Towards Mirror Symmetry on Noncompact Calabi-Yau Manifolds

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Preprint: hep-th/0409003 (Proceeding of SUSY 2004)

References: Witten, hep-th/9301042
Hori and Vafa, hep-th/0002222
Higashijima, Nitta and TK, hep-th/0110216, 0202064

Metrics on Noncompact Calabi-Yau

(K.Higashijima, M.Nitta and TK, 2001, 2002)

line bundles	total dim. D	dual Coxeter C	"orbifolding" ℓ
$\mathbb{C} \ltimes \left(\mathbb{C}P^{N-1} = \frac{SU(N)}{SU(N-1)\times U(1)}\right)$	$1+(N-1)$	N	N
$\mathbb{C} \ltimes \left(Q^{N-2} = \frac{SO(N)}{SO(N-2)\times U(1)}\right)$	$1+(N-2)$	$N-2$	$N-2$
$\mathbb{C} \ltimes E_6/[SO(10)\times U(1)]$	$1+16$	12	12
$\mathbb{C} \ltimes E_7/[E_6\times U(1)]$	$1+27$	18	18
$\mathbb{C} \ltimes \left(G_{N,M} = \frac{U(N)}{U(N-M)\times U(M)}\right)$	$1+M(N-M)$	N	MN
$\mathbb{C} \ltimes SO(2N)/U(N)$	$1+\frac{1}{2}N(N-1)$	$N-1$	$N(N-1)$
$\mathbb{C} \ltimes Sp(N)/U(N)$	$1+\frac{1}{2}N(N+1)$	$N+1$	$N(N+1)$

$$\mathcal{K}'_{\rm noncompact}(\rho,\varphi) \;=\; \left(e^{CX}+b\right)^{1/D}, \qquad X \;=\; \log|\rho^{1/\ell}|^2 + K_{\rm compact}(\varphi)$$

$$K_{\mathbb{C}P^{N-1}}(\varphi) = r\log\left(1+\sum_{i=1}^{N-1}|\varphi_i|^2\right)$$

What are the mirror geometries of them?

String theory on noncompact CY

- **Higashijima-Nitta-TK (2001∼2002):**

  SUSY nonlinear sigma models on canonical line bundles on compact Kähler G/H

How about SCFT description?

- **Eguchi-Sugawara (2000∼) and many people:**

  $\mathcal{N} = 2$ Liouville $\otimes$ Landau-Ginzburg = SCFT description of noncompact CY

Why direct product? (holographic picture of $\mathbb{R}^{d-1,1}\times$ (non)singular CY)

↓

- **T. Kimura (2002∼): GLSM for $\mathcal{O}(-N+\ell)$ bundle on $\mathbb{CP}^{N-1}[\ell]$**

Gaugedlinear sigma model

E. Witten (1993), K. Hori and C. Vafa (2000)

$\mathcal{N} = (2,2)$ SUSY gauge theory with matters $(\mathrm{FI}: \; t \equiv r - i\theta)$

$$\mathscr{L} = \int \mathrm{d}^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_i \overline{\Phi}_i \, \mathrm{e}^{2Q_i V} \Phi_i \Big\} + \Big(\frac{1}{\sqrt{2}} \int \mathrm{d}^2\widetilde{\theta} \, (-\Sigma \, t) + c.c. \Big) + \Big(\int \mathrm{d}^2\theta \, W_{\mathrm{GLSM}}(\Phi_i) + c.c. \Big)$$

- field theory realization of toric variety
- two phases in classical vacua:

  $\mathrm{FI} > 0$: differential-geometric phase $\longrightarrow$ SUSY NLSM

  $\mathrm{FI} < 0$: algebro-gemetric phase $\longrightarrow$ LG, orbifold, SCFT

- CY/LG correspondence between the above phases

  harmonic forms $\longleftrightarrow$ NS-NS chiral primary states

- "Mirror" geometry appears in the IR limit of T-dualized theory

Gauged Linear Sigma Model for $\mathcal{O}(-N+\ell)$ bundle on $\mathbb{C}\mathrm{P}^{N-1}[\ell]$

chiral superfield	S_1	$\cdots$	S_N	P_1	P_2
$U(1)$ charge	1	$\cdots$	1	$-\ell$	$-N+\ell$

$$W_{\text{GLSM}} = P_1 \cdot G_\ell(S_i)$$

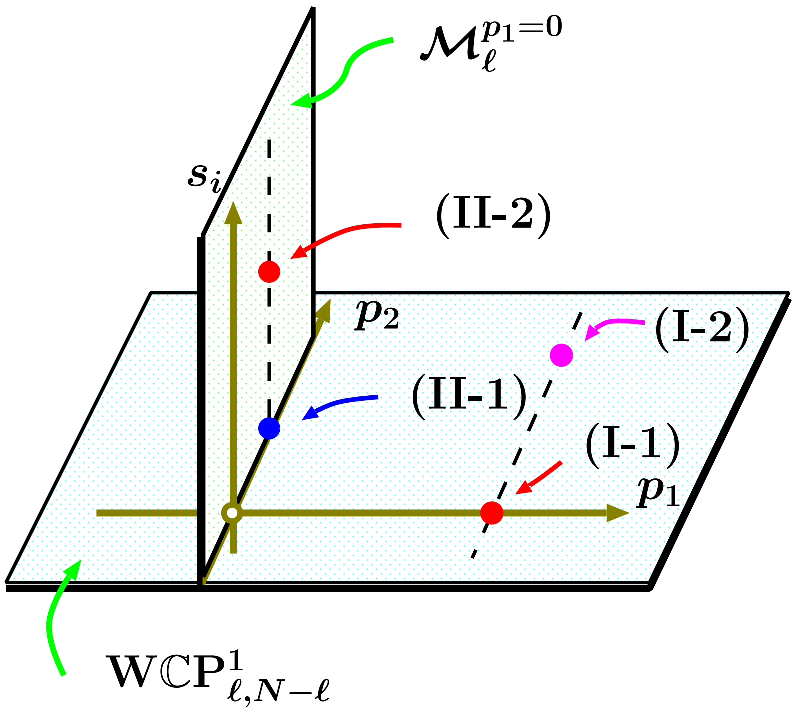
$G_\ell(S_i)$: homogeneous polynomial of degree ℓ

potential energy density:

$$\begin{aligned}\mathcal{U} &= \frac{e^2}{2}\Big[r - \sum_{i=1}^N |s_i|^2 + \ell|p_1|^2 + (N-\ell)|p_2|^2\Big]^2 \\ &\quad + 2|\sigma|^2\Big\{\sum_{i=1}^N |s_i|^2 + \ell^2|p_1|^2 + (N-\ell)^2|p_2|^2\Big\} \\ &\quad + \big|G_\ell(s_i)\big|^2 + |p_1|^2 \cdot \sum_{i=1}^N \big|\partial_i G_\ell(s_j)\big|^2\end{aligned}$$

Let us analyze classical SUSY vacuum manifold $\mathcal{U}=0$ and perturbation theories

▼ classical SUSY vacuum manifold in $r < 0$ phase $(3 \leq \ell \leq N-1)$:



(I-1) : at $\langle p_1 \rangle \neq 0$, $\langle p_2 \rangle = 0$, $\langle s_i \rangle = 0$, massless DOF $= N+1$

$$\left\{ \begin{array}{l} (\text{sigma model on } \mathbb{C}^1) \\ \otimes\ (\text{LG theory with } W_{\rm LG} = \langle p_1 \rangle G_\ell(S)) \end{array} \right\} \Big/ \mathbb{Z}_\ell$$

(I-2) : at $\langle p_1 \rangle \neq 0$, $\langle p_2 \rangle \neq 0$, $\langle s_i \rangle = 0$, massless DOF $= N+1$

$$\left\{ \begin{array}{l} (\text{sigma model on } \mathbb{C}^1) \\ \otimes\ (\text{LG theory with } W_{\rm LG} = \langle p_1 \rangle G_\ell(S)) \end{array} \right\} \Big/ \mathbb{Z}_\alpha$$

$\alpha = \text{GCM}\{\ell, N-\ell\}$

(II-1) : at $\langle p_1 \rangle = 0$, $\langle p_2 \rangle \neq 0$, $\langle s_i \rangle = 0$, massless DOF $= N+1$

$\{\text{LG theory with } W_{\rm LG} = P_1 \cdot G_\ell(S)\} / \mathbb{Z}_{N-\ell}$

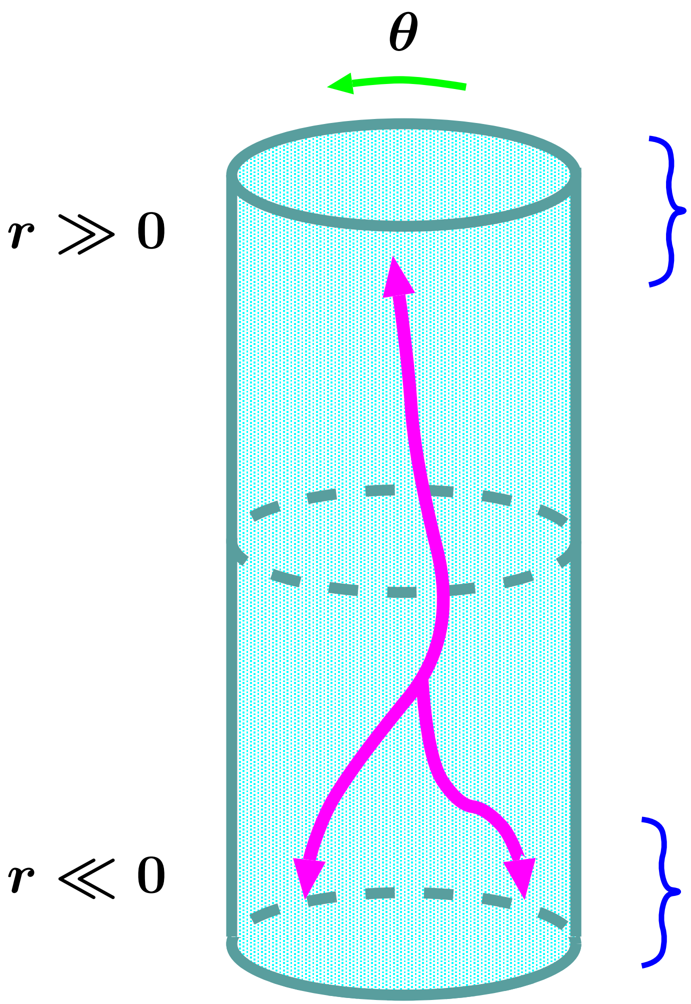
(II-2) : at $\langle p_1 \rangle = 0$, $\langle p_2 \rangle \neq 0$, $\langle s_i \rangle \neq 0$, massless DOF $= N-1$

sigma model on $\mathcal{M}_\ell^{p_1=0}$

$$W\mathbb{C}P^1_{\ell,N-\ell} \ = \ \Big\{ (p_1, p_2) \in \mathbb{C}^2 \,\Big|\, -r = \ell |p_1|^2 + (N-\ell)|p_2|^2 \Big\} \ \simeq \ \mathbb{C}P^1$$

$$\mathcal{M}_\ell^{p_1=0} \ = \ \Big\{ (p_2, s_i) \in \mathbb{C}^* \times \mathbb{C}^N \,\Big|\, r - \sum_{i=1}^N |s_i|^2 + (N-\ell)|p_2|^2 = 0 \ , \ G_\ell(s_i) = 0 \Big\}$$

CY/LG correspondence



CY sigma model on

$\mathcal{O}(-N+\ell)$ **bundle on** $\mathbb{C}\mathrm{P}^{N-1}[\ell]$

classically, not one-to-one (except for $\ell = 1, N$)

one-to-one correspondence via quantum effects?

(now in progress)

massless effective theories on

(I-1), (I-2), (II-1), (II-2)

But, we can construct "mirror geometries" via T-duality of GLSM.

T-dualized Theory

$$\mathscr{L} = \int d^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma - \sum_a \Big(\frac{1}{2}(Y_a + \overline{Y}_a)\log(Y_a + \overline{Y}_a)\Big)\Big\} + \Big(\frac{1}{\sqrt{2}}\int d^2\widetilde{\theta}\,\widetilde{W} + c.c.\Big)$$

$$\widetilde{W} = \Sigma\Big(\sum_{i=1}^{N} Y_i - \ell Y_{P_1} - (N-\ell)Y_{P_2} - t\Big) + \sum_{i=1}^{N} e^{-Y_i} + e^{-Y_{P_1}} + e^{-Y_{P_2}}$$

chiral superfield	Φ_a	S_1	S_2	$\cdots$	S_N	P_1	P_2
$U(1)$ charge	Q_a	1	1	$\cdots$	1	$-\ell$	$-N+\ell$
twisted chiral	Y_a	Y_1	Y_2	$\cdots$	Y_N	Y_{P_1}	Y_{P_2}

relation between chiral superfields $\{\Phi_a\}$ and twisted chiral superfields $\{Y_a\}$:

$$2\overline{\Phi}_a e^{2Q_a V}\Phi_a = Y_a + \overline{Y}_a$$

$U(1)$ phase rotation symmetry on Φ_a ➡ shift symmetry on Y_a:

$$Y_a \longrightarrow Y_a + 2\pi i$$

$\mathbb{Z}_\ell$ orbifold LG theory

Solve Y_{P_1} by using the constraint derived from integrating out Σ:

$$Y_{P_1} = \frac{1}{\ell}\Big\{t - \sum_{i=1}^{N} Y_i + (N-\ell)Y_{P_2}\Big\}$$

Field re-definition preserving canonical measure in $\widehat{\Pi}$:

$$X_i \equiv \mathrm{e}^{-\frac{1}{\ell}Y_i}\,, \quad X_{P_2} \equiv \mathrm{e}^{\frac{N-\ell}{\ell}Y_{P_2}}\,, \quad X_i \to \omega_i X_i\,, \quad X_{P_2} \to \omega_{P_2} X_{P_2}\,, \quad (\mathbb{Z}_\ell)^N \text{ symmetry}$$

Thus we obtain the twisted LG superpotential:

$$\Big\{\widetilde{W}_\ell = X_1^\ell + \cdots + X_N^\ell + X_{P_2}^{-\frac{\ell}{N-\ell}} + \mathrm{e}^{t/\ell} X_1 \cdots X_N X_{P_2}\Big\}\Big/(\mathbb{Z}_\ell)^N$$

negative power term = interpreted as $\mathcal{N}=2$ Liouville potential with

$$\frac{\ell}{N-\ell} = k = \frac{2}{Q^2}$$

Thus the IR effective superpotential becomes

$$\mathcal{N}=2 \text{ Liouville potential} \otimes \Big\{\widetilde{W}_\ell = X_1^\ell + \cdots + X_N^\ell + \mathrm{e}^{t/\ell} X_1 \cdots X_N\, \mathrm{e}^{\frac{N-\ell}{\ell}Y_{P_2}}\Big\}\Big/(\mathbb{Z}_\ell)^{N-1}$$

But, the Kähler potential of Liouville sector vanishes in the asymptotic region.

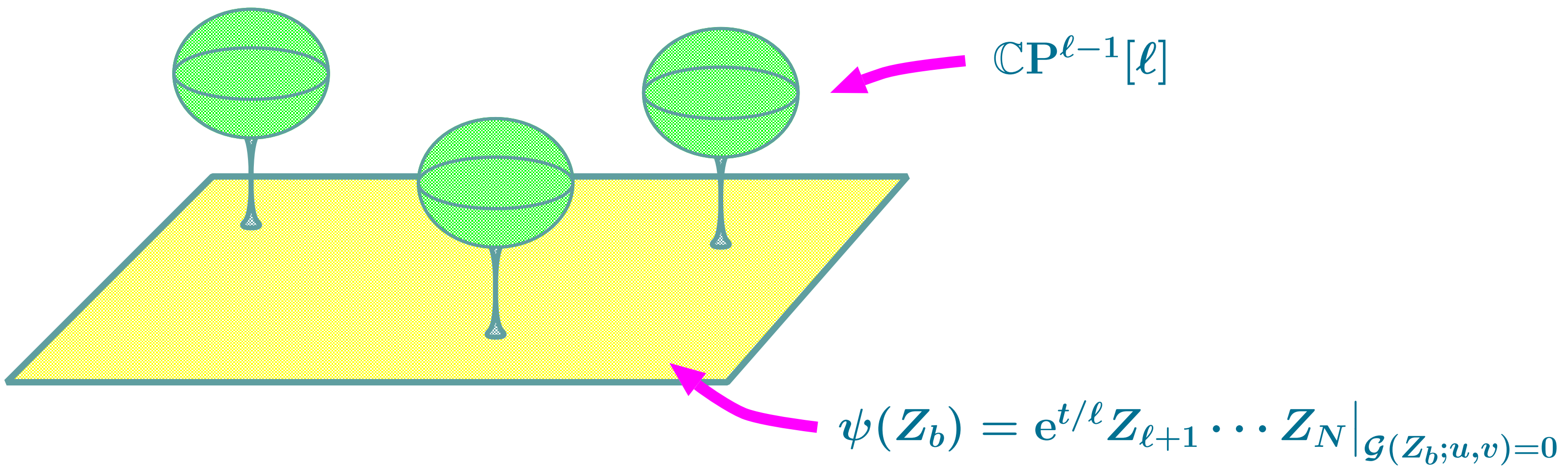
$\mathbb{Z}_\ell$ orbifold geometry

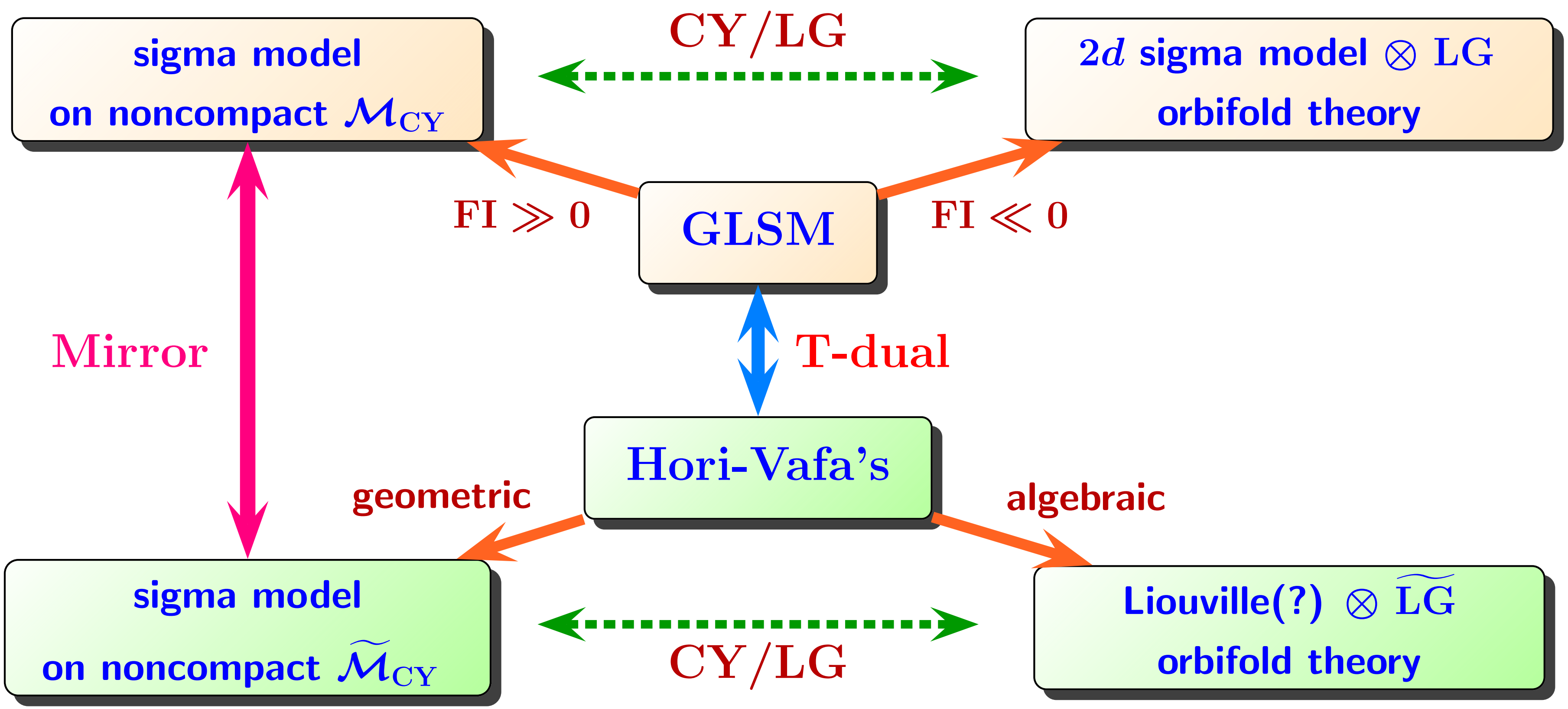
$$\widetilde{\mathcal{M}}_\ell = \Big\{ \{\mathcal{F}(Z_i) = 0\}/\mathbb{C}^* \,,\; \mathcal{G}(Z_b; u, v) = 0 \Big\} \Big/ (\mathbb{Z}_\ell)^{N-2}$$
$$\mathcal{F}(Z_i) = Z_1^\ell + \cdots + Z_\ell^\ell + \mathrm{e}^{t/\ell} Z_1 \cdots Z_N$$
$$\mathcal{G}(Z_b; u, v) = Z_{\ell+1}^\ell + \cdots + Z_N^\ell + 1 - uv$$

$Z_a \mapsto \lambda \omega_a Z_a$ **for** $a = 1, \cdots, \ell$ (homogeneous coordinates of $\mathbb{CP}^{\ell-1}[\ell]$)

$Z_b \mapsto \omega_b Z_b$ **for** $b = \ell+1, \cdots, N$ (homogeneous coordinates of $\mathbb{C}^{N-\ell}$)

$$\omega_a^\ell = \omega_b^\ell = \omega_1 \cdots \omega_N = 1 \,, \qquad \lambda : \mathbb{C}^*\text{-value}$$

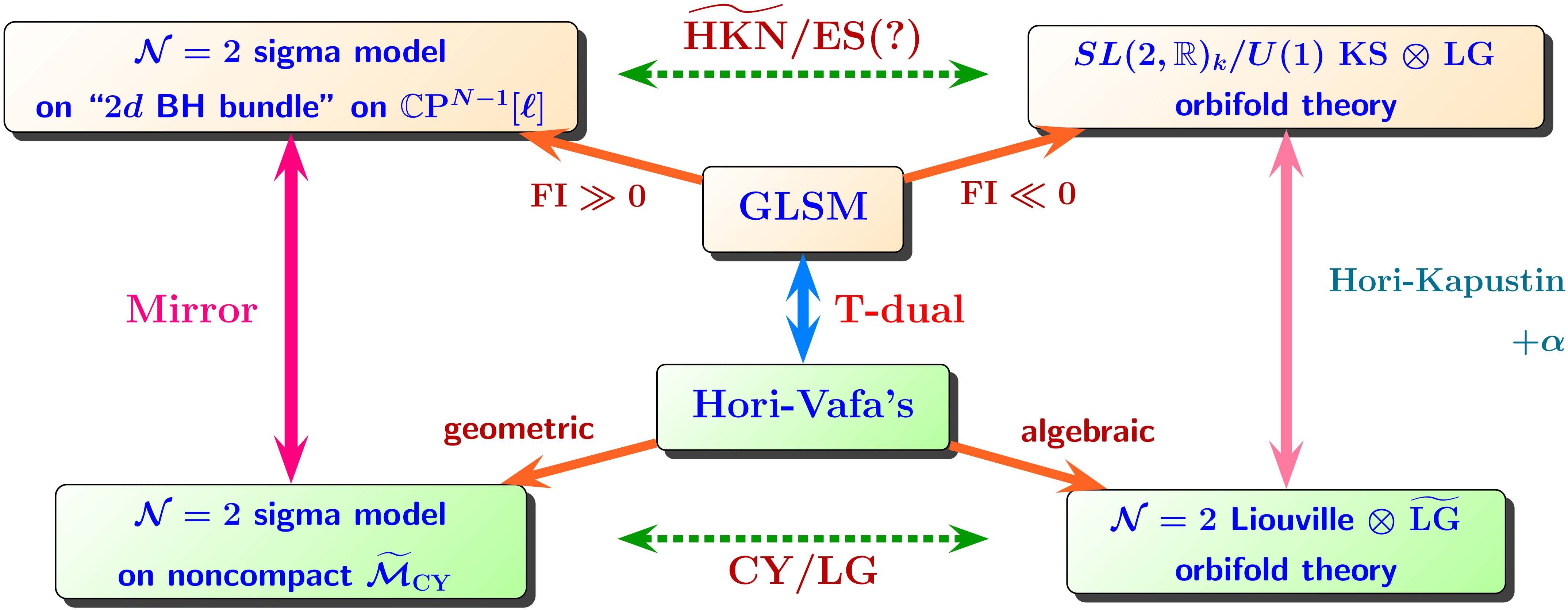




Some problems:

- There are no precise methods to calculate invariants (Gromov-Witten, etc.) for general CY
- There appear many vacua... (← now studying whether they disappear via nonperturbative effects)

One Goal: squashed GLSM and its T-duality



$$\frac{\ell}{N-\ell} = k = \frac{2}{Q^2}$$