\delta_Q^{\rm group}(\varepsilon) R_{\mu\nu}(Q) &= (\tfrac{i}{4} R_{\mu\nu}(M)\cdot\sigma + \tfrac{1}{2}R_{\mu\nu}(D) - \tfrac{3}{4} i\gamma_5 R_{\mu\nu}(A))\varepsilon
\end{aligned} \tag{61}
$$

After some calculations, we find this leads to:

$$
\begin{aligned}
\delta'_Q(\varepsilon)\varphi_\mu &= -\frac{i}{2}(\delta^\nu_\mu - \tfrac{1}{6}\gamma_\mu\gamma^\nu)\mathcal{R}_\nu\varepsilon = -\frac{i}{2}(\mathcal{R}_\mu - \tfrac{1}{6}\gamma_\mu\gamma\cdot\mathcal{R})\varepsilon \\
\mathcal{R}_\mu &\equiv \tfrac{i}{4}\gamma^\mu\sigma^{mn}\varepsilon R^{\rm cov.}_{\mu\nu\, mn}(M) + \tfrac{1}{2}\gamma^\mu\varepsilon R_{\mu\nu}(D) - \tfrac{3}{4} i\gamma^\mu\gamma_5\varepsilon R_{\mu\nu}(A).
\end{aligned} \tag{62}
$$

This quantity $(\mathcal{R}_\mu - \frac{1}{6}\gamma_\mu\gamma\cdot\mathcal{R})\varepsilon$ can be much simplified if we use the Bianchi identity.

The Bianchi identity

$$
\begin{aligned}
0 &= \varepsilon^{\mu\nu\rho\sigma}[D_\nu,\, R_{\rho\sigma}] = \varepsilon^{\mu\nu\rho\sigma}(\partial_\nu R_{\rho\sigma} - [h_\mu,\, R_{\rho\sigma}]). \\
&\quad \to \varepsilon^{\mu\nu\rho\sigma}(\partial_\nu R^A_{\rho\sigma} - h^B_\mu R^C_{\rho\sigma} f_{BC}{}^A) = 0.
\end{aligned} \tag{63}
$$

for $A = P_m$ leads to identities like

$$
\varepsilon_m{}^{abc} R^{\rm cov.}_{na\, bc}(M) = -2\widetilde{R}_{mn}(D). \tag{64}
$$

$$
R^{\rm cov.}_{\mu\nu}(M)|_{\rm antisymm.\ part} \equiv \tfrac{1}{2}(R^{\rm cov.}_{\mu\nu}(M) - R^{\rm cov.}_{\nu\mu}(M)) = -R_{\mu\nu}(D), \tag{65}
$$

where

$$
R^{\rm cov.}_{\mu\nu}(M) \equiv R^{\rm cov.\, mn}_{\mu\rho}(M) e_m{}^\rho e_{n\nu}. \tag{66}
$$

Using those, we eventually find that the extra Q transformation $\delta'_Q(\varepsilon)\varphi_\mu$ is given by

$$
\begin{aligned}
\delta'_Q(\varepsilon)\varphi_\mu &= -\frac{i}{2}(\mathcal{R}_\mu - \tfrac{1}{6}\gamma_\mu\gamma\cdot\mathcal{R})\varepsilon \\
&= -\frac{i}{2}[\gamma^m\varepsilon\,(-\tfrac{1}{12}e_{m\mu}R^{\rm cov.}(M)^\rho_\rho + \tfrac{1}{2}R^{\rm cov.}_{\mu m}(M) + \tfrac{1}{4}\widetilde{R}_{\mu m}(A)) \\
&\qquad + i\gamma^m\gamma_5\varepsilon\,\tfrac{1}{2}R_{\mu m}(A)]
\end{aligned} \tag{67}
$$

5.3 On A_μ and b_μ

Noting

$$\begin{aligned}\delta_Q(\varepsilon)A_\mu &= 4i\bar\varepsilon\gamma_5\varphi_\mu \\ \delta_Q(\varepsilon)b_\mu &= -2\bar\varepsilon\varphi_\mu,\end{aligned} \tag{68}$$

and that φ_μ now receives an extra Q transformation $\delta'_Q(\varepsilon)$ in addition to the original group transformation $\delta_Q^{\rm group}(\varepsilon)$, we find that the $[\delta_Q,\,\delta_Q]$ commutator on A_μ and b_μ now reads

$$\begin{aligned}
[\delta_Q(\varepsilon_2),\,\delta_Q(\varepsilon_1)]A_\mu &= [\delta_Q^{\rm group}(\varepsilon_2),\,\delta_Q^{\rm group}(\varepsilon_1)]A_\mu + 4i(\bar\varepsilon_1\gamma_5(\delta'_Q(\varepsilon_2)\varphi_\mu) - (1\leftrightarrow 2)) \\
&= \delta_{\widetilde P}(\xi)A_\mu - \xi^m R_{\mu m}(A) + 4i((-\tfrac{i}{2})(\bar\varepsilon_1\gamma_5 i\gamma^m\gamma_5\varepsilon_2)\,\tfrac12 R_{\mu m}(A) - (1\leftrightarrow 2)) \\
&= \delta_{\widetilde P}(\xi)A_\mu - \xi^m R_{\mu m}(A) + 4i(-\tfrac{i}{2})\xi^m\tfrac12 R_{\mu m}(A) = \delta_{\widetilde P}(\xi)A_\mu \quad \text{OK!} \\
[\delta_Q(\varepsilon_2),\,\delta_Q(\varepsilon_1)]b_\mu &= [\delta_Q^{\rm group}(\varepsilon_2),\,\delta_Q^{\rm group}(\varepsilon_1)]b_\mu - 2(\bar\varepsilon_1(\delta'_Q(\varepsilon_2)\varphi_\mu) - (1\leftrightarrow 2)) \\
&= \delta_{\widetilde P}(\xi)b_\mu - \xi^m R_{\mu m}(D) \\
&\quad -2(-\tfrac{i}{2})\times 2(\bar\varepsilon_1\gamma^m\varepsilon_2)\big(-\tfrac{1}{12}e_{m\mu}R^{\rm cov.}(M)^\rho_\rho + \tfrac12 R^{\rm cov.}_{\mu m}(M) + \tfrac14\widetilde R_{\mu m}(A)\big) \\
&= \delta_{\widetilde P}(\xi)b_\mu - \xi^m R_{\mu m}(D) \\
&\quad -\xi^m\big(-\tfrac{1}{12}e_{m\mu}R^{\rm cov.}(M)^\rho_\rho + \tfrac12 R^{\rm cov.}_{\mu m}(M) + \tfrac14\widetilde R_{\mu m}(A)\big)
\end{aligned} \tag{69}$$

Thus, the $[\delta_Q,\,\delta_Q]$ commutator on A_μ requires no constraint but that on b_μ requires a condition

$$-\tfrac{1}{12}e_{m\mu}R^{\rm cov.}(M)^\rho_\rho + \tfrac12 R^{\rm cov.}_{\mu m}(M) + \tfrac14\widetilde R_{\mu m}(A) = -R_{\mu m}(D) \tag{70}$$

which leads, by separating the symmetric and antisymmetric parts and using Eq. (65), to

$$\begin{aligned}&R^{\rm cov.}_{\mu m}(M)|_{\rm symm.\ part} = 0, \\ &-\tfrac12 R_{\mu m}(D) + \tfrac14\widetilde R_{\mu m}(A) = -R_{\mu m}(D).\end{aligned} \tag{71}$$

The latter condition is rewritten into

$$R_{\mu m}(D) = -\tfrac12\widetilde R_{\mu m}(A) \quad \text{or } \rightarrow \quad \widetilde R_{\mu m}(D) = +\tfrac12 R_{\mu m}(A). \tag{72}$$

If Eq. (65) is used, these two conditions can be rewritten into a constraint

$$\boxed{R^{\rm cov.}_{\nu\mu}(M) + \tfrac12\widetilde R_{\mu\nu}(A) = 0.} \tag{73}$$

This is the necessary and sufficient condition for the $[\delta_Q,\,\delta_Q]$ algebra Eq. (47) to hold on b_μ. Then the extra Q transformation Eq. (67) of φ_μ is simplified into

$$\boxed{\delta'_Q(\varepsilon)\varphi_\mu = -\frac{i}{4}\gamma^m(\widetilde R_{\mu m}(A) + i\gamma_5 R_{\mu m}(A))\varepsilon.} \tag{74}$$

The constraint Eq. (73) can be solved by the K_m gauge field $f^m_{\ \mu}$, which now becomes a dependent field:

$$f^m_{\ \mu} = f^m_{\ \mu}(e, \psi, b, A). \tag{75}$$

Since the constraint Eq. (73) is not Q-invariant and so $f^m_{\ \mu}$ also receives an extra Q-transformation, which can be derived in the same way as above:

$$\boxed{\delta'_Q(\varepsilon) f^m_{\ \mu} = -\frac{i}{2}\bar{\varepsilon}(\sigma^{m\nu} R^{\rm cov.}_{\mu\nu}(S) + e^{m\nu}\widetilde{R}^{\rm cov.}_{\mu\nu}(S)).} \tag{76}$$

5.4 Resultant modified $SU(2,2|1)$ algebra

Now that the M_{mn}, S and K_m gauge fields $\omega_\mu^{\ mn}$, φ_μ and $f^m_{\ \mu}$ have become dependent fields, there no longer remain other independent gauge fields. Thus the desired $[\delta_Q, \delta_Q]$ algebra (47)

$$[\delta_Q(\varepsilon_2), \delta_Q(\varepsilon_1)] = \delta_{\widetilde{P}}(\xi^m), \quad \text{with} \quad \xi^m \equiv -2i\bar{\varepsilon}_1\gamma^m\varepsilon_2 \tag{77}$$

already holds on all the independent gauge fields $e^m_{\ \mu}$, ψ_μ, A_μ and b_μ.

This implies that

Proposition: *For all the transformations other than $\widetilde{P}_m$ transformation, (which we denote by primed index X' henceforth, $X' \in \{Q, M_{mn}, D, A, S, K_m\}$), the commutators*

$$\boxed{[\delta_{Y'}(\varepsilon^{Y'}), \delta_{X'}(\varepsilon^{X'})] = \sum_C \delta_C(\varepsilon^{X'}\varepsilon^{Y'} f_{X'Y'}{}^C)} \tag{78}$$

of the same form as the original $SU(2,2|1)$ algebra, still hold. Note that when P_m appears in the C sum, it is always understood to stand for $\widetilde{P}_m$.

Proof) Almost trivial.

Proposition:

$$\boxed{[\delta_{\widetilde{P}}(\xi^m), \delta_{A'}(\varepsilon^{A'})] = \sum_{B\ \text{all}} \delta_B(\varepsilon^{A'}\xi^m f_{A'P_m}{}^B) + \delta^Q_{A'} \sum_{B'=M,S,K} \delta_{B'}(\xi^m \delta'_Q(\varepsilon^{A'}) h^{B'}_m)} \tag{79}$$

Proof) Straightforward calculation using

$$\delta_{\widetilde{P}}(\xi^m) = \delta_{\rm GC}(\xi^\lambda = \xi^m e_m^{\ \lambda}) - \sum_{B'} \delta_{B'}(\xi \cdot h^{B'}). \tag{80}$$

Proposition:

$$\boxed{[\delta_{\widetilde{P}}(\xi_1), \delta_{\widetilde{P}}(\xi_2)] = \sum_A \delta_A(\xi_1^m \xi_2^n R^A_{mn}) + \sum_{B'=M,S,K} \delta_{B'}(\delta'_Q(\xi_1\cdot\psi)\xi_2\cdot h^{B'} - \delta'_Q(\xi_2\cdot\psi)\xi_1\cdot h^{B'})} \tag{81}$$

or, equivalently,

$$\boxed{[\delta_{\widetilde{P}}(\xi_1),\, \delta_{\widetilde{P}}(\xi_2)] = \sum_A \delta_A(\xi_1^m \xi_2^n R_{mn}^{\mathrm{cov.}\,A})} \tag{82}$$

Proof) Straightforward calculation.
Also note

$$R_{mn}^{\mathrm{cov.}\,A} = R_{mn}^A - (\delta'_Q(\psi_n) h_m^A - \delta'_Q(\psi_m) h_n^A) \tag{83}$$

5.5 Final transformation rule of the gauge fields

The resultant $\boldsymbol{Q}$, $\boldsymbol{S}$, $\boldsymbol{K}_a$ and $\boldsymbol{A}$ transformation laws of the gauge fields are given as follows, from which one can read the final form of the structure functions of the local superconformal algebra. With $\delta = \delta_Q(\varepsilon) + \delta_S(\zeta) + \delta_K(\xi_K^a) + \delta_A(\theta)$,

$$\begin{aligned}
\delta e_\mu{}^a &= -2i\bar{\varepsilon}\gamma^a\psi_\mu, \\
\delta\psi_\mu &= \mathcal{D}_\mu\varepsilon + i\gamma_\mu\zeta + \tfrac{3}{4}\theta i\gamma_5\psi_\mu, \\
\delta b_\mu &= -2\bar{\varepsilon}\varphi_\mu + 2\bar{\zeta}\psi_\mu - 2\xi_{K\mu}, \\
\delta A_\mu &= 4i\bar{\varepsilon}\gamma_5\varphi_\mu - 4i\bar{\zeta}\gamma_5\psi_\mu + \partial_\mu\theta, \\
\delta\omega_\mu{}^{ab} &= 2\bar{\varepsilon}\gamma^{ab}\varphi_\mu - 2i\bar{\varepsilon}\gamma_\mu\hat{R}^{ab}(Q) + 2\bar{\zeta}\gamma^{ab}\psi_\mu - 4\xi_K^{[a}e_\mu{}^{b]}, \\
\delta\varphi_\mu &= \mathcal{D}_\mu\zeta + i\gamma_a\varepsilon f_\mu{}^a - i\xi_K^a\gamma_a\psi_\mu + \tfrac{i}{4}\gamma^a\varepsilon\,\widetilde{\hat{R}}_{\mu a}(A) - \tfrac{1}{4}\gamma^a\gamma_5\varepsilon\,\hat{R}_{\mu a}(A) - \tfrac{3}{4}\theta i\gamma_5\varphi_\mu, \\
\delta f_\mu{}^a &= \mathcal{D}_\mu\xi_K^a - 2i\bar{\zeta}\gamma^a\varphi_\mu - i\bar{\varepsilon}\gamma_\mu\hat{\mathcal{D}}_b\hat{R}^{ab}(Q),
\end{aligned} \tag{84}$$

where the covariant derivatives of transformation parameters are defined by

$$\begin{aligned}
\mathcal{D}_\mu\varepsilon &= \left(\partial_\mu - \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \tfrac{1}{2}b_\mu - \tfrac{3}{4}i\gamma_5 A_\mu\right)\varepsilon, \\
\mathcal{D}_\mu\zeta &= \left(\partial_\mu - \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \tfrac{1}{2}b_\mu + \tfrac{3}{4}i\gamma_5 A_\mu\right)\zeta, \\
\mathcal{D}_\mu\xi_K^a &= (\partial_\mu - b_\mu)\,\xi_K^a - \omega_\mu{}^{ab}\xi_{Kb}.
\end{aligned} \tag{85}$$

6 $N=1$ Superconformal Tensor Calculus

6.1 Matter multiplets

The general, or so-called vector, (complex, unconstrained) superconformal multiplet $\mathcal{V}$ corresponding to the superfield

$$\begin{aligned}
\mathcal{V}(x,\theta) &= \mathcal{C}(x) + \bar{\theta}\mathcal{Z}(x) + \bar{\theta}\theta\mathcal{H}(x) + \bar{\theta}i\gamma_5\theta\mathcal{K}(x) + \tfrac{1}{2}\bar{\theta}i\gamma^m\gamma_5\theta\mathcal{B}_m(x) \\
&\quad + (\bar{\theta}\theta)\bar{\theta}(\Lambda(x) + \tfrac{1}{2}\partial\!\!\!/\mathcal{Z}(x)) + \tfrac{1}{4}(\bar{\theta}\theta)^2(\mathcal{D}(x) + \tfrac{1}{2}\Box\mathcal{C}(x))
\end{aligned} \tag{86}$$

in the case of usual supersymmetry, is now denoted by $\mathcal{V}=[\mathcal{C},\mathcal{Z},\mathcal{H},\mathcal{K},\mathcal{B}_m,\Lambda,\mathcal{D}]$. (Real vector multiplet is denoted as $V=[C,Z,H,K,B_m,\lambda,D]$, by using the corresponding roman letters.) The basic quantum numbers of the superconformal matter multiplet are Weyl weight w and chiral weight n, which are defined through the transformation law of the first component $\mathcal{C}$:

$$[\delta_D(\rho)+\delta_A(\theta)]\,\mathcal{C}(x)=(w\rho+\tfrac{1}{2}in\theta)\,\mathcal{C}(x). \tag{87}$$

This vector multiplet $\mathcal{V}$ exists for any Weyl and chiral weights $w,\ n$ (and even $\mathcal{V}_A$ with arbitrary external Lorentz index $A=(\alpha_1,\cdots,\alpha_n;\dot\beta_1,\cdots,\dot\beta_m)$. On the contrary, the constrained type multiplets can exist only for particular values of (w, n) (and for particular external Lorentz indices A). For instance, the chiral multiplets exist only when they carry the same values of Weyl and chiral weights, $w=n$ (and only with purely undotted spinor indices $A=(\alpha_1,\cdots,\alpha_n)$).

Here we do not give the transformation laws for the vector multiplet $\mathcal{V}$, but give those for the chiral multiplet $\phi=[z,\chi,h]$ possessing no external Lorentz index, which is embedded into the vector multiplet as follows:

$$\mathcal{V}(\phi)=[\ z,\ -i\chi_{\rm R},\ -h,\ ih,\ iD^c_m z,\ 0,\ 0\ ]. \tag{88}$$

The chiral multiplet transforms under Q, S, D and A as

$$\begin{aligned}
\delta_{QSDA}\,z &\equiv (\delta_Q(\varepsilon)+\delta_S(\zeta)+\delta_D(\rho)+\delta_A(\theta))\,z=\tfrac{1}{2}\bar\varepsilon_{\rm R}\chi_{\rm R}+(w\rho+\tfrac{1}{2}iw\theta)z\\
\delta_{QSDA}\,\chi_{\rm R} &= \not{D}^c z\cdot\varepsilon_{\rm L}+h\varepsilon_{\rm R}+2wz\zeta_{\rm R}+[(w+\tfrac{1}{2})\rho+i(\tfrac{1}{2}w-\tfrac{3}{4})\theta]\chi_{\rm R}\\
\delta_{QSDA}\,h &= \tfrac{1}{2}\bar\varepsilon_{\rm L}\not{D}^c\chi_{\rm R}+(1-w)\bar\zeta_{\rm R}\chi_{\rm R}+[(w+1)\rho+i(\tfrac{1}{2}w-\tfrac{3}{2})\theta]h,
\end{aligned} \tag{89}$$

and inert under K_m, where D^c_m denotes conformal covariant derivative:

$$\begin{aligned}
D^c_m z &= (\partial_m-wb_m-\tfrac{1}{2}iwA_m)z-\tfrac{1}{2}\bar\psi_{\rm R}\chi_{\rm R}\\
D^c_m\chi_{\rm R} &= (D^\omega_m-(w+\tfrac{1}{2})b_m-i(\tfrac{1}{2}w-\tfrac{3}{4})A_m)\chi_{\rm R}\\
&\quad -(\not{D}^c z\cdot\psi_{{\rm L}m}+h\psi_{{\rm R}m})-2wz\varphi_{{\rm R}m}
\end{aligned} \tag{90}$$

with local Lorentz covariant derivative D^ω_m.

6.2 Invariant action formula

F-term formula: applicable to chiral multiplet with weight $w=n=3$, $\phi_{w=n=3}=[\,z{=}\tfrac{1}{2}(A{+}iB),\ \chi_{\rm R},\ h{=}\tfrac{1}{2}(F{+}iG)\,]$

$$\begin{aligned}
I_F &= \int d^4x\ \big[\phi_{(w=n=3)}\big]_F=\int d^4x\,e\Big[h+\tfrac{1}{2}\bar\psi_{{\rm L}m}\gamma^m\chi_{\rm R}+\bar\psi_{{\rm L}m}\sigma^{mn}z\,\psi_{{\rm L}n}+{\rm h.c.}\Big]\\
&= \int d^4x\,e\Big[F+\tfrac{1}{2}\bar\psi_m\gamma^m\chi+\tfrac{1}{2}\bar\psi_m\sigma^{mn}(A-i\gamma_5B)\psi_n\Big]
\end{aligned} \tag{91}$$

The next action formula can be derived from this. Since the chiral projection (analogue of $\bar{D}\bar{D}V$) of real vector multiplet V with Weyl weight $w = 2$ gives a chiral multiplet ΠV with weight $w = n = 3$:

$$\Pi V = [\, \tfrac{1}{2}(H - iK),\ i\not{D}^c Z_{\rm L} + \Lambda_{\rm R},\ -\tfrac{1}{2}(D + \Box^c C + iD^c_m B^m)\,] \tag{92}$$

We can apply the above F-term formula to this chiral multiplet ΠV and obtain
D-term formula: applicable to real vector multiplet $V = [\,C,\, Z,\, H,\, K,\, B_m,\, \lambda,\, D\,]$ with weight $w = 2$ $n = 0$:

$$\begin{aligned} I_D &= \int d^4x\ \left[V_{(w=2,\,n=0)}\right]_D = \int d^4x\ [-\Pi V]_F \\ &= \int d^4x\, e[D - \tfrac{1}{2}\bar{\psi}_m\gamma^m i\gamma_5\lambda - \bar{\varphi}_m\gamma^m i\gamma_5 Z + \tfrac{1}{3}C\left(R + e^{-1}\bar{\psi}_\mu R^\mu\right) \\ &\qquad + \tfrac{1}{4}i\varepsilon^{mnkl}\bar{\psi}_m\gamma_n\psi_k\left(B_l - A_l C - \tfrac{1}{2}\bar{\psi}_l Z\right)] \end{aligned} \tag{93}$$

where

$$R = R_{\mu\nu}{}^{mn}(M)e_m{}^\nu e_n{}^\mu, \qquad R^\mu = \varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu D^\omega_\rho\psi_\sigma\,. \tag{94}$$

6.3 $N = 1$ SUGRA Lagrangian

One may have wondered in the above why we consider such a superconformal framework possessing rather large local symmetry while we want supergravity which has only local Poincaré invariance. We can now answer to this question. All the possible theories of Poincaré supergravity can be obtained from our superconformal framework simply fixing the gauges for the extraneous gauge symmetries, dilatation D, chiral A, conformal supersymmetry S and special conformal K_m symmetries. Then, we need special matter multiplet(s) called *compensator*, whose component fields are used to fix those extraneous gauges. Choosing different type of multiplet as the compensator yields a different formulation of Poincaré supergravity: namely, chiral multiplet compensator leads to (old) minimal formulation, (real) linear multiplet compensator to new-minimal formulation and complex linear multiplet compensator to Breitenlohner formulation. One of the virtue of the superconformal framework is that all those different formulations of Poincaré supergravity can be dreived in a unified way from this unique framework. There is another and more important advantage in the superconformal tensor calculus actually, which we explain shortly.

We explain only the (old) minimal formulation of Poincaré supergravity. Pure (Poincaré) supergravity Lagrangian is given by

$$\mathcal{L}_{\text{pure SUGRA}} = [\Sigma\bar{\Sigma}]_D \tag{95}$$

where Σ is a chiral multiplet with weight $w = n = 1$, the compensator of the (old) minimal formulation. Denoting the components of this compensator as $\Sigma = [\mathcal{A}, \psi_{\rm R}, \mathcal{F}]$, the extraneous D, A, S, K_m gauges are fixed by the following conditions:

$$\begin{array}{llll} D: & {\rm Re}\mathcal{A} = \sqrt{3}, & A: & {\rm Im}\mathcal{A} = 0, \\ S: & \psi_{\rm R} = 0, & K_m: & b_\mu = 0, \end{array} \tag{96}$$

where the last b_μ is the Weyl (D) gauge field. Then, writing $\mathcal{F} = \frac{1}{\sqrt{3}}(S - iP)$ and $A_\mu = -\frac{2}{3}A_\mu^{\rm aux}$, $\Sigma\bar{\Sigma}$ takes the form

$$\Sigma\bar{\Sigma} = [\,3,\ 0,\ -2S,\ 2P,\ -2A_m^{\rm aux},\ 0,\ -\tfrac{1}{3}(S^2 + P^2 - A_m^{\rm aux\,2})\,] \tag{97}$$

Substituting this components expression into Eq. (95) and applying the D-term formula, we actually obtain the following action of pure supergravity:

$$\mathcal{L}_{\rm pure\ SUGRA} = e[R + e^{-1}\bar{\psi}_\mu R^\mu - \tfrac{1}{3}(S^2 + P^2 - A_m^{\rm aux\,2})]. \tag{98}$$

S, P and $A_\mu^{\rm aux}$ constitute the well-known minimal set of auxiliary fields, hence the name of minimal Poincaré supergravity.

If one considers more general matter coupled system, the Lagrangian would take the form

$$\mathcal{L} = [\Sigma\bar{\Sigma}\,\tilde{\Phi}(\phi, \bar{\phi})]_D + [\Sigma^3 W(\phi)]_F\ , \tag{99}$$

omitting the possible gauge fields. Here ϕ denotes a set of matter multiplets $\{\,\phi_i\,\}$. Now we can explain another virtue of our superconformal tensor calculus, as promised above.

First, we note that we can eliminate the superpotential term by redefining the compensator as $W^{1/3}(\phi)\Sigma \to \Sigma$, and rewrite the Lagrangian into the following form using $\Phi \equiv \tilde{\Phi}/\,|W|^{2/3}$:

$$\mathcal{L} = [\Sigma\bar{\Sigma}\,\Phi(\phi, \bar{\phi})]_D + [\Sigma^3]_F\ , \tag{100}$$

In this matter coupled system, the multiplet $\Sigma\bar{\Sigma}\,\Phi(\phi, \bar{\phi}) \equiv V$ in the D-term has the following first two components:

$$\begin{aligned} C(V) &= |\mathcal{A}|^2\ \Phi(z, z^*) \\ \tfrac{1}{2}Z(V) &= i\,|\mathcal{A}|^2\,(\Phi_i\chi_{\rm L}^i - \Phi^i\chi_{{\rm R}i}) + i\Phi\,(\mathcal{A}\psi_{\rm L} - \mathcal{A}^*\psi_{\rm R})\ , \end{aligned} \tag{101}$$

with notation $\Phi^i \equiv \partial\Phi(z, z^*)/\partial z_i$, $\Phi_i \equiv \partial\Phi(z, z^*)/\partial z^{*\,i}$. Therefore, to obtain the canonical form of Einstein-Hilbert as well as Rarita-Schwinger action $R + e^{-1}\bar{\psi}_\mu R^\mu$, it would be best to take the gauge conditions for the extraneous gauges D, A, S, K_m as[2]

$$\begin{array}{llll} D: & {\rm Re}\mathcal{A} = \sqrt{3}\Phi^{-1/2}, & A: & {\rm Im}\mathcal{A} = 0, \\ S: & \psi_{\rm R} = -\mathcal{A}\Phi^{-1}\Phi^i\chi_{{\rm R}i}, & K_m: & b_\mu = 0. \end{array} \tag{102}$$

Indeed, in this superconformal gauge, we have $C(V) = 3$ and $Z(V) = 0$, yielding the desired canonical Einstein-Hilbert and Rarita-Schwinger action $R + e^{-1}\bar{\psi}_\mu R^\mu$ from the beginning, as is seen from the D-term action formula. Note that this is really the power of superconformal tensor calculus. In the Poincaré tensor calculus, there is no freedom of choosing those gauges! From the superconformal viewpoint, the Poincaré tensor calculus is just the tensor calculus obtained from the superconformal one by choosing the Poincaré gauge fixing conditions Eq. (96). It is a good gauge conditions for pure supergravity system, but is ridiculous one for the matter coupled system. There is, however, no other way in the Poincaré tensor calculus, since there are no extraneous gauge freedom. Compare this simplification with the big calculation performed by Cremmer, Ferrara, Girardello and Van Proeyen[3] using the Poincaré tensor calculus. The first thing the latter authors had to do was 1) Weyl rescaling of the vierbein and other fields, 2) chiral rotations of the fermion fields, and 3) recombination of $\tilde{\Phi}$ and the superpotential W into the Kähler potential $\frac{1}{3}K = \ln(\tilde{\Phi}/\left|W\right|^{2/3})$. The first and second tasks are simply bypassed here by the above D and A gauge conditions and the third was the task performed in one line already in Eq. (100).

References

[1] T. Kugo and S. Uehara, *Nucl. Phys.* **B226** (1983) 49; *Prog. Theor. Phys.* **73** (1985) 235.

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Table 2: Kugo-Uehara との換算表

Kugo-Uehara	Ours
$x_m = (x_k,\ x_4)$	$(x^k,\ it) \to$ write x^μ or $-x_\mu$
$\partial_m = (\partial_k,\ \partial_4)$	$(\partial_k,\ -i\partial_t) \to$ write ∂_μ or $-\partial^\mu$
$\partial_m\partial_m \equiv \Box$	$-\partial_\mu\partial^\mu = -\Box$
δ_{mn}	$-\eta_{\mu\nu}$
$\gamma_m = (\gamma_k,\ \gamma_4)$	$(-i\gamma^k,\ \gamma^0) \to$ write $-i\gamma^\mu$ or $i\gamma_\mu$
$\gamma_m\partial_m \equiv \not\partial$	$-i\gamma^\mu\partial_\mu = -i\not\partial$
$\gamma_m B_m \equiv \not B$	$-i\gamma^\mu(-V_\mu) = +i\not V$
a_{mn}	$ia_{\mu\nu}$
γ_5	γ_5
$\sigma_{mn} \equiv (1/4)[\gamma_m,\ \gamma_n]$	$(1/4)[-i\gamma^\mu,\ -i\gamma^\nu] = (i/2)\sigma^{\mu\nu}$
$\mathcal{C},\ \mathcal{Z},\ \mathcal{H},\ \mathcal{K},\ \mathcal{B}_m,\ \Lambda,\ \mathcal{D}$	$C,\ \chi,\ N,\ -M,\ -V_\mu,\ \lambda,\ D$
$\mathcal{A},\ \mathcal{P}_{\mathrm{R}}\chi,\ \mathcal{F}$	$\varphi,\ \sqrt{2}\psi,\ \mathcal{F}$
$(\bar\psi_{\mathrm{R}}, \bar\psi_{\mathrm{L}}),\ \begin{pmatrix}\psi_{\mathrm{R}}\\ \psi_{\mathrm{L}}\end{pmatrix}$	$(\psi, \bar\psi),\ \begin{pmatrix}\psi\\ \bar\psi\end{pmatrix}$
$P_m,\ A,\ D,\ Q$	$P_\mu/i,\ A/i,\ D/i,\ (1/2)Q/i$
$M_{mn},\ K_m,\ S$	$-M_{\mu\nu}/i,\ -K_\mu/i,\ -(1/2)S/i$ (**negative signs**!)
$x_m\ \to\ x^\mu$ or $-x_\mu$	の置き換えをする space 優先のルールでは
ε_{mnrs}	$-i\varepsilon_{\mu\nu\rho\sigma}$、$-i\varepsilon^{\mu\nu\rho\sigma}$ (どちらも $-i$)
super 変換 parameter ε	2α