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Gauge Theoretical Construction of Non-compact Calabi-Yau Manifolds

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1. Introduction

$$\left. \begin{array}{l} \text{Superstring} = 10 \\ \text{Standard Model} = 4 \end{array} \right\} \Rightarrow \text{時空のコンパクト化?}$$

まだ誰もその証拠を見付けていない

↓

$$\mathcal{M}_{10} \longrightarrow \mathcal{M}_4 \times K_6 \text{ と (とりあえず) 仮定}$$

\mathcal{M}_4 : flat Minkowski 時空 ($D = 4$)

K_6 : (compact) six-dimensional space

Supergravity 極限 ($\alpha' \rightarrow 0$ 極限) で

\mathcal{M}_4 に $\mathcal{N} = 1$ supersymmetry が残ることを要請

↓

$$K_6 = \{SU(3) \text{ holonomy を持つ}\}$$

$$= \text{Ricci-flat Kähler} \simeq \text{Calabi-Yau 多様体}$$

このときの String world-sheet theory

$$\implies \underline{D = 2, \mathcal{N} = 2 \text{ Supersymmetric NLSM}}$$

Non-compact で Ricci-flat な Kähler 多様体

Non-compact Calabi-Yau 多様体を構成

SNLSM as Gauge Theories (compact Kähler) を応用



Complex Line Bundle の出現, 特に

Hermitian Symmetric Spaces

+

complex line

特異点が回避されている多様体とみなされる

回避の方法は

deformation とともに small resolution と異なる

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2. Compact Kähler manifolds

projective space: $\mathbb{C}P^{N-1} = SU(N)/[SU(N-1) \times U(1)]$

$\vec{\phi} \in \mathbb{C}^N$: $\vec{\phi} \sim \lambda \vec{\phi}$, $\lambda \in \mathbb{C}^1$ で同一視

$\vec{\phi}^T = (1, \varphi^i)$, $(i = 1, 2, \dots, N-1)$ 座標系を選択

同一視の下で不変な Kähler potential Ψ :

$$\Psi = c \log \vec{\phi}^\dagger \vec{\phi} = c \log \{1 + |\varphi^i|^2\}, \quad c = \text{constant}$$

quadric surface: $Q^{N-2} = SO(N)/[SO(N-2) \times U(1)]$

$\mathbb{C}P^{N-1} + [\vec{\phi}^2 = 0]$:

$\vec{\phi}^T = (1, \varphi^i, -\frac{1}{2}(\varphi^i)^2)$, $(i = 1, 2, \dots, N-2)$ 座標系を選択

$$\Psi = c \log \vec{\phi}^\dagger \vec{\phi} = c \log \left\{ 1 + |\varphi^i|^2 + \frac{1}{4}(\varphi^i)^2(\varphi^{*j})^2 \right\}$$

例外群: $E_6/[SO(10) \times U(1)]$, $E_7/[E_6 \times U(1)]$

$\mathbb{C}P^{26} + [\Gamma_{ijk}\phi^j\phi^k = 0]$, $\mathbb{C}P^{55} + [d_{\alpha\beta\gamma\delta}\phi^\beta\phi^\gamma\phi^\delta = 0]$:

Γ_{ijk} : rank-3 symmetric tensor invariant under E_6

$d_{\alpha\beta\gamma\delta}$: rank-4 symmetric tensor invariant under E_7

$[E_6]$: $\vec{\phi}^T = (1, \varphi_\alpha, -\frac{1}{2\sqrt{2}}\varphi C \sigma_A^\dagger \varphi)$, $(\alpha = 1, 2, \dots, 16; A = 1, 2, \dots, 10)$

$$\Psi = c \log \left\{ 1 + |\varphi_\alpha|^2 + \frac{1}{8}|\varphi C \sigma_A^\dagger \varphi|^2 \right\}$$

$[E_7]$: $\vec{\phi}^T = (1, \varphi^i, \frac{1}{2}\Gamma_{ijk}\varphi^j\varphi^k, \frac{1}{6}\Gamma_{ijk}\varphi^i\varphi^j\varphi^k)$, $(i = 1, 2, \dots, 27)$

$$\Psi = c \log \left\{ 1 + |\varphi^i|^2 + \frac{1}{4}|\Gamma_{ijk}\varphi^j\varphi^k|^2 + \frac{1}{36}|\Gamma_{ijk}\varphi^i\varphi^j\varphi^k|^2 \right\}$$

Grassmannian: $G_{N,M} = U(N)/[U(N-M) \times U(M)]$

$\Phi : N \times M$ matrix, $\Phi \sim \Phi V$ [$V \in U(M)$] で同一視

$$\Phi = \begin{pmatrix} 1_M \\ \varphi_{Aa} \end{pmatrix}, \quad (A = 1, 2, \dots, N-M; a = 1, 2, \dots, M) \text{ の座標系}$$

同一視の下で不変な Kähler potential Ψ :

$$\Psi = c \log \det \Phi^\dagger \Phi = c \log \det \{1_M + \varphi^\dagger \varphi\}$$

$Sp(N)/U(N)$

$G_{2N,N} + [\varphi^T + \varphi = 0]$:

$$\Phi = \begin{pmatrix} 1_N \\ \varphi_{ab} \end{pmatrix}, \quad (1 \leq a \leq b \leq N) \text{ の座標系}$$

$$\Psi = c \log \det \Phi^\dagger \Phi = c \log \det \{1_N + \varphi^\dagger \varphi\}$$

$SO(2N)/U(N)$

$G_{2N,N} + [\varphi^T - \varphi = 0]$:

$$\Phi = \begin{pmatrix} 1_N \\ \varphi_{ab} \end{pmatrix}, \quad (1 \leq a < b \leq N) \text{ の座標系}$$

$$\Psi = c \log \det \Phi^\dagger \Phi = c \log \det \{1_N + \varphi^\dagger \varphi\}$$

Hermitian symmetric spaces

type	G/H	$\dim_{\mathbb{C}}(G/H)$	Kähler potential Ψ
AIII ₁	$\mathbb{C}P^{N-1}$	$N - 1$	$c \log\{1 + \varphi^i ^2\}$
AIII ₂	$G_{N,M}$	$M(N - M)$	$c \log \det\{1_M + \varphi^\dagger \varphi\}$
BDI	Q^{N-2}	$N - 2$	$c \log\{1 + \varphi^i ^2 + \frac{1}{4}(\varphi^i)^2(\varphi^{*j})^2\}$
CI	$\frac{Sp(N)}{U(N)}$	$\frac{1}{2}N(N + 1)$	$c \log \det\{1_N + \varphi^\dagger \varphi\}$
DIII	$\frac{SO(2N)}{U(N)}$	$\frac{1}{2}N(N - 1)$	$c \log \det\{1_N + \varphi^\dagger \varphi\}$
EIII	$\frac{E_6}{SO(10) \times U(1)}$	16	$c \log\{1 + \varphi_\alpha ^2 + \frac{1}{8} \varphi C \sigma_A^\dagger \varphi ^2\}$
EVII	$\frac{E_7}{E_6 \times U(1)}$	27	$c \log\{1 + \varphi^i ^2 + \frac{1}{4} \Gamma_{ijk}\varphi^j\varphi^k ^2 + \frac{1}{36} \Gamma_{ijk}\varphi^i\varphi^j\varphi^k ^2\}$

3. Complex line bundles

line bundle over $\mathbb{C}P^{N-1}$: Proto type

Kähler 構造を壊さずに複素 1 次元追加する最も簡単な方法:

$$\vec{\phi}^T \equiv \sigma(1, \varphi^i), \sigma \in \mathbb{C}^1$$

Assumption: Kähler potential \mathcal{K} (non-compact manifold)

$$\mathcal{K} = \mathcal{K}(X), \quad X \equiv \log \vec{\phi}^\dagger \vec{\phi} = \log |\sigma|^2 + \Psi$$

X : non-compact direction σ と compact な Ψ ($c = 1$) から成る

$$\mathbb{C} \times \frac{SU(N)}{SU(N-1) \times U(1)}$$

Ricci-flat Kähler manifold を求める $\rightarrow \mathcal{K}$ を定める

Ricci-flat condition

holomorphic coordinates: $\phi^\mu = (\sigma, \varphi^i)$

metric: $g_{\mu\nu^*} = \partial_\mu \partial_{\nu^*} \mathcal{K}(X)$

$$\text{Ricci tensor : } (Ric)_{\mu\nu^*} = -\partial_\mu \partial_{\nu^*} \log \det g_{\kappa\lambda^*}$$

Ricci-flat condition:

$$(Ric)_{\mu\nu^*} = 0 \longrightarrow \begin{cases} \det g_{\mu\nu^*} = (\text{constant}) \times |F|^2 \\ F = \text{holomorphic function} \end{cases}$$

具体的には ... ($\sigma \neq 0$ 領域で考える)

$$g_{\sigma\sigma^*} = \frac{d^2\mathcal{K}}{dX^2} \cdot \frac{\partial X}{\partial\sigma} \frac{\partial X}{\partial\sigma^*}, \quad g_{\sigma j^*} = \frac{d^2\mathcal{K}}{dX^2} \cdot \frac{\partial X}{\partial\sigma} \frac{\partial X}{\partial\varphi^{*j}},$$

$$g_{ij^*} = \frac{d^2\mathcal{K}}{dX^2} \cdot \frac{\partial X}{\partial\varphi^i} \frac{\partial X}{\partial\varphi^{*j}} + \frac{d\mathcal{K}}{dX} \cdot \frac{\partial^2 X}{\partial\varphi^i \partial\varphi^{*j}}$$

determinant:

$$\det g_{\mu\nu^*} = \frac{1}{|\sigma|^2} \frac{d^2\mathcal{K}}{dX^2} \cdot \det \left\{ \frac{d\mathcal{K}}{dX} \cdot \frac{\partial^2 X}{\partial\varphi^i \partial\varphi^{*j}} \right\}$$

$$= \frac{1}{|\sigma|^2} \frac{d^2\mathcal{K}}{dX^2} \left(\frac{d\mathcal{K}}{dX} \right)^{N-1} \cdot \det \tilde{g}_{ij^*} \quad \left(\partial_i \partial_{j^*} X = \partial_i \partial_{j^*} \Psi \equiv \tilde{g}_{ij^*} \right)$$

Ricci-flat condition は偏微分方程式

一般には解析不能

しかし, isotropy group を用いて常微分方程式にできる

isotropy $SU(N-1) \times U(1) \rightarrow$ このうち $SU(N-1)$ で変換:

$$\varphi^i \rightarrow \varphi'^i = h^i_j \varphi^j, \quad h^i_j \in SU(N-1)$$

metric とその determinant の変換:

$$\tilde{g}'_{ij^*} = (h^{-1})_i^k (h^{*-1})_{j^*}{}^{l^*} \tilde{g}_{kl^*}$$

$$\det \tilde{g}'_{ij^*} = \det(h^{-1})_i^k \cdot \det(h^{*-1})_{j^*}{}^{l^*} \cdot \det \tilde{g}_{kl^*}$$

$$= \det \tilde{g}_{kl^*} \quad \leftarrow \text{invariant!}$$

isotropy group を用いて φ^i を変換

↓

\tilde{g}_{ij^*} を対角化して $\det \tilde{g}_{ij^*}$ を導出

$SU(N - 1)$ 変換 $\rightarrow \varphi^1 \neq 0, \varphi^i = 0 (i = 2, \dots, N - 1)$:

$$\det \tilde{g}_{ij^*} = \det \left\{ \text{diag.} \left(e^{-2\Psi}, \underbrace{e^{-\Psi}, e^{-\Psi}, \dots, e^{-\Psi}}_{N-2 \text{ 個}} \right) \right\} = \exp(-N\Psi)$$

compact Kähler manifold は Einstein-Kähler:

$$\begin{aligned} -\partial_i \partial_{j^*} \log \det \tilde{g}_{kl^*} &= (\widetilde{Ric})_{ij^*} = \mathcal{C} \tilde{g}_{ij^*} = \mathcal{C} \partial_i \partial_{j^*} \Psi \\ &\rightarrow \det \tilde{g}_{ij^*} = \exp(-\mathcal{C}\Psi) \end{aligned}$$

\mathcal{C} : それぞれの compact Kähler manifold に依存する数
つまり $\det \tilde{g}_{ij^*}$ がその Kähler potential Ψ のみで書かれる

Ricci-flat condition:

$$(\text{constant}) = e^{-NX} \frac{d}{dX} \left(\frac{d\mathcal{K}}{dX} \right)^N$$

$$\frac{d\mathcal{K}}{dX} = (\lambda e^{NX} + b)^{\frac{1}{N}}$$

λ : positive real parameter

b : integration constant, 非常に重要な parameter

$b = 0, b \neq 0$ による多様体の性質の違いを検討

$\sigma = 0$ まで拡張可能かを検討

特徴

$b \neq 0$ での metric: $(ds^2 = g_{\sigma\sigma^*} d\sigma d\sigma^* + \dots)$

$$g_{\sigma\sigma^*} = \lambda(\lambda e^{NX} + b)^{\frac{1-N}{N}} e^{N\Psi} |\sigma|^{2N-2}$$

$\sigma = 0$ で metric が潰れる $\iff \sigma \rightarrow 0$ 極限で curvature は有限

$z^\mu = (\sigma, \varphi^i)$ は座標特異点 ($\sigma = 0$) を持つ

↓

$$\text{座標変換 : } \rho \equiv \frac{\sigma^N}{N}$$

この変換後の metric:

$$g_{\rho\rho^*} = \lambda(\lambda e^{NX} + b)^{\frac{1-N}{N}} e^{N\Psi}$$

$$g_{\rho j^*} = \lambda N (\lambda e^{NX} + b)^{\frac{1-N}{N}} e^{N\Psi} \rho^* \partial_{j^*} \Psi$$

$$g_{ij^*} = \lambda N^2 (\lambda e^{NX} + b)^{\frac{1-N}{N}} e^{N\Psi} |\rho|^2 \partial_i \Psi \partial_{j^*} \Psi + (\lambda e^{NX} + b)^{\frac{1}{N}} \partial_i \partial_{j^*} \Psi$$

$\rho = 0$ ($d\rho = 0$) 部分多様体の metric:

$$g_{ij^*} \Big|_{\rho=0} = b^{\frac{1}{N}} \partial_i \partial_{j^*} \Psi$$

↑

$\mathbb{C}P^{N-1}$ の Fubini-Study metric そのもの

任意の $\rho \neq 0$ ($d\rho = 0$) 部分多様体も $\mathbb{C}P^{N-1}$ で構成される

[$\sigma (\sim \rho)$ を $\vec{\phi}$ に導入する方法からすぐに分かる]

$b = 0$ での Kähler potential: $d\mathcal{K}/dX = \lambda^{\frac{1}{N}} e^X$

$$\mathcal{K}|_{b=0} = \lambda^{\frac{1}{N}} e^X = \lambda^{\frac{1}{N}} |\sigma|^2 (1 + |\varphi^i|^2)$$

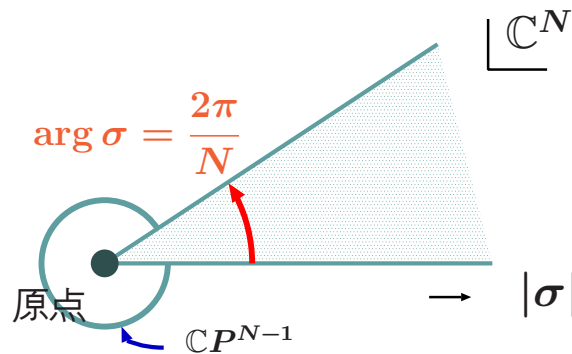
$\phi^1 = \sigma, \phi^i = \sigma \varphi^{i-1}$ と戻すと $\{\vec{\phi}^T = \sigma(1, \varphi^i)\}$

$$\mathcal{K} = \lambda^{\frac{1}{N}} \vec{\phi}^\dagger \vec{\phi} \quad \Leftarrow \quad \text{flat metric?}$$

$\rho = \sigma^N / N$ の座標変換

↓

Orbifold $\mathbb{C}^N / \mathbb{Z}_N$



したがってこの Kähler 多様体は

- $b \neq 0$ という特異点回避を示す parameter を持ち,
- 特異点は $\mathbb{C}P^{N-1}$ 多様体で回避され,
- $b = 0$ 極限で Orbifold $\mathbb{C}^N / \mathbb{Z}_N$ が出現する

↓

complex line bundle over $\mathbb{C}P^{N-1}$

line bundle over Q^{N-2} :

line bundle over $\mathbb{C}P^{N-1}$ と同様の設定:

$$\vec{\phi}^T = \sigma \left(1, \varphi^i, -\frac{1}{2}(\varphi^i)^2 \right), \quad \sigma \in \mathbb{C}^1$$

line bundle over $\mathbb{C}P^{N-1}$ 同様, Kähler potential を次のように仮定:

$$\mathcal{K} = \mathcal{K}(X), \quad X = \log \vec{\phi}^\dagger \vec{\phi} = \log |\sigma|^2 + \Psi$$

Ψ : Kähler potential of Q^{N-2} ($c = 1$)

Ricci-flat condition の解:
$$\frac{d\mathcal{K}}{dX} = (\lambda e^{(N-2)X} + b)^{\frac{1}{N-1}}$$

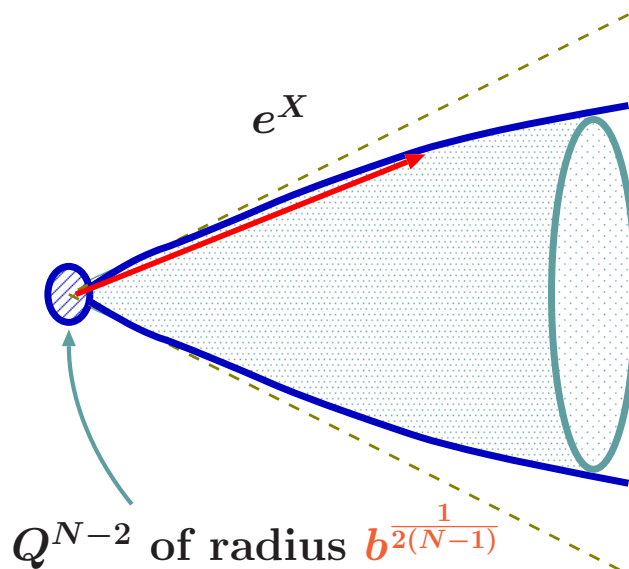
$b \neq 0$ での metric: ($ds^2 = g_{\sigma\sigma^*} d\sigma d\sigma^* + \dots$)

$$g_{\sigma\sigma^*} = \frac{N-2}{N-1} \lambda (\lambda e^{(N-2)X} + b)^{-\frac{N-2}{N-1}} e^{(N-2)\Psi} |\sigma|^{2N-6}$$

座標特異点を除くための座標変換: $\rho \sim \sigma^{N-2}$

$\rho = 0$ ($d\rho = 0$) 部分多様体:

$$g_{ij^*}|_{\rho=0} = b^{\frac{1}{N-1}} \partial_i \partial_{j^*} \Psi \Leftarrow Q^{N-2} \text{ metric そのもの}$$



$N = 3$ 解 : Eguchi-Hanson gravitational instanton

$$\mathcal{K}(X) = 2\sqrt{\lambda e^X + b} + \sqrt{b} \log \left(\frac{\sqrt{\lambda e^X + b} - \sqrt{b}}{\sqrt{\lambda e^X + b} + \sqrt{b}} \right)$$

$\varrho^4 \equiv 4(\lambda e^X + b)$, $a^4 \equiv 4b$ と再定義

$$\mathcal{K} = \varrho^2 + \frac{a^2}{2} \log \left(\frac{\varrho^2 - a^2}{\varrho^2 + a^2} \right)$$

Eguchi-Hanson の Kähler potential そのもの
特異点 (原点) は $Q^1 = SO(3)/U(1) = S^2$ で回避

$N = 4$ 解 : Q^2 Resolved Conifold

特異点回避は $Q^2 \simeq S^2 \times S^2$ (同半径) で行われる

deformation (S^3) でも **small resolution** (S^2) でもない

cf. deformation:

$$\mathbb{C}P^{N-1} + [\vec{\phi}^2 = a^2] + [\sigma \in \mathbb{C}^1]$$

a : deformation parameter

$a \rightarrow 0$ 極限と $b \rightarrow 0$ 極限は一致する (conifold)

exceptional groups:

line bundle over $\mathbb{C}P^{26} + [\Gamma_{ijk}\phi^j\phi^k = 0]$

\implies line bundle over $E_6/[SO(10) \times U(1)]$

line bundle over $\mathbb{C}P^{55} + [d_{\alpha\beta\gamma\delta}\phi^\beta\phi^\gamma\phi^\delta = 0]$

\implies line bundle over $E_7/[E_6 \times U(1)]$

Ricci-flat condition の解と座標変換:

$$\frac{d\mathcal{K}}{dX} = (\lambda e^{cX} + b)^{\frac{1}{D}}, \quad \rho = \sigma^n/n$$

line bundle	\mathcal{C}	D	n
$\mathbb{C} \times \frac{E_6}{SO(10) \times U(1)}$	12	17	12
$\mathbb{C} \times \frac{E_7}{E_6 \times U(1)}$	18	28	18

4. Non-Abelian Gauge Theories

line bundle over $\mathbb{C}P^{N-1}$ 同様の構成もあるが、
ここでは途中まで別の構成 (Gauge Theory) を用いる

\implies Supersymmetric Nonlinear Sigma Models を応用

Grassmannian $G_{N,M}$:

$\Phi : N \times M$ matrix-valued chiral superfield

$U(N) \times U(M)$ 群による変換:

$$\Phi \rightarrow \Phi' = g_L \Phi g_R, \quad (g_L, g_R) \in (U(N), U(M)).$$

$U(M)$ gauge transformation:

$$\Phi \rightarrow \Phi' = \Phi e^{-i\Lambda}, \quad e^V \rightarrow e^{V'} = e^{i\Lambda} e^V e^{-i\Lambda^\dagger}.$$

$U(N) \times U(M)$ 不変な Lagrangian:

$$\mathcal{L} = \int d^4\theta \left\{ \text{tr}(\Phi^\dagger \Phi e^V) - c \text{tr} V \right\}.$$

V : 補助場, vector superfield ($U(M)^\mathbb{C}$ gauge group)

\implies 積分すると非線形模型, target 空間が Grassmannian $G_{N,M}$

gauge-fixing: $\Phi = \begin{pmatrix} 1_M \\ \varphi_{Aa} \end{pmatrix},$

$\varphi_{Aa} : (N - M) \times M$ matrix-valued chiral superfield

$$\mathcal{K} = c \log \det (1_M + \varphi^\dagger \varphi)$$

$$G_{N,M} = \frac{U(N)}{U(N - M) \times U(M)}$$

$G_{2N,N} + \text{F-term constraints:}$

$G_{2N,N}$ Lagrangian + Superpotential term: $W = \text{tr}(\Phi_0 \Phi^T J' \Phi)$

$\Phi_0 : N \times N$ matrix-valued auxiliary superfield

$$J' = \begin{pmatrix} 0 & 1_N \\ \epsilon 1_N & 0 \end{pmatrix}, \quad \epsilon = \pm 1$$

$$\mathcal{L} = \int d^4\theta \mathcal{K}(\Phi, \Phi^\dagger, V) + \left(\int d^2\theta W(\Phi_0, \Phi) + \text{c.c.} \right)$$

補助場 V, Φ_0 を積分して非線形模型

$$\mathcal{K} = c \log \det \{1_N + \varphi^\dagger \varphi\}$$

$$\frac{Sp(N)}{U(N)} \quad (\epsilon = +1, \varphi^T + \varphi = 0), \quad \frac{SO(2N)}{U(N)} \quad (\epsilon = -1, \varphi^T - \varphi = 0)$$

line bundle over $G_{N,M}$:

$U(M)$ gauge symmetric Lagrangian:

$$\mathcal{K}_0(\Phi, \Phi^\dagger, V) = f(\text{tr}(\Phi^\dagger \Phi e^V)) - c \text{tr} V$$

$f : \text{tr}(\Phi^\dagger \Phi e^V)$ の任意関数, $V = V^a T_a$, $T_a \in U(M)$

$c : \text{FI constant} \rightarrow$ **vector superfield** C に格上げ

equation of motion for V and C :

$$\partial \mathcal{L} / \partial V = f' \cdot \Phi^\dagger \Phi e^V - C \cdot 1_M = 0$$

$$\partial \mathcal{L} / \partial C = -\text{tr} V = 0$$

$\text{tr} V = 0$ のもと, 第1式の trace, determinant をとる:

$$f' \cdot \text{tr}(\Phi^\dagger \Phi e^V) = M \cdot C, \quad (f')^M \cdot \det \Phi^\dagger \Phi = C^M$$

$$C \text{ を消去: } \text{tr}(\Phi^\dagger \Phi e^V) = M [\det \Phi^\dagger \Phi]^{\frac{1}{M}}$$

↓

$\partial \mathcal{L} / \partial C = 0$ ($U(M) \rightarrow SU(M)$ gauge group) のもとで

$$\mathcal{K}_0 = f(M [\det \Phi^\dagger \Phi]^{\frac{1}{M}}) \equiv \mathcal{K}(\underbrace{\log \det \Phi^\dagger \Phi}_{\equiv X})$$

C を導入 = gauge 群のうち $U(1)$ 部分を ungauged

line bundle を構成する際最も簡単な 1 次元追加方法:

$$\Phi = \sigma \begin{pmatrix} 1_M \\ \varphi_{Aa} \end{pmatrix}, \sigma \in \mathbb{C}^1 \implies X = M^2 \log |\sigma|^2 + \Psi$$

以後の解析方法は line bundle over $\mathbb{C}P^{N-1}$ と同じ

Ricci-flat condition の解と座標変換:

$$\frac{d\mathcal{K}}{dX} = (\lambda e^{cX} + b)^{\frac{1}{D}}, \quad \rho = \sigma^n/n$$

line bundle	\mathcal{C}	D	n
$\mathbb{C} \times G_{N,M}$	N	$1 + M(N - M)$	MN
$\mathbb{C} \times \frac{Sp(N)}{U(N)}$	$N + 1$	$1 + \frac{1}{2}N(N + 1)$	$N(N + 1)$
$\mathbb{C} \times \frac{SO(2N)}{U(N)}$	$N - 1$	$1 + \frac{1}{2}N(N - 1)$	$N(N - 1)$

すべて特異点はそれぞれの compact な多様体で回避
 やはり **deformation** でも **small resolution** でもない

Isomorphism and duality:

base manifold にある同相性が line bundle にも存在する

Eguchi-Hanson space (complex two-dimensions)

$$\mathbb{C}P^1 \simeq \frac{SO(4)}{U(2)} \simeq \frac{Sp(1)}{U(1)} \simeq Q^1 = \frac{SO(3)}{U(1)}$$

Complex line bundle over $\mathbb{C}P^3$ (complex four-dimensions)

$$\mathbb{C}P^3 \simeq \frac{SO(6)}{U(3)}$$

Another four-dimensional manifolds

$$\frac{Sp(2)}{U(2)} \simeq Q^3 = \frac{SO(5)}{SO(3) \times U(1)}$$

Line bundle over the Klein quadric (complex five-dimensions)

$$G_{4,2} \simeq Q^4 = \frac{SO(6)}{SO(4) \times U(1)}$$

Grassmannian, line bundle 共に次の duality が存在する

Duality between Grassmannians

$$G_{N,M} \simeq G_{N,N-M}$$

Ricci-flat solution and coordinate transformation:

$$\frac{d\mathcal{K}}{dX} = (\lambda e^{cX} + b)^{\frac{1}{D}}, \quad \rho = \sigma^n/n.$$

Hermitian symmetric spaces:

type	$\mathbb{C} \times G/H$	D	\mathcal{C}	n
AIII ₁	$\mathbb{C} \times \mathbb{C}P^{N-1}$	$1 + (N - 1)$	N	N
AIII ₂	$\mathbb{C} \times G_{N,M}$	$1 + M(N - M)$	N	MN
BDI	$\mathbb{C} \times Q^{N-2}$	$1 + (N - 2)$	$N - 2$	$N - 2$
CI	$\mathbb{C} \times Sp(N)/U(N)$	$1 + \frac{1}{2}N(N + 1)$	$N + 1$	$N(N + 1)$
DIII	$\mathbb{C} \times SO(2N)/U(N)$	$1 + \frac{1}{2}N(N - 1)$	$N - 1$	$N(N - 1)$
EIII	$\mathbb{C} \times E_6/[SO(10) \times U(1)]$	$1 + 16$	12	12
EVII	$\mathbb{C} \times E_7/[E_6 \times U(1)]$	$1 + 27$	18	18

$$D = \dim_{\mathbb{C}}(\mathbb{C} \times G/H), \quad \mathcal{C} = \frac{1}{2}C_2(G)$$

$$Q^1 \simeq \mathbb{C}P^1 \simeq SO(4)/U(2) \simeq Sp(1)/U(1) \quad \mathbb{C}P^3 \simeq SO(6)/U(3)$$

$$Q^2 \simeq \mathbb{C}P^1 \times \mathbb{C}P^1 \quad Q^4 \simeq G_{4,2}$$

$$Q^3 \simeq Sp(2)/U(2) \quad G_{N,M} \simeq G_{N,N-M}$$

5. Summary and Discussions

Gauge theory を用いた compact な Kähler 多様体



$U(1)$ ungauged \Rightarrow non-compact Kähler 多様体の導出

isotropy 群変換 \Rightarrow 「Ricci-flat 条件 = 常微分方程式」

座標変換 $\rho \sim \sigma^n \Rightarrow$ 座標特異点消失

積分定数 $b \neq 0 \Rightarrow$ 特異点消失

$\rho = 0$ 部分多様体 = 「compact Kähler 多様体」



Complex line bundle over compact Kähler manifolds



Non-compact Calabi-Yau manifolds

課題:

多様体の大域的な構造/無限遠の構造

他の Kähler 多様体への応用

deformation の構成 (F-term constraint models)

超共形場理論の構成

Supergravity/Superstring, D-branes への応用

付録

NLSM = Non-Linear Sigma Model

対称性の破れを記述 (ex. $D = 4$)

Riemann 多様体, coset space ($M = G/H$)

$$\mathcal{L} = g_{ab}(\varphi) \partial_\mu \varphi^a(x) \partial^\mu \varphi^b(x)$$

μ : 時空の添字

φ^a : Nambu-Goldstone 場, Riemann 多様体の座標

$g_{ab}(\varphi)$: Riemann 多様体の計量

SNLSM and Kähler Potential

SNLSM = Supersymmetric NLSM

Kähler 多様体 ($D = 4, \mathcal{N} = 1$)

$$\begin{aligned} \mathcal{L} &= g_{ab^*} \partial_\mu \varphi^a \partial^\mu \varphi^{*b} + i g_{ab^*} \bar{\psi}^b (\not{D}\psi)^a + \frac{1}{4} R_{ab^*cd^*} \psi^a \psi^c \bar{\psi}^b \bar{\psi}^d \\ &= \int d^4\theta \mathcal{K}(\Phi, \Phi^\dagger) \end{aligned}$$

$\Phi^a = \varphi^a + \sqrt{2}\theta\psi^a + \theta\theta F^a$: chiral superfield

$\mathcal{K}(\Phi, \Phi^\dagger)$: Kähler potential

計量, 曲率は Kähler potential で表現される

$$g_{ab^*} = \partial_a \partial_{b^*} \mathcal{K}, \quad (\partial_a \equiv \partial/\partial\varphi^a)$$

Kähler potential の一般解と積分因子:

$$\frac{d\mathcal{K}}{dX} = (\lambda e^{cX} + b)^{\frac{1}{D}}$$

$$\mathcal{K}(X) = \frac{D}{c} \left[(\lambda e^{cX} + b)^{\frac{1}{D}} + b^{\frac{1}{D}} \cdot I(b^{-\frac{1}{D}} (\lambda e^{cX} + b)^{\frac{1}{D}}; D) \right]$$

$$\begin{aligned} I(y; D) \equiv \int^y \frac{dt}{t^D - 1} &= \frac{1}{D} \left[\log(y - 1) - \frac{1 + (-1)^D}{2} \log(y + 1) \right] \\ &+ \frac{1}{D} \sum_{r=1}^{\lfloor \frac{D-1}{2} \rfloor} \cos \frac{2r\pi}{D} \log \left(y^2 - 2y \cos \frac{2r\pi}{D} + 1 \right) \\ &+ \frac{2}{D} \sum_{r=1}^{\lfloor \frac{D-1}{2} \rfloor} \sin \frac{2r\pi}{D} \arctan \left[\frac{\cos(2r\pi/D) - y}{\sin(2r\pi/D)} \right] \end{aligned}$$

Deformation, Small resolution

Six-dimensional manifold:

$$\sum_{A=1}^4 (w^A)^2 = 0, \quad \sum_{A=1}^4 |w^A|^2 = r^2$$



$$\mathbb{R} \times S^2 \times S^3$$

