基研研究会「場の量子論の基礎的諸問題と応用」 *December 17, 2004*

ポスターセッション B7

Gauged Linear Sigma Models

for Noncompact Calabi-Yau Varieties

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Abstract

We study gauged linear sigma models for noncompact Calabi-Yau manifolds described as a line bundle on a hypersurface in a projective space. This gauge theory has a unique phase if the Fayet-Iliopoulos parameter is positive, while there exist two distinct phases if the parameter is negative. We find four massless effective theories in the infrared limit, which are related to each other under the CY/LG correspondence and the topology change. In the T-dual theory, on the other hand, we obtain two types of exact massless effective theories: One is the sigma model on a newly obtained Calabi-Yau geometry as a mirror dual, while the other is given by a Landau-Ginzburg theory with a negative power term, indicating $\mathcal{N} = 2$ superconformal field theory on $SL(2, \mathbb{R})/U(1)$. We argue that the effective theories in the original gauged linear sigma model are exactly realized as $\mathcal{N} = 2$ Liouville theories coupled to well-defined Landau-Ginzburg minimal models.

Introduction

Two-dimensional field theory is a powerful framework

- toy model to investigate dynamical symmetry breaking
- applied to string theory as (ultimately) unified theory

  combining conformal invariance
- coupled to a gauge field, this has been applied to more complicated physics

  compactification of target space via Higgs mechanism
- Mirror Symmetry between string theories on different CY's

  unification of various SCFTs and (algebraic) geometries

Gauged linear sigma model E. Witten (1993), K. Hori and C. Vafa (2000)

$\mathcal{N} = (2,2)$ SUSY gauge theory with matters $(\mathrm{FI}: \ t \equiv r - i\theta)$

$$\mathscr{L} = \int \mathrm{d}^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_a \overline{\Phi}_a \, \mathrm{e}^{2Q_a V} \Phi_a \Big\} + \Big(\frac{1}{\sqrt{2}} \int \mathrm{d}^2\widetilde{\theta} \, (-\Sigma \, t) + c.c. \Big) + \Big(\int \mathrm{d}^2\theta \, W_{\mathrm{GLSM}}(\Phi_a) + c.c. \Big)$$

- Φ_a : charged chiral superfield, $\overline{D}_{\pm}\Phi_a = 0$
  Σ : twisted chiral superfield, $\overline{D}_{+}\Sigma = D_{-}\Sigma = 0$, $\Sigma = \frac{1}{\sqrt{2}}\overline{D}_{+}D_{-}V$
- There exist at least two phases:

  $\mathrm{FI} \gg 0$: differential-geometric phase $\longrightarrow$ SUSY NLSM

  $\mathrm{FI} \ll 0$: algebro-gemetric phase $\longrightarrow$ LG, orbifold, SCFT

- Calabi-Yau/Landau-Ginzburg correspondence

  harmonic forms $\longleftrightarrow$ NS-NS chiral primary states

- "Mirror geometry" appears in the T-dual theory in terms of twisted chiral superfields Y_a

$$Y_a + \overline{Y}_a \equiv 2\overline{\Phi}_a \, \mathrm{e}^{2Q_a V} \Phi_a$$

Effective theories

The potential energy density is given by

$$\mathcal{U}(\phi,\sigma) = \frac{e^2}{2}\mathcal{D}^2 + \sum_a |F_a|^2 + \mathcal{U}_\sigma(\phi,\sigma)$$

$$\mathcal{D} = \frac{1}{e^2}D = r - \sum_a Q_a|\phi_a|^2 , \quad \overline{F}_a = -\frac{\partial}{\partial\phi_a}W_{\mathrm{GLSM}}(\phi) , \quad \mathcal{U}_\sigma(\phi,\sigma) = 2|\sigma|^2 \sum_a Q_a^2|\phi_a|^2$$

The supersymmetric vacuum manifold $\mathcal{M}$ is defined by

$$\mathcal{M} = \Big\{(\phi_a,\sigma) \in \mathbb{C}^n \Big| \mathcal{D} = F_a = \mathcal{U}_\sigma = 0\Big\}\Big/ U(1)$$

In the IR limit $e \to \infty$, there appears the supersymmetric NLSM on $\mathcal{M}$ whose coupling is

$$r = \frac{1}{g^2}$$

Renormalization of the FI parameter is $$\boxed{r_0 = r_R + s \cdot \log\Big(\frac{\Lambda_{\mathrm{UV}}}{\mu}\Big), \quad s = \sum_a Q_a}$$

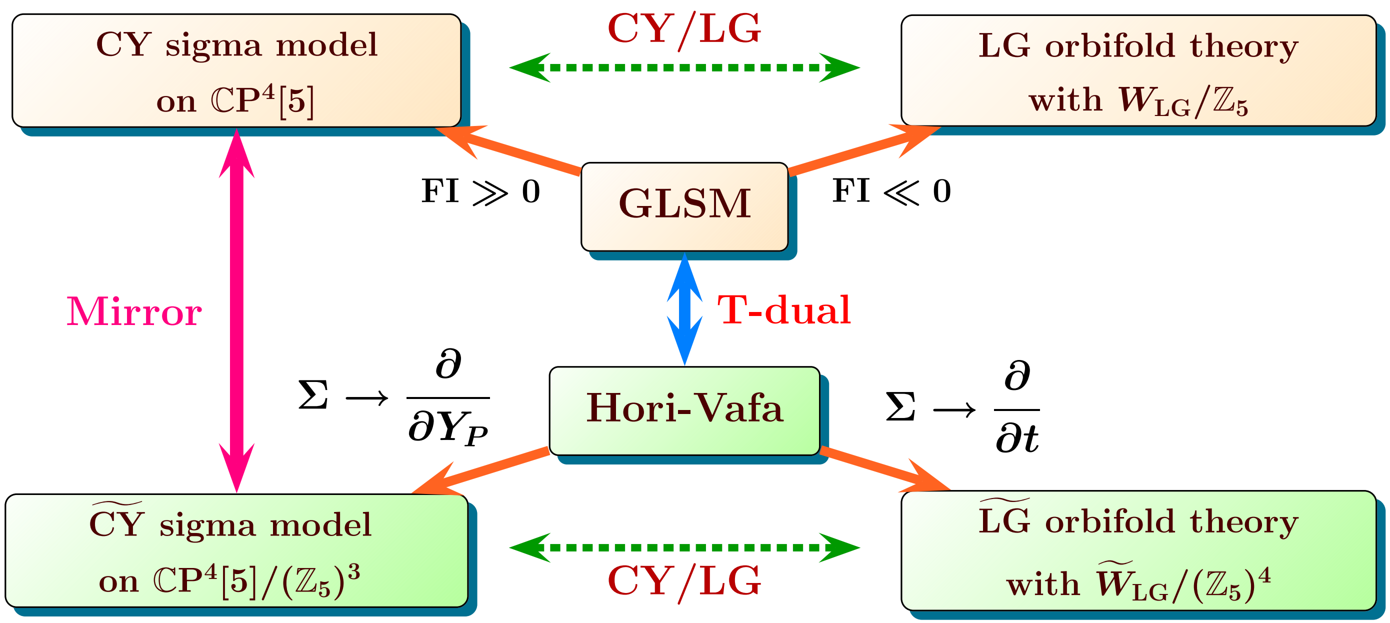
Thus we find that

$s > 0 \quad \longrightarrow \quad$ the theory is asymptotic free

$s = 0 \quad \longrightarrow \quad$ the theory is conformal

$s < 0 \quad \longrightarrow \quad$ the theory is infrared free

Example: quintic hypersurface and its mirror



where

GLSM :

Φ_a	S_1	S_2	$\ldots$	S_5	P
Q_a	1	1	$\ldots$	1	-5

$$W_{\mathrm{GLSM}} = P \cdot G_5(S)$$

Hori-Vafa : $\widehat{\Pi} \equiv \int d\Sigma \prod_a dY_a (5\Sigma) \exp(-\widetilde{W}), \quad \widetilde{W} = \Sigma\Big(\sum_a Q_a Y_a - t\Big) + \sum_a \mathrm{e}^{-Y_a}$

$$W_{\mathrm{LG}} = G_5(S)$$

$$\widetilde{W}_{\mathrm{LG}} = X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 + \mathrm{e}^{t/5} X_1 X_2 X_3 X_4 X_5 \,, \quad X_i^5 = \mathrm{e}^{-Y_i}$$

$$h_{21}(\mathbb{C}\mathrm{P}^4[5]) = h_{11}(\mathbb{C}\mathrm{P}^4[5]/(\mathbb{Z}_5)^3) = 101 \,, \qquad h_{11}(\mathbb{C}\mathrm{P}^4[5]) = h_{21}(\mathbb{C}\mathrm{P}^4[5]/(\mathbb{Z}_5)^3) = 1$$

Metrics on Noncompact Calabi-Yau

(K.Higashijima, M.Nitta and TK, 2001, 2002)

line bundles	total dim. D	dual Coxeter C	"orbifolding" ℓ
$\mathbb{C} \ltimes \left(\mathbb{C}P^{N-1} = \frac{SU(N)}{SU(N-1)\times U(1)}\right)$	$1+(N-1)$	N	N
$\mathbb{C} \ltimes \left(Q^{N-2} = \frac{SO(N)}{SO(N-2)\times U(1)}\right)$	$1+(N-2)$	$N-2$	$N-2$
$\mathbb{C} \ltimes E_6/[SO(10)\times U(1)]$	$1+16$	12	12
$\mathbb{C} \ltimes E_7/[E_6\times U(1)]$	$1+27$	18	18
$\mathbb{C} \ltimes \left(G_{N,M} = \frac{U(N)}{U(N-M)\times U(M)}\right)$	$1+M(N-M)$	N	MN
$\mathbb{C} \ltimes SO(2N)/U(N)$	$1+\frac{1}{2}N(N-1)$	$N-1$	$N(N-1)$
$\mathbb{C} \ltimes Sp(N)/U(N)$	$1+\frac{1}{2}N(N+1)$	$N+1$	$N(N+1)$

$$\mathcal{K}'_{\text{noncompact}}(\rho,\varphi) = \left(e^{CX}+b\right)^{1/D}, \qquad X = \log|\rho^{1/\ell}|^2 + K_{\text{compact}}(\varphi)$$

$$K_{\mathbb{C}P^{N-1}}(\varphi) = r\log\left(1+\sum_{i=1}^{N-1}|\varphi_i|^2\right)$$

What are their mirror geometries?

Another motivation

- Higashijima-Nitta-TK (2001$\sim$2002):

  SUSY nonlinear sigma models = geometric description of noncompact CY

How about SCFT description?

- Giveon-Kutasov, Eguchi-Sugawara (1999$\sim$), etc.:

  $\mathcal{N}=2$ Liouville $\otimes$ Landau-Ginzburg = SCFT description of noncompact CY

Why direct product? (holographic picture of $\mathbb{R}^{d-1,1}\times$ (non)singular CY)

↓

- T. Kimura (2002$\sim$): GLSM for $\mathcal{O}(-N+\ell)$ bundle on $\mathbb{C}\mathrm{P}^{N-1}[\ell]$

Gauged Linear Sigma Model for $\mathcal{O}(-N+\ell)$ bundle on $\mathbb{C}\mathrm{P}^{N-1}[\ell]$

chiral superfield	S_1	$\cdots$	S_N	P_1	P_2
$U(1)$ charge	1	$\cdots$	1	$-\ell$	$-N+\ell$

$$W_{\mathrm{GLSM}} = P_1 \cdot G_\ell(S_i)$$

$G_\ell(S_i)$: homogeneous polynomial of degree ℓ

potential energy density:

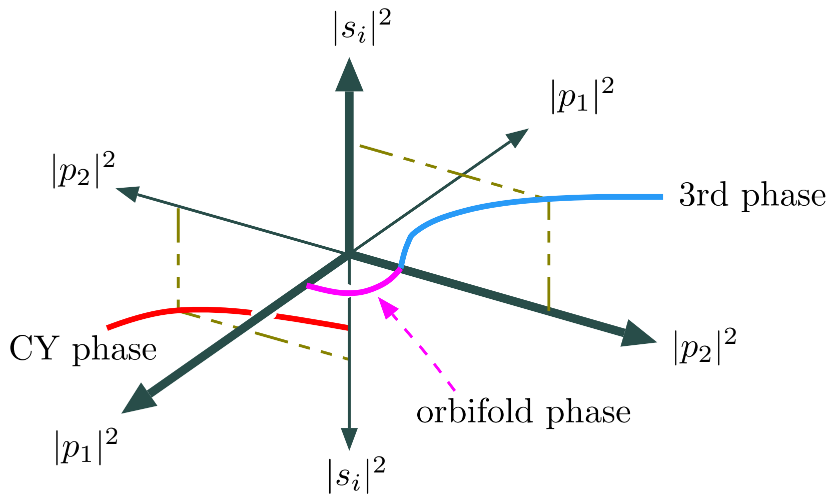
$$\mathcal{U} = \frac{e^2}{2}\mathcal{D}^2 + \big|G_\ell(s)\big|^2 + \sum_{i=1}^{N} \big|p_1 \partial_i G_\ell(s)\big|^2 + \mathcal{U}_\sigma$$

$$\mathcal{D} = r - \sum_{i=1}^{N} |s_i|^2 + \ell|p_1|^2 + (N-\ell)|p_2|^2$$

$$\mathcal{U}_\sigma = +2|\sigma|^2 \Big\{ \sum_{i=1}^{N} |s_i|^2 + \ell^2|p_1|^2 + (N-\ell)^2|p_2|^2 \Big\}$$

Let us analyze SUSY vacuum manifold $\mathcal{U}=0$ and massless effective theories

Supersymmetric vacua



- **CY phase on $\mathcal{M}_{\rm CY}$**

  conformal sigma model on $\mathcal{M}_{\rm CY}$

- **orbifold phase on $\mathcal{M}^1_{r<0}$**

  $\big\{$CFT on $\mathbb{C}^1 \otimes$ LG with $W_{\rm LG} = \langle p_1 \rangle G_\ell(S)\big\}/\mathbb{Z}_\ell$

  $\big\{$"LG" with $W_{\rm LG} = P_1 \cdot G_\ell(S)\big\}/\mathbb{Z}_{N-\ell}$

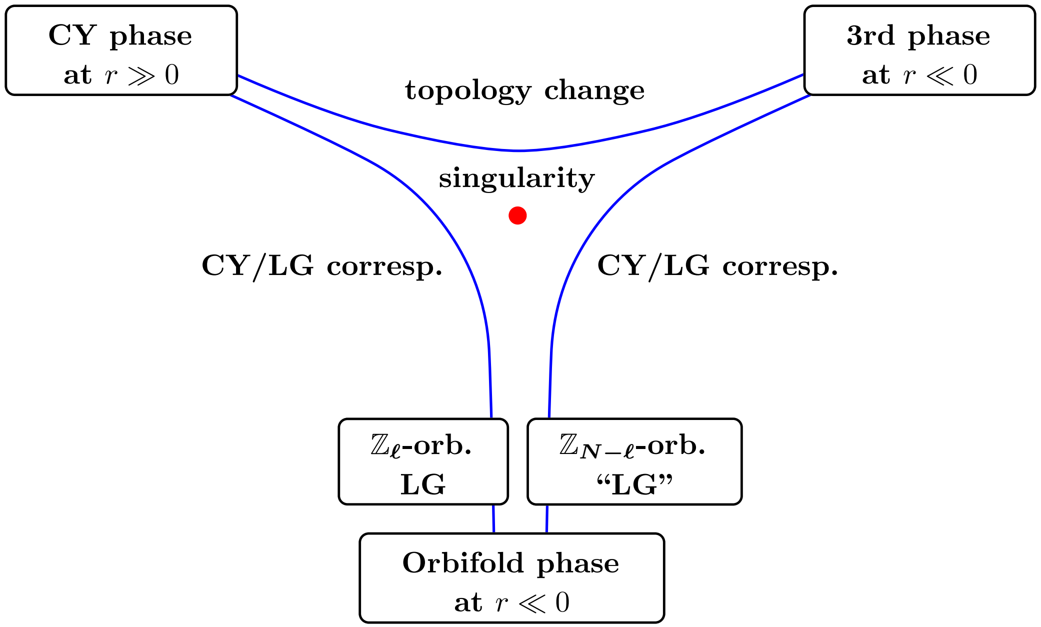
- **3rd phase on $\mathcal{M}^2_{r<0}$** NEW!

  conformal sigma model on $\mathcal{M}^2_{r<0}$

$$
\begin{aligned}
\mathcal{M}_{\rm CY} &= \Big\{(s_i; p_2) \in \mathbb{C}^{N+2} \,\Big|\, \mathcal{D} = G_\ell = 0,\ r > 0\Big\}\Big/U(1) \quad \equiv \quad \mathcal{O}(-N+\ell) \text{ bundle on } \mathbb{CP}^{N-1}[\ell] \\
\mathcal{M}^1_{r<0} &= \Big\{(p_1, p_2) \in \mathbb{C}^2 \,\Big|\, \mathcal{D} = 0, r < 0\Big\}\Big/U(1) \quad \equiv \quad \mathrm{W}\mathbb{CP}^1_{\ell, N-\ell} \\
\mathcal{M}^2_{r<0} &= \Big\{(s_i; p_2) \in \mathbb{C}^{N+2} \,\Big|\, \mathcal{D} = G_\ell = 0,\ r < 0\Big\}\Big/U(1)
\end{aligned}
$$

CY/LG correspondence and topology change

The four theories are related to each other via CY/LG correspondence and topology change:



Notice that the above relation is just a conjectured one because we still have no mathematical techniques to check the topological aspects on the noncompact CY.

We also notice that we have obtained various massless effective theories by decomposing all massive modes. Thus they are just approximate descriptions. However the T-dual theory of the GLSM is so powerful to obtain the exact effective theories. Analyzing them exact theories we will re-investigate the massless effective theories in the original GLSM.

T-dual Theory

$$\mathscr{L} = \int d^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma - \sum_a \Big(\frac{1}{2}(Y_a + \overline{Y}_a)\log(Y_a + \overline{Y}_a)\Big)\Big\} + \Big(\frac{1}{\sqrt{2}}\int d^2\widetilde{\theta}\,\widetilde{W} + c.c.\Big)$$

$$\widetilde{W} = \Sigma\Big(\sum_{i=1}^{N} Y_i - \ell Y_{P_1} - (N-\ell)Y_{P_2} - t\Big) + \sum_{i=1}^{N} e^{-Y_i} + e^{-Y_{P_1}} + e^{-Y_{P_2}}$$

Period integral : $\widehat{\Pi} \equiv \int d\Sigma \prod_{i=1}^{N} dY_i\, dY_{P_1}\, dY_{P_2}\, (\ell\Sigma)\, \exp\big(-\widetilde{W}\big)$

chiral superfield	S_1	S_2	$\cdots$	S_N	P_1	P_2
$U(1)$ charge	1	1	$\cdots$	1	$-\ell$	$-N+\ell$
twisted chiral	Y_1	Y_2	$\cdots$	Y_N	Y_{P_1}	Y_{P_2}

$$2\overline{\Phi}_a\, e^{2Q_a V}\, \Phi_a = Y_a + \overline{Y}_a$$

$U(1)$ phase rotation symmetry on Φ_a ➡ shift symmetry on Y_a: $Y_a \equiv Y_a + 2\pi i$

In the IR limit $e \to \infty$, the gauge field Σ is no longer dynamical and should be integrated out.

in order to obtain LG theory or geometry, we replace Σ to $\Sigma \to \dfrac{\partial}{\partial t}$ or $\Sigma \to \dfrac{\partial}{\partial Y_P}$

Twisted Landau-Ginzburg theory: There exist two consistent solutions

— Solution one: $\mathbb{Z}_\ell$-type orbifold symmetry —

Solve Y_{P_1} by using the constraint derived from integrating out Σ:

$$Y_{P_1} = \frac{1}{\ell}\Big\{t - \sum_{i=1}^{N} Y_i + (N-\ell)Y_{P_2}\Big\}$$

Field re-definition preserving canonical measure in $\widehat{\Pi}$:

$$X_i \equiv e^{-\frac{1}{\ell}Y_i}, \quad X_{P_2} \equiv e^{\frac{N-\ell}{\ell}Y_{P_2}}, \quad X_i \to \omega_i X_i, \quad X_{P_2} \to \omega_{P_2} X_{P_2}, \quad (\mathbb{Z}_\ell)^N \text{ symmetry}$$

Thus we obtain the twisted LG superpotential:

$$\Big\{\widetilde{W}_\ell = X_1^\ell + \cdots + X_N^\ell + X_{P_2}^{-\frac{\ell}{N-\ell}} + e^{t/\ell} X_1 \cdots X_N X_{P_2}\Big\}\Big/(\mathbb{Z}_\ell)^N$$

The negative power term describes $\mathcal{N}=2$ Kazama-Suzuki model on $SL(2,\mathbb{R})_k/U(1)$:

$$\frac{\ell}{N-\ell} = k = \frac{2}{Q^2}$$

Thus we argue that

this effective theory is the LG minimal model coupled to the KS model with $(\mathbb{Z}_\ell)^N$ orbifold symmetry

— Solution two: $\mathbb{Z}_{N-\ell}$-type orbifold symmetry —

Solve Y_{P_2} by using the constraint derived from integrating out Σ:

$$Y_{P_2} = \frac{1}{N-\ell}\Big\{t - \sum_{i=1}^{N} Y_i + \ell Y_{P_1}\Big\}$$

Field re-definition preserving canonical measure in $\widehat{\Pi}$:

$$X_i \equiv e^{-\frac{1}{N-\ell}Y_i} , \quad X_{P_1} \equiv e^{\frac{\ell}{N-\ell}Y_{P_1}} , \quad X_i \to \omega_i X_i , \quad X_{P_1} \to \omega_{P_1} X_{P_1} , \quad (\mathbb{Z}_{N-\ell})^N \text{ symmetry}$$

Thus we obtain the twisted LG superpotential:

$$\Big\{\widetilde{W}_{N-\ell} = X_1^{N-\ell} + \cdots + X_N^{N-\ell} + X_{P_1}^{-\frac{N-\ell}{\ell}} + e^{t/\ell} X_1 \cdots X_N X_{P_1}\Big\}\Big/ (\mathbb{Z}_{N-\ell})^N$$

negative power term = interpreted as $\mathcal{N}=2$ Kazama-Suzuki model on $SL(2,\mathbb{R})_k/U(1)$:

$$\frac{N-\ell}{\ell} = k = \frac{2}{Q^2}$$

Thus we argue that

this effective theory is the LG minimal model coupled to the KS model with $(\mathbb{Z}_{N-\ell})^N$ orbifold symmetry

▼ Twisted mirror geometry: There also exist **two** consistent solutions

— Solution one: $\mathbb{Z}_\ell$-type orbifold symmetry —

We replace $\ell\Sigma$ to $\dfrac{\partial}{\partial Y_{P_1}}$ and obtain

$$\widehat{\Pi} \;=\; \int \prod_{i=1}^{N} \mathrm{d}Y_i \, (\mathrm{e}^{-Y_{P_1}} \mathrm{d}Y_{P_1}) \, \mathrm{d}Y_{P_2} \, \delta\Big(\sum_i Y_i - \ell Y_{P_1} - (N-\ell) Y_{P_2} - t \Big) \exp\Big(- \sum_i \mathrm{e}^{-Y_i} - \mathrm{e}^{-Y_{P_1}} - \mathrm{e}^{-Y_{P_2}} \Big)$$

Re-defining the variables in order to obtain the canonical measure, we obtain

$$\widetilde{\mathcal{M}}_\ell \;=\; \Big\{ \{ \mathcal{F}(Z_i) \;=\; 0 \} / \mathbb{C}^* \,,\; \mathcal{G}(Z_b; u, v) \;=\; 0 \Big\} \Big/ (\mathbb{Z}_\ell)^{N-2}$$

$$\mathcal{F}(Z_i) \;=\; Z_1^\ell + \cdots + Z_\ell^\ell + \psi Z_1 \cdots Z_\ell \,, \quad \psi \;=\; \mathrm{e}^{t/\ell} Z_{\ell+1} \cdots Z_N$$

$$\mathcal{G}(Z_b; u, v) \;=\; Z_{\ell+1}^\ell + \cdots + Z_N^\ell + 1 - uv$$

$Z_a \;\mapsto\; \lambda\, \omega_a\, Z_a$ for $a \;=\; 1, \cdots, \ell$ (homogeneous coordinates of $\mathbb{C}\mathrm{P}^{\ell-1}[\ell]$)

$Z_b \;\mapsto\; \omega_b\, Z_b$ for $b \;=\; \ell + 1, \cdots, N$ (homogeneous coordinates of $\mathbb{C}^{N-\ell}$)

$$\omega_a^\ell \;=\; \omega_b^\ell \;=\; \omega_1 \cdots \omega_N \;=\; 1 \,, \qquad \lambda \,:\, \mathbb{C}^*\text{-value}$$

— Solution one: $\mathbb{Z}_{N-\ell}$-type orbifold symmetry —

We replace $\ell\Sigma$ to $\dfrac{\partial}{\partial Y_{P_2}}$ and obtain

$$\widehat{\Pi} \;=\; \int \prod_{i=1}^{N} \mathrm{d}Y_i \, \mathrm{d}Y_{P_1} \, (\mathrm{e}^{-Y_{P_2}} \mathrm{d}Y_{P_2}) \, \delta\Big(\sum_i Y_i - \ell Y_{P_1} - (N-\ell) Y_{P_2} - t \Big) \exp\Big(- \sum_i \mathrm{e}^{-Y_i} - \mathrm{e}^{-Y_{P_1}} - \mathrm{e}^{-Y_{P_2}} \Big)$$

Re-defining the variables in order to obtain the canonical measure, we obtain

$$\widetilde{\mathcal{M}}_{N-\ell} \;=\; \Big\{ F(Z_a; u, v) \;=\; 0 \,,\; \big\{ G(Z_i) \;=\; 0 \big\} / \mathbb{C}^* \Big\} \Big/ (\mathbb{Z}_{N-\ell})^{N-2}$$

$$F(Z_a; u, v) \;=\; Z_1^{N-\ell} + \cdots + Z_{\ell}^{N-\ell} + 1 - uv$$

$$G(Z_i) \;=\; Z_{\ell+1}^{N-\ell} + \cdots + Z_N^{N-\ell} + \psi Z_{\ell+1} \cdots Z_N \,, \qquad \psi = \mathrm{e}^{t/(N-\ell)} Z_1 \cdots Z_{\ell}$$

$Z_a \;\mapsto\; \omega_a \, Z_a$ for $a \;=\; 1, \cdots, \ell$ (homogeneous coordinates of $\mathbb{C}^{\ell}$)

$Z_b \;\mapsto\; \lambda \, \omega_b \, Z_b$ for $b \;=\; \ell + 1, \cdots, N$ (homogeneous coordinates of $\mathbb{CP}^{N-\ell-1}[N-\ell]$)

$$\omega_a^{N-\ell} \;=\; \omega_b^{N-\ell} \;=\; \omega_1 \cdots \omega_N \;=\; 1 \,, \qquad \lambda \;:\; \mathbb{C}^*\text{-value}$$

Return to the original GLSM

Recall the following two arguments:

- $\mathcal{N}=2$ SCFT on $SL(2,\mathbb{R})_k/U(1)$ is equivalent to $\mathcal{N}=2$ Liouville theory via T-duality
- If a CFT $\mathcal{C}$ has an abelian discrete symmetry group Γ, the orbifold CFT $\mathcal{C}'=\mathcal{C}/\Gamma$ has a symmetry group Γ' which is isomorphic to Γ. Furthermore a new orbifold CFT $\mathcal{C}'/\Gamma'$ is identical to the original CFT $\mathcal{C}$.

Thus we insist that

"$\{$CFT on $\mathbb{C}^1$ $\otimes$ LG with $W_{\text{LG}}=\langle p_1\rangle G_\ell(S)\}/\mathbb{Z}_\ell$" in the original GLSM

is described by

$\{\mathcal{N}=2$ Liouville theory coupled to the LG minimal model with $W_{\text{LG}}\}/\mathbb{Z}_\ell$

as an exact effective theory

The $\mathbb{Z}_{N-\ell}$-type orbifolded effective theory is also deformed in the same analogy.

Summary

- **We found three non-trivial phases and four effective theories in the GLSM**

  two CY sigma models

  two orbifolded LG theories coulped to 1-dim. SCFT

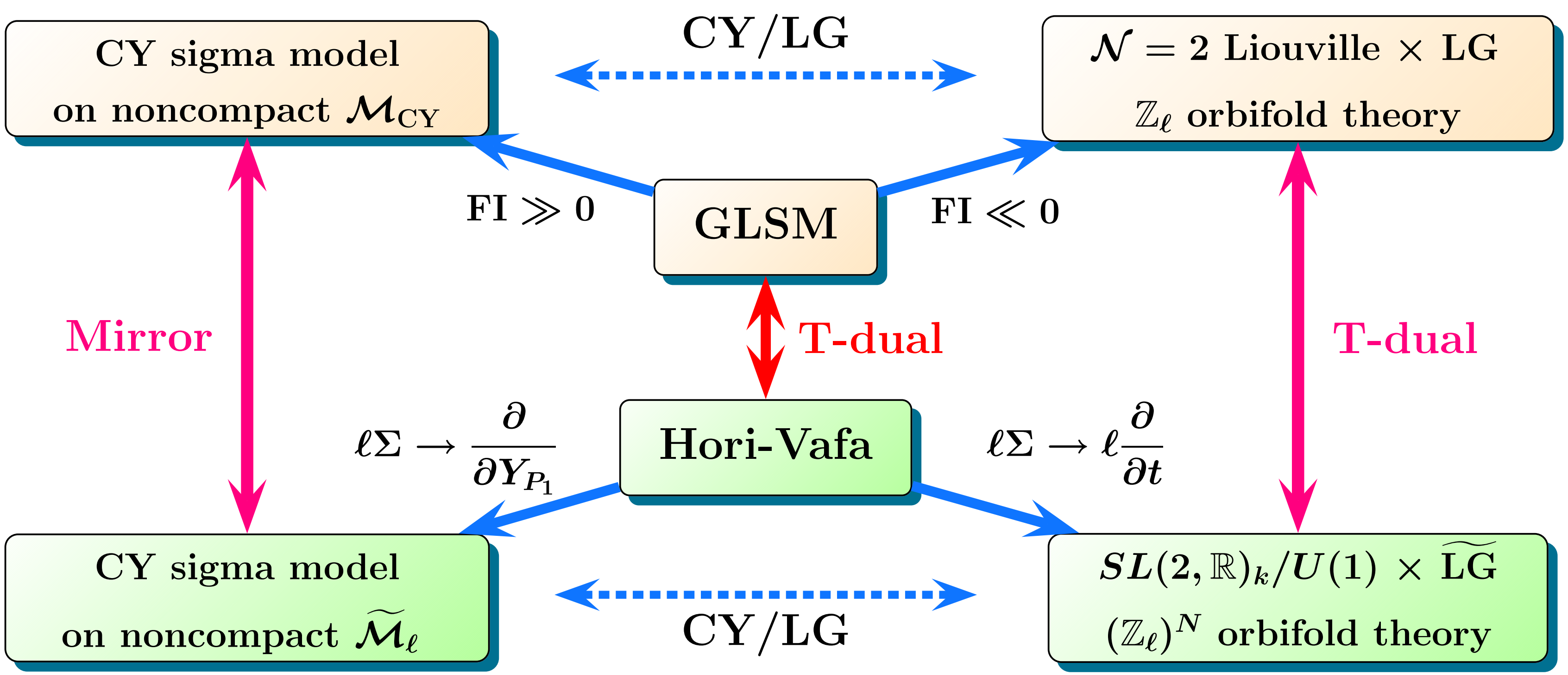
- **We constructed four exact effective theories in the T-dual theory**

  two NLSMs on mirror CY geometries

  two orbifolded LG theories including a term with negative power $-k$

  This term represents a gauged WZW model on $SL(2,\mathbb{R})_k/U(1)$ at level k

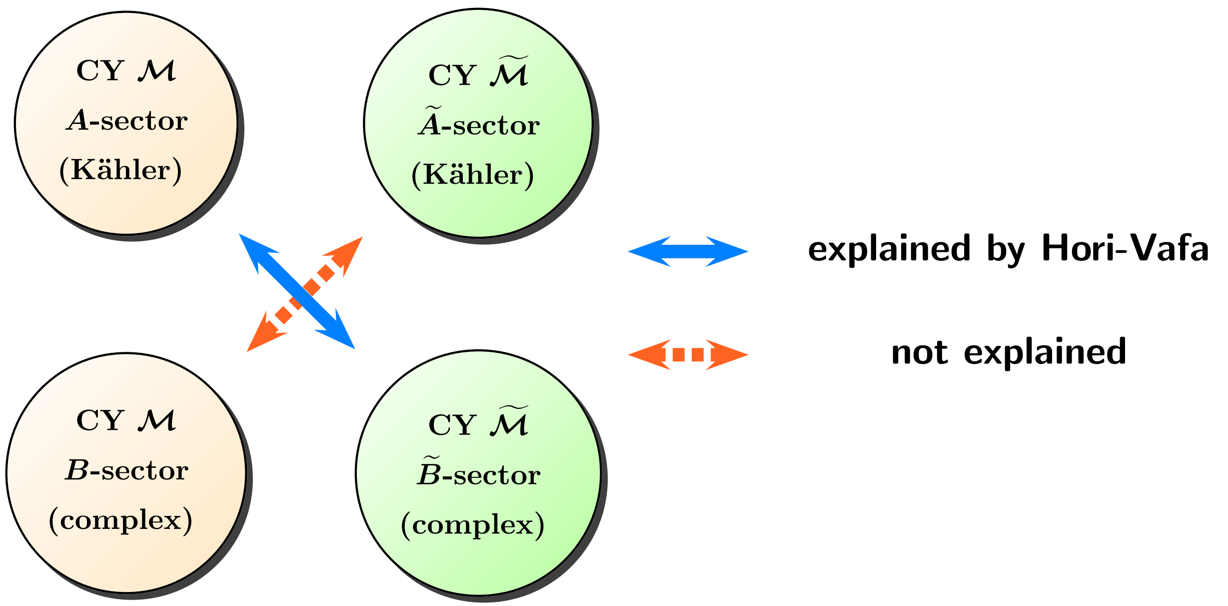
- **We argue that the LG theories in the orignal GLSM can be interpreted as $\mathcal{N}=2$ Liouville theories coupled to LG minimal models**


CY sigma model
on noncompact $\mathcal{M}_{\rm CY}$
CY/LG
$\mathcal{N}=2$ Liouville $\times$ LG
$\mathbb{Z}_\ell$ orbifold theory
FI $\gg 0$
GLSM
FI $\ll 0$
Mirror
T-dual
T-dual
$\ell\Sigma \to \frac{\partial}{\partial Y_{P_1}}$
Hori-Vafa
$\ell\Sigma \to \ell\frac{\partial}{\partial t}$
CY sigma model
on noncompact $\widetilde{\mathcal{M}}_\ell$
CY/LG
$SL(2,\mathbb{R})_k/U(1) \times \widetilde{\rm LG}$
$(\mathbb{Z}_\ell)^N$ orbifold theory

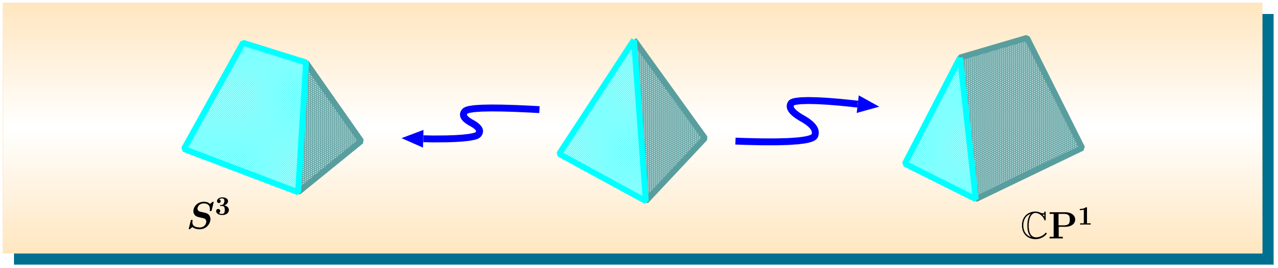
Discussions

- In the case of compact CY manifolds, we have already understood that the local rings in the LG theory are identified with the chiral rings of the SCFT and these chiral rings are related to the harmonic forms on such manifolds. However we have no proof that this relation is also satisfied in the case of noncompact CY manifolds. Thus we must investigate the spectra of the above effective theories as a future problem.

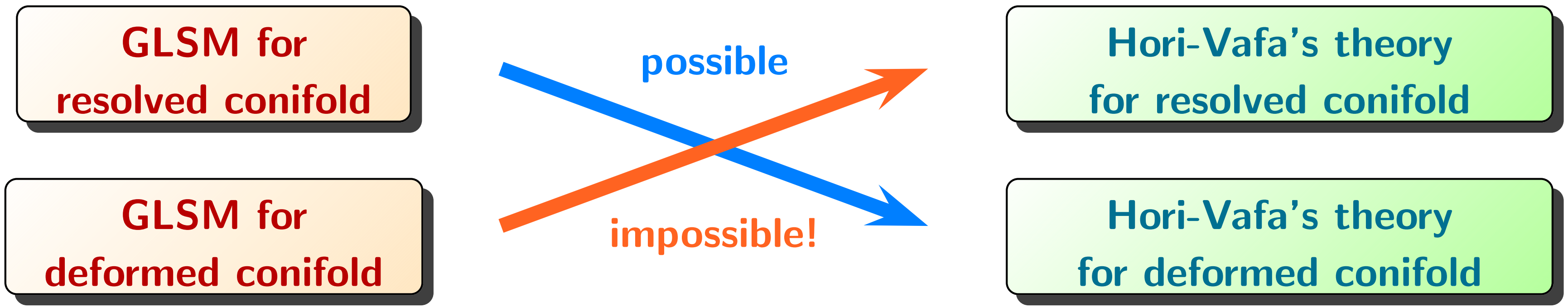
- Hori-Vafa's T-dual theory is only valid when we consider the GLSM without a superpotential or with a superpotential given simply by a homogeneous polynomial such as $W_{\text{GLSM}} = P \cdot G_\ell(S)$. Even though the polynomial $G_\ell(S)$ has an additional symmetry, the period integral $\widehat{\Pi}$ cannot recognize the existence of this additional symmetry. Thus the T-dual theory does not map all structures of the CY $\mathcal{M}$ to the mirror geometry completely.



Example: resolved/deformed conifold



- deformed conifold: deformation of complex moduli
- resolved conifold: deformation of Kähler moduli



Derivation of the T-dual Lagrangian

Here we briefly review the T-duality of a generic GLSM without any superpotentials. We start from

$$\mathscr{L}' = \int d^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_a \Big(e^{2Q_a V + B_a} - \frac{1}{2}(Y_a + \overline{Y}_a) B_a \Big) \Big\} + \Big(\frac{1}{\sqrt{2}} \int d^2\widetilde{\theta}\, (-\Sigma\, t) + (c.c.) \Big) , \qquad (1)$$

where Y_a and B_a are twisted chiral superfields and a real superfields B_a.

Integrating out twisted chiral superfields Y_a, we obtain $\overline{D}_+ D_- B_a = D_+ \overline{D}_- B_a = 0$, whose solutions are written in terms of chiral superfields Ψ_a and $\overline{\Psi}_a$ such as $B_a = \Psi_a + \overline{\Psi}_a$. When we substitute them into (1), $\mathscr{L}_{\mathrm{GLSM}}$ appears:

$$\mathscr{L}'\Big|_{B_a = \Psi_a + \overline{\Psi}_a} = \int d^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_a \overline{\Phi}_a\, e^{2Q_a V} \Phi_a \Big\} + \Big(\frac{1}{\sqrt{2}} \int d^2\widetilde{\theta}\, (-\Sigma\, t) + (c.c.) \Big) \equiv \mathscr{L}_{\mathrm{GLSM}} , \qquad (2)$$

where we re-wrote $\Phi_a = e^{\Psi_a}$.

On the other hand, when we first integrate out B_a in $\mathscr{L}'$, we obtain $B_a = -2Q_a V + \log\Big(\frac{Y_a + \overline{Y}_a}{2}\Big)$. Let us insert these solutions into (1). By using a deformation $\int d^4\theta\, Q_a V Y_a = -\frac{Q_a}{2} \int d^2\widetilde{\theta}\, \overline{D}_+ D_- V Y_a = -\frac{Q_a}{\sqrt{2}} \int d^2\widetilde{\theta}\, \Sigma Y_a$, we find that a Lagrangian of twisted chiral superfields appears:

$$\begin{aligned} \mathscr{L}_{\mathrm{T}} &= \int d^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma - \sum_a \Big(\frac{1}{2}(Y_a + \overline{Y}_a) \log(Y_a + \overline{Y}_a) \Big) \Big\} + \Big(\frac{1}{\sqrt{2}} \int d^2\widetilde{\theta}\, \widetilde{W} + (c.c.) \Big) , \\ \widetilde{W} &= \Sigma \Big(\sum_a Q_a Y_a - t \Big) + \mu \sum_a e^{-Y_a} . \end{aligned}$$

Notice that the twisted superpotential $\widetilde{W}$ is corrected by instanton effects where the instantons are the vortices of the gauge theory. In attempt to analyze a model satisfying $\sum_a Q_a = 0$, the scale parameter μ is omitted by field re-definitions.

linear dilaton CFT and Liouville theory

$$\mathbb{R}^{9,1} = \underbrace{\mathbb{R}^{d-1,1}}_{\text{free SCFT}} \times \underbrace{X^{2n}}_{\text{singular CY}} \sim \mathbb{R}^{d-1,1} \times \underbrace{\mathbb{R}_\phi \times S^1}_{\text{linear dilaton SCFT}} \times \underbrace{\mathcal{M}/U(1)}_{\mathcal{N}=2\text{ Landau-Ginzburg}}$$

linear dilaton: $\Phi = -\frac{Q}{2}\phi$

Landau-Ginzburg: $W_{\text{LG}} = F(Z_a), \quad F(\lambda^{r_a} Z_a) = \lambda F(Z_a)$

$$c_{\text{total}} = c_d + c_{\text{dilaton}} + c_{\text{LG}} \quad \longrightarrow \quad 15 = \frac{3}{2}d + \left(\frac{3}{2} + 3Q^2\right) + 3\sum_{a=1}^{n+1}(1 - 2r_a)$$

$$\mathcal{N}=2 \text{ “LG” on } \mathbb{R}_\phi \times S^1 \times \mathcal{M}/U(1)\text{: } W = -\mu Z_0^{-k} + F(Z_a)$$

$$k = \frac{1}{r_\Omega} = \frac{2}{Q^2}, \qquad r_\Omega \equiv \textstyle\sum_a r_a - 1$$

linear dilaton SCFT on $\mathbb{R}_\phi \times S^1$ $\equiv$ “LG” with 「$W = -\mu Z_0^{-k}$」

$\equiv$ Kazama-Suzuki model on $SL(2,\mathbb{R})_k/U(1)$

$\overset{\text{T-dual}}{\equiv}$ Liouville theory of charge Q

Strictly, we consider the Euclidean black hole: $SL(2,\mathbb{R})_k/U(1) \longrightarrow [SL(2,\mathbb{C})_k/SU(2)]/U(1)$

Derivation of the twisted geometry

Let us study how to obtain the geometry with $\mathbb{Z}_\ell$-type orbifold symmetry. Replacing $\ell\Sigma$ in $\widehat{\Pi}$ to

$$\ell\Sigma \;\to\; \frac{\partial}{\partial Y_{P_1}} \,,$$

we can perform the integration of Σ and obtain

$$\widehat{\Pi} \;=\; \int \prod_{i=1}^{N} \mathrm{d}Y_i \, (\mathrm{e}^{-Y_{P_1}} \mathrm{d}Y_{P_1}) \, \mathrm{d}Y_{P_2} \, \delta\Big(\sum_i Y_i - \ell Y_{P_1} - (N-\ell) Y_{P_2} - t \Big) \exp\Big(- \sum_i \mathrm{e}^{-Y_i} - \mathrm{e}^{-Y_{P_1}} - \mathrm{e}^{-Y_{P_2}} \Big) \,. \tag{3}$$

We perform the re-definitions of the variables Y_i, Y_{P_1} and Y_{P_2}:

$$\mathrm{e}^{-Y_{P_1}} \;=\; \widetilde{P}_1 \,, \quad \mathrm{e}^{-Y_a} \;=\; \widetilde{P}_1 U_a \quad \text{for } a = 1, \dots, \ell \,, \qquad \mathrm{e}^{-Y_{P_2}} \;=\; \widetilde{P}_2 \,, \quad \mathrm{e}^{-Y_b} \;=\; \widetilde{P}_2 U_b \quad \text{for } b = \ell+1, \dots, N \,.$$

Substituting these re-defined variables into (3), we continue the calculation:

$$\begin{aligned}
\widehat{\Pi} \;&=\; \int \prod_{i=1}^{N} \Big(\frac{\mathrm{d}U_i}{U_i} \Big) \mathrm{d}\widetilde{P}_1 \Big(\frac{\mathrm{d}\widetilde{P}_2}{\widetilde{P}_2} \Big) \delta\Big(\log \Big(\prod_i U_i \Big) + t \Big) \exp\Big\{ - \widetilde{P}_1 \Big(\sum_{a=1}^{\ell} U_a + 1 \Big) - \widetilde{P}_2 \Big(\sum_{b=\ell+1}^{N} U_b + 1 \Big) \Big\} \\
&=\; \int \prod_i \Big(\frac{\mathrm{d}U_i}{U_i} \Big) \mathrm{d}\widetilde{P}_2 \, \mathrm{d}u \, \mathrm{d}v \, \delta\Big(\log \Big(\prod_i U_i \Big) + t \Big) \, \delta\Big(\sum_a U_a + 1 \Big) \exp\Big\{ - \widetilde{P}_2 \Big(\sum_b U_b + 1 - uv \Big) \Big\} \\
&=\; \int \prod_i \Big(\frac{\mathrm{d}U_i}{U_i} \Big) \mathrm{d}u \, \mathrm{d}v \, \delta\Big(\log \Big(\prod_i U_i \Big) + t \Big) \, \delta\Big(\sum_a U_a + 1 \Big) \, \delta\Big(\sum_b U_b + 1 - uv \Big) \,,
\end{aligned} \tag{4}$$

where we introduced new variables u and v taking values in $\mathbb{C}$ and used a following equation

$$\frac{1}{\widetilde{P}_2} \;=\; \int \mathrm{d}u \, \mathrm{d}v \, \exp\big(\widetilde{P}_2 \, uv \big) \,.$$

It is obvious that (4) still includes a non-canonical integral measure. Thus we perform further re-definitions such as

$$U_a \;=\; \mathrm{e}^{-t/\ell}\frac{Z_a^\ell}{Z_1\cdots Z_N} \,, \qquad U_b \;=\; Z_b^\ell \,.$$

Note that the period integral (4) is invariant under the following transformations acting on the new variables Z_i:

$$Z_a \;\mapsto\; \lambda\,\omega_a\,Z_a \,, \quad Z_b \;\mapsto\; \omega_b\,Z_b \,, \qquad \omega_a^\ell \;=\; \omega_b^\ell \;=\; \omega_1\cdots\omega_N \;=\; 1 \,,$$

where λ is an arbitrary number taking in $\mathbb{C}^*$. The ω_i come from the shift symmetry of the original variables $Y_i \equiv Y_i + 2\pi i$. Combining these transformations we find that $\widehat{\Pi}$ has $\mathbb{C}^* \times (\mathbb{Z}_\ell)^{N-2}$ symmetries. Substituting Z_i into (4), we obtain

$$\widehat{\Pi} \;=\; \int \frac{1}{\mathrm{vol.}(\mathbb{C}^*)} \prod_{i=1}^N \mathrm{d}Z_i\,\mathrm{d}u\,\mathrm{d}v\,\delta\Big(\sum_{a=1}^\ell Z_a^\ell + \mathrm{e}^{t/\ell}Z_1\cdots Z_N\Big)\,\delta\Big(\sum_{b=\ell+1}^N Z_b^\ell + 1 - uv\Big) \,,$$

which indicates that the resulting mirror geometry is described by

$$\widetilde{\mathcal{M}}_\ell \;=\; \Big\{(Z_i;u,v)\in\mathbb{C}^{N+2}\,\Big|\,\big\{\mathcal{F}(Z_i) \;=\; 0\big\}/\mathbb{C}^* \,,\;\; \mathcal{G}(Z_b;u,v) \;=\; 0\Big\}\Big/(\mathbb{Z}_\ell)^{N-2} \,,$$

$$\mathcal{F}(Z_i) \;=\; \sum_{a=1}^\ell Z_a^\ell + \psi Z_1\cdots Z_\ell \,, \qquad \mathcal{G}(Z_b;u,v) \;=\; \sum_{b=\ell+1}^N Z_b^\ell + 1 - uv \,, \qquad \psi \;=\; \mathrm{e}^{t/\ell}Z_{\ell+1}\cdots Z_N \,.$$

This is an $(N-1)$-dimensional complex manifold.

The equation $\mathcal{F}(Z_i) = 0$ denotes that the complex variables Z_a consist of the degree ℓ hypersurface in the projective space: $\mathbb{CP}^{\ell-1}[\ell]$. This subspace itself is a compact CY manifold, which is parametrized by a parameter ψ which is subject to the equation $\mathcal{G}(Z_b;u,v) = 0$. Moreover we can also interpret that the total space is a noncompact CY manifold whose compact directions are described by Z_i, while the variables u and v run in the noncompact directions under the equations.