

— *Prologue* —

My research interests:

- Two-dimensional Field Theory and $1/N$ expansion (M2)
- Non-compact Calabi-Yau manifolds (D1)
- Mirror symmetry and topological string (D2, and now)

SUSY2004 at Tsukuba(...?)

- **SUPERGRAVITY** and matrix theory (D3, and now)

my Doctor thesis (Osaka University)



Zero-mode Spectrum of Eleven-dimensional Theory on the Plane-wave Background

木村 哲士 (KEK)

©RIKEN (2004 5/18)

based on my Doctor thesis ([hep-th/0402054](https://arxiv.org/abs/hep-th/0402054))

— *Introduction* —

Towards quantum theory of Gravity

→ higher dim. Supergravity and theories of Extended Objects

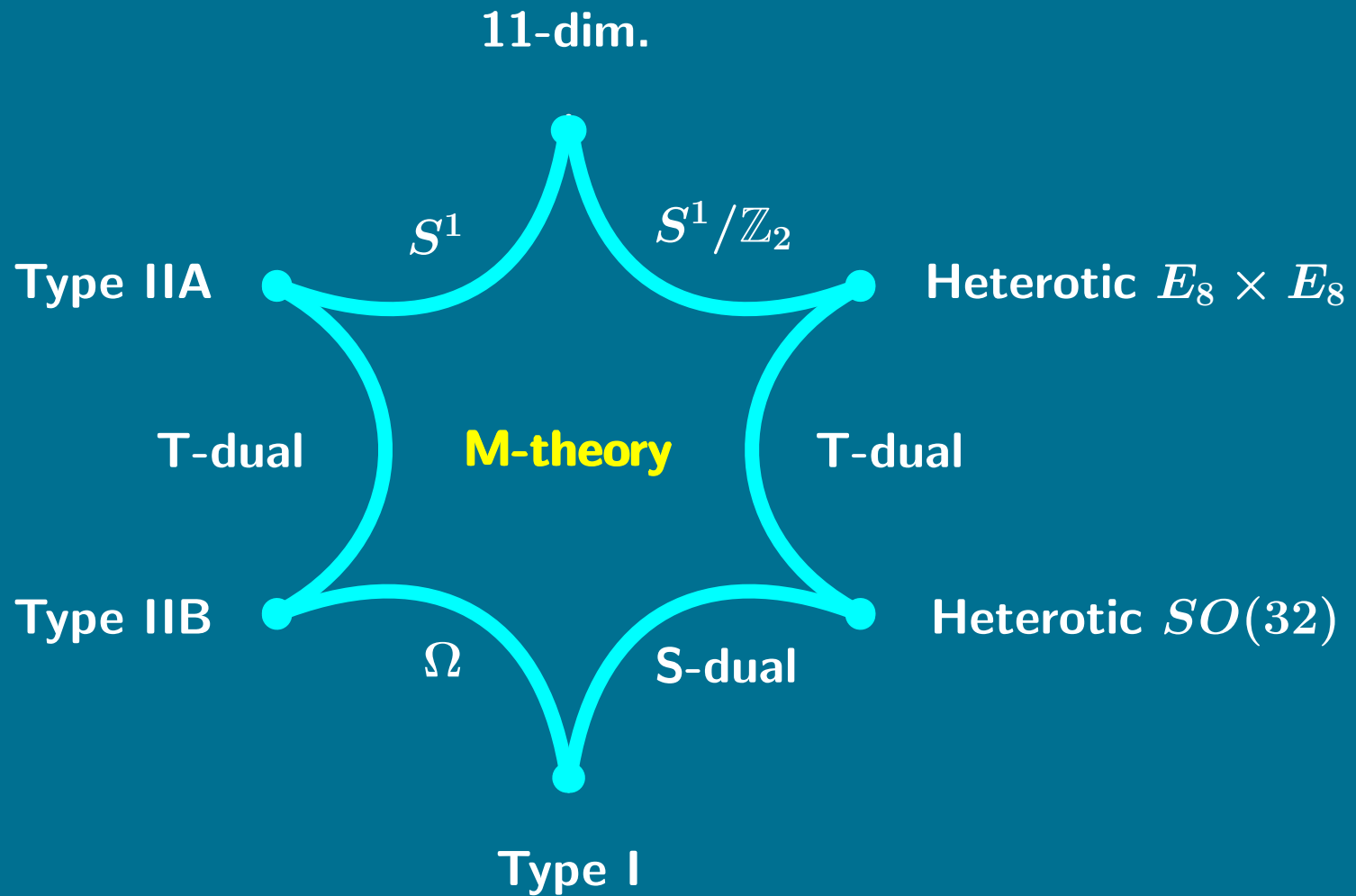
- 10-dimensions:

type I, IIA, IIB, heterotic Supergravities and Superstrings

- 11-dimensions:

Supergravity and Supermembrane

difficulties = $\left\{ \begin{array}{l} \text{supergravities} \quad : \quad \text{classical theories} \\ \text{superstrings} \quad : \quad \text{more than one consistent theories} \\ \text{supermembrane} \quad : \quad \text{unstable} \end{array} \right.$



M(atr ix) Theory conjecture

M-theory compactified on circle of radius R

→ type IIA string (M-theory conjecture)

N D0-branes ← (0 + 1)-dim. reduction of 10D $U(N)$ SYM

(fields = $N \times N$ matrix variables)

In **flat** background,

- Hamiltonian corresponds to that of supermembrane
- new interpretation for supermembrane instability
- graviton amplitudes agree
with those of 11-dim. supergravity
- *etc.*

How about on **curved** background??

(Model should be well-defined on **any** background.)

On general curved background,
it is difficult or impossible
to solve the dynamics in terms of today's technique.

\therefore highly non-trivial interactions

But,

Plane-wave background is a useful spacetime!

- Maximally Supersymmetric (32 supercharges)
- Penrose Limit of $AdS_p \times S^q$ (considering around a null geodesic)
- IIB string is solved as a massive free model (beyond AdS_5/CFT_4)

Matrix Model \Rightarrow Berenstein, Maldacena and Nastase

Supermembrane \Rightarrow $\left\{ \begin{array}{l} \bullet \text{ Dasgupta, Sheikh-Jabbari} \\ \text{and Van Raamsdonk} \\ \bullet \text{ Sugiyama and Yoshida} \end{array} \right.$

Supergravity \Rightarrow **UNKNOWN YET!**

It is important to study supergravity on the plane-wave
because of the lack of investigation
whether BMN matrix model includes the supergraviton modes or not.

Contents

- Geometry of the Plane-wave
- Matrix Theory
- Supergravity
- Conclusion

In this talk only the zero-mode spectrum will be considered.

— Background Geometry —

Metric

$$ds^2 = -2 dx^+ dx^- + G_{++} \cdot (dx^+)^2 + \sum_{I=1}^9 (dx^I)^2$$

$$G_{++} = -\left(\frac{\mu}{3}\right)^2 \sum_{\tilde{I}=1}^3 (x^{\tilde{I}})^2 - \left(\frac{\mu}{6}\right)^2 \sum_{I'=4}^9 (x^{I'})^2$$

Freund-Rubin Ansatz

$$F_{123+} = \mu > 0$$

Vielbeins, Spin Connections, etc.

$$e_+^+ = e_-^- = 1 \quad e_+^- = -\frac{1}{2}G_{++}$$

$$E_+^+ = E_-^- = 1 \quad E_+^- = \frac{1}{2}G_{++}$$

$$\omega_+^{I-} = \frac{1}{2}\partial_I G_{++}$$

$$\Gamma_{++}^I = \Gamma_{+I}^- = -\frac{1}{2}\partial_I G_{++}$$

$$R^I_{+J+} = -\frac{1}{2}\partial_I\partial_J G_{++} \quad \mathcal{R}_{++} = \frac{1}{2}\mu^2 \quad \mathcal{R} = 0$$

— *Matrix Theory on the Plane-wave* —

D. Berenstein, J.M. Maldacena and H. Nastase:

“Strings in flat space and pp-wave from $\mathcal{N}=4$ super Yang-Mills”

JHEP 0204 (2002) 013, hep-th/0202021.

Lagrangian

N D0-branes' effective action

arising from the dimensional reduction of 10-dim. $U(N)$ SYM

	+	1	2	...	8	9	-
M-theory	●	●	●	●	●	●	●
type IIA	●	●	●	●	●	●	S^1
D0-branes	●	-	-	-	-	-	(S^1)

N D0-branes on the plane-wave ($x^- \sim x^- + 2\pi R$)

$$\begin{aligned} L = & \text{Tr} \left\{ \frac{1}{2R} D_\tau X^I D_\tau X^I + \frac{R}{4} [X^I, X^J]^2 + \frac{i}{2} \Psi^\dagger D_\tau \Psi + \frac{1}{2} \Psi^\dagger \gamma^I [X^I, \Psi] \right\} \\ & + R \text{Tr} \left\{ -\frac{1}{2} \left[\left(\frac{\mu}{3R} \right)^2 (X^{\tilde{I}})^2 + \left(\frac{\mu}{6R} \right)^2 (X^{I'})^2 \right] - \frac{i\mu}{8R} \Psi^\dagger \gamma^{123} \Psi \right\} \\ & + R \text{Tr} \left\{ -\frac{i\mu}{3R} \epsilon_{\tilde{I}\tilde{J}\tilde{K}} X^{\tilde{I}} X^{\tilde{J}} X^{\tilde{K}} \right\} \end{aligned}$$

Additional terms are derived from

- **supervielbeins on the plane-wave**
- **Myers' effect: Dielectric Phenomenon**

Hamiltonian

$$H = R \text{Tr} \left\{ \frac{1}{2} P_I^2 - \frac{1}{4} [X^I, X^J]^2 - \frac{1}{2} \Psi^\dagger \gamma^I [X^I, \Psi] \right. \\ \left. + \frac{1}{2} \left[\left(\frac{\mu}{3R} \right)^2 (X^{\tilde{I}})^2 + \left(\frac{\mu}{6R} \right)^2 (X^{I'})^2 \right] + \frac{i\mu}{8R} \Psi^\dagger \gamma^{123} \Psi \right. \\ \left. + \frac{i\mu}{3R} \epsilon_{\tilde{I}\tilde{J}\tilde{K}} X^{\tilde{I}} X^{\tilde{J}} X^{\tilde{K}} \right\}$$

This corresponds to the **Light-cone** Hamiltonian

of **matrix regularized supermembrane** on the plane-wave.

Lagrangian has 32 supersymmetries:

- 16 dynamical SUSY: Q
nonlinearly realized
- 16 kinematical SUSY: q
linearly realized

16 Dynamical Supersymmetry

$$\begin{aligned}
 \delta_\epsilon X^I &= \sqrt{R} \Psi^\dagger \gamma^I \epsilon(\tau) & \delta_\epsilon \omega &= \sqrt{R} \Psi^\dagger \epsilon(\tau) \\
 \delta_\epsilon \Psi &= \sqrt{R} \left(-\frac{i}{R} D_\tau X^I \gamma^I \epsilon(\tau) + \frac{1}{2} [X^I, X^J] \gamma^{IJ} \epsilon(\tau) \right. \\
 &\quad \left. - \frac{i\mu}{3R} X^{\tilde{I}} \gamma^{\tilde{I}} \gamma^{123} \epsilon(\tau) + \frac{i\mu}{6R} X^{I'} \gamma^{I'} \gamma^{123} \epsilon(\tau) \right) \\
 \epsilon(\tau) &= \exp \left(-\frac{\mu}{12} \gamma^{123} \tau \right) \epsilon_0
 \end{aligned}$$

$$\begin{aligned}
 Q &= \sqrt{R} \text{Tr} \left\{ P^I \gamma^I \Psi - \frac{i}{2} [X^I, X^J] \gamma^{IJ} \Psi \right. \\
 &\quad \left. - \frac{\mu}{3R} X^{\tilde{I}} \gamma^{\tilde{I}} \gamma^{123} \Psi - \frac{\mu}{6R} X^{I'} \gamma^{I'} \gamma^{123} \Psi \right\}
 \end{aligned}$$

This supersymmetry acts on the $SU(N)$ interaction part of the theory.

16 Kinematical Supersymmetry

$$\begin{aligned}\delta_\eta X^I &= 0 & \delta_\eta \omega &= 0 & \delta_\eta \Psi &= \frac{1}{\sqrt{R}} \eta(\tau) \\ \eta(\tau) &= \exp\left(\frac{\mu}{4} \gamma^{123} \tau\right) \eta_0 \\ q &= \frac{1}{\sqrt{R}} \text{Tr}(\Psi)\end{aligned}$$

This supercharge acts only $U(1)$ part of the theory.

(center of mass)

From now on we consider only the $U(1)$ part of the theory.

Commutation Relations

$$[H, q_\alpha] = -\frac{\mu}{4}(i\gamma^{123}q)_\alpha$$

$SU(4) \times SU(2)$ Decomposition

$$q = \begin{pmatrix} q_{i\alpha} \\ \epsilon_{\alpha\beta} q^{\dagger i\beta} \end{pmatrix} \quad i\gamma^{123} = \begin{pmatrix} \mathbf{1}_8 & 0 \\ 0 & -\mathbf{1}_8 \end{pmatrix}$$

Thus the above commutation relation reduces to

$$[H, q_{i\alpha}] = -\frac{\mu}{4}q_{i\alpha} \quad [H, q^{\dagger i\alpha}] = \frac{\mu}{4}q^{\dagger i\alpha}$$

Construction of Zero-mode Supermultiplet

$$q_{i\alpha} |\Lambda\rangle = 0 \quad \text{for } \forall i, \forall \alpha$$

“Fermionic” Raising Operator

$$q^{\dagger i\alpha} \sim \left(\square, \square \right)$$

We will construct the ground state supermultiplet.

free $U(1)$ part of the matrix model

degrees of center of mass = superparticles

Floor 0 (ground state):

$$|\Lambda\rangle = |1, 1\rangle$$

Floor 1:

$$(\square, \square) \otimes |1, 1\rangle = |\square, \square\rangle$$

Floor 2:

$$(\square, \square) \otimes |\square, \square\rangle$$

$$= \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \rangle \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \rangle \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \rangle \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \rangle$$

Floor 3:

$$(\square, \square) \otimes \left\{ \left| \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right\rangle \oplus \left| \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right\rangle \right\}$$

$$= \left| \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right\rangle \oplus \left| \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right\rangle \oplus \left| \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right\rangle$$

$$\oplus \left| \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right\rangle \oplus \left| \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right\rangle \oplus \left| \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \right\rangle \oplus \left| \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right\rangle$$

Floor N	$SU(4) \times SU(2)$ Representation			Energy
8		$(\mathbf{1}, \mathbf{1})$		2μ
7		$(\bar{\mathbf{4}}, \mathbf{2})$		$7\mu/4$
6	$(\bar{\mathbf{6}}, \mathbf{3})$		$(\bar{\mathbf{10}}, \mathbf{1})$	$3\mu/2$
5	$(\mathbf{4}, \mathbf{4})$		$(\bar{\mathbf{20}}, \mathbf{2})$	$5\mu/4$
4	$(\mathbf{1}, \mathbf{5})$	$(\mathbf{15}, \mathbf{3})$	$(\mathbf{20}', \mathbf{1})$	μ
3		$(\bar{\mathbf{4}}, \mathbf{4})$	$(\mathbf{20}, \mathbf{2})$	$3\mu/4$
2		$(\mathbf{6}, \mathbf{3})$	$(\mathbf{10}, \mathbf{1})$	$\mu/2$
1		$(\mathbf{4}, \mathbf{2})$		$\mu/4$
0		$(\mathbf{1}, \mathbf{1})$		0

The multiplet grouped into irreducible representations of $SU(4) \times SU(2)$ on each floor of equal energies.

— *Supergravity on the Plane-wave* —

T.K. and K. Yoshida:

“Spectrum of eleven-dimensional supergravity on a pp-wave background”

Phys. Rev. D68 (2003) 125007, hep-th/0307193.

Field Contents

e_M^A : vielbein , E_A^M : inverse vielbein

Ψ_M : gravitino (vectorial Majorana spinor)

C_{MNP} : three-form gauge field

ω_M^{AB} : spin connection

Lagrangian

$$\begin{aligned} \mathcal{L} = & e \mathcal{R} - \frac{1}{2} e \bar{\Psi}_M \Gamma^{MNP} D_N \Psi_P - \frac{1}{48} e F_{MNPQ} F^{MNPQ} \\ & - \frac{1}{192} e \bar{\Psi}_M \tilde{\Gamma}^{MNPQRS} \Psi_N F_{PQRS} \\ & - \frac{1}{(144)^2} \epsilon^{MNPQRSUVWXY} F_{MNPQ} F_{RSUV} C_{WXY} \end{aligned}$$

Light-cone Hamiltonian $H = i\partial_+$:

$$\begin{aligned} H &= \frac{1}{-2p_-} \left\{ (p_I)^2 - \tilde{G}_{++} \cdot (p_-)^2 - \alpha \mu p_- \right\} \\ &= \frac{1}{3} \mu \sum_{\tilde{I}} \bar{a}^{\tilde{I}} a^{\tilde{I}} + \frac{1}{6} \mu \sum_{I'} \bar{a}^{I'} a^{I'} + \frac{1}{2} \mu (2 + \alpha) \end{aligned}$$

$$\begin{aligned} a^{\tilde{I}} &\equiv \frac{1}{\sqrt{2\tilde{m}}} \{ p_{\tilde{I}} + \tilde{m} \partial_{p_{\tilde{I}}} \} & a^{I'} &\equiv \frac{1}{\sqrt{2m'}} \{ p_{I'} + m' \partial_{p_{I'}} \} & [a^I, \bar{a}^J] &= \delta^{IJ} \\ \tilde{m} &\equiv -\frac{1}{3} \mu p_- & m' &\equiv -\frac{1}{6} \mu p_- \end{aligned}$$

Zero Point Energy

$$(\square + \alpha \mu i\partial_-)\phi = 0 \quad \longrightarrow \quad E_0 = \frac{1}{2} \mu (2 + \alpha)$$

Classical Field Equations

$$\begin{aligned}\mathcal{R}_{MN} &= -\frac{1}{144} g_{MN} F_{PQRS} F^{PQRS} + \frac{1}{12} F_{MPQR} F_N{}^{PQR} \\ 0 &= \Gamma^{MNP} D_N \Psi_P + \frac{1}{96} \tilde{\Gamma}^{MNPQRS} \Psi_N F_{PQRS} \\ 0 &= \nabla^Q \{ e F_{QMNP} \} \\ &\quad - \frac{18}{(144)^2} \varepsilon_{MNP}{}^{QRSUVWXY} F_{QRSU} F_{VWXY}\end{aligned}$$

Fluctuations

$$\begin{aligned}g_{MN} &\rightarrow g_{MN} + h_{MN} & \Psi_M &\rightarrow 0 + \psi_M \\ F_{MNPQ} &\rightarrow F_{MNPQ} + \mathcal{F}_{MNPQ}\end{aligned}$$

Light-cone Gauge Fixing and Derived Constraints:

$$h_{-M} = 0 \quad \mathcal{C}_{-MN} = 0 \quad \psi_- = 0$$

Constraints for Bosons:

$$h_M{}^M = 0 \quad \partial^M h_{IM} = 0 \quad \partial^M h_{+M} = \frac{1}{3} \mu \mathcal{C}_{123}$$
$$\partial^M \mathcal{C}_{IJM} = 0 \quad \partial^M \mathcal{C}_{+IM} = 0$$

Constraints for Fermions:

$$\Gamma^M \psi_M = 0 \quad \partial^M \psi_M = 0$$
$$\Gamma^+ \Gamma^M D_M \psi_I = 0$$

Non-trivial Equations for Bosonic Fields

$$0 = \square h_{\tilde{I}\tilde{J}} - \frac{2}{3} \mu \delta_{\tilde{I}\tilde{J}} \partial_- \mathcal{C}$$

$$0 = \square h_{\tilde{I}J'} - \mu \partial_- \mathcal{C}_{\tilde{I}J'}$$

$$0 = \square h_{I'J'} + \frac{1}{3} \mu \delta_{I'J'} \partial_- \mathcal{C}$$

$$0 = \square \mathcal{C} + 2 \mu \partial_- h_{\tilde{I}\tilde{I}}$$

$$0 = \square \mathcal{C}_{\tilde{I}J'} + \mu \partial_- h_{\tilde{I}J'}$$

$$0 = \square \mathcal{C}_{\tilde{I}J'K'}$$

$$0 = \square \mathcal{C}_{I'J'K'} - \frac{1}{6} \mu \epsilon^{I'J'K'W'X'Y'} \partial_- \mathcal{C}_{W'X'Y'}$$

$$\mathcal{C}_{\tilde{I}J'} \equiv \frac{1}{2} \epsilon_{\tilde{I}K\tilde{L}} \mathcal{C}_{\tilde{K}\tilde{L}J'} \quad \mathcal{C} \equiv 2 \mathcal{C}_{123}$$

$$0 = (\square + \mu i \partial_-) H_{\tilde{I}J'}$$

$$0 = \square h_{\tilde{I}\tilde{J}}^\perp$$

$$0 = (\square + 2\mu i \partial_-) h$$

$$0 = (\square + \mu i \partial_-) \mathcal{C}_{I'J'K'}^\oplus$$

$$0 = (\square - \mu i \partial_-) \bar{H}_{\tilde{I}J'}$$

$$0 = \square h_{I'J'}^\perp$$

$$0 = (\square - 2\mu i \partial_-) \bar{h}$$

$$0 = (\square - \mu i \partial_-) \mathcal{C}_{I'J'K'}^\ominus$$

$$H_{\tilde{I}J'} = h_{\tilde{I}J'} + i\mathcal{C}_{\tilde{I}J'}$$

$$\bar{H}_{\tilde{I}J'} = h_{\tilde{I}J'} - i\mathcal{C}_{\tilde{I}J'}$$

$$h_{\tilde{I}\tilde{J}}^\perp = h_{\tilde{I}\tilde{J}} - \frac{1}{3}\delta_{\tilde{I}\tilde{J}}h_{\tilde{K}\tilde{K}}$$

$$h_{I'J'}^\perp = h_{I'J'} - \frac{1}{6}\delta_{I'J'}h_{K'K'}$$

$$h = h_{\tilde{K}\tilde{K}} + i\mathcal{C}$$

$$\bar{h} = h_{\tilde{K}\tilde{K}} - i\mathcal{C}$$

$$\mathcal{C}_{I'J'K'}^\oplus = \frac{i}{6}\epsilon^{I'J'K'W'X'Y'}\mathcal{C}_{W'X'Y'}^\oplus$$

$$\mathcal{C}_{I'J'K'}^\ominus = -\frac{i}{6}\epsilon^{I'J'K'W'X'Y'}\mathcal{C}_{W'X'Y'}^\ominus$$

Field Equations for Fermions

$$0 = \left(\square + \frac{1}{2} \mu i \partial_- \right) \psi_{\tilde{I}\mathbf{R}}^{\oplus\perp}$$

$$0 = \left(\square - \frac{1}{2} \mu i \partial_- \right) \psi_{\tilde{I}\mathbf{L}}^{\oplus\perp}$$

$$0 = \left(\square - \frac{1}{2} \mu i \partial_- \right) \psi_{I'\mathbf{R}}^{\oplus\perp}$$

$$0 = \left(\square + \frac{1}{2} \mu i \partial_- \right) \psi_{I'\mathbf{L}}^{\oplus\perp}$$

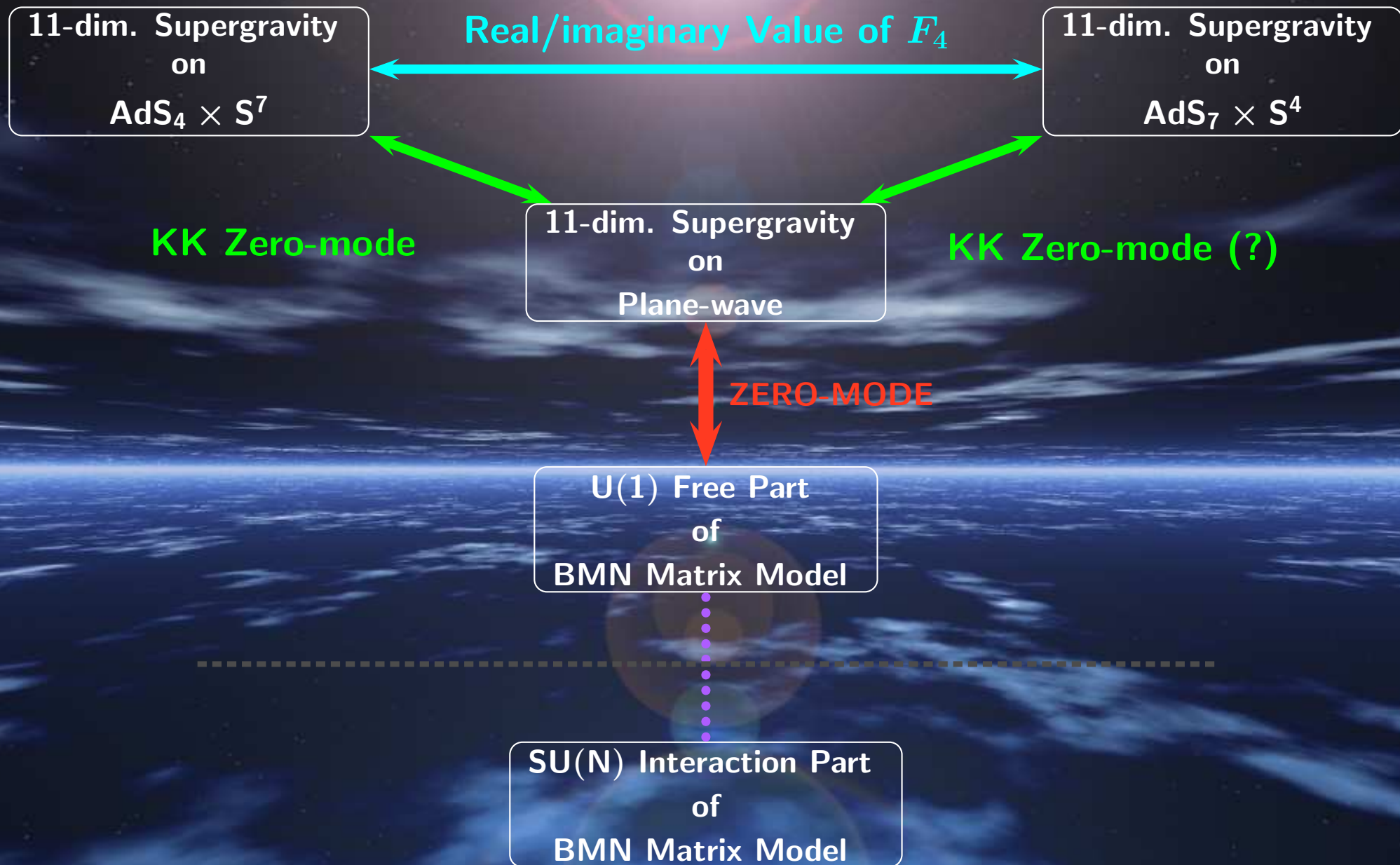
$$0 = \left(\square - \frac{3}{2} \mu i \partial_- \right) \psi_{\mathbf{R}}^{\oplus\parallel}$$

$$0 = \left(\square + \frac{3}{2} \mu i \partial_- \right) \psi_{\mathbf{L}}^{\oplus\parallel}$$

energy E_0	bosonic/fermionic fields				degrees of freedom
2μ			h		1
$7\mu/4$			$\psi_L^{\oplus\parallel}$		8
$3\mu/2$		$H_{\tilde{I}J'}$		$\mathcal{C}_{I'J'K'}^\ominus$	18 + 10
$5\mu/4$		$\psi_{\tilde{I}R}^{\oplus\perp}$		$\psi_{I'L}^{\oplus\perp}$	16 + 40
μ	$h_{\tilde{I}\tilde{J}}^\perp$		$\mathcal{C}_{\tilde{I}J'K'}$	$h_{I'J'}^\perp$	5 + 45 + 20
$3\mu/4$		$\psi_{\tilde{I}L}^{\oplus\perp}$		$\psi_{I'R}^{\oplus\perp}$	16 + 40
$\mu/2$		$\overline{H}_{\tilde{I}J'}$		$\mathcal{C}_{I'J'K'}^\oplus$	18 + 10
$\mu/4$			$\psi_R^{\oplus\parallel}$		8
0			\overline{h}		1

Zero point energy spectrum of all the physical fields of the linearized supergravity on the plane-wave background.

— *Conclusion* —



— *Discussions* —

- (Vacua have already been investigated...)
- Connection to linearized SUGRA on $AdS_{4(7)} \times S^7(4)$
- Graviton Amplitudes on the Plane-wave
- Non-zero-mode Interactions
- (Transverse) M5-brane Formulation

$SO(10, 1)$ Dirac Gamma Matrix

$$\begin{aligned}\{\Gamma^A, \Gamma^B\} &= 2\eta^{AB} \\ (\Gamma^A)^\dagger &= \Gamma_A = -\Gamma^0 \Gamma^A (\Gamma^0)^{-1} \\ C\Gamma^A C^{-1} &= -(\Gamma^A)^T\end{aligned}$$

$SO(9)$ Representation

$$\begin{aligned}\Gamma^0 &= \begin{pmatrix} 0 & i\mathbf{1}_{16} \\ i\mathbf{1}_{16} & 0 \end{pmatrix} & \Gamma^{10} &= \begin{pmatrix} 0 & -i\mathbf{1}_{16} \\ i\mathbf{1}_{16} & 0 \end{pmatrix} \\ \Gamma^I &= \begin{pmatrix} -(\gamma^I)^T & 0 \\ 0 & \gamma^I \end{pmatrix} \\ \{\gamma^I, \gamma^J\} &= 2\delta^{IJ} & (\gamma^I)^\dagger &= \gamma^I\end{aligned}$$

Myers' Effect (Dielectric Phenomenon)

The single Dp -brane action

$$\begin{aligned} S &= S_{\text{DBI}} + S_{\text{WZ}} \\ S_{\text{DBI}} &= -T_p \int d^{p+1} \sigma \left(e^{-\phi} \sqrt{-\det \{ P[G + B]_{ab} + \lambda F_{ab} \}} \right) \\ S_{\text{WZ}} &= \mu_p \int P \left[\sum C^{(n)} e^B \right] e^{\lambda F} \\ \lambda &= 2\pi\alpha' \end{aligned}$$

Wess-Zumino term should be added
in order for the consistency of **T-duality**.

N coincident Dp -branes' action

$$\begin{aligned}
 S_{\text{DBI}} &= -T_p \int d^{p+1}\sigma \text{STr} \left(e^{-\phi} \sqrt{\det(Q^i_j)} \right. \\
 &\quad \left. \times \sqrt{-\det \{ P[E_{ab} + E_{ai}(Q^{-1} - \delta)^{ij}E_{jb}] + \lambda F_{ab} \}} \right) \\
 S_{\text{WZ}} &= \mu_p \int \text{STr} \left(P \left[e^{i\lambda i_X i_X} \left(\sum C^{(n)} e^B \right) \right] e^{\lambda F} \right)
 \end{aligned}$$

where

$$\begin{aligned}
 Q^i_j &= \delta^i_j + i\lambda [X^i, X^k] E_{kj} \\
 E_{\mu\nu} &= G_{\mu\nu} + B_{\mu\nu} \\
 i_X i_X C^{(n)} &= \frac{1}{2(n-2)!} [X^i, X^j] C_{ji\mu_3 \dots \mu_n} dx^{\mu_3} \wedge \dots \wedge dx^{\mu_n}
 \end{aligned}$$

X^i : adjoint scalar matrix in the transverse direction to Dp -branes

WZ term for D0-branes:

$$\begin{aligned}
 S_{\text{WZ}} &= i\lambda\mu_0 \int \text{Tr} \left(\mathbf{P} \left[(\mathbf{i}_X \mathbf{i}_X) C^{(3)} \right] \right) \\
 &= \frac{i\lambda}{2} \mu_0 \int dt \text{Tr} \left(C_{tij} [X^i, X^j] + \lambda C_{ijk} D_t X^i [X^k, X^j] \right)
 \end{aligned}$$

with $B = F = 0$ and $C^{(1)} = C^{(5)} = C^{(7)} = C^{(9)} = 0$

Existence of non-vanishing constant RR field strength:

$$F_{t\tilde{I}\tilde{J}\tilde{K}} = -\mu \epsilon_{\tilde{I}\tilde{J}\tilde{K}}$$

Then, the WZ term reduces to

$$S_{\text{WZ}} = \frac{i}{3} \lambda^2 \mu_0 \int dt \text{Tr} (X^{\tilde{I}} X^{\tilde{J}} X^{\tilde{K}}) F_{t\tilde{I}\tilde{J}\tilde{K}}$$

Relation

11D : $F^{(4)} = dC^{(3)}$ ← couples to M2-brane

↓

10D : $F_{RR}^{(4)} = dC_{RR}^{(3)}$ ← couples to D2-brane

↓

$SU(4) \times SU(2)$ Decomposition

$$\mathbf{16} \rightarrow (\mathbf{4}, \mathbf{2}) \oplus (\bar{\mathbf{4}}, \mathbf{2}) : \quad \Psi \rightarrow \{ \psi_{i\alpha}, \psi^{\dagger i\alpha} \}$$

$$\Psi = \begin{pmatrix} \psi_{i\alpha} \\ \epsilon_{\alpha\beta} \psi^{\dagger i\beta} \end{pmatrix}$$

$$\gamma^{\tilde{I}} = \begin{pmatrix} -\sigma^{\tilde{I}} \otimes \mathbf{1}_4 & 0 \\ 0 & \sigma^{\tilde{I}} \otimes \mathbf{1}_4 \end{pmatrix} \quad \gamma^{I'} = \begin{pmatrix} 0 & \mathbf{1}_2 \otimes \mathbf{g}^{I'} \\ \mathbf{1}_2 \otimes (\mathbf{g}^{I'})^\dagger & 0 \end{pmatrix}$$

$$\{ \sigma^{\tilde{I}}, \sigma^{\tilde{J}} \} = 2\delta^{\tilde{I}\tilde{J}} \cdot \mathbf{1}_2$$

$$\mathbf{g}^{I'} (\mathbf{g}^{J'})^\dagger + \mathbf{g}^{J'} (\mathbf{g}^{I'})^\dagger = 2\delta^{I'J'} \cdot \mathbf{1}_4$$

Commutation Relations

$$\{Q^{\dagger i\alpha}, Q_{j\beta}\} = 2\delta_j^i \delta_\beta^\alpha H + \frac{\mu}{3} \epsilon^{\tilde{I}\tilde{J}\tilde{K}} (\sigma^{\tilde{K}})_\beta^\alpha \delta_j^i \Sigma^{\tilde{I}\tilde{J}} \\ + \frac{i\mu}{6} \delta_\beta^\alpha (\mathbf{g}^{I'J'})_{j^i} \Sigma^{I'J'}$$

$$\{q_{i\alpha}, Q_{j\beta}\} = -i\sqrt{\frac{\mu}{3}} (\mathbf{g}^{I'})_{ij} \epsilon_{\alpha\beta} a^{I'}$$

$$\{q^{\dagger i\alpha}, Q_{j\beta}\} = -i\sqrt{\frac{2\mu}{3}} (\sigma^{\tilde{I}})_\beta^\alpha \delta_j^i a^{\dagger\tilde{I}}$$

$$\{q^{\dagger i\alpha}, q_{j\beta}\} = \delta_\beta^\alpha \delta_j^i P^+$$

$$P^+ = \frac{1}{R} \text{Tr}(1)$$

$$a^{\tilde{I}} = \frac{1}{\sqrt{R}} \text{Tr} \left(\sqrt{\frac{\mu}{6R}} X^{\tilde{I}} + i \sqrt{\frac{3R}{2\mu}} P^{\tilde{I}} \right)$$

$$a^{I'} = \frac{1}{\sqrt{R}} \text{Tr} \left(\sqrt{\frac{\mu}{12R}} X^{I'} + i \sqrt{\frac{3R}{\mu}} P^{I'} \right)$$

$$\Sigma^{\tilde{I}\tilde{J}} = \text{Tr} \left(P^{\tilde{I}} X^{\tilde{J}} - P^{\tilde{J}} X^{\tilde{I}} - i \epsilon^{\tilde{I}\tilde{J}\tilde{K}} \psi^\dagger \sigma^{\tilde{K}} \psi \right)$$

$$\Sigma^{I'J'} = \text{Tr} \left(P^{I'} X^{J'} - P^{J'} X^{I'} - \frac{1}{2} \psi^\dagger \mathbf{g}^{I'J'} \psi \right)$$

Supervielbeins on the Plane-wave

Isometry superalgebra on $AdS_{4(7)} \times S^{7(4)}$

$$F_{\widetilde{M}\widetilde{N}\widetilde{P}\widetilde{Q}} = f \varepsilon_{\widetilde{M}\widetilde{N}\widetilde{P}\widetilde{Q}} \quad \text{Freund-Rubin Ansatz}$$

$$f : \text{real} \rightarrow AdS_4 \times S^7 \quad \text{imaginary} \rightarrow AdS_7 \times S^4$$

$$[P_{\widetilde{A}}, P_{\widetilde{B}}] = \frac{i}{9} f^2 \Sigma_{\widetilde{A}\widetilde{B}} \quad [P_{A'}, P_{B'}] = -\frac{i}{36} f^2 \Sigma_{A'B'}$$

$$[P_A, \Sigma_{CD}] = i(\eta_{AC} P_D - \eta_{AD} P_C)$$

$$i[\Sigma_{AB}, \Sigma_{CD}] = \eta_{AC} \Sigma_{BD} + \eta_{BD} \Sigma_{AC} - \eta_{AD} \Sigma_{BC} - \eta_{BC} \Sigma_{AD}$$

$$[P_A, \overline{Q}] = i\overline{Q} T_A^{BCDE} F_{BCDE}$$

$$[\Sigma_{AB}, \overline{Q}] = \frac{i}{2} \overline{Q} \Gamma_{AB}$$

$$\{Q, \overline{Q}\} = 2i \Gamma_A P^A - \frac{i}{144} \{ \Gamma^{ABCDEF} F_{CDEF} + 24 \Gamma_{CD} F^{ABCD} \} \Sigma_{AB}$$

Representative

$$L(Z) = \ell(x) \cdot \hat{L}(\theta)$$
$$\ell(x) = \exp(i x^A P_A) \quad \hat{L}(\theta) = \exp(i \bar{\theta} Q)$$

Maurer-Cartan one-form

$$\alpha = i^{-1} L^{-1} dL = \hat{E} + \hat{\Omega}$$
$$\hat{E} = E^A P_A + \bar{Q} E \quad \hat{\Omega} = \frac{1}{2} \Omega^{AB} \Sigma_{AB}$$
$$d\alpha + i \alpha \wedge \alpha = 0$$

$$E^A(x, \theta) = e^A + 2 \sum_{n=0}^{15} \frac{1}{(2n+2)!} \bar{\theta} \Gamma^A \mathcal{M}^{2n} D\theta$$

$$E(x, \theta) = \sum_{n=0}^{16} \frac{1}{(2n+1)!} \mathcal{M}^{2n} D\theta$$

$$\Omega^{AB}(x, \theta) = -\omega^{AB} - \frac{1}{72} \sum_{n=0}^{15} \frac{1}{(2n+2)!} \bar{\theta} \{ \Gamma^{ABCDEF} F_{CDEF} + 24 \Gamma_{CD} F^{ABCD} \} \mathcal{M}^{2n} D\theta$$

$$D\theta = d\theta - e^A T_A^{BCDE} \theta F_{BCDE} + \frac{1}{4} \omega^{AB} \Gamma_{AB} \theta$$

$$\mathcal{M}^2 = -2(T_A^{BCDE} \theta) F_{BCDE} (\bar{\theta} \Gamma^A) + \frac{1}{288} (\Gamma_{AB} \theta) \left(\bar{\theta} \left[\Gamma^{ABCDEF} F_{CDEF} + 24 \Gamma_{CD} F^{ABCD} \right] \right)$$

Penrose Limit

$$ds^2_{AdS_4 \times S^7} = R_A^2 \left\{ -\cosh^2 \rho \cdot dt^2 + d\rho^2 + \sinh^2 \rho \cdot d\Omega_2^2 \right\} \\ + R_S^2 \left\{ \cos^2 \theta \cdot d\varphi^2 + d\theta^2 + \sin^2 \theta \cdot d\Omega_5'^2 \right\}$$

Choose a **null geodesic**

$$R_S = 2R_A \quad t = 2\varphi \quad \rho = \theta = 0$$

Take $R_A \rightarrow \infty$ (Penrose Limit):

$$ds^2 = -2dx^+ dx^- + G_{++} \cdot (dx^+)^2 + \sum_{I=1}^9 (dx^I)^2$$

where

$$x = R_A \rho \quad y = 2R_A \theta \\ x^+ = \frac{1}{2}(t + 2\varphi) \cdot \frac{3}{\mu} \quad x^- = R_A^2 (t - 2\varphi) \cdot \frac{\mu}{3}$$

Field Equations for Fluctuations

δg_{MN} :

$$\delta \mathcal{R}_{MN} = -\frac{1}{2} \left\{ \nabla_N \nabla_M h_P{}^P - \nabla_N \nabla^P h_{MP} - \nabla_M \nabla^P h_{NP} \right\} + \frac{1}{2} \hat{\Delta} h_{MN}$$

$$\hat{\Delta} h_{MN} = -\nabla_P \nabla^P h_{MN} - 2 R_{MPNQ} h^{PQ} + \mathcal{R}_M{}^P h_{PN} + \mathcal{R}_N{}^P h_{PM}$$

$\hat{\Delta}$: Lichnerowicz operator

$$\begin{aligned} \delta \mathcal{R}_{MN} = & -\frac{1}{144} h_{MN} F_{PQRS} F^{PQRS} - \frac{1}{72} g_{MN} F^{PQRS} \mathcal{F}_{PQRS} \\ & + \frac{1}{36} h^{PU} g_{MN} F_{PQRS} F_U{}^{QRS} - \frac{1}{4} h^{PU} F_{MPQR} F_{NU}{}^{QR} \\ & + \frac{1}{12} \left(\mathcal{F}_{MPQR} F_N{}^{PQR} + \mathcal{F}_{NPQR} F_M{}^{PQR} \right) \end{aligned}$$

$\delta\Psi_M$:

$$0 = \Gamma^{MNP} D_N \psi_P + \frac{1}{96} \tilde{\Gamma}^{MNPQRS} F_{PQRS} \psi_N$$

δC_{MNP} :

$$\begin{aligned} 0 = & e \nabla^Q \mathcal{F}_{QMNP} \\ & - e \left\{ F_{SMNP} \left(\nabla^Q h_Q^S - \frac{1}{2} \partial^S h_Q^Q \right) + F_{QSNP} \nabla^Q h_M^S \right. \\ & \quad \left. + F_{QMSP} \nabla^Q h_N^S + F_{QMNS} \nabla^Q h_P^S \right\} \\ & - \frac{1}{576} \varepsilon_{MNP}{}^{QRSUVWXY} \mathcal{F}_{QRSU} F_{VWXY} \end{aligned}$$