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Towards Mirror Symmetry on Noncompact Calabi-Yau Manifolds

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Introduction

Two-dimensional SUSY theories are very important

- They have some properties similar to ones of four-dim. SUSY theories

  asymptotic free, mass gap in SUSY gauge theories, SUSY NLSM, etc.

- We apply them to superstring worldsheet theories

  Neveu-Schwarz/Ramond string = superconformal field theory

  strings on curved spacetime = SUSY conformal sigma models

SYM, SUSY NLSM, LG theory, SCFT, etc. have been investigated for more than 25 years.

Noncompact Calabi-Yau (K.Higashijima, M.Nitta and TK, 2001, 2002)

Ricci-flat metrics on line bundles on $[\otimes_a (G_a/H_a)]$

base G/H	total dimension
$\mathbb{C}\mathrm{P}^{N-1} = \frac{SU(N)}{SU(N-1)\times U(1)}$	$1 + (N-1)$
$Q^{N-2} = \frac{SO(N)}{SO(N-2)\times U(1)}$	$1 + (N-2)$
$E_6/[SO(10) \times U(1)]$	$1 + 16$
$E_7/[E_6 \times U(1)]$	$1 + 27$
$\mathrm{G}_{N,M} = \frac{U(N)}{U(N-M)\times U(M)}$	$1 + M(N-M)$
$SO(2N)/U(N)$	$1 + \frac{1}{2}N(N-1)$
$Sp(N)/U(N)$	$1 + \frac{1}{2}N(N+1)$

We consider the sigma model on line bundle on Q^{N-2} and its mirror dual.

- There exists a nontrivial holomorphic constraint on the base manifold
  - $\mathbb{C}\mathrm{P}^{N-1}$ with holomorphic constraint of degree $d \Rightarrow \mathbb{C}\mathrm{P}^{N-1}[d]$
  - when $d = 2$, isometry group of surface can be enhanced to $SO(N)$

    $\Rightarrow$ quadric surface Q^{N-2}
- Strictly, this CY is the $\mathcal{O}(-N+2)$ bundle on Q^{N-2}
- Conifold appears in the zero volume limit of base Q^{N-2} manifold

Can we read geometrical aspects of CY from field theory?

$\Downarrow$

Gauged Linear Sigma Models and their T-dualized Theories

- E. Witten: "Phases of $\mathcal{N} = 2$ Theories in Two Dimensions"
- K. Hori and C. Vafa: "Mirror Symmetry"

- study orbifold (or LG) description of these non-compact CYs (CY/LG correspondence)
- construct Mirror Pairs of CYs

Example Mirror dual for compact CY 3-fold:

CY sigma model on $\mathbb{C}\mathrm{P}^4[5]$	CY/LG ⟷	LG orbifold theory with $W/\mathbb{Z}_5$	← Witten
	↕ T-duality		
$\widetilde{\mathrm{CY}}$ sigma model on $\mathbb{C}\mathrm{P}^4[5]/(\mathbb{Z}_5)^3$	CY/LG ⟷	$\widetilde{\mathrm{LG}}$ orbifold theory with $\widetilde{W}/(\mathbb{Z}_5)^4$	← Hori-Vafa

Hori-Vafa's T-dualized theory re-constructs a CY mirror pair discussed by Greene and Plesser!

Let us apply this relation to the case of line bundle on Q^{N-2}

Gauged Linear Sigma Model

(c,c) superfields	S_1	$\cdots$	S_N	P_1	P_2
$U(1)$ charge	1	$\cdots$	1	-2	$-N+2$

$$W_{\text{GLSM}} = P_1 \cdot G_2(S_i)$$

$G_2(S_i) = \sum_{i=1}^{N} S_i^2$: $SO(N)$ symmetric polynomial of degree 2

($\subset$ quasi-homogeneous polynomial of degree 2)

- **FI $\gg 0$:** sigma model on $\mathcal{O}(-N+2)$ bundle on Q^{N-2}
- **FI $\ll 0$:** $\left\{(S_1, \cdots, S_N) \in \mathbb{C}^N \,\middle|\, G_2(S_i) = 0\right\} \Big/ \mathbb{Z}_{N-2}$ orbifold theory

$$\Downarrow$$

sigma model on $(N-1)$-dim. **conifold**

T-dualized Theory

Low energy effective theory = LG twisted superpotential $(N \equiv 2k+2)$:

$$\widetilde{W} \;=\; \sum_{i=1}^{N} \mathrm{e}^{-Y_i} + \mathrm{e}^{-Y_{P_1}} + \mathrm{e}^{-Y_{P_2}} + \Sigma\Big(\sum_{i=1}^{N} Y_i - 2Y_{P_1} - 2k\, Y_{P_2} - t\Big)$$

Introduce the "period integral": (depending on the existence of W_{GLSM})

$$\begin{cases} \Pi_{\text{without } W_{\mathrm{GLSM}}} \;=\; \displaystyle\int \mathrm{d}\Sigma \prod_{i=1}^{N} \mathrm{d}Y_i\, \mathrm{d}Y_{P_1}\, \mathrm{d}Y_{P_2}\, \exp(-\widetilde{W}) \\ \widehat{\Pi}_{\text{with } W_{\mathrm{GLSM}}} \;\equiv\; 2\dfrac{\partial}{\partial t} \Pi_{\text{without } W_{\mathrm{GLSM}}} \end{cases}$$

Does $\widehat{\Pi}$ feel the $SO(N)$ symmetry of $G_2(S_i)$ in W_{GLSM}?
(knows only the quasi-homogeneity?)

Field re-definition for suitable measure of $\widehat{\Pi}$ for LG description ($N \equiv 2k+2$):

$$e^{-Y_i} = \boldsymbol{X}_i^{2k}\,, \quad e^{-Y_{P_1}} = \boldsymbol{X}_{P_1}^{-k}$$

$$e^{-Y_{P_2}} = e^{t/2k}\boldsymbol{X}_1\boldsymbol{X}_2\cdots\boldsymbol{X}_N\cdot\boldsymbol{X}_{P_1}$$

$$\Rightarrow \quad \widetilde{W} = \sum_{i=1}^{N}\boldsymbol{X}_i^{2k} + \boldsymbol{X}_{P_1}^{-k} + e^{t/2k}\boldsymbol{X}_1\boldsymbol{X}_2\cdots\boldsymbol{X}_N\cdot\boldsymbol{X}_{P_1}$$

[Remarks] on $\widetilde{W}$:

- One massless vacuum
- $(\mathbb{Z}_{2k})^{2k} \times \mathbb{Z}_k$ orbifold symmetry (when $k=1$, symmetry becomes $(\mathbb{Z}_2)^3$)
- This is still a mirror dual of $\mathcal{O}(-N+2)$ on $\mathbb{C}\mathrm{P}^{N-1}[2]$!

Investigate twisted RR ground states related to (c,c)-ring structure

$$\Downarrow$$

We can determine the correct field re-definitions of Y_i

We construct a mirror CY geometry of $\mathcal{O}(-N+2)$ on $\mathbb{CP}^{N-1}[2]$:

$$\widehat{\Pi} = \int \prod_{i=1}^{N} \mathrm{d}Y_i \, (\mathrm{e}^{-Y_{P_1}} \mathrm{d}Y_{P_1}) \, \mathrm{d}Y_{P_2} \, \delta\Big(\sum_{i=1}^{N} Y_i - 2Y_{P_1} - 2kY_{P_2} - t \Big) \exp\Big(- \sum_{i=1}^{N} \mathrm{e}^{-Y_i} - \mathrm{e}^{-Y_{P_1}} - \mathrm{e}^{-Y_{P_2}} \Big)$$

solving Y_N and field re-definition:

$$\mathrm{e}^{-Y_{P_1}} = \widetilde{P}_1 \qquad \mathrm{e}^{-Y_{P_2}} = \widetilde{P}_2$$

$$\begin{aligned} \mathrm{e}^{-Y_a} &= e^{-t/N} \widetilde{P}_2 \, Z_a && \text{for } a = 1, 2, \cdots, N-3 \\ \mathrm{e}^{-Y_b} &= \mathrm{e}^{-t/N} \widetilde{P}_1 \, Z_b && \text{for } b = N-2, N-1 \end{aligned}$$

Introducing $\mathbb{C}$-valued fields u and v and integrating out $\widetilde{P}_1$ and $\widetilde{P}_2$:

$$\widehat{\Pi} = \int \prod_{i=1}^{N-1} \Big(\frac{\mathrm{d}Z_i}{Z_i} \Big) \mathrm{d}u \, \mathrm{d}v \, \delta\Big(\sum_b Z_b + \mathrm{e}^{t/N} \Big) \, \delta\Big(\sum_a Z_a + \mathrm{e}^{t/N} + \frac{1}{Z_1 \cdots Z_{N-1}} - uv \Big)$$

CY algebraic equation in $(\mathbb{C}^*)^{N-2} \times \mathbb{C}^2$ space:

$$1 - Z_1 \cdots Z_{N-2} \Big(Z_{N-2} + \mathrm{e}^{t/N} \Big) \Big(\sum_{a=1}^{N-3} Z_a + \mathrm{e}^{t/N} - uv \Big) = 0$$

Result CY 3-fold case ($N = 2k + 2 = 4$):

CY sigma model on $\mathcal{O}(-2) \to Q^2$	CY/LG ⟷	$\mathbb{Z}_2$ orbifold theory as conifold sigma model
	↕ T-dualize (incomplete)	
$\widetilde{\mathrm{CY}}$ sigma model on $\widetilde{M}$	CY/LG ⟷	$\widetilde{\mathrm{LG}}$ orbifold theory with $\widetilde{W}/(\mathbb{Z}_2)^3$

$$\widetilde{W} = X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_{P_1}^{-1} + \mathrm{e}^{t/2} X_1 X_2 X_3 X_4 \cdot X_{P_1}$$

$$\widetilde{M} \ : \ 1 - Z_1 Z_2 \big(Z_1 + \mathrm{e}^{t/4} - uv\big)\big(Z_2 + \mathrm{e}^{t/4}\big) = 0$$

for the mirror dual of the line bundle on $\mathbb{C}\mathrm{P}^3[2]$

Next, we must investigate the twisted RR ground states for $SO(N)$ symmetry:

$$\mathbb{C}\mathrm{P}^{N-1}[2] \ \Rightarrow \ Q^{N-2}$$

Applications and Discussions

- **New aspects of conifold transition in string theory**
- **CY sigma models with nonabelian gauge symmetries and their mirror duals**
- **New kinds of CY mirror pairs which have been unknown cohomology rings**
- **SCFT on noncompact CY which differs from Liouville theory**

  Liouville theory describes CFT on direct product spaces

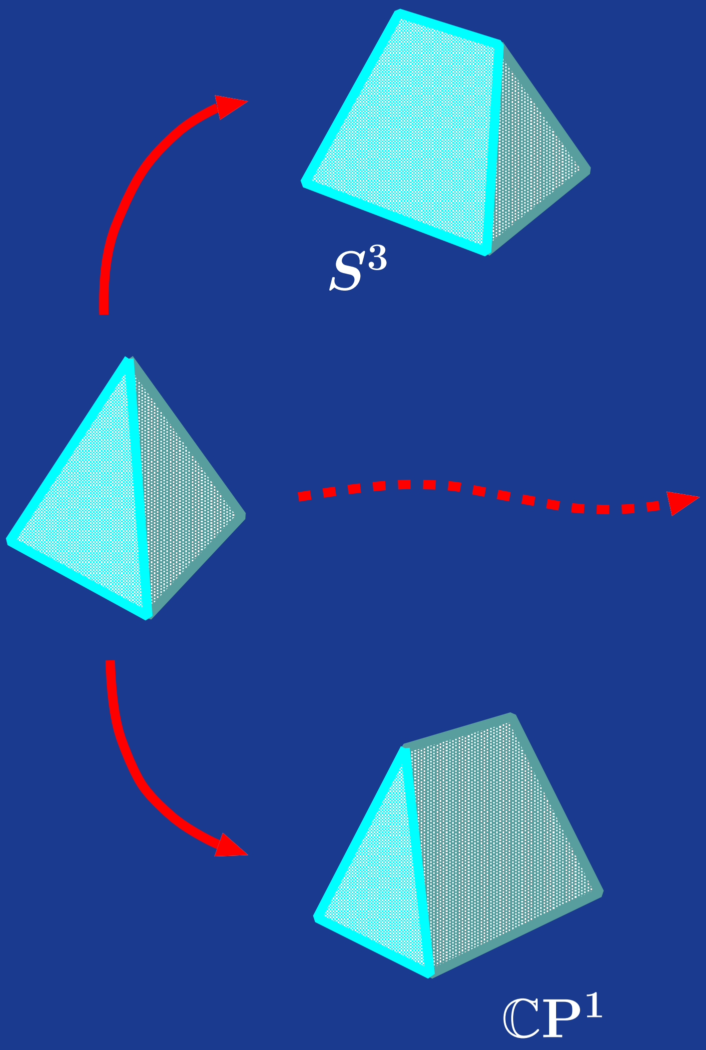
- **Mirror duals of asymptotic free sigma models (sigma models on HSS)**

(ex.)

sigma model on $\mathbb{C}\mathrm{P}^{N-1}$	$\Longleftrightarrow$	$\widehat{A}_{N-1}$ **Toda field theory**
sigma model on Q^{N-2}	$\Longleftrightarrow$	**???**

Line bundle on Q^2 = noncompact CY$_3$

Q^2 shrinks to zero size $\longrightarrow$ singular conifold



deformation:

related to complex moduli of CY$_3$

"Q^2 deformation":

singularity is resolved by $Q^2 = \dfrac{SO(4)}{SO(2)\times U(1)}$

small resolution:

related to Kähler moduli of CY$_3$

[Sigma model on Q^{N-2}] (K. Higashijima, M. Nitta, M. Tsuzuki and TK, 2000)

Kähler potential of sigma model with gauge symmetry is

$$\mathscr{L} = \int \mathrm{d}^4\theta \left(\sum_{i=1}^{N} \overline{S}_i \, \mathrm{e}^{2V} S_i - 2r\, V \right) + \left(\int \mathrm{d}^2\theta \, P_1 \sum_{i=1}^{N} S_i^2 + h.c. \right)$$

sigma model on $\mathbb{C}\mathrm{P}^{N-1}$ with F-term constraint $\Rightarrow$ $SO(N)$ symmetry

integrating out the auxiliary fields V and P_1 (Φ_j: local "coordinates"):

$$\mathscr{L} = r \int \mathrm{d}^4\theta \, \log \left\{ 1 + \sum_{j=1}^{N-2} |\Phi_j|^2 + \left| 1 + \sum_{k=1}^{N-2} \Phi_k^2 \right| \right\}$$

- sigma model on $\frac{SO(N)}{SO(N-2)\times U(1)}$ ($\equiv$ "Q^{N-2}")
- asymptotically free
- mass gap
- There exist Coulomb and Higgs phases in $1/N$ analysis