Note on the $O(s,t)$ γ-matrix

Taichiro KUGO*)

§1. Euclidean Case: $SO(d=2n)$

1.1. *Clifford algebra $\mathcal{C}$*

The Clifford algebra $\mathcal{C}$ is generated by γ_μ $(\mu = 1, \cdots, 2n)$:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}, \qquad \gamma_\mu^\dagger = \gamma_\mu. \tag{1·1}$$

Define creation and annihilation operators of $n = d/2$ fermions:

$$\begin{aligned} a_k^\dagger &= \frac{1}{2}(\gamma_{2k-1} + i\gamma_{2k}) \qquad & \gamma_{2k-1} &= a_k^\dagger + a_k \\ a_k &= \frac{1}{2}(\gamma_{2k-1} - i\gamma_{2k}) \qquad & \gamma_{2k} &= (a_k^\dagger - a_k)/i \end{aligned} \tag{1·2}$$

For the case of a single spece of fermion, the creation and annihilation operators a, $a^\dagger$ are represented by the Pauli matrices as follows:

$$\begin{aligned} \Big\{|1\rangle,\ |2\rangle\Big\} &\equiv \Big\{|+\rangle,\ |-\rangle\ \Big|\ a^\dagger|+\rangle = 0,\ a^\dagger|-\rangle = |+\rangle\Big\} \\ a^\dagger\Big(|+\rangle,\ |-\rangle\Big) &= \Big(|+\rangle,\ |-\rangle\Big)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \Big(|+\rangle,\ |-\rangle\Big)\frac{1}{2}(\sigma_1 + i\sigma_2) \\ a\Big(|+\rangle,\ |-\rangle\Big) &= \Big(|+\rangle,\ |-\rangle\Big)\frac{1}{2}(\sigma_1 - i\sigma_2) \end{aligned} \tag{1·3}$$

The representation space in this case is, therefore, given by:

$$\Big\{|\pm, \pm, \cdots, \pm\rangle\Big\} = \Big\{|s_1, s_2, \cdots, s_n\rangle = a_1^{\frac{1-s_1}{2}} a_2^{\frac{1-s_2}{2}} \cdots a_n^{\frac{1-s_n}{2}} |+, +, \cdots, +\rangle\Big\} \tag{1·4}$$

On this basis, γ matrices are represented as (Standard Representation)

$$\begin{aligned} \gamma_1 &= \sigma_1 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \\ \gamma_2 &= \sigma_2 \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \\ \gamma_3 &= \sigma_3 \otimes \sigma_1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \\ \gamma_4 &= \sigma_3 \otimes \sigma_2 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \\ &\vdots \\ \gamma_{2n-1} &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1 \\ \gamma_{2n} &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2 \end{aligned} \tag{1·5}$$

*) kugo@gauge.scphys.kyoto-u.ac.jp

Then $\mathit{\Gamma}_5 = \gamma_{2n+1}$ is defined by

$$\begin{aligned} \mathit{\Gamma}_5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \\ &= i^{-n}\gamma_1\gamma_2\cdots\gamma_{2n} \equiv \gamma_{2n+1}. \end{aligned} \tag{1·6}$$

1.2. *Charge conjugation matrix*

$$\begin{aligned} C^{-1}\gamma_\mu C = \eta'\gamma_\mu^{\mathrm{T}} \qquad (\eta' = \pm 1) \\ C^{\mathrm{T}} = \varepsilon' C\ , \qquad C^\dagger C = 1\ . \end{aligned} \tag{1·7}$$

In even dimension, either sign for η' can be chosen, but it is *determined* in the *odd* dimension $d = 2n+1$: indeed, the relatin $C^{-1}\gamma_\mu C = \eta'\gamma_\mu^{\mathrm{T}}$ should hold also for $\mu = 2n+1$, so

$$\begin{aligned} C^{-1}\gamma_{2n+1}C &= (\eta')^{2n} i^{-n}\gamma_1^{\mathrm{T}}\gamma_2^{\mathrm{T}}\cdots\gamma_{2n}^{\mathrm{T}} = (\eta')^{2n} i^{-n}(\gamma_{2n}\gamma_{2n-1}\cdots\gamma_1)^{\mathrm{T}} \\ &= (\eta')^{2n} i^{-n}(-1)^{n(2n-1)}(\gamma_1\gamma_2\cdots\gamma_{2n})^{\mathrm{T}} = (-1)^n\gamma_{2n+1}^{\mathrm{T}} \end{aligned} \tag{1·8}$$

so that

$$\eta' = (-1)^n = (-1)^{[\frac{d}{2}]} \qquad \text{in } d = 2n+1 \text{ dimension.} \tag{1·9}$$

Noting

$$\begin{aligned} \sigma_1\sigma_i\sigma_1 = +\sigma_i^{\mathrm{T}} \quad &\text{and} \quad \sigma_2\sigma_i\sigma_2 = -\sigma_i^{\mathrm{T}} \quad \text{for} \quad i = 1, 2 \\ \sigma_1\sigma_3\sigma_1 = -\sigma_3^{\mathrm{T}} \quad &\text{and} \quad \sigma_2\sigma_3\sigma_2 = -\sigma_3^{\mathrm{T}}\ , \end{aligned} \tag{1·10}$$

We see that C is explicitly given by in the Standard Representation:

$$\begin{cases} C = \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \cdots & \text{for} \quad \eta' = +1 \\ C = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \cdots & \text{for} \quad \eta' = -1 \end{cases} \tag{1·11}$$

Eqs. (1·10) and (1·6) imply that the last factor of C has to be σ_2 in the case of odd dimensions, and this requires again $\eta' = (-1)^n$, i.e., Eq. (1·9). Note that this explicit C in the standard repr. is clearly *unitary*. The transpose is found to be:

$$\begin{cases} C^{\mathrm{T}} = \sigma_1 \otimes -\sigma_2 \otimes \sigma_1 \otimes \cdots & \text{for} \quad \eta' = +1 \\ C^{\mathrm{T}} = -\sigma_2 \otimes \sigma_1 \otimes -\sigma_2 \otimes \cdots & \text{for} \quad \eta' = -1 \end{cases} \tag{1·12}$$

so that

$$C^{\mathrm{T}} = C \times \begin{cases} + & \text{for } n = 1 \\ - & \text{for } n = 2 \\ - & \text{for } n = 3 \\ + & \text{for } n = 4 \end{cases} \quad \text{for } \eta' = +1 \qquad C^{\mathrm{T}} = C \times \begin{cases} - & \text{for } n = 1 \\ - & \text{for } n = 2 \\ + & \text{for } n = 3 \\ + & \text{for } n = 4 \end{cases} \quad \text{for } \eta' = -1 \tag{1·13}$$

Thus the sign ε' of $C^{\mathrm{T}} = \varepsilon' C$ is given in dimension $d = 2n$ and $2n+1$ by

$$\varepsilon' = \cos\frac{\pi}{2}n + \eta' \sin\frac{\pi}{2}n \ . \tag{1·14}$$

The symmetry property of rank r gamma tensor $\gamma_{\mu_1\mu_2\cdots\mu_r}C$ can be seen as follows:

$$\begin{aligned}\gamma_{\mu_1\mu_2\cdots\mu_r}C &= \gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_r}C = (\eta')^r C\gamma^{\mathrm{T}}_{\mu_1}\gamma^{\mathrm{T}}_{\mu_2}\cdots\gamma^{\mathrm{T}}_{\mu_r} \\ &= (\eta')^r\varepsilon' C^{\mathrm{T}}\gamma^{\mathrm{T}}_{\mu_1}\gamma^{\mathrm{T}}_{\mu_2}\cdots\gamma^{\mathrm{T}}_{\mu_r} = (\eta')^r\varepsilon'\left(\gamma_{\mu_r}\cdots\gamma_{\mu_2}\gamma_{\mu_1}C\right)^{\mathrm{T}} \\ &= (\eta')^r\varepsilon'(-)^{[\frac{r}{2}]}\left(\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_r}C\right)^{\mathrm{T}} = (\eta')^r\varepsilon'(-)^{[\frac{r}{2}]}\left(\gamma_{\mu_1\mu_2\cdots\mu_r}C\right)^{\mathrm{T}}\end{aligned} \tag{1·15}$$

Table I. Possible signs for η' and ε': $C^{-1}\gamma_\mu C = \eta'\gamma^{\mathrm{T}}_\mu$, $C^{\mathrm{T}} = \varepsilon' C$

dimension (mod 8)	0		1	2		3	4		5	6		7
η'	+	−	+	+	−	−	+	−	+	+	−	−
ε'	+	+	+	+	−	−	−	−	−	−	+	+

Table II. The rank r of $\gamma_{\mu_1\mu_2\cdots\mu_r}$ for which $\gamma_{\mu_1\mu_2\cdots\mu_r}C$ are symmetric and anti-symmetric matrices.

dimension d	η'	ε'	r of Symmetric $\gamma_{\mu_1\mu_2\cdots\mu_r}C$	r of Anti-symmetric $\gamma_{\mu_1\mu_2\cdots\mu_r}C$
1	+	+	0	
2	+	+	0, 1	2
	−	−	1, 2	0
3	−	−	1	0
4	+	−	2, 3	0, 1, 4
	−	−	1, 2	0, 3, 4
5	+	−	2	0, 1
6	+	−	2, 3, 6	0, 1, 4, 5
	−	+	0, 3, 4	1, 2, 5, 6
7	−	+	0, 3	1, 2
8	+	+	0, 1, 4, 5, 8	2, 3, 6, 7
	−	+	0, 3, 4, 7, 8	1, 2, 5, 6
9	+	+	0, 1, 4	2, 3
10	+	+	0, 1, 4, 5, 8, 9	2, 3, 6, 7, 10
	−	−	1, 2, 5, 6, 9, 10	0, 3, 4, 7, 8
11	−	−	1, 2, 5	0, 3, 4
12	+	−	2, 3, 6, 7, 10, 11	0, 1, 4, 5, 8, 9, 12
	−	−	1, 2, 5, 6, 9, 10	0, 3, 4, 7, 8, 11, 12

§2. Clifford 代数の表現およびη', ε'の一意性

任意の表現のγ行列を持ってきたとき、それから fermion 演算子 a_k, $a_k^\dagger$ を 式 (1·2) のように作れば、そのすべての生成演算子で消える状態 $|+,+,\cdots,+\rangle$ を必ず作れる。これがいくつもあれば、直交化して独立にしておく。そのそれぞれの上で、式 (1·4) の部分空間を作れて、その base に関しては、元のγ行列は、標準表示の (1·5) で表現される。よって、既約表現では、状態 $|+,+,\cdots,+\rangle$ は一意的である。この時、base (1·4) は正規直交系であるから、あるユニタリ行列 U が存在して、

$$\gamma_\mu^{\rm std} = U^{-1}\gamma_\mu U \tag{2·1}$$

と書ける。

この式より、また、η', ε'の一意性が言える。実際、一般の表示のγ行列に対する、charge conjugation matrix C と、標準表示のそれ $C^{\rm std}$との関係は、それぞれの定義を比較して

$$C = UC^{\rm std}U^{\rm T} \tag{2·2}$$

となり、η', ε'は、両者で共通である事がわかる。また、$C^{\rm std}$ のユニタリ性から C のユニタリ性もでる。

From the transformation property $C = UC^{\rm std}U^{\rm T}$ under the change of the basis, we note that C *can always be taken to be 1 whenever* $\varepsilon' = 1$*, i.e.,* $C^{\rm T} = C$. Indeed, for such cases, $C^{\rm std}$ is symmetric and cntains even number of σ_2 factors. So it is real symmetric matrix and can be be diagonalized by an orthogonal matrix O. But the eigenvalues are 1 and -1:

$$OC^{\rm std}O^{\rm T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = J\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}J$$
$$J \equiv \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = J^{\rm T}\ . \tag{2·3}$$

Therefore $(JO)C^{\rm std}(JO)^{\rm T}$ becomes a unit matrix 1.

The explicit construction of such γ matrx representation for which C becomes unit matrix is as follows: in 9 dimensions, γ matrices have to be symmetric by themselves if $C = 1$.

$$\begin{aligned}
\gamma_1 &= \sigma_3 \otimes 1 \otimes 1 \otimes 1 \\
\gamma_2 &= \sigma_1 \otimes 1 \otimes 1 \otimes 1 \\
\gamma_3 &= \sigma_2 \otimes \sigma_3 \otimes 1 \otimes \sigma_2 \\
\gamma_4 &= \sigma_2 \otimes \sigma_1 \otimes 1 \otimes \sigma_2 \\
\gamma_5 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \otimes 1 \\
\gamma_6 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \otimes 1
\end{aligned}$$

$$
\begin{aligned}
\gamma_7 &= \sigma_2 \otimes \ 1 \ \otimes \sigma_2 \otimes \sigma_3 \\
\gamma_8 &= \sigma_2 \otimes \ 1 \ \otimes \sigma_2 \otimes \sigma_1 \\
\gamma_9 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\end{aligned}
\tag{2·4}
$$

We can go down to 7 dimension, by throwing away the first two gamma matrices $\gamma_1,\ \gamma_2$ and simultaneously the first column of the tensor product. Then the resultant seven γ matrices become antisymmetric by losing their first σ_2 factors, being in accord with $\eta' = -1$ in seven dimension. Clearly these can be repeated for $8n+1$ dimensions by using these $\gamma_1 - \gamma_8$ blocks and $\otimes^4\sigma_2$ and $\otimes^4 1$ as a building blocks of tensor product.

§3. (General) Minkowski Case: $SO(t,s)$

3.1. *Clifford algebra $\mathcal{C}$*

The Clifford algebra $\mathcal{C}$ in this $SO(t,s)$ case is:

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\eta_{\mu\nu},\qquad \eta_{\mu\nu}=\mathrm{diag}(\overbrace{+,\cdots,+}^{t},\overbrace{-,\cdots,-}^{s})$$
$$\gamma_\mu^\dagger=\begin{cases}\gamma_\mu & \text{for } \mu=1,\cdots,t\\ -\gamma_\mu & \text{for } \mu=t+1,\cdots,t+s\end{cases}. \tag{3·1}$$

The standard representation in this case is simply given by

$$\gamma_\mu\equiv\begin{cases}\gamma_\mu^{\mathrm{E}} & \text{for } \mu=1,\cdots,t\\ i^{-1}\gamma_\mu^{\mathrm{E}} & \text{for } \mu=t+1,\cdots,t+s\end{cases}. \tag{3·2}$$

by putting i^{-1} to the space components from the previous Euclidean one γ_μ^{E}. Since the change from Euclidean to Minkowskian cases is only the multiplicatin of the *non-matrix* factor i, the *same* charge conjugation matrix C as before satisfies

$$C^{-1}\gamma_\mu C=\eta'\gamma_\mu^{\mathrm{T}}\qquad(\eta'=\pm1). \tag{3·3}$$

So the signs of η' and ε' in Table I and the symmetry properties of gamma matrices in Table II are also valid in the general Minkowskian cases.

3.2. *B-Conjugation*

With a matrix

$$\varGamma_0\equiv\gamma_1\gamma_2\cdots\gamma_t,\qquad \varGamma_0\varGamma_0^\dagger=1. \tag{3·4}$$

we define the Dirac conjugate field $\bar\psi$ by

$$\bar\psi=\psi^\dagger\varGamma_0^{-1} \tag{3·5}$$

and Dirac conjugation by

$$(\bar\psi\gamma_\mu\chi)^\dagger\equiv\bar\chi\bar\gamma_\mu\psi\quad\Rightarrow\quad\bar\gamma_\mu=\varGamma_0\gamma_\mu^\dagger\varGamma_0 \tag{3·6}$$

for which we have

$$\varGamma_0\gamma_\mu^\dagger\varGamma_0=(-1)^{[\frac{t}{2}]+t+1}\gamma_\mu. \tag{3·7}$$

For the existence of Majorana(-Weyl) spinor, however, more important than the charge conjugation matrix C is the following matrix B:

$$B^{-1}\gamma_\mu B=\eta\gamma_\mu^*\qquad(\eta=\pm1),$$
$$B^{\mathrm{T}}=\varepsilon B,\qquad B^\dagger B=1. \tag{3·8}$$

Indeed, we define the charge conjugation by

$$\psi^c = C\bar{\psi}^{\mathrm{T}} \ (= C\Gamma_0^* \psi^*) \tag{3·9}$$

and also write it into the form

$$\psi^c = B\psi^* \ . \tag{3·10}$$

Then, comparing the two expressions, we find the relation between B and C as

$$B = C\Gamma_0^* \ . \tag{3·11}$$

Indeed, then, using the unitarity of C and Γ_0, we have the properties Eq. (3·8) of B:

$$\begin{aligned} B^\dagger B &= \Gamma_0^{\mathrm{T}} C^\dagger C \Gamma_0^* = (\Gamma_0^\dagger \Gamma_0)^* = 1, \\ B^{-1}\gamma_\mu B &= \Gamma_0^{\mathrm{T}} C^{-1} \gamma_\mu C \Gamma_0^* = \eta' \Gamma_0^{\mathrm{T}} \gamma_\mu^{\mathrm{T}} \Gamma_0^* \\ &= \eta' (\Gamma_0^\dagger \gamma_\mu^\dagger \Gamma_0)^* = \eta' (-1)^{[\frac{t}{2}]} (\Gamma_0 \gamma_\mu^\dagger \Gamma_0)^* = \eta' (-1)^{t+1} \gamma_\mu^* \ , \\ B^{\mathrm{T}} &= \Gamma_0^\dagger C^{\mathrm{T}} = \varepsilon' \Gamma_0^\dagger C = \varepsilon' \gamma_t \cdots \gamma_1 C \\ &= \varepsilon' (\eta')^t C \gamma_t^{\mathrm{T}} \cdots \gamma_1^{\mathrm{T}} = \varepsilon' (\eta')^t C \gamma_t^* \cdots \gamma_1^* = \varepsilon' (\eta')^t C (\gamma_t \cdots \gamma_1)^* \\ &= \varepsilon' (\eta')^t (-1)^{[\frac{t}{2}]} C \Gamma_0^* = \varepsilon' (\eta')^t (-1)^{[\frac{t}{2}]} B \end{aligned} \tag{3·12}$$

so that we find

$$\eta = \eta'(-1)^{t+1}, \qquad \varepsilon = \varepsilon'(\eta')^t (-1)^{[\frac{t}{2}]} = \varepsilon'(\eta)^t (-1)^{[\frac{t}{2}]} \ . \tag{3·13}$$

ε is a mod 4 function of t. Examining all cases by using the expression Eq. (1·14), we find

$$\varepsilon = \cos \frac{\pi}{2} \frac{s-t}{2} - \eta \sin \frac{\pi}{2} \frac{s-t}{2} \ . \tag{3·14}$$

As the previous Eq. (1·14) does, this applies only to even dimensions $d = s + t = 2n$ for which $(s-t)/2$ is an integer. This is again a mod 4 function of $(s-t)/2$.

We thus obtain the results for the signs η and ε as follows:

$$\begin{aligned} &\varepsilon = +1, \ \eta = -1 : \quad s - t = 1, \ 2, \ 8, \ \mathrm{mod}\, 8 \\ &\varepsilon = +1, \ \eta = +1 : \quad s - t = 6, \ 7, \ 8, \ \mathrm{mod}\, 8 \\ &\varepsilon = -1, \ \eta = -1 : \quad s - t = 4, \ 5, \ 6, \ \mathrm{mod}\, 8 \\ &\varepsilon = -1, \ \eta = +1 : \quad s - t = 2, \ 3, \ 4, \ \mathrm{mod}\, 8 \end{aligned} \tag{3·15}$$

This is summarized more explicitly in Table III.

Table III. Possible signs for η and ε: $B^{-1}\gamma_\mu B = \eta\gamma_\mu^*$, $B^{\mathrm{T}} = \varepsilon B$, together with the signs for η' and ε': $C^{-1}\gamma_\mu C = \eta'\gamma_\mu^{\mathrm{T}}$, $C^{\mathrm{T}} = \varepsilon' C$.

dim d (mod 8)→		0		1	2		3	4		5	6		7
η'		+	−	+	+	−	−	+	−	+	+	−	−
ε'		+	+	+	+	−	−	−	−	−	−	+	+
$t=0$	η	−	+	−	−	+	+	−	+	−	−	+	+
	ε	+	+	+	+	−	−	−	−	−	−	+	+
$t=1$	η	+	−	+	+	−	−	+	−	+	+	−	−
	ε	+	−	+	+	+	+	−	+	−	−	−	−
$t=2$	η	−	+	−	−	+	+	−	+	−	−	+	+
	ε	−	−	−	−	+	+	+	+	+	+	−	−
$t=3$	η	+	−	+	+	−	−	+	−	+	+	−	−
	ε	−	+	−	−	−	−	+	−	+	+	+	+

Before closing this subsection, we note that the charge conjugation ψ^c can also be written in the form:

$$\bar{\psi^c} = \psi^{\mathrm{T}}\Gamma_0^{\mathrm{T}}C^\dagger \cdot \Gamma_0^{-1} = (\eta')^t(-)^{[\frac{t}{2}]}C^{-1}\Gamma_0 \cdot \Gamma_0^{-1} \ . \tag{3·16}$$

Namely,

$$\bar{\psi^c} = \tau\psi^{\mathrm{T}}C^{-1} \ , \qquad \tau \equiv (\eta')^t(-)^{[\frac{t}{2}]} = \eta^t(-)^{[\frac{t}{2}]} \ . \tag{3·17}$$

We are now in a position to discuss separately when Weyl, Majorana and Majorana-Weyl spinors can exist.

3.3. *Weyl spinor*

When the dimension $d = s + t$ is even, we can define Γ_5 in this general Minkowskian case as

$$\Gamma_5 = i^{-(d/2)}i^s\gamma_1\gamma_2\cdots\gamma_d = i^{(s-t)/2}\gamma_1\gamma_2\cdots\gamma_d \ ,$$
$$\Gamma_5^2 = 1 \qquad (\Gamma_5^\dagger = \Gamma_5) \tag{3·18}$$

This Γ_5 has the B conjugation property:

$$B^{-1}\Gamma_5 B = (-)^{(s-t)/2}\Gamma_5^* \tag{3·19}$$

(For odd dimension $d+1$, η, and hence ε also, is fixed: the anti-hermitian γ_{d+1} should be $\gamma_{d+1} = i^{-1}\Gamma_5$, for which Eq. (3·4) holds with $\eta = -(-)^{(s-t)/2}$.)

Whenever d is even, we have Weyl spinors: we can construct the chiral projection operator

$$\mathcal{P}_{\pm} = \frac{1}{2}(1 \pm \Gamma_5) \ , \tag{3·20}$$

and $\mathcal{P}_{\pm}\psi \equiv \psi_{\pm}$ gives the Weyl spinors.

3.4. *Majorana and Pseudo-Majorana*

The Dirac equation is

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \tag{3·21}$$

and its hermitian conjugation and multiplication of $(-\eta)B$ gives

$$\Big(i\gamma^{\mu}\partial_{\mu} - (-\eta)m\Big)B\psi^{*} = 0 \tag{3·22}$$

So $B\psi^*$ satisfies the same equation as ψ if $(-\eta)m = m$. If in addition $\varepsilon = +1$, we can equate $B\psi^*$ with ψ:

$$\text{Majorana:} \qquad B\psi^* = \psi \ , \qquad (\text{if} \quad \varepsilon = +1\ ) \tag{3·23}$$

This is possible only when $\varepsilon = +1$ because Eq. (3·23) implies $BB^* = 1$. If $\varepsilon = -1$ and we have two spinors ψ_i $(i = 1, 2)$, then we can impose instead an "SU(2) reality" condition

$$\text{SU(2) Majorana:} \qquad \varepsilon_{ij}B\psi^{*j} = \psi_i \qquad (\text{if} \quad \varepsilon = -1\ ) \tag{3·24}$$

where ε_{ij} is the SU(2) invariant anti-symmetric tensor. If we have $2N$ spinors, we can instead impose USp(2N) reality condition $\Omega_{ij}B\psi^{*j} = \psi_i$ by replacing the SU(2) metric ε_{ij} by USp(2N) invariant (real anti-symmetric) metric Ω_{ij}. In both cases of Majorana and USp(2N) Majorana, the condition $(-\eta)m = m$ means that if $\eta = +1$ we must have $m = 0$. So we put the term *pseudo* for $\eta = +1$.

Thus we can have the following four types of Majorana spinors for the combination of the signs of η and ε:

$$\begin{array}{rl} \text{Majorana} & \text{for } \ \varepsilon = +1,\ \eta = -1 \ \leftrightarrow \ s - t = 1,\ 2,\ 8,\ \text{mod}\, 8 \\ \text{pseudo-Majorana} & \text{for } \ \varepsilon = +1,\ \eta = +1 \ \leftrightarrow \ s - t = 6,\ 7,\ 8,\ \text{mod}\, 8 \\ \text{USp(2N) Majorana} & \text{for } \ \varepsilon = -1,\ \eta = -1 \ \leftrightarrow \ s - t = 4,\ 5,\ 6,\ \text{mod}\, 8 \\ \text{USp(2N) pseudo-Majorana} & \text{for } \ \varepsilon = -1,\ \eta = +1 \ \leftrightarrow \ s - t = 2,\ 3,\ 4,\ \text{mod}\, 8 \end{array} \tag{3·25}$$

3.5. *Majorana-Weyl Spinors*

The Weyl spinors $\psi_{\pm} \equiv \mathcal{P}_{\pm}\psi$ satisfying

$$\Gamma_5\psi_{\pm} = \pm\psi_{\pm} \tag{3·26}$$

always exist for even dimension d. But this is compatible with the [SU(2)] Majorana condition, Eq. (3·23) or Eq. (3·24), only if

$$\sigma \equiv (-1)^{(s-t)/2} = 1 \quad \Longrightarrow \quad s-t = 0 \bmod 4 \tag{3·27}$$

This is because we have from the B conjugation property Eq. (3·19) of Γ_5

$$B^{-1}\mathcal{P}_{\pm}B = \mathcal{P}^*_{\pm\sigma} \tag{3·28}$$

with $\sigma = (-1)^{(s-t)/2}$. So, applying the chiral projection $\mathcal{P}_{\pm}$ to the Majorana spinor condition Eq. (3·23), for instance, we would get

$$\begin{aligned} \psi = B\psi^* \quad \longrightarrow \quad & \mathcal{P}_{\pm}\psi = \mathcal{P}_{\pm}B\psi^* \\ & \quad \psi_{\pm} = B\mathcal{P}^*_{\pm\sigma}\psi^* = B(\psi_{\pm\sigma})^* \end{aligned} \tag{3·29}$$

We, therefore, see that such a reality condition on Weyl spinors can be a closed condition only if $\sigma = +1$.

Noting that $\varepsilon = -1$ for $s-t = 4 \bmod 8$, and $\varepsilon = +1$ for $s-t = 8 \bmod 8$, we thus find that we can have

$$\begin{aligned} \text{(pseudo-) Majorana-Weyl} \quad &\text{for} \quad s-t = 8 \bmod 8 \\ \text{USp(2N) (pseudo-) Majorana-Weyl} \quad &\text{for} \quad s-t = 4 \bmod 8 \end{aligned} \tag{3·30}$$