Towards Mirror Symmetry on Noncompact Calabi-Yau Manifolds

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Introduction

String theory is a powerful framework

- F-string, NS5-brane, Dp-brane, NS-NS/R-R flux, etc.
- M-theory, supergravities, super Yang-Mills, etc.
- (Ultimately) unified theory

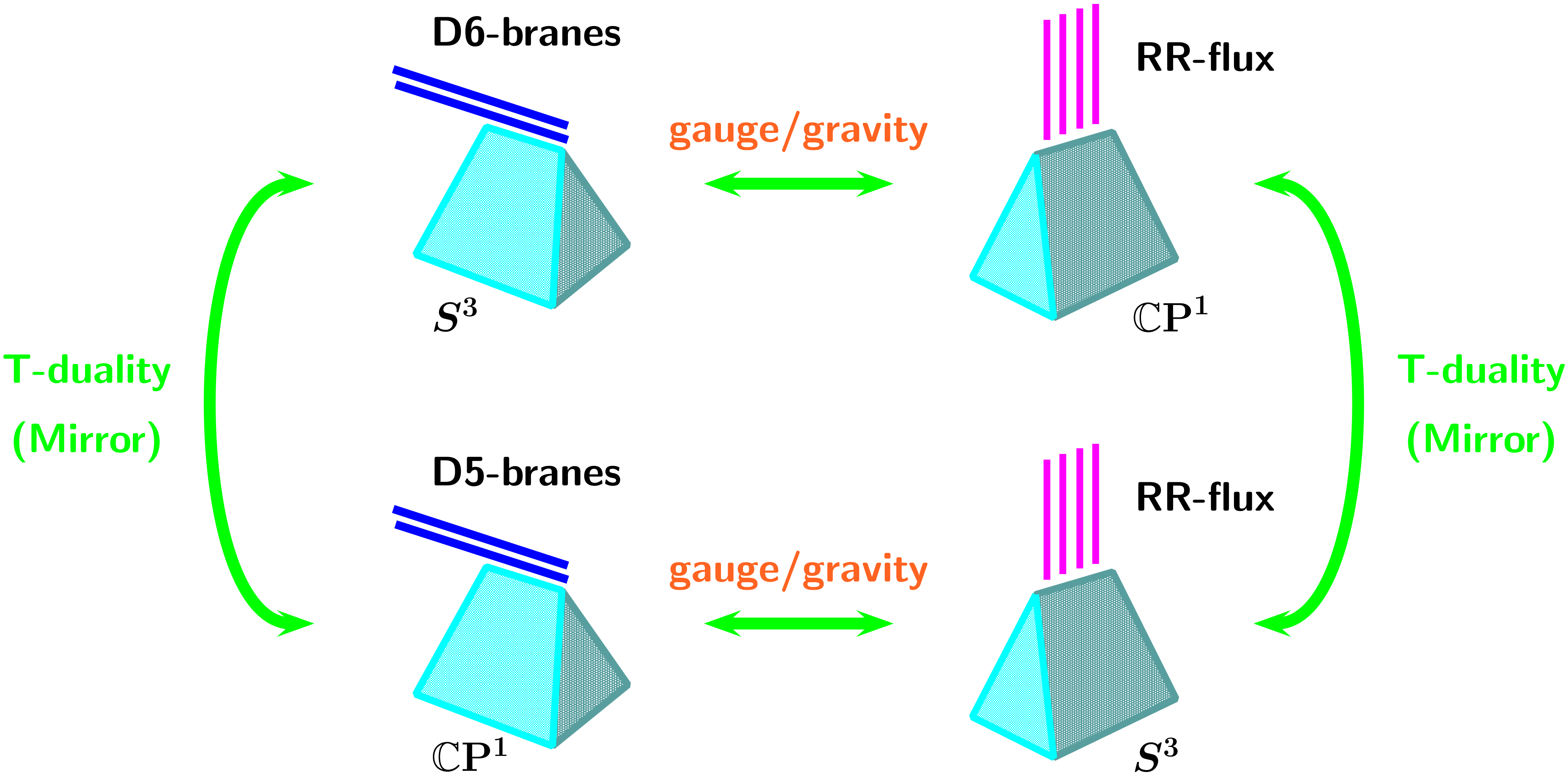
  gravitational/strong/weak/electromagnetic interactions

- Powerful tool to understand nonperturbative effects in field theories

  through various dualities, holography, etc.

Gauge/Gravity duality in string theory

on deformed/resolved conifold



Noncompact Calabi-Yau

(K.Higashijima, M.Nitta and TK, 2001, 2002)

line bundles (ex.)	total dimension
$\mathbb{C} \ltimes \left(\mathbb{C}\mathrm{P}^{N-1} = \frac{SU(N)}{SU(N-1)\times U(1)} \right)$	$1 + (N-1)$
$\mathbb{C} \ltimes \left(Q^{N-2} = \frac{SO(N)}{SO(N-2)\times U(1)} \right)$	$1 + (N-2)$
$\mathbb{C} \ltimes E_6/[SO(10)\times U(1)]$	$1 + 16$
$\mathbb{C} \ltimes E_7/[E_6\times U(1)]$	$1 + 27$
$\mathbb{C} \ltimes \left(\mathrm{G}_{N,M} = \frac{U(N)}{U(N-M)\times U(M)} \right)$	$1 + M(N-M)$
$\mathbb{C} \ltimes SO(2N)/U(N)$	$1 + \frac{1}{2}N(N-1)$
$\mathbb{C} \ltimes Sp(N)/U(N)$	$1 + \frac{1}{2}N(N+1)$

これらの

Mirror Pairs

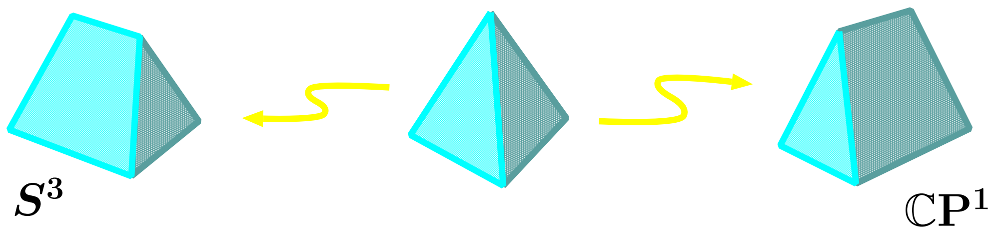
はどんなものか?

$\mathcal{O}(-N+k)$ bundle on $\mathbb{C}\mathrm{P}^{N-1}[k]$ の mirror dual を構成する

理由

- line bundle on $\mathbb{C}\mathrm{P}^{N-1}$ もしくは line bundle on $\mathbb{C}\mathrm{P}^{N-1}[k]$ 以上の情報を持つ
- line bundles on HSS の mirror dual の試金石
- Noncompact CY 3-fold ($N=4$) を含む

cf. noncompact CY 3-fold の典型例: deformed/singular/resolved conifold



2-dim. field theory から幾何学的性質を読み取れるか?

Gauged Linear Sigma Models and their T-dualized Theories

- E. Witten: "Phases of $\mathcal{N} = 2$ Theories in Two Dimensions"
- K. Hori and C. Vafa: "Mirror Symmetry"

A SUSY Gauge Theory

$U(1)$ gauge theory written by V, B and Y $\quad (Y \equiv Y + 2\pi i)$:

$$\begin{aligned} \mathscr{L}' &= \int \mathrm{d}^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \mathrm{e}^{2QV+B} - \frac{1}{2}(Y+\overline{Y})B \Big\} \\ &\qquad + \Big(\frac{1}{\sqrt{2}} \int \mathrm{d}^2\widetilde{\theta}\, \widetilde{W}(\Sigma) + h.c. \Big) \\ \Sigma &= \frac{1}{\sqrt{2}}\overline{D}_+ D_- V \,, \qquad \widetilde{W}(\Sigma) = -\Sigma\, t \,, \qquad t = r - i\theta \end{aligned}$$

- Integrating out Y, we obtain $B = \Psi + \overline{\Psi}$ ($\Phi \equiv \mathrm{e}^{\Psi}$ is a chiral superfield)

  ➡ Gauged Linear Sigma Model $\mathscr{L}_{\mathrm{GLSM}}$

- Integrating out B, we obtain $B = -2QV + \log\Big(\frac{Y+\overline{Y}}{2}\Big)$

  ➡ T-dualized theory $\mathscr{L}_{\mathrm{T}}$

Gauged Linear Sigma Model

一般的な 2-dim. $\mathcal{N} = (2,2)$ SUSY $U(1)$ gauge theory Lagrangian:

$$\mathscr{L}_{\mathrm{GLSM}} = \int \mathrm{d}^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_i \overline{\Phi}_i \, \mathrm{e}^{2Q_i V} \Phi_i \Big\} + \Big(\frac{1}{\sqrt{2}} \int \mathrm{d}^2\widetilde{\theta} \, \widetilde{W}(\Sigma) + h.c. \Big) + \Big(\int \mathrm{d}^2\theta \, W_{\mathrm{GLSM}}(\Phi) + h.c. \Big)$$

$$W_{\mathrm{GLSM}}(\Phi) = \Phi_0 \cdot G_K(\Phi_j)$$

$$G_K(\Phi_i) = \text{quasi-homogeneous polynomial degree } K$$

$$\widetilde{W}(\Sigma) = -\Sigma \, t \, , \qquad t = r - i\theta$$

potential energy density (SUSY vacua の探索):

$$\mathcal{U} = \frac{e^2}{2}\Big(r - \sum_i Q_i |\phi_i|^2 \Big)^2 + \sum_i \Big| \frac{\partial W}{\partial \phi_i} \Big|^2 + 2|\sigma|^2 \sum_i Q_i^2 |\phi_i|^2$$

T-dualized Theory

Lagrangian written by twisted chiral $Y_i \equiv Y_i + 2\pi i$:

$$\mathscr{L}_{\mathrm{T}} = \int d^4\theta \Big\{ -\mathcal{K}(\Sigma, \overline{\Sigma}, Y_i, \overline{Y}_i) \Big\} + \Big(\frac{1}{\sqrt{2}} \int d^2\widetilde{\theta}\, \widetilde{W}(\Sigma, Y_i) + h.c. \Big)$$

$$\widetilde{W}(\Sigma, Y_i) = \Sigma \Big(\sum_i Q_i Y_i - t \Big) + \mu \sum_i e^{-Y_i}$$

- $\mathcal{K}(\Sigma, \overline{\Sigma}, Y_i, \overline{Y}_i)$: Kähler potential including the information of W_{GLSM}
- $\widetilde{W}(\Sigma, Y_i)$: twisted superpotential $\longrightarrow$ Landau-Ginzburg superpotential

[GLSM]

CY phase		orbifold phase
FI $\gg 0$	$\longleftrightarrow$ CY/LG	FI $\ll 0$
SNLSM on M		LG$_{cc}$ theory

untwsited	twisted
(cc-ring)	(ac-ring)
$h_{cc}(M)$	$h_{ac}(M)$

[T-dualized theory]

"period" integral		period integral
$\Pi = \int \exp(-\widetilde{W})$	$\longleftrightarrow$ CY/LG	$\Pi = \int \Omega$
LG$_{ac}$ theory		CY geometry $\widetilde{M}$

untwsited	twisted
(ac-ring)	(cc-ring)
$h_{ac}(\widetilde{M})$	$h_{cc}(\widetilde{M})$

Mirror dual

$$h_{cc}(M) = h_{ac}(\widetilde{M}), \quad h_{ac}(M) = h_{cc}(\widetilde{M})$$

Example

Mirror dual for compact CY 3-fold:

CY sigma model on $\mathbb{C}\mathrm{P}^4[5]$ ⟷ (CY/LG) **LG orbifold theory with $W/\mathbb{Z}_5$**

↕ **T-duality**

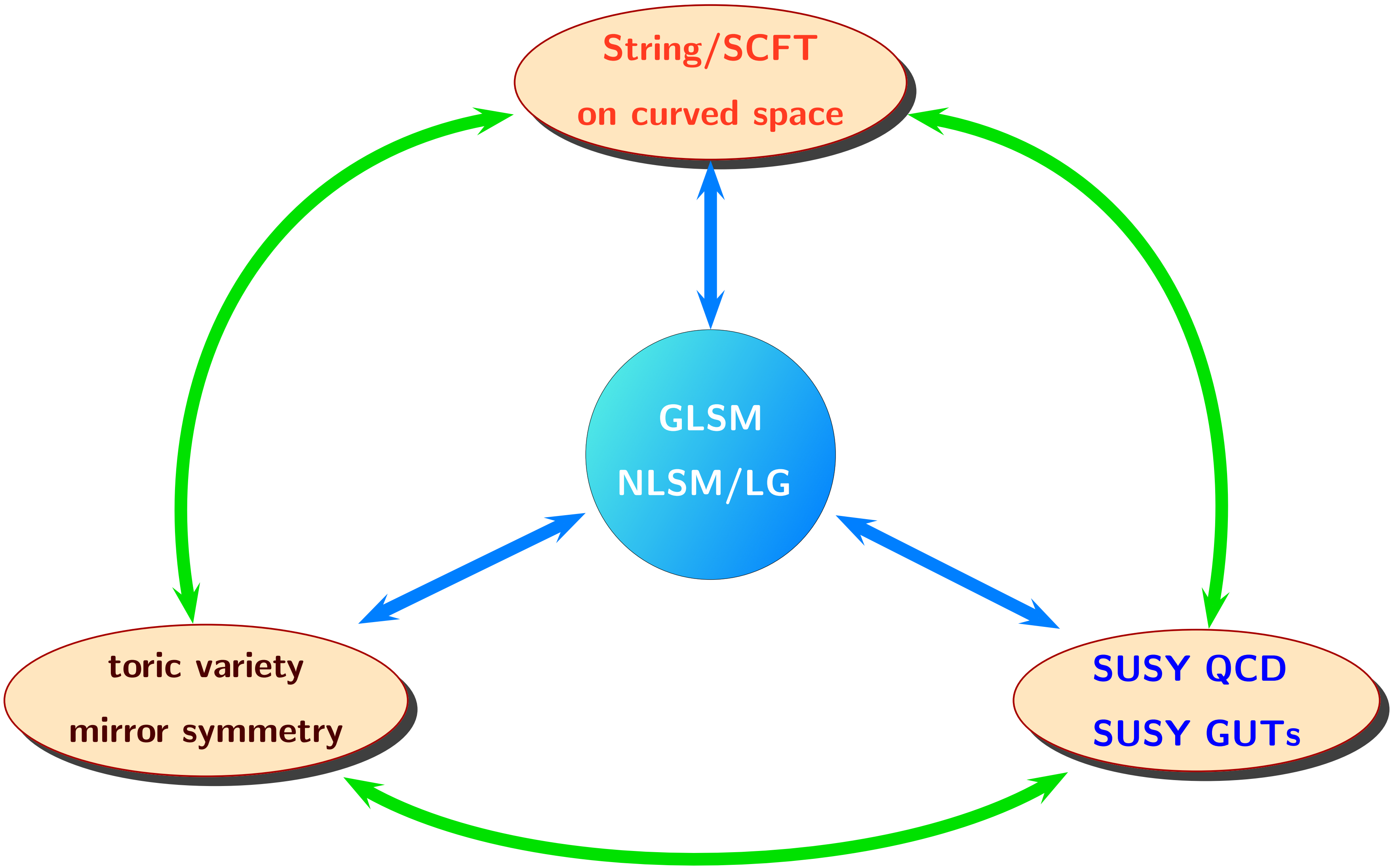
$\widetilde{\mathrm{CY}}$ sigma model on $\mathbb{C}\mathrm{P}^4[5]/(\mathbb{Z}_5)^3$ ⟷ (CY/LG) **$\widetilde{\mathrm{LG}}$ orbifold theory with $\widetilde{W}/(\mathbb{Z}_5)^4$**

$$W = X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5$$

$$\widetilde{W} = \widetilde{X}_1^5 + \widetilde{X}_2^5 + \widetilde{X}_3^5 + \widetilde{X}_4^5 + \widetilde{X}_5^5 + \mathrm{e}^{t/5}\widetilde{X}_1\widetilde{X}_2\widetilde{X}_3\widetilde{X}_4\widetilde{X}_5$$

$$h_{21}(\mathbb{C}\mathrm{P}^4[5]) = h_{11}(\mathbb{C}\mathrm{P}^4[5]/(\mathbb{Z}_5)^3) = 101 \,, \qquad h_{11}(\mathbb{C}\mathrm{P}^4[5]) = h_{21}(\mathbb{C}\mathrm{P}^4[5]/(\mathbb{Z}_5)^3) = 1$$

GLSMと T-dualized theoryは, Greene-Plesser構成法を簡単に再現する!


String/SCFT
on curved space
GLSM
NLSM/LG
toric variety
mirror symmetry
SUSY QCD
SUSY GUTs

Gauged Linear Sigma Model for $\mathcal{O}(-N+k)$ bundle on $\mathbb{CP}^{N-1}[k]$

chiral superfields	S_1	$\cdots$	S_N	P_1	P_2
$U(1)$ charge	1	$\cdots$	1	$-k$	$-N+k$

$$W_{\text{GLSM}} = P_1 \cdot G_k(S_i)$$

$G_k(S_i)$: quasi-homogeneous polynomial of degree k

potential energy:

$$\begin{aligned}\mathcal{U}(s_i, p_a, \sigma) &= \frac{e^2}{2}\Big[r - \sum_{i=1}^{N} |s_i|^2 + k|p_1|^2 + (N-k)|p_2|^2\Big]^2 \\ &\quad + 2|\sigma|^2\Big\{\sum_{i=1}^{N} |s_i|^2 + k^2|p_1|^2 + (N-k)^2|p_2|^2\Big\} \\ &\quad + \big|G_k(s_i)\big|^2 + |p_1|^2 \cdot \sum_{i=1}^{N} \big|\partial_i G_k(s_j)\big|^2\end{aligned}$$

真空 $\mathcal{U}(s_i, p_a, \sigma) = 0$

▼ $r \gg 0$: vacuum manifold $M_{\rm CY}$ を調べると...

$$M_{\rm CY} = \left\{ (S_i, P_2) \in \mathbb{C}^{N+1} \,\middle|\, r = \sum_{i=1}^{N} |S_i|^2 - (N-k)|P_2|^2, \ \ G_k(S_i) = 0 \right\} \Big/ U(1)$$

⇨ $\mathcal{O}(-N+k)$ bundle on $\mathbb{CP}^{N-1}[k]$

この phase では, massless theory は

CY sigma model on $M_{\rm CY}$

で与えられる

($M_{\rm CY}$ に沿わないモードは全て massive fields)

▼ $r \ll 0$: vacuum manifold $M_{<0}$ を調べると...

$$M_{<0} = \left\{ (p_1, p_2; s_i) \in \mathbb{C}^{N+2} \;\middle|\; \begin{array}{l} F(p_a, s_i) = 0 \\ G_k(s_i) = 0, \quad p_1 \partial_i G_k = 0 \end{array} \right\} \Big/ U(1)$$

$$F(p_a, s_i) = k|p_1|^2 + (N-k)|p_2|^2 - \sum_{i=1}^{N} |s_i|^2 + r$$

$\therefore$ $\oplus_{i=1}^{N} \mathcal{O}_i(-1)$ bundle on $\mathrm{W}\mathbb{C}\mathrm{P}^1_{k,N-k}$
with constraints $G_k(s_i) = 0$ and $p_1 \partial_i G_k = 0$

$p_1 \partial_i G_k = 0$ が vacuum manifold の均一性を壊す!

↓

$$M_{<0} = \{ M_{N-k} = (p_1 = 0 \text{ region}) \} \oplus \{ M_k = (p_1 \neq 0 \text{ region}) \}$$

Orbifold phase of GLSM

- $(p_1 = 0, p_2 \neq 0) \in \mathbf{WCP}^1_{k,N-k}$ **region:**

$$M_{N-k} = \left\{ (p_1, p_2; s_i) \in \mathbb{C}^{N+2} \,\middle|\, \begin{array}{l} F(p_a, s_i) = G_k(s_i) = 0 \\ p_1 \partial_j G_k = p_1 = 0 \end{array} \right\} \Big/ U(1)$$

$$= \left\{ (p_2; s_i) \in \mathbb{C}^* \times \mathbb{C}^N \,\middle|\, F(p_1 = 0) = 0 \;,\; G_k(s_i) = 0 \right\} \Big/ \mathbb{Z}_{N-k}$$

massless modes: s_i **tangent to** M_{N-k} **and** p_1

⇓

orbifold sigma model on M'_{N-k}

$$M'_{N-k} = \left\{ (p_1; s_i) \in \mathbb{C}^{N+1} \,\middle|\, G_k(s_i) = p_1 \partial_j G_k = 0 \right\} \Big/ \mathbb{Z}_{N-k}$$

▼ $(p_1 \neq 0, p_2) \in \mathbf{WCP}^1_{k,N-k}$ **region:**

$$M_k = \left\{ (p_1, p_2; s_i) \in \mathbb{C}^{N+2} \;\middle|\; \begin{array}{l} F(p_a, s_i) = G_k(s_i) = 0 \\ p_1 \partial_j G_k = 0 \;,\; p_1 \neq 0 \end{array} \right\} \Big/ U(1)$$

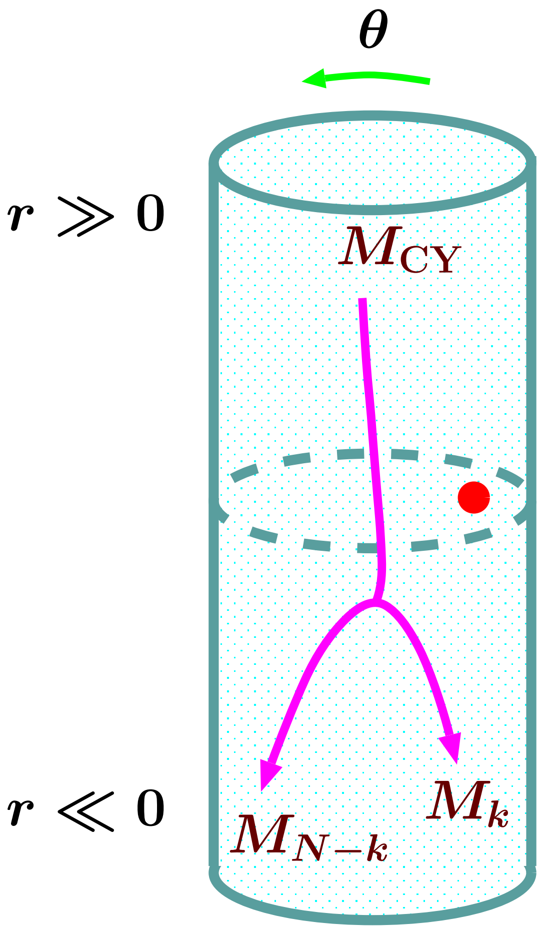
$$= \left\{ (p_1, p_2) \in \mathbb{C}^* \times \mathbb{C} \;\middle|\; F(s_i = 0) = 0 \right\} \Big/ \mathbb{Z}_k$$

massless modes: p_2 **tangent to** M_k **and** s_i

↓

$\Big\{$**(sigma model on** $\mathbb{C}^1$**)** $\oplus$ **(**$W_{\mathrm{LG}} = G_k(S_i)$ **LG theory)**$\Big\} \Big/ \mathbb{Z}_k$

$\mathcal{N} = 2$ **orbifolded Liouville Field Theory?**



on $M_{\rm CY}$:

CY sigma model

on $\mathcal{O}(-N+k)$ bundle on $\mathbb{C}\mathrm{P}^{N-1}[k]$

---

on M_{N-k}:

orbifold sigma model

on $\Big\{ (p_1; s_i) \in \mathbb{C}^{N+1} \Big| G_k(S_i) = p_1 \partial_j G_k = 0 \Big\} \Big/ \mathbb{Z}_{N-k}$

on M_k:

$$\left\{ \begin{array}{l} \text{(sigma model on } \mathbb{C}^1) \\ \oplus (W_{\rm LG} = G_k(S_i) \text{ LG theory}) \end{array} \right\} \Big/ \mathbb{Z}_k$$

$\mathcal{N} = 2$ **orbifolded Liouville Field Theory?**

T-dualized Theory

$$Y_i + \overline{Y}_i \equiv \overline{\Phi}_i \, \mathrm{e}^{2Q_i V} \Phi_i$$

LG twisted superpotential:

$$\widetilde{W} \;=\; \sum_{i=1}^{N} \mathrm{e}^{-Y_i} + \mathrm{e}^{-Y_{P_1}} + \mathrm{e}^{-Y_{P_2}} + \Sigma \Big(\sum_{i=1}^{N} Y_i - k Y_{P_1} - (N-k)\, Y_{P_2} - t \Big)$$

"period integral" を導入 (W_{GLSM} の存在の有無を感知)

$$\Pi_{\text{without } W_{\mathrm{GLSM}}} \;\equiv\; \int \mathrm{d}\Sigma \prod_{i=1}^{N} \mathrm{d}Y_i \, \mathrm{d}Y_{P_1} \, \mathrm{d}Y_{P_2} \; \exp(-\widetilde{W})$$

$$\widehat{\Pi}_{\text{with } W_{\mathrm{GLSM}}} \;\equiv\; \int \mathrm{d}\Sigma \prod_{i=1}^{N} \mathrm{d}Y_i \, \mathrm{d}Y_{P_1} \, \mathrm{d}Y_{P_2} \, (k\Sigma) \; \exp(-\widetilde{W})$$

$\widehat{\Pi}$ から mirror geometry を読み取る操作: $k\Sigma \to \dfrac{\partial}{\partial Y_{P_1}}$

$\mathbb{Z}_{N-k}$ orbifold solution

$$\widetilde{M}_{N-k} = \Big\{F(Z_a) = 0\,,\; G(Z_i;u,v) = 0\Big\}\Big/\big\{\mathbb{C}^* \times (\mathbb{Z}_{N-k})^{N-2}\big\}$$

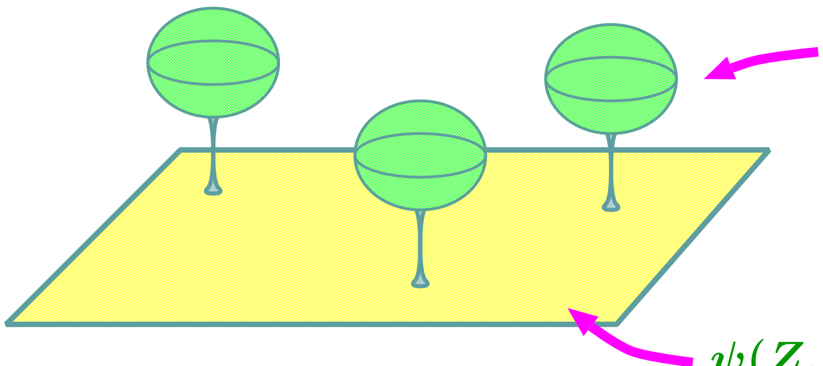
$$F(Z_a) = Z_1^{N-k} + \cdots + Z_k^{N-k} + 1$$

$$G(Z_i;u,v) = Z_{k+1}^{N-k} + \cdots + Z_N^{N-k} + Z_1 \cdots Z_N(\mathrm{e}^{t/(N-k)} - uv)$$

$Z_a \mapsto \omega_a Z_a$ for $a = 1, \cdots, k$ ($\mathbb{C}^k$-plane 上に住む)

$Z_b \mapsto \lambda \omega_b Z_b$ for $b = k+1, \cdots, N$ ($\mathbb{CP}^{N-k-1}[N-k]$ 上に住む)

$$\omega_a^{N-k} = \omega_b^{N-k} = \omega_1 \cdots \omega_N = 1\,, \qquad \lambda : \mathbb{C}^*\text{-value}$$



$\mathbb{CP}^{N-k-1}[N-k]$

$\psi(Z_a,u,v) = (\mathrm{e}^{t/(N-k)} - uv)Z_1 \cdots Z_k\big|_{F(Z_a)=0}$

$\mathbb{Z}_k$ orbifold solution

$$\widetilde{M}_k = \left\{ \mathcal{F}(Z_i) = 0 \,,\ \mathcal{G}(Z_b; u, v) = 0 \right\} \Big/ \{\mathbb{C}^* \times (\mathbb{Z}_k)^{N-2}\}$$

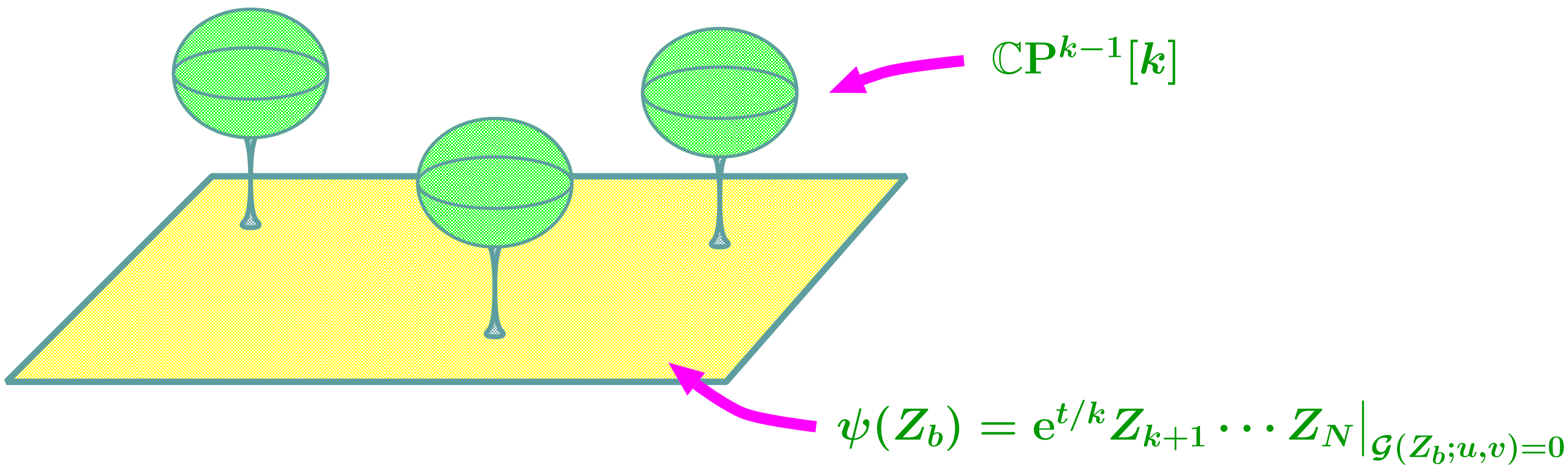
$$\mathcal{F}(Z_i) = Z_1^k + \cdots + Z_k^k + \mathrm{e}^{t/k} Z_1 \cdots Z_N$$

$$\mathcal{G}(Z_b; u, v) = Z_{k+1}^k + \cdots + Z_N^k + 1 - uv$$

$Z_a \mapsto \lambda \omega_a Z_a$ for $a = 1, \cdots, k$ ($\mathbb{C}\mathbf{P}^{k-1}[k]$ 上に住む)

$Z_b \mapsto \omega_b Z_b$ for $b = k+1, \cdots, N$ ($\mathbb{C}^{N-k}$-plane 上に住む)

$$\omega_a^k = \omega_b^k = \omega_1 \cdots \omega_N = 1 \,, \qquad \lambda : \mathbb{C}^*\text{-value}$$



$\widehat{\Pi}$ から LG twisted superpotential を読み取る操作: $k\Sigma \to k\dfrac{\partial}{\partial t}$

Y_{P_2} を解く　$\mathrm{e}^{-Y_i} = X_i^k$, etc.

$$\widetilde{W}_{N-k} = \sum_{i=1}^{N} X_i^{N-k} + X_{P_1}^{-(N-k)/k} + \mathrm{e}^{t/(N-k)} X_1 X_2 \cdots X_N \cdot X_{P_1}$$

$$X_i \to \omega_i X_i\,, \quad X_{P_1} \to \omega_{P_1} X_{P_1}$$
$$\omega_k^{N-k} = \omega_{P_1}^{-\frac{N-k}{k}} = \omega_1\omega_2\cdots\omega_N\omega_{P_1} = 1$$

⇒ $(\mathbb{Z}_{N-k})^N$ orbifold symmetry

$\sim$ GLSM の M_{N-k} に関連

Y_{P_1} を解く　$\mathrm{e}^{-Y_i} = X_i^k$, etc.

$$\widetilde{W}_{k} = \sum_{i=1}^{N} X_i^{k} + X_{P_2}^{-k/(N-k)} + \mathrm{e}^{t/k} X_1 X_2 \cdots X_N \cdot X_{P_2}$$

$$X_i \to \omega_i X_i\,, \quad X_{P_2} \to \omega_{P_2} X_{P_2}$$
$$\omega_k^{k} = \omega_{P_2}^{-\frac{k}{N-k}} = \omega_1\omega_2\cdots\omega_N\omega_{P_2} = 1$$

⇒ $(\mathbb{Z}_{k})^N$ orbifold symmetry

$\sim$ GLSM の M_{k} に関連

$\widehat{\Pi}$ が canonical measure を持ちつつ $\widetilde{W}$ が整数ベキとなるべし:

- $1 \leq k \leq \frac{1}{2}N$ の時 $\widetilde{W}_{N-k}$ が選ばれる:

$$k = \frac{1}{m+1}N \, , \quad m \in \mathbb{Z}_{>0}$$

- $\frac{1}{2}N \leq k \leq N-1$ の時 $\widetilde{W}_k$ が選ばれる:

$$k = \frac{\ell}{\ell+1}N \, , \quad \ell \in \mathbb{Z}_{>0}$$

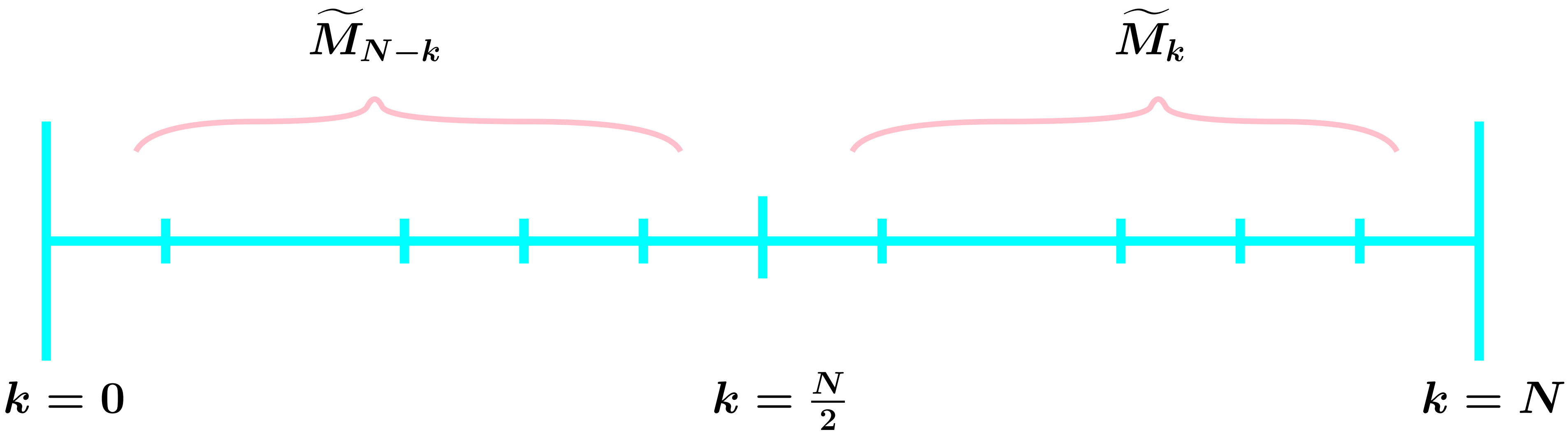
(N,k) 毎に適切な orbifolded LG theory が一意に決まる

これらの orbifolded LG theory に対応する (mirror) CY geometry も構成できる!

Relation

mirror geometry of $\mathcal{O}(-N+k)$ bundle on $\mathbb{C}\mathrm{P}^{N-1}[k]$

(但し $1 \leq k \leq N-1$)



$k = 0$ case : $\mathcal{O}(-N)$ bundle on $\mathbb{C}\mathrm{P}^{N-1}$

$k = N$ case : CY hypersurface $\mathbb{C}\mathrm{P}^{N-1}[N]$

Result $N = 2m+2,\ k = 2$ case:

CY sigma model on $\mathcal{O}(-N+2)$ bundle on $\mathbb{C}\mathrm{P}^{N-1}[2]$	CY/LG ⟷	**orbifold sigma model** on M'_{N-2}
	T-duality ↕	
$\widetilde{\mathrm{CY}}$ **sigma model** on $\widetilde{M}_{N-2}$	CY/LG ⟷	$\widetilde{\mathrm{LG}}$ **orbifold theory** with $\widetilde{W}_{N-2}/(\mathbb{Z}_{N-2})^N$

$$M'_{N-2} = \Big\{ G_2(s_i) = p_1 \partial_j G_2 = 0 \Big\} \Big/ \mathbb{Z}_{N-2}$$

$$\widetilde{W}_{N-2} = \sum_{i=1}^{N} X_i^{N-2} + X_{P_1}^{-m} + \mathrm{e}^{t/(N-2)} X_1 \cdots X_N \cdot X_{P_1}$$

$$\widetilde{M}_{N-2} = \left\{ \begin{aligned} 0 &= 1 + Z_1^{N-2} + Z_2^{N-2} \\ 0 &= 1 + \sum_{b=3}^{N-1} Z_b^{N-2} + Z_1 \cdots Z_{N-1} (\mathrm{e}^{t/(2m)} - uv) \end{aligned} \right\} \Big/ (\mathbb{Z}_{N-2})^{N-2}$$

Summary and Discussions

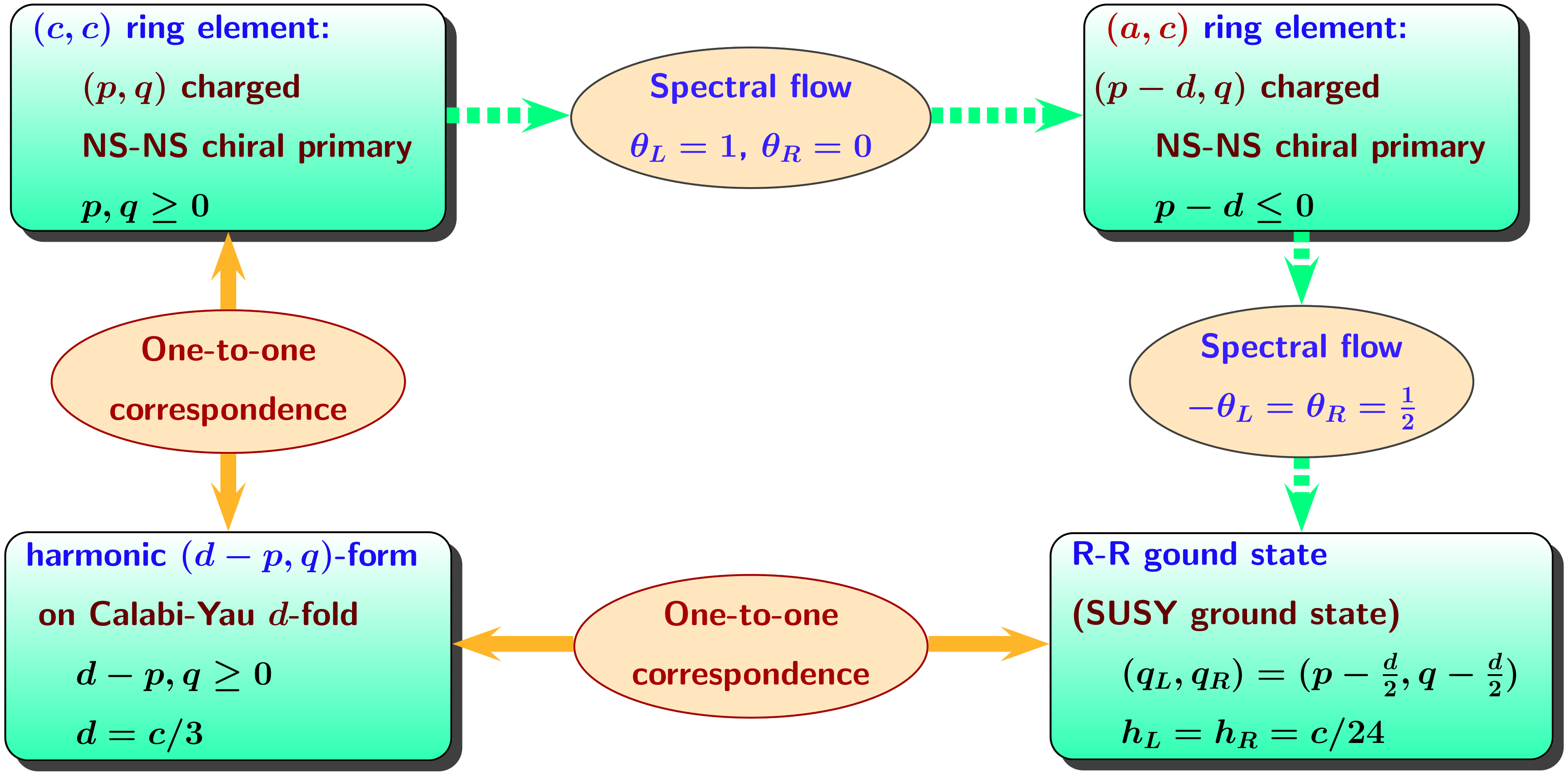
- $\mathcal{O}(-N+k)$ bundle on $\mathbb{C}\mathrm{P}^{N-1}[k]$ のミラー多様体, LG superpotential が得られた

- $\widetilde{W}$ で与えられる LG orbifolded theory から chiral ring などを構成すべし

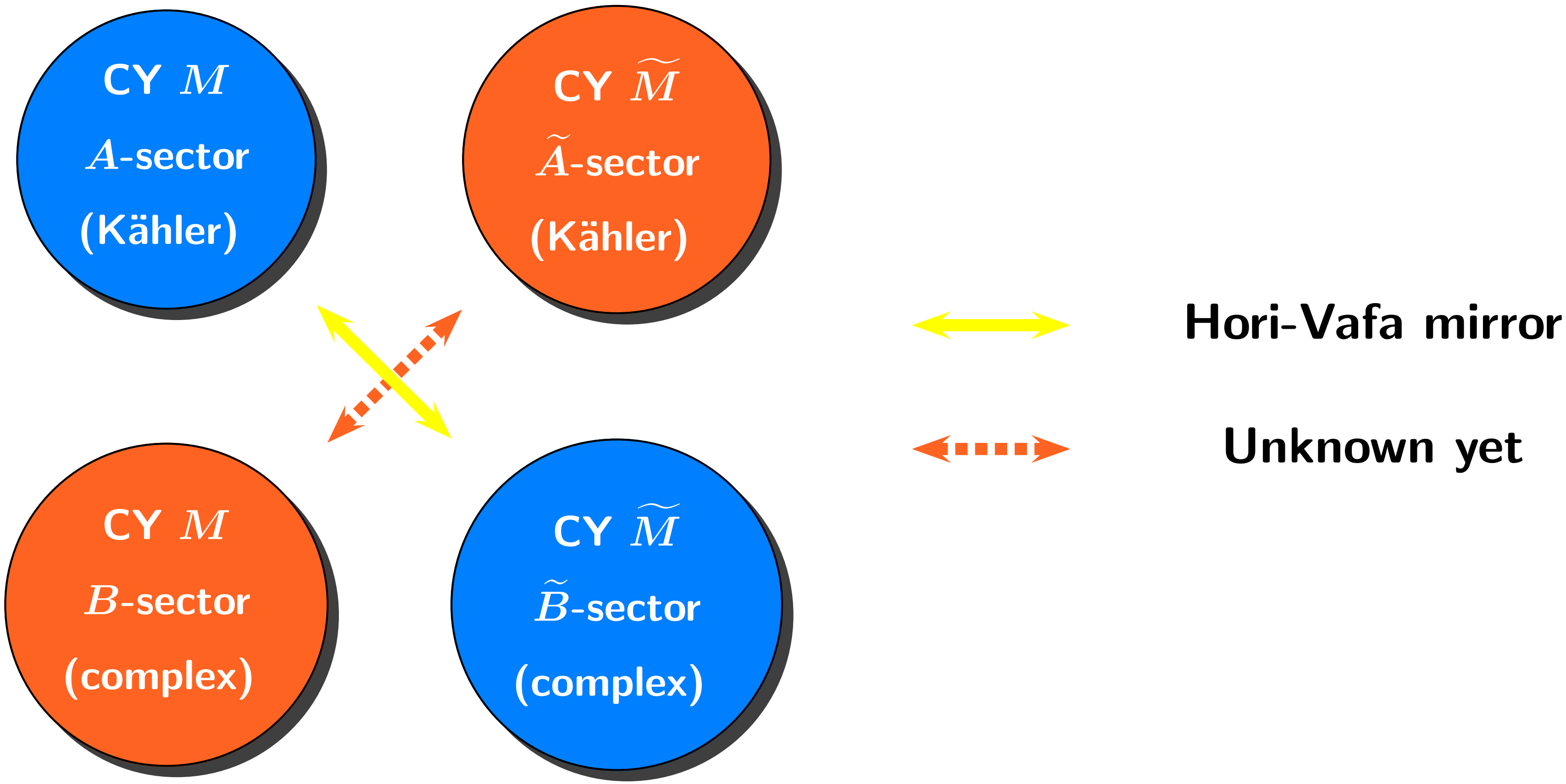
- Noncompact である (しかも fiber bundle) 事からの問題
  - ⇒ 既存の方法で (co)homology や CFT などを評価, 構築できるのか?
  - ⇒ $\mathcal{N}=2$ Liouville theory description (?) が有力な道具

- line bundle on Q^{N-2} の mirror dual を構成するにあたって...
  - $SO(N)$ symmetry の情報は D-term (Kähler potential) が担う
    - ⇒ topological sectors のみでは mirror dual の議論は不完全のはず
    - ⇒ Witten, Hori-Vafa formulation のより深い追跡が必要

$\mathcal{N} = (2,2)$ SCFT and compact Calabi-Yau geometry



NS-NS chiral primary states, R-R ground states と harmonic formsの相関


CY M
A-sector
(Kähler)
CY $\widetilde{M}$
$\widetilde{A}$-sector
(Kähler)
CY M
B-sector
(complex)
CY $\widetilde{M}$
$\widetilde{B}$-sector
(complex)
Hori-Vafa mirror
Unknown yet

Applications

- New kinds of CY mirror pairs which have been unknown cohomology rings
- Mirror dual description for $\mathcal{N} = 2$ Liouville theory on noncompact CY

  Liouville theory tells us the partition function

- Mirror duals of asymptotic free sigma models (sigma models on HSS)

ex.

sigma model on $\mathbb{C}\mathrm{P}^{N-1}$ $\Longleftrightarrow$ $\widehat{A}_{N-1}$ Toda field theory

sigma model on Q^{N-2} $\Longleftrightarrow$ $\widehat{D}$-type Toda field theory?

Sigma model on Q^{N-2}

(K.Higashijima, M.Nitta, M.Tsuzuki and TK, 2000)

Kähler potential (Lagrangian) of quadric surface:

$$\mathscr{L} = r \int d^4\theta \, \log \left\{ 1 + \sum_{j=1}^{N-2} |\Phi_j|^2 + \left| 1 + \sum_{k=1}^{N-2} \Phi_k^2 \right| \right\}$$

Introduce auxiliary fields V and P_1:

$$\mathscr{L} = \int d^4\theta \left(\sum_{i=1}^{N} \overline{S}_i \, e^{2V} S_i - 2r\, V \right) + \left(\int d^2\theta \, P_1 \sum_{i=1}^{N} S_i^2 + h.c. \right)$$

sigma model on $\mathbb{C}\mathrm{P}^{N-1}$ with F-term constraint $\Rightarrow$ $SO(N)$ symmetry

Remarks

- 漸近的自由性
- mass gap
- $\underbrace{\text{Coulomb相 } (\langle V \rangle \neq 0)}_{\text{sigma model on } \mathbb{C}\mathrm{P}^{N-1}}$ / $\underbrace{\text{Higgs相 } (\langle P_1 \rangle \neq 0)}_{\text{NEW}}$ の存在 [$1/N$ analysis]

$\mathcal{O}(-N)$ bundle on $\mathbb{C}\mathrm{P}^{N-1}$

homogeneous coordinates

$$(\phi_0, \phi_1, \cdots, \phi_{N-1})$$

local coordinates

$$\phi_0(1, \varphi_1, \cdots, \varphi_{N-1})$$

$$\varphi_i \equiv \phi_i / \phi_0$$

$$\rho \sim (\phi_0)^N$$

local coordinates

$$\phi_{N-1}(\widetilde{\varphi}_0, \cdots, \widetilde{\varphi}_{N-2}, 1)$$

$$\widetilde{\varphi}_i \equiv \phi_i / \phi_{N-1}$$

$$\widetilde{\rho} \sim (\phi_{N-1})^N$$

$$\varphi^i = \widetilde{\varphi}^i \cdot P \,, \qquad P = \frac{\phi^{N-1}}{\phi^0}$$

$$\rho = \widetilde{\rho} \cdot P^{-N}$$

φ^i and $\widetilde{\varphi}^i$ are local coordinates of the $\mathbb{C}\mathrm{P}^{N-1}$

ρ and $\widetilde{\rho}$ are fiber coordinates

LG twisted superpotentials

Period integral $\widehat{\Pi}$

$$\widehat{\Pi} = k\frac{\partial}{\partial t}\int\prod_{i=1}^{N} dY_i\, dY_{P_1}\, dY_{P_2}\, \delta\Big(\sum_{i=1}^{N} Y_i - kY_{P_1} - (N-k)Y_{P_2} - t\Big) \times \exp\Big\{-\sum_{i=1}^{N} e^{-Y_i} - e^{-Y_{P_1}} - e^{-Y_{P_2}}\Big\}$$

▼ $-Y_{P_2} = \frac{1}{N-k}\Big(t - \sum_{i=1}^{N} Y_i + kY_{P_1}\Big)$:

$$\widehat{\Pi} = e^{t/(N-k)}\int\prod_{i=1}^{N}\Big(e^{-\frac{1}{N-k}Y_i} dY_i\Big)\Big(e^{\frac{k}{N-k}Y_{P_1}} dY_{P_1}\Big) \times \exp\Big\{-\sum_{i=1}^{N} e^{-Y_i} - e^{-Y_{P_1}} - e^{t/(N-k)}\prod_{i=1}^{N} e^{-\frac{1}{N-k}Y_i} e^{\frac{k}{N-k}Y_{P_1}}\Big\}$$

$$e^{-\frac{1}{N-k}Y_i} \equiv X_i\,, \qquad e^{\frac{k}{N-k}Y_{P_1}} \equiv X_{P_1}$$

$$\therefore \quad \widehat{\Pi} = (\text{factors})\int\prod_{i=1}^{N} dX_i\, dY_{P_1}\, \exp\Big\{\underbrace{-\sum_{i=1}^{N} X_i^{N-k} - X_{P_1}^{-\frac{N-k}{k}} - e^{t/(N-k)} X_1\cdots X_N X_{P_1}}_{-\widetilde{W}_{N-k}}\Big\}$$

$$-Y_{P_1} = \frac{1}{k}\Big(t - \sum_{i=1}^{N} Y_i + (N-k) Y_{P_2}\Big):$$

$$\widehat{\Pi} \;=\; e^{t/k} \int \prod_{i=1}^{N} \Big(e^{-\frac{1}{k}Y_i} dY_i\Big) \Big(e^{\frac{N-k}{k}Y_{P_2}} dY_{P_2}\Big) \exp\Big\{ - \sum_{i=1}^{N} e^{-Y_i} - e^{-Y_{P_2}} - e^{t/k} \prod_{i=1}^{N} e^{-\frac{1}{k}Y_i} e^{\frac{N-k}{k}Y_{P_2}} \Big\}$$

$$e^{-\frac{1}{k}Y_i} \;\equiv\; X_i \,, \qquad e^{\frac{N-k}{k}Y_{P_2}} \;\equiv\; X_{P_2}$$

$$\therefore \qquad \widehat{\Pi} \;=\; \text{(factors)} \int \prod_{i=1}^{N} dX_i \, dY_{P_2} \, \exp\Big\{ \underbrace{- \sum_{i=1}^{N} X_i^k - X_{P_2}^{-\frac{k}{N-k}} - e^{t/k} X_1 \cdots X_N X_{P_2}}_{-\widetilde{W}_k} \Big\}$$

Mirror geometries **Period integral $\widehat{\Pi}$**

$$\widehat{\Pi} = \int \prod_{i=1}^{N} \mathrm{d}Y_i \, \mathrm{d}Y_{P_1} \, \mathrm{d}Y_{P_2} \, \mathrm{e}^{-Y_{P_1}} \, \delta\Big(\sum_{i=1}^{N} Y_i - kY_{P_1} - (N-k)Y_{P_2} - t \Big)$$
$$\times \exp\Big\{ - \sum_{i=1}^{N} \mathrm{e}^{-Y_i} - \mathrm{e}^{-Y_{P_1}} - \mathrm{e}^{-Y_{P_2}} \Big\}$$

field re-definition:

$$\mathrm{e}^{-Y_{P_1}} \equiv \widetilde{P}_1 \, , \qquad \mathrm{e}^{-Y_a} \equiv \widetilde{P}_1 U_a \qquad \text{for } a = 1, \cdots, k$$
$$\mathrm{e}^{-Y_{P_2}} \equiv \widetilde{P}_2 \, , \qquad \mathrm{e}^{-Y_b} \equiv \widetilde{P}_2 U_b \qquad \text{for } b = k+1, \cdots, N$$

$$\widehat{\Pi} = \int \prod_{i=1}^{N} \Big(\frac{\mathrm{d}U_i}{U_i} \Big) \mathrm{d}\widetilde{P}_1 \Big(\frac{\mathrm{d}\widetilde{P}_2}{\widetilde{P}_2} \Big) \delta\Big(\log\Big(\prod_{i=1}^{N} U_i \Big) + t \Big) \exp\Big\{ - \widetilde{P}_1 \Big(\sum_a U_a + 1 \Big) - \widetilde{P}_2 \Big(\sum_b U_b + 1 \Big) \Big\}$$
$$= \int \prod_{i=1}^{N} \Big(\frac{\mathrm{d}U_i}{U_i} \Big) \mathrm{d}u \, \mathrm{d}v \, \delta\Big(\log\Big(\prod_{i=1}^{N} U_i \Big) + t \Big) \delta\Big(\sum_a U_a + 1 \Big) \delta\Big(\sum_b U_b + 1 - uv \Big)$$

$$\text{where} \qquad \frac{1}{\widetilde{P}_2} = \mathrm{d}u \, \mathrm{d}v \, \exp\Big(\widetilde{P}_2 \, uv \Big)$$

for $\widetilde{M}_{N-k}$:

$$U_a \equiv Z_a^{N-k} , \qquad U_b \equiv \mathrm{e}^{-t/(N-k)} \frac{Z_b^{N-k}}{Z_1 \cdots Z_N}$$

$$\widehat{\Pi} = \int \frac{1}{\mathrm{vol.}(\mathbb{C}^*)} \prod_{i=1}^{N} \mathrm{d}Z_i \, \mathrm{d}u \, \mathrm{d}v \, \delta\Big(\sum_a Z_a^{N-k} + 1 \Big) \, \delta\Big(\sum_b Z_b^{N-k} + Z_1 \cdots Z_N \big(\mathrm{e}^{t/(N-k)} - uv \big) \Big)$$

for $\widetilde{M}_k$:

$$U_a \equiv \mathrm{e}^{-t/k} \frac{Z_a^k}{Z_1 \cdots Z_N} , \qquad U_b \equiv Z_b^k$$

$$\widehat{\Pi} = \int \frac{1}{\mathrm{vol.}(\mathbb{C}^*)} \prod_{i=1}^{N} \mathrm{d}Z_i \, \mathrm{d}u \, \mathrm{d}v \, \delta\Big(\sum_a Z_a^k + \mathrm{e}^{t/k} Z_1 \cdots Z_N \Big) \, \delta\Big(\sum_b Z_b^k + 1 - uv \Big)$$

Algebraic equations

▼ mirror dual of $\mathcal{O}(-N) \to \mathbb{C}\mathrm{P}^{N-1}$:

$$\widetilde{M} = \left\{ (Z_i; u, v) \in \mathbb{C}^N \times \mathbb{C}^2 \,\middle|\, \widetilde{G}(Z_i; u, v) = 0 \right\} \Big/ \{\mathbb{C}^* \times (\mathbb{Z}_N)^{N-2}\}$$

$$\widetilde{G}(Z_i; u, v) = Z_1^N + \cdots + Z_N^N + Z_1 \cdots Z_N (\mathrm{e}^{t/N} - uv)$$

▼ mirror dual of $\mathbb{C}\mathrm{P}^{N-1}[N]$:

$$\widetilde{M} = \left\{ (Z_1, \cdots, Z_N) \in \mathbb{C}^N \,\middle|\, G(Z_i) = 0 \right\} \Big/ \{\mathbb{C}^* \times (\mathbb{Z}_N)^{N-2}\}$$

$$G(Z_i) = Z_1^N + \cdots + Z_N^N + \mathrm{e}^{t/N} Z_1 \cdots Z_N$$

Z_i: homogeneous coordinates of mirror dual space (also $\mathbb{C}\mathrm{P}^{N-1}$)