Seminar at HES, Kyoto University (September 20th, 2002)

Towards Mirror Symmetry on Non-compact Calabi-Yau Manifolds

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(*Sorry, this work is in progress...*)

References

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Phys. Lett. **B518** (2001) 301, hep-th/0107100.

"*Gauge Theoretical Construction of Non-compact Calabi-Yau Manifolds*",

Ann. Phys. **296** (2002) 347, hep-th/0110216.

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Introduction

Recently, in type IIA and type IIB string theories,

singular Calabi-Yau manifolds have been studied
in order to understand
the **nonperturbative aspects** of
$\mathcal{N}=1$ super-Yang–Mills theories in four-dimensions

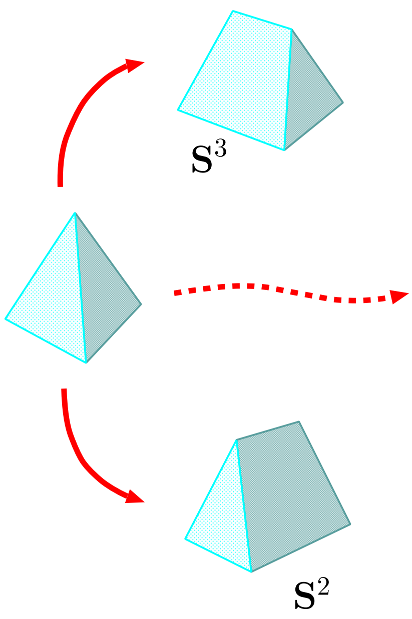
$$\left[\begin{array}{c}\text{appearance of the new massless spectrum,}\\ \text{dynamical generations of the superpotential,}\\ \text{gauge/gravity, large-}N\text{ duality, ...}\end{array}\right]$$

by

flop, conifold transition,
D-branes wrapped around the SUSY-cycles, etc.

Near the singular point of such Calabi-Yau's,
these Calabi-Yau manifolds look like **non-compact**

conifold transition



deformation:

related to complex moduli of CY_3

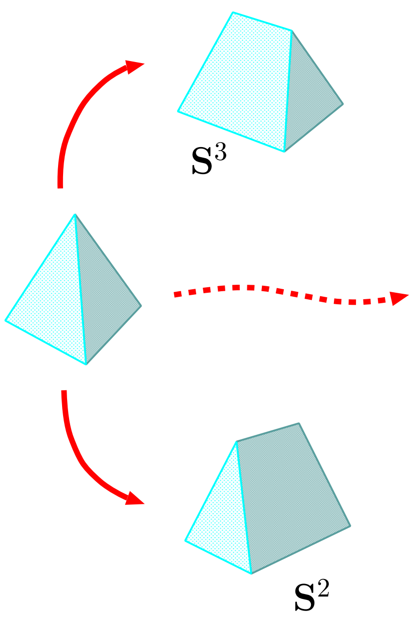
??

singularity is resolved by $\mathbf{Q}^2 \simeq \mathbf{P}^1 \times \mathbf{P}^1$

small resolution:

related to Kähler moduli of CY_3

geometric transtion in IIB string (Vafa's large-N duality)



gravity side:

N RR-fluxes through $\mathbf{S}^3$

??

D7-branes wrapping around $\mathbf{P}^2$, $\mathbf{Q}^2 \simeq \mathbf{P}^1 \times \mathbf{P}^1$

gauge side:

N D5-branes wrapping around $\mathbf{S}^2$

Now we obtain the metrics of non-compact Calabi-Yau n-folds

$\Rightarrow$ generalizations of the conifold

$$\left(\begin{array}{l} \text{conifold : topologically } \mathbb{R}_+ \times \mathbf{S}^2 \times \mathbf{S}^3 \\ \qquad\qquad \text{appears in } \textit{conifold transition, Vafa's large-}N\textit{ dual, etc.} \end{array} \right)$$

Our manifolds are constructed as

$$\mathbb{C}^1 \times G/H$$

G/H = Hermitian Symmetric Spaces (HSS), and more.

At large distance

the metrics tend to Ricci-flat “cones” over HSS.

But at the small distance

the “nodes” are repaired by the HSS and become smooth!

Next,

We would like to know global structures on these CY!

homology cycles, cohomology classes, Euler numbers, etc...

BUT, we know these manifolds only locally (metrics only...).

Thus we should represent these manifolds in a different way.

Gauged Linear Sigma Models (GLSM)

global description of toric varieties

and hypersurfaces in these spaces

not differential geometry, but algebraic geometry

If we construct their **T-dualized models**,

We can understand mirror manifolds of those non-compact CY!

Organization

Introduction

Canonical Line Bundle over Projective Space

Conifold

Towards mirror symmetry

Discussions

Appendix

Hermitian symmetric spaces

type	G/H	$\dim_{\mathbb{C}}(G/H)$	**Kähler potential** K
AIII_1	$\mathbf{P}^{N-1}$	$N-1$	$r\log\{1+\lvert\varphi^i\rvert^2\}$
AIII_2	$\mathbf{G}_{N,M}$	$M(N-M)$	$r\log\det\{\mathbf{1}_M+\varphi^\dagger\varphi\}$
BDI	$\mathbf{Q}^{N-2}$	$N-2$	$r\log\{1+\lvert\varphi^i\rvert^2+\frac{1}{4}(\varphi^i)^2(\varphi^{*j})^2\}$
CI	$\dfrac{Sp(N)}{U(N)}$	$\dfrac{1}{2}N(N+1)$	$r\log\det\{\mathbf{1}_N+\varphi^\dagger\varphi\}$
DIII	$\dfrac{SO(2N)}{U(N)}$	$\dfrac{1}{2}N(N-1)$	$r\log\det\{\mathbf{1}_N+\varphi^\dagger\varphi\}$
EIII	$\dfrac{E_6}{SO(10)\times U(1)}$	16	$r\log\{1+\lvert\varphi_\alpha\rvert^2+\frac{1}{8}\lvert\varphi C\sigma_A^\dagger\varphi\rvert^2\}$
EVII	$\dfrac{E_7}{E_6\times U(1)}$	27	$r\log\{1+\lvert\varphi^i\rvert^2+\frac{1}{4}\lvert\Gamma_{ijk}\varphi^j\varphi^k\rvert^2+\frac{1}{36}\lvert\Gamma_{ijk}\varphi^i\varphi^j\varphi^k\rvert^2\}$

Canonical Line Bundle over Projective Space

the simplest construction:

$$\vec{\phi}^T \equiv \sigma\big(1, \varphi^i\big), \quad \sigma \in \mathbb{C}^1$$

Assumption: Kähler potential $\mathcal{K}$ (non-compact manifold)

$$\mathcal{K} = \mathcal{K}(X) \ , \quad X \equiv \log \vec{\phi}^\dagger \vec{\phi} = \log |\sigma|^2 + \Psi$$

X : non-compact direction σ $\times$ compact Kähler potential Ψ $(= K|_{r=1})$

$$\mathbb{C} \times \frac{SU(N)}{SU(N-1) \times U(1)}$$

Ricci-flatness condition

holomorphic coordinates: $\phi^\mu = \{\sigma, \varphi^i\}$

metric: $g_{\mu\nu^*} = \partial_\mu \partial_{\nu^*} \mathcal{K}(X)$

$$\text{Ricci tensor} : \quad \mathcal{R}_{\mu\nu^*} = -\partial_\mu \partial_{\nu^*} \log \det g_{\kappa\lambda^*}$$

Ricci-flatness condition:

$$\mathcal{R}_{\mu\nu^*} = 0 \ \longrightarrow \ \begin{cases} \det g_{\mu\nu^*} = (\text{constant}) \times |F|^2 \\ F = \text{holomorphic function} \end{cases}$$

consider in $\sigma \neq 0$ region:

$$g_{\sigma\sigma^*} = \frac{\mathrm{d}^2\mathcal{K}}{\mathrm{d}X^2} \cdot \frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \sigma^*} \, , \quad g_{\sigma j^*} = \frac{\mathrm{d}^2\mathcal{K}}{\mathrm{d}X^2} \cdot \frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \varphi^{*j}} \, ,$$
$$g_{ij^*} = \frac{\mathrm{d}^2\mathcal{K}}{\mathrm{d}X^2} \cdot \frac{\partial X}{\partial \varphi^i} \frac{\partial X}{\partial \varphi^{*j}} + \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}X} \cdot \frac{\partial^2 X}{\partial \varphi^i \partial \varphi^{*j}}$$

determinant:

$$\det g_{\mu\nu^*} = \frac{1}{|\sigma|^2} \frac{\mathrm{d}^2\mathcal{K}}{\mathrm{d}X^2} \cdot \det \left\{ \frac{\mathrm{d}\mathcal{K}}{\mathrm{d}X} \cdot \frac{\partial^2 X}{\partial \varphi^i \partial \varphi^{*j}} \right\}$$
$$= \frac{1}{|\sigma|^2} \frac{\mathrm{d}^2\mathcal{K}}{\mathrm{d}X^2} \left(\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}X} \right)^{N-1} \cdot \det \widetilde{g}_{ij^*} \quad \Big(\partial_i \partial_{j^*} X = \partial_i \partial_{j^*} \Psi \equiv \widetilde{g}_{ij^*} \Big)$$

Ricci-flatness condition is a Partial Differential Equation

not soluble in general...

BUT fortunately,

$\mathbf{P}^{N-1}$ is Einstein-Kähler

The condition reduces to an Ordinary Differential Equation (ODE)!

$$-\partial_i \partial_{j^*} \log \det \widetilde{g}_{kl^*} = \widetilde{\mathcal{R}}_{ij^*} = \mathcal{C} \, \widetilde{g}_{ij^*} = \mathcal{C} \, \partial_i \partial_{j^*} \Psi$$
$$\rightarrow \det \widetilde{g}_{ij^*} \sim \exp(-\mathcal{C}\Psi\big) \sim \exp(-\mathcal{C}X)$$

(up to holomorphic functions)

Ricci-flatness condition:

$$\mathrm{e}^{-NX} \frac{\partial}{\partial X} \left(\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}X} \right)^N \equiv (\text{constant})$$

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}X} = \left(\lambda \mathrm{e}^{NX} + b\right)^{\frac{1}{N}}$$

λ : positive real parameter

b : integration constant, very important parameter

$b \neq 0$ case:

metric: $\left(\mathrm{d}s^2 = g_{\sigma\sigma^*}\mathrm{d}\sigma\mathrm{d}\sigma^* + \cdots \right)$

$$g_{\sigma\sigma^*} = \lambda\left(\lambda \mathrm{e}^{NX} + b\right)^{\frac{1-N}{N}} \mathrm{e}^{N\Psi}|\sigma|^{2N-2}$$

The metric is ill-defined (degenerate) at $\sigma = 0$

But the curvature is still finite in the $\sigma \longrightarrow 0$ limit

The coordinates $\phi^\mu = \{\sigma, \varphi^i\}$ has a coordiate singularity at $\sigma = 0$

$$\Downarrow$$

transformation : $\rho \equiv \dfrac{\sigma^N}{N}$

metric after transformation:

$$g_{\rho\rho^*} = \lambda\left(\lambda \mathrm{e}^{NX} + b\right)^{\frac{1-N}{N}} \mathrm{e}^{N\Psi}$$

$$g_{\rho j^*} = \lambda N\left(\lambda \mathrm{e}^{NX} + b\right)^{\frac{1-N}{N}} \mathrm{e}^{N\Psi}\rho^*\partial_{j^*}\Psi$$

$$g_{ij^*} = \lambda N^2\left(\lambda \mathrm{e}^{NX} + b\right)^{\frac{1-N}{N}} \mathrm{e}^{N\Psi}|\rho|^2\partial_i\Psi\partial_{j^*}\Psi + \left(\lambda \mathrm{e}^{NX} + b\right)^{\frac{1}{N}}\partial_i\partial_{j^*}\Psi$$

metric of $\rho = 0$ ($\mathrm{d}\rho = 0$) sub-manifold

$$g_{ij^*}\Big|_{\rho=0} = b^{\frac{1}{N}}\partial_i\partial_{j^*}\Psi \;\Rightarrow\; \mathbf{P}^{N-1} \text{ metric!}$$

The new coordinate system $\{\rho, \varphi^i\}$ is **well-defined** at $\rho = 0$.

$b = 0$ case:

Kähler potential is very simple:

$$\mathcal{K}\Big|_{b=0} = \lambda^{\frac{1}{N}} \mathrm{e}^{X} = \lambda^{\frac{1}{N}} |\sigma|^2 \left(1 + |\varphi^i|^2\right)$$

If we re-write the coordinates as $\phi^1 = \sigma$, $\phi^i = \sigma \varphi^{i-1}$: $\{\vec{\phi}^T = \sigma(1, \varphi^i)\}$

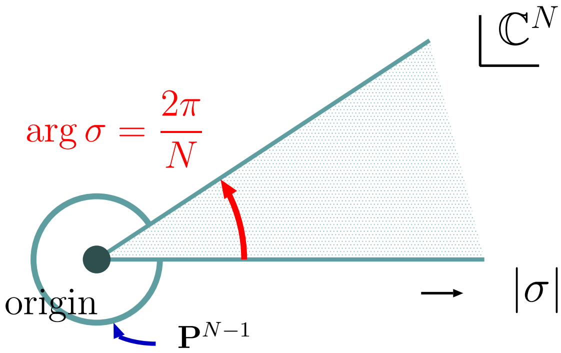
$$\mathcal{K} = \lambda^{\frac{1}{N}} \vec{\phi}^\dagger \vec{\phi} \quad \Leftarrow \quad \text{flat metric?}$$

But we need to transform $\rho = \sigma^N / N$

in order to avoid the coordinate singularity

$\Downarrow$

Orbifold $\mathbb{C}^N / \mathbb{Z}_N$



Result:

- This manifold has a parameter b.
- If $b \neq 0$, this is $\mathbb{C}^1 \times \mathbf{P}^{N-1}$ and has no singularity at the origin.
- Orbifold singularity $\mathbb{C}^N / \mathbb{Z}_N$ appears in $b = 0$ limit.

$\Downarrow$

canonical line bundle over $\mathbf{P}^{N-1}$

Conifold

the similar set-up to the line bundle over $\mathbf{P}^{N-1}$:

$$\vec{\phi}^T = \sigma\big(\ 1,\ \varphi^i,\ -\frac{1}{2}(\varphi^i)^2\ \big)\ ,\quad \sigma \in \mathbb{C}^1$$

Assumption:

$$\mathcal{K} = \mathcal{K}(X)\ ,\quad X = \log \vec{\phi}^\dagger \vec{\phi} = \log |\sigma|^2 + \Psi$$

Ψ: Kähler potential of Q^{N-2} $(r=1)$

the solution of Ricci-flatness condition: $\dfrac{\mathrm{d}\mathcal{K}}{\mathrm{d}X} = \big(\lambda \mathrm{e}^{(N-2)X} + b\big)^{\frac{1}{N-1}}$

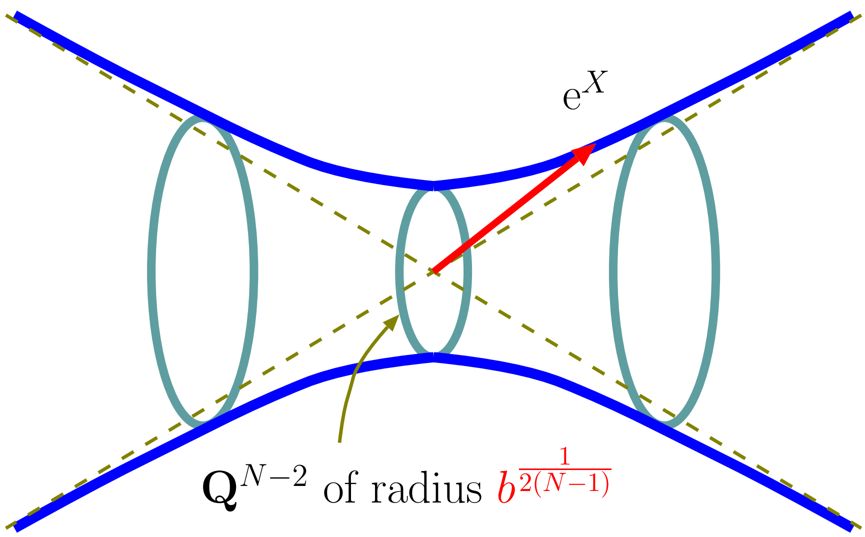
metric with $b \neq 0$: $\big(\mathrm{d}s^2 = g_{\sigma\sigma^*}\mathrm{d}\sigma\mathrm{d}\sigma^* + \cdots\big)$

$$g_{\sigma\sigma^*} = \frac{N-2}{N-1}\lambda\big(\lambda \mathrm{e}^{(N-2)X} + b\big)^{-\frac{N-2}{N-1}} \mathrm{e}^{(N-2)\Psi}|\sigma|^{2N-6}$$

cooridnate transformation: $\rho \sim \sigma^{N-2}$

$\rho = 0$ $(\mathrm{d}\rho = 0)$ sub-manifold:

$$g_{ij^*}\big|_{\rho=0} = b^{\frac{1}{N-1}}\partial_i\partial_{j^*}\Psi \ \Leftarrow\ \mathbf{Q}^{N-2} \text{ metric!}$$



$N = 3$ solution : Eguchi-Hanson gravitational instanton

$$\mathcal{K}(X) = 2\sqrt{\lambda \mathrm{e}^X + b} + \sqrt{b} \log \left(\frac{\sqrt{\lambda \mathrm{e}^X + b} - \sqrt{b}}{\sqrt{\lambda \mathrm{e}^X + b} + \sqrt{b}} \right)$$

re-definition: $\varrho^4 \equiv 4(\lambda \mathrm{e}^X + b)$, $a^4 \equiv 4b$

$$\rightarrow \quad \mathcal{K} \;=\; \varrho^2 + \frac{a^2}{2} \log \left(\frac{\varrho^2 - a^2}{\varrho^2 + a^2} \right)$$

Kähler potential of Eguchi-Hanson (Gibbons-Pope, 1979)

the node is repaired by $\mathbf{Q}^1 = SO(3)/U(1) \simeq \mathbf{S}^2$

$N = 4$ solution : $\mathbf{Q}^2$ Resolved Conifold

the node is repaired by

$$\mathbf{Q}^2 = SO(4)/[SO(2) \times U(1)] \simeq \mathbf{P}^1 \times \mathbf{P}^1 \text{ (same radius)}$$

neither deformation ($\mathbf{S}^3$) nor small resolution ($\mathbf{S}^2$)

Metric on Non-compact Calabi-Yau

non-compact CY = line bundle over coset spaces $\otimes_a(G_a/H_a)$

Kähler potential $\mathcal{K}(X)$ (local description)

$$X \;\equiv\; \log|\rho|^2 + \sum_a h_a K_a$$

$K_a = r_a \log(1+\cdots)$: Kähler potential of base manifold (local description)

r_a: radius of G_a/H_a

h_a: some parameter

related to an isometry group G_a (dual Coxeter)

Key point: Einstein-Kähler metric on G_a/H_a

$$\widetilde{\mathcal{R}}^a_{m\overline{n}} \;=\; h_a\, \widetilde{g}^a_{m\overline{n}}$$

$$\widetilde{\mathcal{R}}^a_{m\overline{n}} \;\equiv\; -\partial_m\partial_{\overline{n}} \log\det \widetilde{g}^a_{k\overline{\ell}}\,, \qquad \widetilde{g}^a_{m\overline{n}} \;=\; \partial_m\partial_{\overline{n}} K_a$$

$$\therefore \quad \det \widetilde{g}^a_{m\overline{n}} \;=\; \exp\left(-h_a K_a\right) \times |\mathrm{hol.}|^2$$

Ricci-flatness condition of the line bundle:

$$\mathcal{R}_{\mu\overline{\nu}} \;=\; 0 \quad \longrightarrow \quad \det g_{\mu\overline{\nu}} \;=\; (\mathrm{constant}) \times |\mathrm{hol.}|^2$$

more explicitly,

$$\begin{aligned} \det g_{\mu\overline{\nu}} \;&=\; \frac{1}{|\rho|^2}\frac{\mathrm{d}^2\mathcal{K}}{\mathrm{d}X^2}\left(\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}X}\right)^d \cdot \prod_a \det\left(h_a g^a_{m\overline{n}}\right) \\ &=\; \mathrm{e}^{-X}\frac{\mathrm{d}^2\mathcal{K}}{\mathrm{d}X^2}\left(\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}X}\right)^d \times (\mathrm{constant}) \times |\mathrm{hol.}|^2 \end{aligned}$$

$d = \dim_{\mathbb{C}}[\otimes_a(G_a/H_a)]$: complex dimension of coset spaces

Solution:

$$e^{-X}\frac{d}{dX}\left(\frac{d\mathcal{K}}{dX}\right)^{D} = (\text{constant}) , \qquad D = d+1$$

$$\frac{d\mathcal{K}}{dX} = \left(\lambda e^{X}+b\right)^{1/D}$$

Example 1: G/H is hermitian symmetric spaces ($h = h_g/r$)

G/H	h_g	D
$\mathbf{P}^{N-1} = \frac{SU(N)}{SU(N-1)\times U(1)}$	N	$(N-1)+1$
$\mathbf{Q}^{N-2} = \frac{SO(N)}{SO(N-2)\times U(1)}$	$N-2$	$(N-2)+1$
$E_6/[SO(10)\times U(1)]$	12	$16+1$
$E_7/[E_6\times U(1)]$	18	$27+1$
$\mathbf{G}_{N,M} = \frac{U(N)}{U(N-M)\times U(M)}$	N	$M(N-M)+1$
$SO(2N)/U(N)$	$N-1$	$\frac{1}{2}N(N-1)+1$
$Sp(N)/U(N)$	$N+1$	$\frac{1}{2}N(N+1)+1$

Example 2: $\mathbf{P}^{N-1}\otimes\mathbf{P}^{M-1}$

$$h_1 = \frac{N}{r_1} , \quad h_2 = \frac{M}{r_2} , \quad D = (N-1)+(M-1)+1$$

Towards Mirror Symmetry

In order to investigate various phenomena in string theory,

We would like to use our metrics on non-compact CY:

$$\text{Complex line bundle over} \begin{cases} \text{HSS} \\ \mathbf{P}^{N-1} \times \mathbf{P}^{M-1} \\ \text{non-symmetric spaces} \end{cases}$$

We would like to know global structures on these CY!

homology cycles, cohomology classes, Euler numbers, etc...

BUT, we know these manifolds only locally (metrics only...).

Thus we should represent these manifolds in a different way.

Gauged Linear Sigma Models (GLSM)

global description of toric varieties

and hypersurfaces in these spaces

not differential geometry, but algebraic geometry

From GLSMs and their **T-dualized models**,

We can understand the global structures of toric varieties

Mirror symmetry on chiral rings

Mirror Symmetry

The same structures of chiral rings

between $\left\{ \begin{array}{c} \text{LG model from GLSM} \simeq \text{cohomology rings of CY} \\ \text{and} \\ \text{LG model from T-dualized theory} \end{array} \right.$

The same theory between $\left\{ \begin{array}{c} \text{sigma model on CY manifold } \mathcal{M} \\ \text{and} \\ \text{T-dualized LG model on manifold } \mathcal{W} \end{array} \right.$

Mirror symmetry on some simple CY's have already been proved.

$\mathcal{O}_{\mathbf{P}^{N-1}}(-N)$; hypersurface, complete intersection in toric, etc.

We would like to investigate mirror symmetry

on more complicated manifolds,

for example,

line bundles over Hermitian symmetric spaces (our model)

In this work,

We would like to construct GLSM and T-dualized model for

line bundle over quadric surface $\mathbf{Q}^{N-2}$

We would like to construct the LG orbifold model

which is dual to CY sigma model

CY/LG correspondence, cc-ring structure

and its T-dualized theory

ac-ring structure

Then, we will check the mirror symmetry on chiral rings,

and will consider applications of these models.

BUT, unfortunately,

I have few knowledge on chiral ring structures and more.

So this work is still in progress...

Discussions

If we construct the chiral ring structure of these model,

we can understand $\left\{ \begin{array}{l} \text{the (co)homology of this CY (NEW)} \\ \text{mirror symmetry of this non-compact CY (NEW)} \end{array} \right.$

and moreover,

we can use them to the study of "*String duality*"

IIA, IIB string/F-theory compactified by these CY

Dp-brane wrapping around non-vanishing cycles in CY

gauge/gravity duality

In particular, CY 3-fold is very interesting manifold.

CY 3-fold = line bundle over $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$, $\mathbf{Q}^2$.

Line bundle over $\mathbf{Q}^2$ has not been investigated yet...

We have a big chance to use this model

to gauge/gravity dual and geometric transition!!

BUT...

$$\left.\begin{array}{c}\text{Line bundle over } \mathbf{Q}^2 \\ \text{and} \\ \text{Line bundle over } \mathbf{P}^1 \times \mathbf{P}^1\end{array}\right\} \text{are globally isomorphic!}$$

probably...

Geometric transition of line bundle over $\mathbf{P}^1 \times \mathbf{P}^1$

has been investigated and used in a lot of topics!!

So, there will be No topics to use the "line bundle over $\mathbf{Q}^2$"...

I found this "bad news" 10 days ago...

My Next Study

- understand the ring structures of LG model
- check mirror symmetry of our non-compact CY
- look for an application to use these CY (CY 4-folds?)

Gauged Linear Sigma Model

GLSM = Field Theory on toric variety

$$\mathcal{L} = \int \mathrm{d}^4\theta \sum_a \Big\{ -\frac{1}{e_a^2} \overline{\Sigma}_a \Sigma_a + \sum_i \overline{\Phi}_i \Phi_i \, \mathrm{e}^{2Q_{i,a}V_a} \Big\} + \Big(\sum_a \frac{t_a}{\sqrt{2}} \int \mathrm{d}^2\widetilde{\theta} \, \Sigma_a + (c.c.) \Big) + \Big(\int \mathrm{d}^2\theta \, W(\Phi_i) + (c.c.) \Big)$$

$t_a = r_a - i\theta_a$: complexified FI parameter

Σ_a: twisted chiral superfield, $\Sigma_a = \dfrac{1}{\sqrt{2}} \overline{D}_+ D_- V_a$

appear only in two-dimensional supersymmetric theory

Two-dimensional $\mathcal{N} = 2$ supersymmetric theory

Various "phases" at low energy

Massless effective theory is written as

supersymmetric Nonlinear Sigma Model on Calabi-Yau,

supersymmetric Landau-Ginzburg Orbifold Model,

or more complicated model

RG flow and asymptotic behavior (one-loop correction):

$$\sum_i Q_{i,a} = \begin{cases} \text{positive}: & \text{asymptotically free} \\ 0: & \text{CY (finite)} \\ \text{negative}: & \text{asymptotically non-free} \end{cases}$$

Construction 0: the simplest example (review)

Contents of chiral superfields and their $U(1)$ charges:

$$\begin{aligned} \Phi_i &= \big(S_1\,,\, S_2\,,\cdots,\, S_{N-1}\,,\, S_N\,,\;\; P\;\; \big) \\ Q_i &= \big(\;1\;\;,\;\;1\;\;,\cdots,\;\;\;1\;\;\;\;,\;\;1\;\;,-N\,\big) \end{aligned}$$

Superpotential:

$$W \;=\; \lambda P \cdot G(S_i) \;\; ; \qquad \text{(ex.)} \;\; G(S_i) \;=\; \sum_{i=1}^{N} S_i^N$$

$\lambda = 0$ or 1

- Supersymmetric vacua

  Potential energy (scalar field only)

$$\begin{aligned} U(s,p,\sigma) \;&=\; \frac{1}{2e^2}D^2 + 2|\sigma|^2\Big(\sum_{i=1}^{N}|s_i|^2 + N^2|p|^2\Big) \\ &\qquad + \lambda^2\left(|p|^2\sum_{i=1}^{N}\left|\frac{\partial G}{\partial S_i}\right|^2 + \left|\sum_{i=1}^{N} G\right|^2\right) \end{aligned}$$

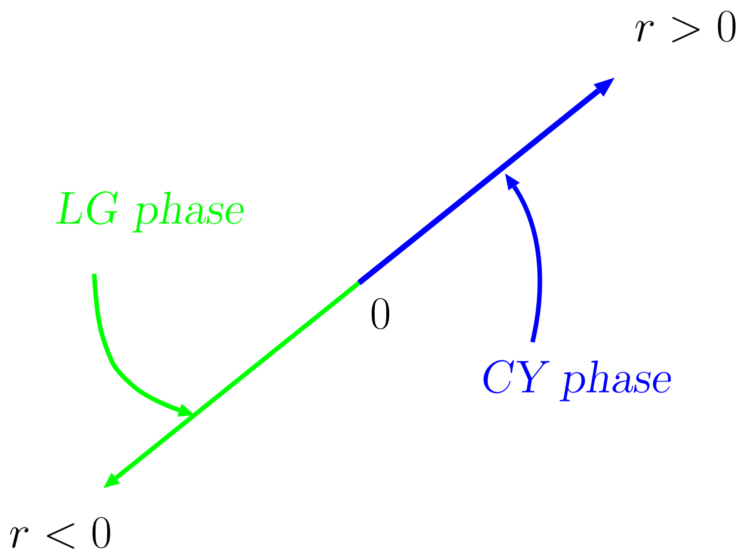
$$\frac{1}{e^2}D \;=\; r - \sum_{i=1}^{N}|s_i|^2 + N|p|^2$$

Condition of SUSY vacua:

$$D \;=\; 0\,, \qquad U(\langle s\rangle,\langle p\rangle,\langle\sigma\rangle) \;=\; 0\,, \qquad \mathrm{d}U(\langle s\rangle,\langle p\rangle,\langle\sigma\rangle) \;=\; 0$$

The solution depends on FI parameter r.

- Low energy theory



CY phase: $r > 0$

sigma model on $\mathcal{O}(-N)$ bundle on $\mathbf{P}^{N-1} \equiv \mathcal{V}$

S_i: $\mathbf{P}^{N-1}$, P: fiber

LG phase: $r < 0$

$\mathbb{Z}_N$ Landau-Ginzburg orbifold model on $\mathcal{M}$

chiral field S_i $(i = 1, 2, \cdots, N)$

superpotential: $W' = G(S_i)$

- CY/LG correspondence

"smoothly" connect between various phases

geometries between CY and LG are topologically equivalent $\mathcal{V} \simeq \mathcal{M}$

$\Rightarrow$ (co)homologies between $\mathcal{V}$ and $\mathcal{M}$ are same!

- Ring structures

  (un)twisted sectors in LG orbifold minimal model.

We can investigate the cohomology class of CY.

In the large radii limit of CY,
chiral rings of LG model correspond to cohomology classes of CY.

Since GLSM has a **suitable** superpotential,
we can investigate *cc*-ring structure easily.
Non-trivial *ac*-ring structure appears when the theory is orbifolded.

Key point: isolated singularity

$$\frac{\partial W'}{\partial S_i} = 0 \qquad \text{if and only if} \quad {}^{\forall}S_i = 0$$

This model has been investigated and used by a lot of people.
Now we try to apply this model to our model.

Construction 1: GLSM for line bundle over $\mathbf{Q}^{N-2}$

Contents of chiral superfields and their $U(1)$ charges:

$$\begin{aligned} \Phi_i &= \big(S_0 \, , \, S_1 \, , \, S_2 \, , \cdots , \, S_{N-1} \, , \, S_N \, , \quad P \quad \big) \\ Q_i &= \big(\; 1 \;\, , \; 1 \;\, , \; 1 \;\, , \cdots , \quad 1 \quad , -2 \, , -N+2 \big) \end{aligned}$$

superpotential:

$$W \;=\; S_N \sum_{i=0}^{N-1} S_i^2 \qquad \Longleftarrow \qquad \text{different from ordinary GLSM!}$$

(cf.) GLSM on $\mathbf{Q}^{N-2}$ *(delete* P *field)*

- Supersymmetric vacua

Potential energy (scalar field only)

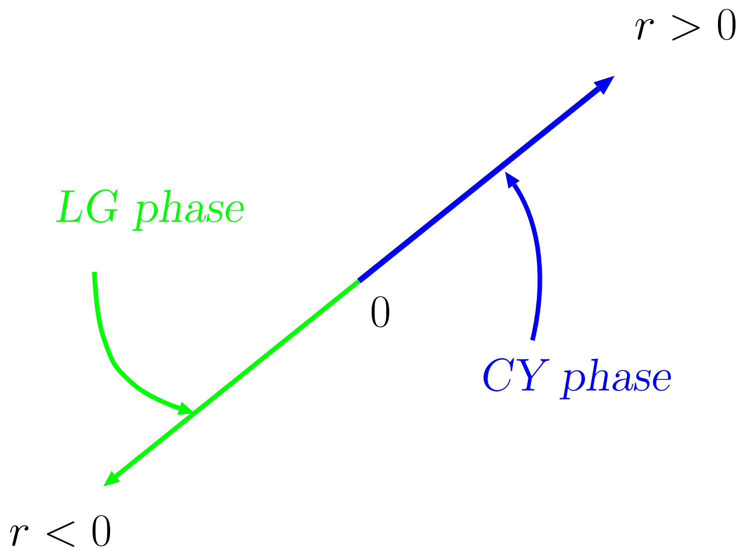
$$\begin{aligned} U(s,p,\sigma) \;&=\; \frac{1}{2e^2} D^2 + 2|\sigma|^2 \Big(\sum_{i=0}^{N-1} |s_i|^2 + 4 |s_N|^2 + (N-2)^2 |p|^2 \Big) \\ &\qquad + 4 |s_N|^2 \sum_{i=0}^{N-1} |s_i|^2 + \Big| \sum_{i=0}^{N-1} s_i^2 \Big|^2 \\ \frac{1}{e^2} D \;&=\; r - \sum_{i=0}^{N-1} |s_i|^2 + 2 |s_N|^2 + (N-2) |p|^2 \end{aligned}$$

Condition of SUSY vacua:

$$D \;=\; 0 \, , \qquad U(\langle s \rangle, \langle p \rangle, \langle \sigma \rangle) \;=\; 0 \, , \qquad \mathrm{d}U(\langle s \rangle, \langle p \rangle, \langle \sigma \rangle) \;=\; 0$$

The solution depends on FI parameter r.

- Low energy theory



CY phase: $r > 0$

sigma model on complex line bundle over $\mathbf{Q}^{N-2} = \mathcal{X}_1$

S_i: $\mathbf{Q}^{N-2}$, P: fiber

LG phase: $r < 0$

$\mathbb{Z}_{N-2}$ "Landau-Ginzburg" orbifold model on $\mathcal{X}_2$

chiral field S_i $(i = 0, 1, \cdots, N-1, N)$

superpotential: $W' = S_N \sum_{i=0}^{N-1} S_i^2$

It's a strange superpotential for LG!!

(No isolated singularity)

Can we treat this as LG minimal model?

Can we understand (co)homology of this CY manifold?

I am studying this point now...

Construction 2: GLSM for line bundle over $\mathbf{P}^{N-1} \times \mathbf{P}^{M-1}$

Contents of chiral superfields and their $U(1) \times U(1)$ charges:

$$
\begin{aligned}
\Phi_i \; &= \big(\, S_0 \, , \, S_1 \, , \, \cdots , \, S_{N-1} \, , \, T_0 \, , \, T_1 \, , \, \cdots , \, T_{M-1} \, , \quad P \quad \big) \\
Q_{i,1} &= \big(\, 1 \, , \, 1 \, , \cdots , \quad 1 \quad , \, 0 \, , \, 0 \, , \cdots , \quad 0 \quad , -N \, \big) \\
Q_{i,2} &= \big(\, 0 \, , \, 0 \, , \cdots , \quad 0 \quad , \, 1 \, , \, 1 \, , \cdots , \quad 1 \quad , -M \, \big)
\end{aligned}
$$

No superpotential

- Supersymmetric vacua

  Potential energy (scalar field only)

$$
\begin{aligned}
U(s,t,p,\sigma) \;&=\; \frac{1}{2e_1^2} D_1^2 + \frac{1}{2e_2^2} D_2^2 \\
&\quad + 2|\sigma_1|^2 \Big(\sum_{i=0}^{N-1} |s_i|^2 + N^2 |p|^2 \Big) + 2|\sigma_2|^2 \Big(\sum_{j=0}^{M-1} |t_j|^2 + M^2 |p|^2 \Big)
\end{aligned}
$$

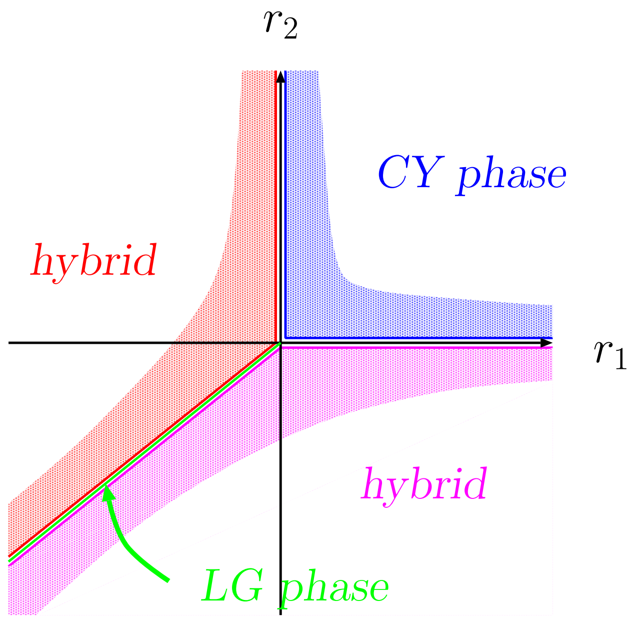
$$
\begin{aligned}
\frac{1}{e_1^2} D_1 \;&=\; r_1 - \sum_{i=0}^{N-1} |s_i|^2 + N|p|^2 \\
\frac{1}{e_2^2} D_2 \;&=\; r_2 - \sum_{j=0}^{M-1} |t_j|^2 + N|p|^2
\end{aligned}
$$

  Condition of SUSY vacua:

$$
D_a \;=\; 0 \, , \qquad U(\langle s \rangle, \langle t \rangle, \langle p \rangle, \langle \sigma \rangle) \;=\; 0 \, , \qquad \mathrm{d}U(\langle s \rangle, \langle t \rangle, \langle p \rangle, \langle \sigma \rangle) \;=\; 0
$$

The solution depends on FI parameters (r_1, r_2).

- Low energy theory



CY phase: $r_1 > 0$, $r_2 > 0$

sigma model on complex line bundle over $\mathbf{P}^{N-1} \times \mathbf{P}^{M-1}$

S_i, T_j: $\mathbf{P}^{N-1} \times \mathbf{P}^{M-1}$, P: fiber

hybrid: $r_1 < 0$, $-Mr_1 + Nr_2 > 0$

sigma model on vector bundle over $\mathbf{P}^{M-1}$

T_j: $\mathbf{P}^{M-1}$, S_i: $\mathbb{C}^N$ fiber

or

Landau-Ginzburg orbifold model

S_i: chiral fields of LG orbifold ($\mathbb{Z}_N$), T_j: $M-1$ free fields

LG phase: $r_1, r_2 < 0$, $Mr_1 = Nr_2$

$U(1) \times \mathbb{Z}_d$ gauged LG orbifold model

d: greatest common divisor of N and M

T-dualized Theory

This theory is written by twisted chiral superfields only:

$$\begin{aligned}\widetilde{\mathcal{L}} &= \int \mathrm{d}^4\theta \Big\{ \sum_a \Big(-\frac{1}{e_a^2}\overline{\Sigma}_a\Sigma_a\Big) - \frac{1}{2}\sum_i (Y_i + \overline{Y}_i)\log(Y_i + \overline{Y}_i)\Big\} \\ &\quad + \Big(\frac{1}{\sqrt{2}}\int \mathrm{d}^2\widetilde{\theta}\,\widetilde{W} + (c.c.)\Big) \\ \widetilde{W} &= \sum_a \Sigma_a\Big(t_a - \sum_i Q_{i,a}Y_i\Big) + \sum_i \mathrm{e}^{-Y_i}\end{aligned}$$

Y_i: twisted chiral superfield, $\overline{D}_+ Y_i = D_- Y_i = 0$

periodicity $\quad Y_i \sim Y_i + 2\pi i$

Relation between chiral superfields and twisted chiral superfields:

$$Y_i + \overline{Y}_i \ = \ 2\sum_a \overline{\Phi}_i\Phi_i\,\mathrm{e}^{2Q_{i,a}V_a}$$

Low energy theory ($e_a \gg 1$): integrating out Σ_a

$\Rightarrow$ LG orbifold model with twisted superpotential:

$$\widetilde{W} \ = \ \sum_i \mathrm{e}^{-Y_i} \qquad \text{with constraints} \quad t_a \ = \ \sum_i Q_{i,a}Y_i$$

Investigating this twisted superfield,

we can understand *ac*-ring structures of this model.

We can also understand cc-ring structure.

Construction 0: review

T-dual of GLSM for $\mathcal{O}(-N)$ bundle over $\mathbf{P}^{N-1}$ $(\lambda = 0)$,

or hypersurface in $\mathbf{P}^{N-1}$ $(\lambda = 1)$

$$\widetilde{W} = \sum_{i=1}^{N} \mathrm{e}^{-Y_i} + \mathrm{e}^{-Y_P} , \qquad \text{constraint:} \quad t = \sum_{i=1}^{N} Y_i - N Y_P$$

$$Y_i + \overline{Y}_i = 2\,\overline{S}_i S_i\,\mathrm{e}^{2V} , \quad Y_P + \overline{Y}_P = 2\,\overline{P} P\,\mathrm{e}^{-2NV}$$

Sokution of the above constraint:

$$\mathrm{e}^{-Y_i/N} = X_i \qquad \mathrm{e}^{-Y_P} = \mathrm{e}^{t/N} X_1 X_2 \cdots X_N$$

Residual symmetry from periodicity:

$$G = (\mathbb{Z}_N)^{N-1}$$

We obtain the T-dualized LG orbifold model on $\mathcal{W}$ (with $\lambda = 0$):

$$\widetilde{\mathrm{LG}} \ : \quad \widetilde{W}//G = \Big(\sum_{i=1}^{N} X_i^N + \mathrm{e}^{t/N} \prod_{i=1}^{N} X_i \Big) \Big/\Big/ (\mathbb{Z}_N)^{N-1}$$

Theories on CY and $\widetilde{\mathrm{LG}}$ are same!

$(\mathcal{V}, \mathcal{W})$ is a mirror pair!

It is convenient to introduce a period

in order to read the "fundamental" representation of fields.

Definition of the period ($\lambda = 0$):

$$\Pi_{\lambda=0} \equiv \langle \widetilde{\gamma} \,|\, 1 \rangle_{\lambda=0} = \int_{\widetilde{\gamma}} \Omega = \int \mathrm{d}\Sigma \, \mathrm{d}Y_P \prod_{i=1}^{N} \mathrm{d}Y_i \, \exp\big(-\widetilde{W}\big)$$

"BPS mass" of D-brane wrapping around SLAG

In the case of $\lambda = 1$:

$$\Pi_{\lambda=1} = \int \mathrm{d}\Sigma \, \mathrm{d}Y_P \prod_{i=1}^{N} \mathrm{d}Y_i \, N\Sigma \, \exp\big(-\widetilde{W}\big)$$

definition of the period

By re-deifining several fields,

we read the "fundamental" representations of twisted chiral fields.

(cf.) relation these periods are derived from:

$$\Pi_{\lambda=1} = \langle \gamma \,|\, 1 \rangle_{\lambda=1} = \langle \widetilde{\gamma} \,|\, \delta \rangle_{\lambda=0}$$

$\langle \gamma \,|$: B-type boundary states (γ is $N-2$-cycle)

$\langle \widetilde{\gamma} \,|$: B-type boundary states ($\widetilde{\gamma}$ is N-cycle)

$|\, \delta \rangle = N\Sigma \,|\, 1 \rangle$: A-type field

D-brane wrapping around SLAG and BPS mass.

Calculation:

$$\Pi_{\lambda=1} \;=\; \int \mathrm{d}Y_P \prod_{i=1}^{N} \mathrm{d}Y_i \, \mathrm{e}^{-Y_P} \, \delta\Big(t - \sum_{i=1}^{N} Y_i + N Y_P\Big) \exp\Big(- \sum_{i=1}^{N} \mathrm{e}^{-Y_i} - \mathrm{e}^{-Y_P}\Big)$$

Re-definition of fields:

$$\mathrm{e}^{-Y_P} \;=\; \widetilde{P} \qquad \mathrm{e}^{-Y_i} \;=\; \widetilde{P} U_i$$

$$\Rightarrow \quad \Pi_{\lambda=1} \;=\; \int \prod_{i=1}^{N} \frac{\mathrm{d}U_i}{U_i} \delta\Big(t + \sum_{i=1}^{N} \log U_i\Big) \delta\Big(1 + \sum_{i=1}^{N} U_i\Big) \exp(-0)$$

The twisted superpotential is trivial!

Target space $\mathcal{W}'$ is submanifold of algebraic torus $(\mathbb{C}^*)^N$:

$$\mathcal{W}' : \quad t \;=\; -\sum_{i=1}^{N} \log U_i \;, \quad 1 + \sum_{i=1}^{N} U_i \;=\; 0 \quad \text{in } (\mathbb{C}^*)^N$$

$(\mathcal{M}, \mathcal{W}')$ is a mirror pair!

Construction 1: T-dual of GLSM for line bundle over $\mathbf{Q}^{N-2}$

$$\widetilde{W} \;=\; \sum_{i=0}^{N} X_i^{N-2} + \mathrm{e}^{t/(N-2)} \Big(\prod_{i=0}^{N-1} X_i \Big) / X_N^2$$

$$Y_i + \overline{Y}_i \;=\; 2\,\overline{\Phi}_i \Phi_i \, \mathrm{e}^{2Q_i V} \;, \qquad X_i^{N-2} \;=\; \mathrm{e}^{-Y_i}$$

Moreover we should mod out some symmetry derived from the periodicity:

$$G \;=\; \big(\mathbb{Z}_{N-2}\big)^N$$

LG orbifold model is written by this twisted superpotential:

$$\widetilde{W} // G$$

Isolated singularity at the origin,

Only one supersymmetric vacuum, etc.

good behavior to study chiral rings

Sorry, further investigations are in progress...

question:

How and where does the effect of chiral superpotentials appear?

Can we construct *ac*-ring structure? (finite? infinite?)

What is the mirror manifold of line bundle over $\mathbf{Q}^{N-2}$?

Review: Hermitian Symmetric Spaces

projective space: $\mathbf{P}^{N-1} = SU(N)/[SU(N-1) \times U(1)]$

$\vec{\phi} \in \mathbb{C}^N$: identification $\vec{\phi} \sim \lambda\vec{\phi}$, $\lambda \in \mathbb{C}^1$

$\vec{\phi}^T = \left(1, \varphi^i\right)$, $(i = 1, 2, \cdots, N-1)$: local coordinates

Kähler potential K:

$$K = r \log \vec{\phi}^\dagger \vec{\phi} = r \log \left\{1 + |\varphi^i|^2\right\}, \quad r = \text{constant}$$

quadric surface: $\mathbf{Q}^{N-2} = SO(N)/[SO(N-2) \times U(1)]$

$\mathbf{P}^{N-1} + [\vec{\phi}^2 = 0]$:

$\vec{\phi}^T = \left(1, \varphi^i, -\frac{1}{2}(\varphi^i)^2\right)$, $(i = 1, 2, \cdots, N-2)$: local coordinates

$$K = r \log \vec{\phi}^\dagger \vec{\phi} = r \log \left\{1 + |\varphi^i|^2 + \frac{1}{4}(\varphi^i)^2(\varphi^{*j})^2\right\}$$

exceptional groups: $E_6/[SO(10) \times U(1)]$, $E_7/[E_6 \times U(1)]$

$\mathbf{P}^{26} + [\Gamma_{ijk}\phi^j\phi^k = 0]$, $\mathbf{P}^{55} + [d_{\alpha\beta\gamma\delta}\phi^\beta\phi^\gamma\phi^\delta = 0]$:

Γ_{ijk}: rank-3 symmetric tensor invariant under E_6

$d_{\alpha\beta\gamma\delta}$: rank-4 symmetric tensor invariant under E_7

$[E_6]$: $\vec{\phi}^T = \left(1, \varphi_\alpha, -\frac{1}{2\sqrt{2}}\varphi C \sigma_A^\dagger \varphi\right)$, $(\alpha = 1, 2, \cdots, 16;\ A = 1, 2, \cdots, 10)$

$$K = r \log \left\{1 + |\varphi_\alpha|^2 + \frac{1}{8}|\varphi C \sigma_A^\dagger \varphi|^2\right\}$$

$[E_7]$: $\vec{\phi}^T = \left(1, \varphi^i, \frac{1}{2}\Gamma_{ijk}\varphi^j\varphi^k, \frac{1}{6}\Gamma_{ijk}\varphi^i\varphi^j\varphi^k\right)$, $(i = 1, 2, \cdots, 27)$

$$K = r \log \left\{1 + |\varphi^i|^2 + \frac{1}{4}|\Gamma_{ijk}\varphi^j\varphi^k|^2 + \frac{1}{36}|\Gamma_{ijk}\varphi^i\varphi^j\varphi^k|^2\right\}$$

Grassmannian: $\mathbf{G}_{N,M} = U(N)/[U(N-M) \times U(M)]$

Φ : $N \times M$ matrix, identification $\Phi \sim \Phi V$ $[V \in U(M)]$

$$\Phi = \begin{pmatrix} \mathbf{1}_M \\ \varphi_{Aa} \end{pmatrix}, \ (A = 1, 2, \cdots, N-M;\ a = 1, 2, \cdots, M)$$

Kähler potential K:

$$K = r \log \det \Phi^\dagger \Phi = r \log \det \left\{ \mathbf{1}_M + \varphi^\dagger \varphi \right\}$$

$Sp(N)/U(N)$

$\mathbf{G}_{2N,N} + [\varphi^T - \varphi = 0]$:

$$\Phi = \begin{pmatrix} \mathbf{1}_N \\ \varphi_{ab} \end{pmatrix}, \ (1 \leq a \leq b \leq N)$$

$$K = r \log \det \Phi^\dagger \Phi = r \log \det \left\{ \mathbf{1}_N + \varphi^\dagger \varphi \right\}$$

$SO(2N)/U(N)$

$\mathbf{G}_{2N,N} + [\varphi^T + \varphi = 0]$:

$$\Phi = \begin{pmatrix} \mathbf{1}_N \\ \varphi_{ab} \end{pmatrix}, \ (1 \leq a < b \leq N)$$

$$K = r \log \det \Phi^\dagger \Phi = r \log \det \left\{ \mathbf{1}_N + \varphi^\dagger \varphi \right\}$$