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Gauge Theoretical Construction of Non-compact Calabi-Yau Manifolds

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Introduction

We want to one-loop (or higher-loops) finite Field Theory

also

We want to obtain **Calabi-Yau metrics**
in order to give $\mathcal{N} = 1$ Supergravity in Four-dimensions
from String Theory (Candelas-Horowitz-Strominger-Witten, 1985)

$\Downarrow$

$\mathcal{N} = 2$ Supersymmetric Nonlinear Sigma Models in Two-dimensions
(= worldsheet in string theory)
with Ricci-flat Kähler (Calabi-Yau) target space
(= extra six-dimensions in superstring theory)

BUT...
It is difficult to obtain the explicit expression
of the metric of **compact** Calabi-Yau

Recently,

singular Calabi-Yau n-folds have been studied

in order to understand

the nonperturbative effect of

$\mathcal{N} = 1$ super-Yang–Mills theories in four-dimensions

$$\left[\begin{array}{c}\text{appearance of the new massless spectrum,}\\ \text{dynamical generations of the superpotential,}\\ \text{gauge/gravity dual, large-}N\text{ dualty, ...}\end{array}\right]$$

by

flop, conifold transition,

D-branes wrapped around the SUSY-cycles, etc.

Near the singular point of such Calabi-Yau,

this Calabi-Yau looks like **non-compact**

and

we can obtain this metric **very easily!**

Now we obtain the metrics of non-compact Calabi-Yau n-folds

$\Rightarrow$ generalizations of the conifold

$$\left(\begin{array}{l} \text{conifold : topologically } \mathbb{R}_+ \times \mathbf{S}^2 \times \mathbf{S}^3 \\ \qquad\qquad \text{appears in } \textit{conifold transition, Vafa's large-}N \textit{ dual, etc.} \end{array} \right)$$

Our manifolds are constructed as

$$\mathbb{C}^1 \times G/H \ \simeq \ \mathbb{R}^1 \times G/H'$$

$$G/H \ = \ \text{Hermitian Symmetric Spaces (HSS)} \ , \quad H \ = \ H' \times U(1)$$

At large distance

the metrics tend to Ricci-flat "cones" over HSS.

But at the small distance

the "nodes" are repaired by the HSS and become smooth!

Organization

Review: Hermitian Symmetric Spaces

projective space: $\mathbb{C}\mathbf{P}^{N-1} = SU(N)/[SU(N-1) \times U(1)]$

$\vec{\phi} \in \mathbb{C}^N$: identification $\vec{\phi} \sim \lambda\vec{\phi}$, $\lambda \in \mathbb{C}^1$

$\vec{\phi}^T = \left(1, \varphi^i\right)$, $(i = 1, 2, \cdots, N-1)$: local coordinates

Kähler potential K:

$$K = c \log \vec{\phi}^\dagger \vec{\phi} = c \log \left\{1 + |\varphi^i|^2\right\}, \quad c = \text{constant}$$

quadric surface: $Q^{N-2} = SO(N)/[SO(N-2) \times U(1)]$

$\mathbb{C}\mathbf{P}^{N-1} + [\vec{\phi}^2 = 0]$:

$\vec{\phi}^T = \left(1, \varphi^i, -\frac{1}{2}(\varphi^i)^2\right)$, $(i = 1, 2, \cdots, N-2)$: local coordinates

$$K = c \log \vec{\phi}^\dagger \vec{\phi} = c \log \left\{1 + |\varphi^i|^2 + \frac{1}{4}(\varphi^i)^2(\varphi^{*j})^2\right\}$$

exceptional groups: $E_6/[SO(10) \times U(1)]$, $E_7/[E_6 \times U(1)]$

$\mathbb{C}\mathbf{P}^{26} + [\Gamma_{ijk}\phi^j\phi^k = 0]$, $\mathbb{C}\mathbf{P}^{55} + [d_{\alpha\beta\gamma\delta}\phi^\beta\phi^\gamma\phi^\delta = 0]$:

Γ_{ijk}: rank-3 symmetric tensor invariant under E_6

$d_{\alpha\beta\gamma\delta}$: rank-4 symmetric tensor invariant under E_7

$[E_6]$: $\vec{\phi}^T = \left(1, \varphi_\alpha, -\frac{1}{2\sqrt{2}}\varphi C \sigma_A^\dagger \varphi\right)$, $(\alpha = 1, 2, \cdots, 16;\ A = 1, 2, \cdots, 10)$

$$K = c \log \left\{1 + |\varphi_\alpha|^2 + \frac{1}{8}|\varphi C \sigma_A^\dagger \varphi|^2\right\}$$

$[E_7]$: $\vec{\phi}^T = \left(1, \varphi^i, \frac{1}{2}\Gamma_{ijk}\varphi^j\varphi^k, \frac{1}{6}\Gamma_{ijk}\varphi^i\varphi^j\varphi^k\right)$, $(i = 1, 2, \cdots, 27)$

$$K = c \log \left\{1 + |\varphi^i|^2 + \frac{1}{4}|\Gamma_{ijk}\varphi^j\varphi^k|^2 + \frac{1}{36}|\Gamma_{ijk}\varphi^i\varphi^j\varphi^k|^2\right\}$$

Grassmannian: $G_{N,M} = U(N)/[U(N-M) \times U(M)]$

Φ : $N \times M$ matrix, identification $\Phi \sim \Phi V$ $[V \in U(M)]$

$$\Phi = \begin{pmatrix} \mathbf{1}_M \\ \varphi_{Aa} \end{pmatrix}, \ (A = 1, 2, \cdots, N-M;\ a = 1, 2, \cdots, M)$$

Kähler potential K:

$$K = c \log \det \Phi^\dagger \Phi = c \log \det \left\{ \mathbf{1}_M + \varphi^\dagger \varphi \right\}$$

$Sp(N)/U(N)$

$G_{2N,N} + [\varphi^T - \varphi = 0]$:

$$\Phi = \begin{pmatrix} \mathbf{1}_N \\ \varphi_{ab} \end{pmatrix}, \ (1 \leq a \leq b \leq N)$$

$$K = c \log \det \Phi^\dagger \Phi = c \log \det \left\{ \mathbf{1}_N + \varphi^\dagger \varphi \right\}$$

$SO(2N)/U(N)$

$G_{2N,N} + [\varphi^T + \varphi = 0]$:

$$\Phi = \begin{pmatrix} \mathbf{1}_N \\ \varphi_{ab} \end{pmatrix}, \ (1 \leq a < b \leq N)$$

$$K = c \log \det \Phi^\dagger \Phi = c \log \det \left\{ \mathbf{1}_N + \varphi^\dagger \varphi \right\}$$

Hermitian symmetric spaces

type	G/H	$\dim_{\mathbb{C}}(G/H)$	**Kähler potential** K
AIII_1	$\mathbb{C}\mathbf{P}^{N-1}$	$N-1$	$c\log\{1+\lvert\varphi^i\rvert^2\}$
AIII_2	$G_{N,M}$	$M(N-M)$	$c\log\det\{\mathbf{1}_M+\varphi^\dagger\varphi\}$
BDI	Q^{N-2}	$N-2$	$c\log\{1+\lvert\varphi^i\rvert^2+\frac{1}{4}(\varphi^i)^2(\varphi^{*j})^2\}$
CI	$\dfrac{Sp(N)}{U(N)}$	$\frac{1}{2}N(N+1)$	$c\log\det\{\mathbf{1}_N+\varphi^\dagger\varphi\}$
DIII	$\dfrac{SO(2N)}{U(N)}$	$\frac{1}{2}N(N-1)$	$c\log\det\{\mathbf{1}_N+\varphi^\dagger\varphi\}$
EIII	$\dfrac{E_6}{SO(10)\times U(1)}$	16	$c\log\{1+\lvert\varphi_\alpha\rvert^2+\frac{1}{8}\lvert\varphi C\sigma_A^\dagger\varphi\rvert^2\}$
EVII	$\dfrac{E_7}{E_6\times U(1)}$	27	$c\log\{1+\lvert\varphi^i\rvert^2+\frac{1}{4}\lvert\Gamma_{ijk}\varphi^j\varphi^k\rvert^2+\frac{1}{36}\lvert\Gamma_{ijk}\varphi^i\varphi^j\varphi^k\rvert^2\}$

Canonical Line Bundle over Projective Space

the simplest construction:

$$\vec{\phi}^T \equiv \sigma\left(1, \varphi^i\right), \quad \sigma \in \mathbb{C}^1$$

Assumption: Kähler potential $\mathcal{K}$ (non-compact manifold)

$$\mathcal{K} = \mathcal{K}(X)\ , \quad X \equiv \log \vec{\phi}^\dagger \vec{\phi} = \log |\sigma|^2 + \Psi$$

X : non-compact direction σ $\times$ compact Kähler potential Ψ $(= K|_{c=1})$

$$\mathbb{C} \times \frac{SU(N)}{SU(N-1) \times U(1)}$$

Ricci-flatness condition

holomorphic coordinates: $\phi^\mu = \{\sigma, \varphi^i\}$

metric: $g_{\mu\nu^*} = \partial_\mu \partial_{\nu^*} \mathcal{K}(X)$

$$\text{Ricci tensor}: \quad \mathcal{R}_{\mu\nu^*} = -\partial_\mu \partial_{\nu^*} \log \det g_{\kappa\lambda^*}$$

Ricci-flatness condition:

$$\mathcal{R}_{\mu\nu^*} = 0 \quad \longrightarrow \quad \begin{cases} \det g_{\mu\nu^*} = (\text{constant}) \times |F|^2 \\ F = \text{holomorphic function} \end{cases}$$

consider in $\sigma \neq 0$ region:

$$g_{\sigma\sigma^*} = \frac{d^2\mathcal{K}}{dX^2} \cdot \frac{\partial X}{\partial \sigma}\frac{\partial X}{\partial \sigma^*} \,, \quad g_{\sigma j^*} = \frac{d^2\mathcal{K}}{dX^2} \cdot \frac{\partial X}{\partial \sigma}\frac{\partial X}{\partial \varphi^{*j}} \,,$$
$$g_{ij^*} = \frac{d^2\mathcal{K}}{dX^2} \cdot \frac{\partial X}{\partial \varphi^i}\frac{\partial X}{\partial \varphi^{*j}} + \frac{d\mathcal{K}}{dX} \cdot \frac{\partial^2 X}{\partial \varphi^i \partial \varphi^{*j}}$$

determinant:

$$\det g_{\mu\nu^*} = \frac{1}{|\sigma|^2}\frac{d^2\mathcal{K}}{dX^2} \cdot \det\left\{\frac{d\mathcal{K}}{dX} \cdot \frac{\partial^2 X}{\partial \varphi^i \partial \varphi^{*j}}\right\}$$
$$= \frac{1}{|\sigma|^2}\frac{d^2\mathcal{K}}{dX^2}\Big(\frac{d\mathcal{K}}{dX}\Big)^{N-1} \cdot \det \widetilde{g}_{ij^*} \quad \Big(\partial_i \partial_{j^*} X = \partial_i \partial_{j^*} \Psi \equiv \widetilde{g}_{ij^*}\Big)$$

Ricci-flatness condition is a Partial Differential Equation

not soluble in general...

BUT fortunately,

$\mathbb{C}\mathbf{P}^{N-1}$ is Einstein-Kähler

The condition reduces to an Ordinary Differential Equation (ODE)!

$$-\partial_i \partial_{j^*} \log\det \widetilde{g}_{kl^*} = \widetilde{\mathcal{R}}_{ij^*} = \mathcal{C}\, \widetilde{g}_{ij^*} = \mathcal{C}\, \partial_i \partial_{j^*} \Psi$$
$$\to \det \widetilde{g}_{ij^*} \sim \exp\big(-\mathcal{C}\Psi\big) \sim \exp(-\mathcal{C}X)$$

(up to holomorphic functions)

Ricci-flatness condition:

$$e^{-NX}\frac{d}{dX}\Big(\frac{d\mathcal{K}}{dX}\Big)^N \equiv (\text{constant})$$

$$\frac{d\mathcal{K}}{dX} = \left(\lambda e^{NX} + b\right)^{\frac{1}{N}}$$

λ : positive real parameter

b : integration constant, very important parameter

$b \neq 0$ case:

metric: $\left(ds^2 = g_{\sigma\sigma^*} d\sigma d\sigma^* + \cdots \right)$

$$g_{\sigma\sigma^*} = \lambda \left(\lambda e^{NX} + b\right)^{\frac{1-N}{N}} e^{N\Psi} |\sigma|^{2N-2}$$

The metric is ill-defined (degenerate) at $\sigma = 0$

But the curvature is still finite in the $\sigma \to 0$ limit

The coordinates $\phi^\mu = \{\sigma, \varphi^i\}$ has a coordiate singularity at $\sigma = 0$

$$\Downarrow$$

$$\text{transformation} : \rho \equiv \frac{\sigma^N}{N}$$

metric after transformation:

$$g_{\rho\rho^*} = \lambda \left(\lambda e^{NX} + b\right)^{\frac{1-N}{N}} e^{N\Psi}$$

$$g_{\rho j^*} = \lambda N \left(\lambda e^{NX} + b\right)^{\frac{1-N}{N}} e^{N\Psi} \rho^* \partial_{j^*} \Psi$$

$$g_{ij^*} = \lambda N^2 \left(\lambda e^{NX} + b\right)^{\frac{1-N}{N}} e^{N\Psi} |\rho|^2 \partial_i \Psi \partial_{j^*} \Psi + \left(\lambda e^{NX} + b\right)^{\frac{1}{N}} \partial_i \partial_{j^*} \Psi$$

metric of $\rho = 0$ $(d\rho = 0)$ sub-manifold

$$g_{ij^*}\Big|_{\rho=0} = b^{\frac{1}{N}} \partial_i \partial_{j^*} \Psi \;\Rightarrow\; \mathbb{C}\mathbf{P}^{N-1} \text{ metric!}$$

The new coordinate system $\{\rho, \varphi^i\}$ is **well-defined** at $\rho = 0$.

$b = 0$ case:

Kähler potential is very simple:

$$\mathcal{K}\big|_{b=0} = \lambda^{\frac{1}{N}} e^X = \lambda^{\frac{1}{N}} |\sigma|^2 \left(1 + |\varphi^i|^2\right)$$

If we re-write the coordinates as $\phi^1 = \sigma$, $\phi^i = \sigma\varphi^{i-1}$: $\{\vec{\phi}^T = \sigma(1, \varphi^i)\}$

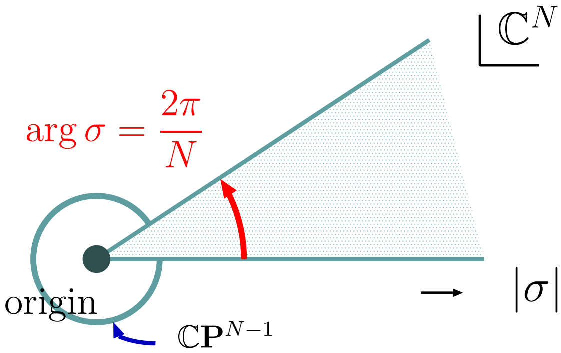
$$\mathcal{K} = \lambda^{\frac{1}{N}} \vec{\phi}^\dagger \vec{\phi} \quad \Leftarrow \quad \text{flat metric?}$$

But we need to transform $\rho = \sigma^N / N$

in order to avoid the coordinate singularity

$\Downarrow$

Orbifold $\mathbb{C}^N / \mathbb{Z}_N$



Result:

- This manifold has a parameter b.
- If $b \neq 0$, this is $\mathbb{C}^1 \times \mathbb{C}\mathbf{P}^{N-1}$ and has no singularity at the origin.
- Orbifold singularity $\mathbb{C}^N / \mathbb{Z}_N$ appears in $b = 0$ limit.

$\Downarrow$

canonical line bundle over $\mathbb{C}\mathbf{P}^{N-1}$

Conifold

the similar set-up to the line bundle over $\mathbb{C}\mathbf{P}^{N-1}$:

$$\vec{\phi}^T = \sigma\big(\, 1, \; \varphi^i, \; -\frac{1}{2}(\varphi^i)^2 \, \big) \, , \quad \sigma \in \mathbb{C}^1$$

Assumption:

$$\mathcal{K} = \mathcal{K}(X) \, , \quad X = \log \vec{\phi}^\dagger \vec{\phi} = \log |\sigma|^2 + \Psi$$

$$\Psi\text{: Kähler potential of } Q^{N-2} \; (c = 1)$$

the solution of Ricci-flatness condition: $\quad \dfrac{d\mathcal{K}}{dX} = \big(\lambda e^{(N-2)X} + b\big)^{\frac{1}{N-1}}$

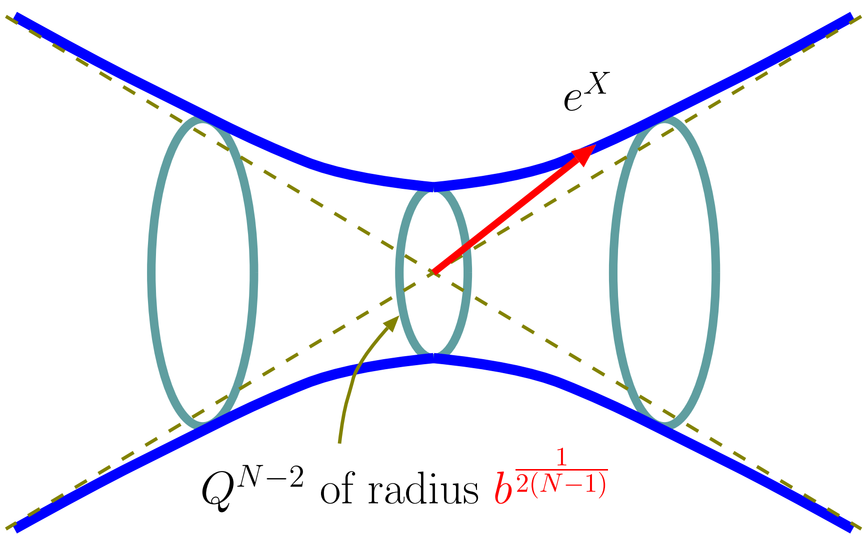
metric with $b \neq 0$: $\big(ds^2 = g_{\sigma\sigma^*} d\sigma d\sigma^* + \cdots\big)$

$$g_{\sigma\sigma^*} = \frac{N-2}{N-1} \lambda \big(\lambda e^{(N-2)X} + b\big)^{-\frac{N-2}{N-1}} e^{(N-2)\Psi} |\sigma|^{2N-6}$$

cooridnate transformation: $\rho \sim \sigma^{N-2}$

$\rho = 0$ $(d\rho = 0)$ sub-manifold:

$$g_{ij^*}\Big|_{\rho=0} = b^{\frac{1}{N-1}} \partial_i \partial_{j^*} \Psi \;\; \Leftarrow \;\; Q^{N-2} \text{ metric!}$$



$N = 3$ solution : Eguchi-Hanson gravitational instanton

$$\mathcal{K}(X) = 2\sqrt{\lambda e^X + b} + \sqrt{b}\log\left(\frac{\sqrt{\lambda e^X + b} - \sqrt{b}}{\sqrt{\lambda e^X + b} + \sqrt{b}}\right)$$

re-definition: $\varrho^4 \equiv 4(\lambda e^X + b)$, $a^4 \equiv 4b$

$$\rightarrow \quad \mathcal{K} = \varrho^2 + \frac{a^2}{2}\log\left(\frac{\varrho^2 - a^2}{\varrho^2 + a^2}\right)$$

Kähler potential of Eguchi-Hanson (Gibbons-Pope, 1979)

the node is repaired by $Q^1 = SO(3)/U(1) \simeq \mathbf{S}^2$

$N = 4$ solution : Q^2 Resolved Conifold

the node is repaired by

$$Q^2 = SO(4)/[SO(2) \times U(1)] \simeq \mathbf{S}^2 \times \mathbf{S}^2 \text{ (same radius)}$$

neither deformation ($\mathbf{S}^3$) nor small resolution ($\mathbf{S}^2$)

exceptional groups:

line bundle over $\mathbb{C}\mathbf{P}^{26} + [\Gamma_{ijk}\phi^j\phi^k = 0]$

$$\Longrightarrow \text{line bundle over } E_6/[SO(10)\times U(1)]$$

line bundle over $\mathbb{C}\mathbf{P}^{55} + [d_{\alpha\beta\gamma\delta}\phi^\beta\phi^\gamma\phi^\delta = 0]$

$$\Longrightarrow \text{line bundle over } E_7/[E_6\times U(1)]$$

the solution of Ricci-flatness condition:

$$\frac{d\mathcal{K}}{dX} = \left(\lambda e^{\mathcal{C}X} + b\right)^{\frac{1}{D}} , \quad \rho = \sigma^n/n$$

line bundle	$\mathcal{C}$	D	n
$\mathbb{C}\times\frac{E_6}{SO(10)\times U(1)}$	12	17	12
$\mathbb{C}\times\frac{E_7}{E_6\times U(1)}$	18	28	18

Gauge Theoretical Construction

Tool: Supersymmetric Nonlinear Sigma Model as Gauge Theory

(introduce the gauge field as an auxiliary field)

Grassmannian $G_{N,M}$:

$$\Phi : N \times M \text{ matrix-valued chiral superfield}$$

$U(N)_L \times U(M)_R$ transformation:

$$\Phi \to \Phi' = g_{\rm L}\Phi g_{\rm R} , \quad (g_{\rm L}, g_{\rm R}) \in (U(N)_L, U(M)_R) .$$

$U(M)_R$ is gauged (introduce a $U(M)$ gauge field):

$$\Phi \to \Phi' = \Phi e^{-i\Lambda} , \quad e^V \to e^{V'} = e^{i\Lambda} e^V e^{-i\Lambda^\dagger} .$$

$U(N)_L \times U(M)_R$ invariant Lagrangian (linear sigma model):

$$\mathcal{L} = \int d^4\theta \left\{ {\rm tr}(\Phi^\dagger \Phi e^V) - c\, {\rm tr} V \right\} .$$

V : vector superfield ($U(M)^{\mathbb{C}}$ gauge group)

integrate out $V \Longrightarrow$ target space becomes Grassmannian $G_{N,M}$

gauge-fixing: $\Phi = \begin{pmatrix} \mathbf{1}_M \\ \varphi_{Aa} \end{pmatrix}$,

φ_{Aa} : $(N-M) \times M$ matrix-valued chiral superfield

$$\mathcal{K} = c \log\det \left(\mathbf{1}_M + \varphi^\dagger \varphi \right)$$

$G_{2N,N}$ + F-term constraints:

$G_{2N,N}$ Lagrangian + Superpotential term: $W = \mathrm{tr}(\Phi_0 \Phi^T J' \Phi)$

Φ_0 : $N \times N$ matrix-valued auxiliary superfield

$$J' = \begin{pmatrix} 0 & \mathbf{1}_N \\ \epsilon \mathbf{1}_N & 0 \end{pmatrix}, \ \epsilon = \pm 1$$

$$\mathcal{L} = \int d^4\theta \, \mathcal{K}(\Phi, \Phi^\dagger, V) + \left(\int d^2\theta \, W(\Phi_0, \Phi) + \mathrm{c.c.} \right)$$

integrate out V, $\Phi_0 \longrightarrow$ NLSM

$$\mathcal{K} = c \log \det \left\{ \mathbf{1}_N + \varphi^\dagger \varphi \right\}$$

$$\frac{Sp(N)}{U(N)} \ \ (\epsilon = -1, \varphi^T - \varphi = 0) \ , \quad \frac{SO(2N)}{U(N)} \ \ (\epsilon = +1, \varphi^T + \varphi = 0)$$

line bundle over $G_{N,M}$:

$U(M)$ gauge symmetric Lagrangian:

$$\mathcal{K}_0(\Phi, \Phi^\dagger, V) = f(\mathrm{tr}(\Phi^\dagger \Phi e^V)) - c\,\mathrm{tr}V$$

f : arbitrary function of $\mathrm{tr}(\Phi^\dagger\Phi e^V)$, $\qquad V = V^a T_a,\, T_a \in U(M)$

c : FI constant $\Rightarrow$ vector superfield C

equation of motion for V and C:

$$\partial\mathcal{L}/\partial V = f' \cdot \Phi^\dagger \Phi e^V - C \cdot \mathbf{1}_M = 0$$

$$\partial\mathcal{L}/\partial C = -\mathrm{tr}V = 0$$

under $\mathrm{tr}V = 0$, perform the trace and determinant of the 1st eq.:

$$f' \cdot \mathrm{tr}(\Phi^\dagger\Phi e^V) = M \cdot C\ , \quad (f')^M \cdot \det \Phi^\dagger \Phi = C^M$$

$$\text{delete } C\text{: } \mathrm{tr}(\Phi^\dagger\Phi e^V) = M\left[\det \Phi^\dagger\Phi\right]^{\frac{1}{M}}$$

$$\Downarrow$$

under eq. of motion $\partial\mathcal{L}/\partial C = 0$ ($U(M) \to SU(M)$ gauge group):

$$\mathcal{K}_0 = f(M[\det\Phi^\dagger\Phi]^{\frac{1}{M}}) \equiv \mathcal{K}(\underbrace{\log\det\Phi^\dagger\Phi}_{\equiv X})$$

introduce C = ungauge the $U(1)$ symmetry

the simplest construction of the line bundle

$$\Phi = \sigma \begin{pmatrix} \mathbf{1}_M \\ \varphi_{Aa} \end{pmatrix}, \sigma \in \mathbb{C}^1 \quad \Longrightarrow \quad X = M^2 \log|\sigma|^2 + \Psi$$

Later discussion is same as the canonical line bundle over $\mathbb{C}\mathbf{P}^{N-1}$.

Solutions of Ricci-flatness condition:

$$\frac{d\mathcal{K}}{dX} = \left(\lambda e^{\mathcal{C}X} + b\right)^{\frac{1}{D}} \ , \quad \rho = \sigma^n / n$$

line bundle	$\mathcal{C}$	D	n
$\mathbb{C} \times G_{N,M}$	N	$1 + M(N-M)$	MN
$\mathbb{C} \times \frac{Sp(N)}{U(N)}$	$N+1$	$1 + \frac{1}{2}N(N+1)$	$N(N+1)$
$\mathbb{C} \times \frac{SO(2N)}{U(N)}$	$N-1$	$1 + \frac{1}{2}N(N-1)$	$N(N-1)$

Isomorphism and duality:

Eguchi-Hanson space (complex two-dimensions)

$$\mathbb{C}^1 \times \mathbb{C}\mathbf{P}^1 \;\simeq\; \mathbb{C}^1 \times \frac{SO(4)}{U(2)} \;\simeq\; \mathbb{C}^1 \times \frac{Sp(1)}{U(1)} \;\simeq\; \mathbb{C}^1 \times Q^1$$

Line bundle over $\mathbb{C}\mathbf{P}^3$ (complex four-dimensions)

$$\mathbb{C}^1 \times \mathbb{C}\mathbf{P}^3 \;\simeq\; \mathbb{C}^1 \times \frac{SO(6)}{U(3)}$$

Another complex four-dimensional manifolds

$$\mathbb{C}^1 \times \frac{Sp(2)}{U(2)} \;\simeq\; \mathbb{C}^1 \times Q^3$$

Line bundle over the Klein quadric (complex five-dimensions)

$$\mathbb{C}^1 \times G_{4,2} \;\simeq\; \mathbb{C}^1 \times Q^4$$

Duality between Grassmannians

$$\mathbb{C}^1 \times G_{N,M} \;\simeq\; \mathbb{C}^1 \times G_{N,N-M}$$

Ricci-flatness solutions:

$$\frac{d\mathcal{K}}{dX} = \left(\lambda e^{\mathcal{C}X} + b\right)^{\frac{1}{D}} , \quad \rho = \sigma^n/n \ .$$

Solutions:

type	$\mathbb{C} \times G/H$	$\mathcal{C}$	D	n
AIII_1	$\mathbb{C} \times \mathbb{C}\mathbf{P}^{N-1}$	N	$1 + (N-1)$	N
AIII_2	$\mathbb{C} \times G_{N,M}$	N	$1 + M(N-M)$	MN
BDI	$\mathbb{C} \times Q^{N-2}$	$N-2$	$1 + (N-2)$	$N-2$
CI	$\mathbb{C} \times Sp(N)/U(N)$	$N+1$	$1 + \frac{1}{2}N(N+1)$	$N(N+1)$
DIII	$\mathbb{C} \times SO(2N)/U(N)$	$N-1$	$1 + \frac{1}{2}N(N-1)$	$N(N-1)$
EIII	$\mathbb{C} \times E_6/[SO(10) \times U(1)]$	12	$1 + 16$	12
EVII	$\mathbb{C} \times E_7/[E_6 \times U(1)]$	18	$1 + 27$	18

$$D = \dim_{\mathbb{C}}(\mathbb{C} \times G/H) \ , \quad \mathcal{C} = \frac{1}{2} C_2(G)$$

$$Q^1 \simeq \mathbb{C}\mathbf{P}^1 \simeq SO(4)/U(2) \simeq Sp(1)/U(1)$$
$$Q^2 \simeq \mathbb{C}\mathbf{P}^1 \times \mathbb{C}\mathbf{P}^1$$
$$Q^3 \simeq Sp(2)/U(2)$$

$$\mathbb{C}\mathbf{P}^3 \simeq SO(6)/U(3)$$
$$Q^4 \simeq G_{4,2}$$
$$G_{N,M} \simeq G_{N,N-M}$$

Generalization

If the base manifolds are Einstein-Kähler,

We can construct the other line bundles:

$$\mathbb{C}^1 \times (\text{non-symmetric spaces}) : \quad \mathbb{C}^1 \times \frac{SU(\ell+m+n)}{S[U(\ell)\times U(m)\times U(n)]}$$

$$\mathbb{C}^1 \times (\text{direct product of HSS}) : \quad \mathbb{C}^1 \times \mathbb{C}\mathbf{P}^{N-1} \times \mathbb{C}\mathbf{P}^{M-1}$$

omit the explanation (mathematically very complicated!),

please see `hep-th/0202064`.

Summary

Supersymmetric Nonlinear Sigma Model as Gauge Theory

$\Downarrow$

$U(1)$ ungauged $\longrightarrow$ non-compact Kähler manifold

G/H is Einstein-Kähler $\longrightarrow$ "Ricci-flatness condition = ODE"

coordinate transformation $\rho \sim \sigma^n$ $\longrightarrow$ "delete coordinate singularity"

integration constant $b \neq 0$ $\longrightarrow$ "delete conical singularity"

(the node is repaired by HSS)

We obtained the canonical line bundle over HSS

$\Downarrow$

"Non-compact Calabi-Yau Manifolds"

Eguchi-Hanson space in real four-dimensions

New deformation of the conifold: $\mathbb{C}^1 \times Q^2 \simeq \mathbb{C}^1 \times \mathbf{S}^2 \times \mathbf{S}^2$

New metric: $\mathbb{C}^1 \times G_{N,M}$, etc.

General Solutions of Kähler Potential:

$$\frac{d\mathcal{K}}{dX} = \left(\lambda e^{\mathcal{C}X} + b\right)^{\frac{1}{D}}$$

$$\mathcal{K}(X) = \frac{D}{\mathcal{C}}\Big[\left(\lambda e^{\mathcal{C}X} + b\right)^{\frac{1}{D}} + b^{\frac{1}{D}} \cdot I\left(b^{-\frac{1}{D}}\left(\lambda e^{\mathcal{C}X} + b\right)^{\frac{1}{D}}; D\right)\Big]$$

$$\begin{aligned} I(y;D) \equiv \int^y \frac{dt}{t^D - 1} = \frac{1}{D}\Big[\log(y-1) - \frac{1+(-1)^D}{2}\log(y+1)\Big] \\ + \frac{1}{D}\sum_{r=1}^{[\frac{D-1}{2}]} \cos\frac{2r\pi}{D}\log\left(y^2 - 2y\cos\frac{2r\pi}{D} + 1\right) \\ + \frac{2}{D}\sum_{r=1}^{[\frac{D-1}{2}]} \sin\frac{2r\pi}{D}\arctan\left[\frac{\cos(2r\pi/D) - y}{\sin(2r\pi/D)}\right] \end{aligned}$$