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Towards Mirror Symmetry on (Generalized) Complex Geometries

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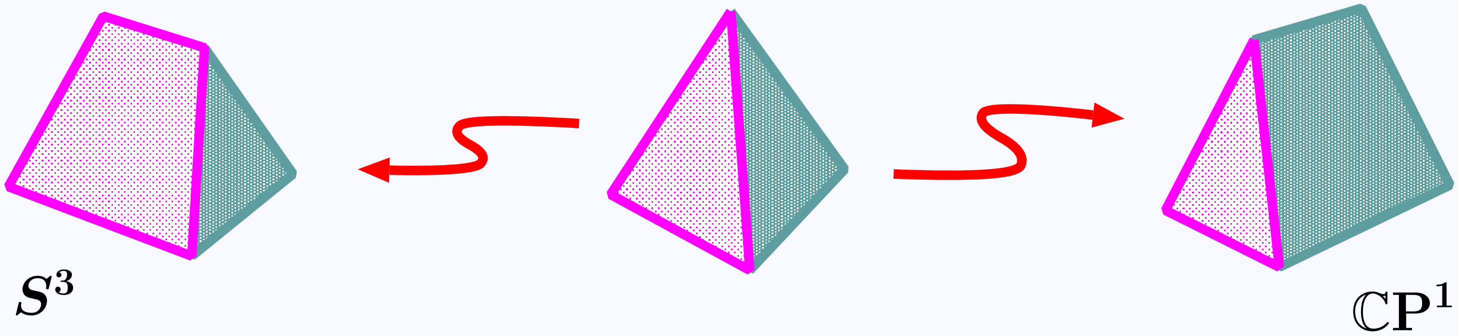
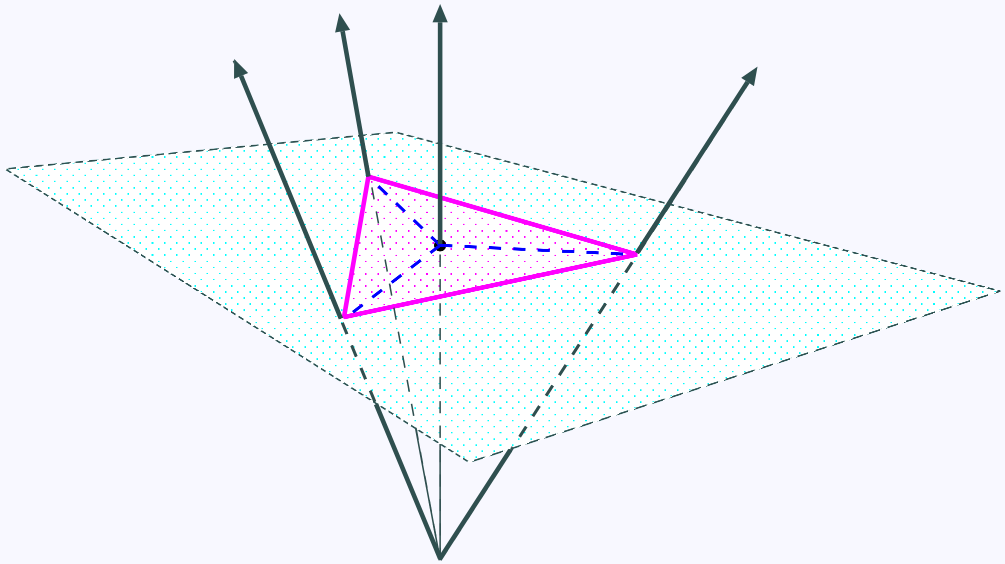
Introduction and Motivation

deformed conifold ← singular conifold → resolved conifold



toric CY cone such as $\mathcal{O}_{\mathbb{CP}^2}(-3)$



Study of string worldsheet theories

describing (non-)perturbative physics:

dualities, mirror symmetry, etc.

nonlinear sigma models	$\Longleftarrow$	differential geometry
CFT, gauged linear sigma models	$\Longleftarrow$	algebraic geometry

Both of them are useful,

Contents

- **NLSMs on noncompact Calabi-Yau manifolds**

  line bundles on Einstein-Kähler coset spaces

- **GLSMs on noncompact Calabi-Yau manifolds**

  line bundles on hypersurfaces in $\mathbb{CP}^{N-1}$

- **Generalization**

  towards $\mathcal{N} = (2,2)'$-theories on generalized complex geometries

SUSY NLSMs on Noncompact CYs

differential geometry

K.Higashijima, M.Nitta and TK

[hep-th/0110216, hep-th/0202064]

SUSY sigma models on noncompact CYs

We constructed $\mathcal{N} = (2,2)$ SUSY sigma models on

$$\mathcal{M}_{\text{CY}} = \mathbb{C} \ltimes \otimes_a (G_a/H_a) \;, \qquad \text{each } G_a/H_a \text{ is an compact Einstein-Kähler}$$

We can obtain the metrics on them by nonlinear realization.

The Kähler potential on $\mathcal{M}_{\text{CY}}$ follows $\dfrac{\mathrm{d}}{\mathrm{d}X} K_{\text{CY}} = (\mathrm{e}^X + b)^{1/D}$

where D is the number of complex dimensions of $\mathcal{M}_{\text{CY}}$ and

$$X = \log |\sigma|^2 + \sum_a h_a \, K_{G_a/H_a}(\varphi_a^i)$$

$$z^\mu = \{\sigma, \varphi_a^i\} \; : \; \text{chiral superfields, coordinates on } (\mathbb{C}, \otimes_a (G_a/H_a))$$

$$h_a \; : \; \text{a real number, } \mathcal{R}_{i\bar{j}} = h_a \, g^a_{i\bar{j}}$$

Examples of G_a/H_a

- **Hermitian symmetric spaces**

$$\frac{SU(N+1)}{S[U(N)\times U(1)]}\,,\qquad \frac{SO(N+2)}{SO(N)\times U(1)}\,,\qquad \frac{E_6}{SO(10)\times U(1)}\,,\qquad \frac{E_7}{E_6\times U(1)}$$

$$\frac{U(M+N)}{U(M)\times U(N)}\,,\qquad \frac{SO(2N)}{U(N)}\,,\qquad \frac{Sp(N)}{U(N)}$$

- **a non-symmetric space**

$$\frac{SU(\ell+m+n)}{S[U(\ell)\times U(m)\times U(n)]}\left(\ni\frac{SU(3)}{U(1)^2}\right)$$

We obtained the explicit Ricci-flat Kähler metrics (CY metrics) on $\mathcal{M}_{\mathsf{CY}}$, but...

global aspects of noncompact geometries?

CFT descriptions (Virasoro- and current-algebras)?

and

mirror geometries?

GLSMs
for Noncompact CYs

algebraic geometry

TK [hep-th/0411258]

CY/LG Correspondence and Mirror Symmetry

CY sigma models are equivalent to Landau-Ginzuburg theories.

- **LG superpotential = polynomial in algebraic geometry**

  generates chiral rings $\longrightarrow$ cohomology in CY

CY/LG Correspondence and Mirror Symmetry

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  generates chiral rings $\longrightarrow$ cohomology in CY

[Compact directions]

Two-dimensional GAUGE theory for (subspaces of) toric varieties

Witten [hep-th/9301042]

Hori and Vafa [hep-th/0002222]

[Noncompact direction]

Linear dilaton theory, Landau-Ginzburg model and Liouville theory

Giveon, Kutasov and Pelc [hep-th/9907178]

Hori and Kapustin [hep-th/0104202]

Gauged linear sigma model

$\mathcal{N} = (2,2)$ **SUSY gauge theory with matters** $(\mathrm{FI}: \; t \equiv r - i\theta)$

$$\mathscr{L} = \int \mathrm{d}^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_a \overline{\Phi}_a \, \mathrm{e}^{2Q_a V} \Phi_a \Big\} + \Big(\frac{1}{\sqrt{2}} \int \mathrm{d}^2\widetilde{\theta} \, (-\Sigma \, t) + c.c. \Big) + \Big(\int \mathrm{d}^2\theta \, W_{\text{gauge}}(\Phi_a) + c.c. \Big)$$

- Φ_a : **charged chiral superfield,** $\overline{D}_{\pm}\Phi_a = 0$
  Σ : **twisted chiral superfield,** $\overline{D}_{+}\Sigma = D_{-}\Sigma = 0 \; , \; \Sigma = \frac{1}{\sqrt{2}}\overline{D}_{+}D_{-}V$

- **There exist at least two phases:**

$\mathrm{FI} \gg 0$: **SUSY NLSM**

$\mathrm{FI} \ll 0$: **LG, orbifold, SCFT**

Gauged Linear Sigma Model for $\mathcal{O}(-N+\ell)$ bundle on $\mathbb{CP}^{N-1}[\ell]$

chiral superfield Φ_a	S_1	$\cdots$	S_N	P_1	P_2
$U(1)$ charge Q_a	1	$\cdots$	1	$-\ell$	$-N+\ell$

$$W_{\text{gauge}} = P_1 \cdot G_\ell(S_i)$$

$G_\ell(S_i)$: homogeneous polynomial of degree $\ell \geq 3$

There are two special cases:

- $\ell = 0$: $\mathcal{M}_{\text{CY}}$ becomes $\mathcal{O}(-N)$ bundle on $\mathbb{CP}^{N-1}$ (no superpotential)
- $\ell = N$: $\mathcal{M}_{\text{CY}}$ becomes $\mathbb{CP}^{N-1}[N]$ (compact)

The potential energy density is given by

$$\mathcal{U}(\phi,\sigma) \;=\; \frac{e^2}{2}\mathcal{D}^2 + \sum_a |F_a|^2 + 2|\sigma|^2 \sum_a Q_a^2 |\phi_a|^2$$

$$\mathcal{D} \;=\; \frac{1}{e^2} D \;=\; r - \sum_a Q_a |\phi_a|^2 \,, \qquad \overline{F}_a \;=\; -\frac{\partial}{\partial \phi_a} W_{\text{gauge}}(\phi)$$

The supersymmetric vacuum manifold $\mathcal{M}$ is defined by

$$\mathcal{M} \;=\; \Big\{ (\phi_a, \sigma) \in \mathbb{C}^n \Big| \mathcal{U}(\phi,\sigma) = 0 \Big\} \Big/ U(1)$$

In the IR limit $e \to \infty$, there appears the NLSM on $\mathcal{M}$ whose coupling is

$$r \;=\; \frac{1}{g^2}$$

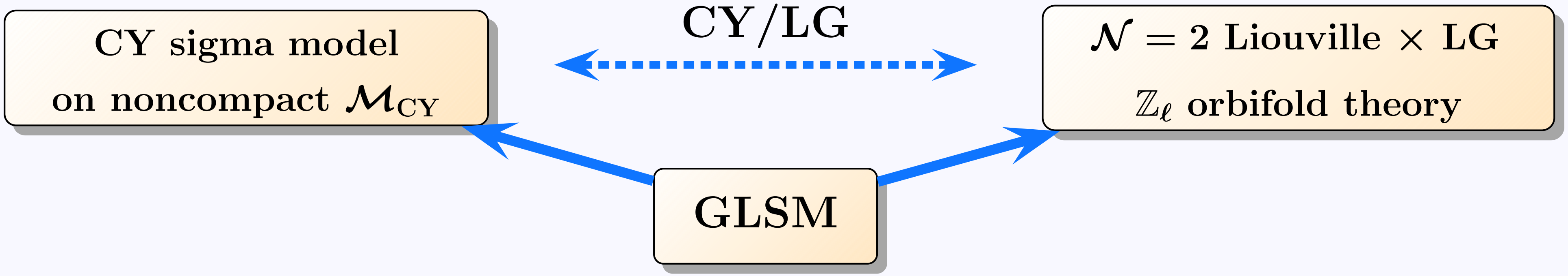
Renormalization of the FI parameter is $r_0 = r_R + s \cdot \log\left(\frac{\Lambda_{\text{UV}}}{\mu}\right)$, $s = \sum_a Q_a$

Thus we find that

$s > 0 \quad \longrightarrow \quad$ the theory is asymptotic free

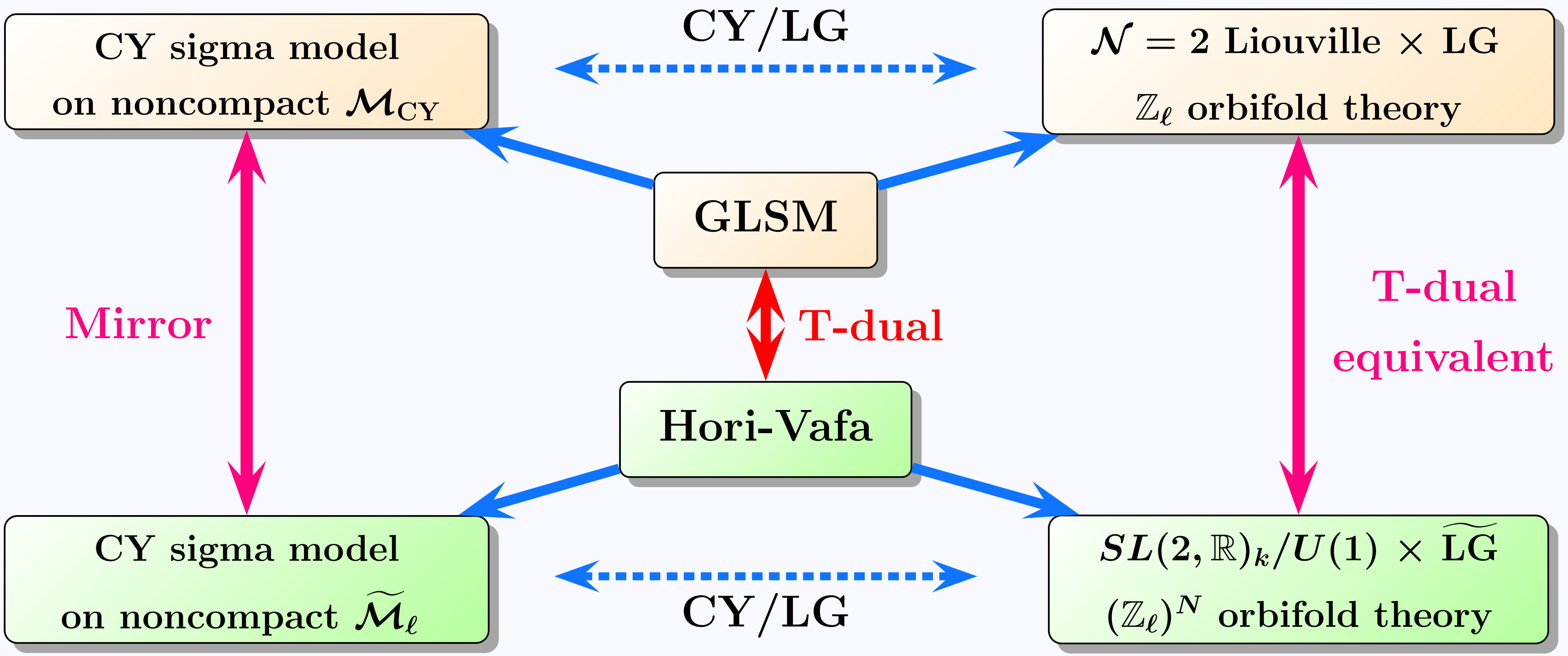
$s = 0 \quad \longrightarrow \quad$ the theory is conformal

$s < 0 \quad \longrightarrow \quad$ the theory is infrared free



$$\mathcal{M}_{\rm CY}: \quad \mathcal{O}(-N+\ell) \text{ bundle on } \mathbb{C}\mathrm{P}^{N-1}[\ell]$$

$$\mathrm{LG}: \quad W_{\rm LG} = \langle P_1 \rangle G_\ell(S_i) \equiv W_{\rm gauge}\Big|_{\langle P_1 \rangle}$$



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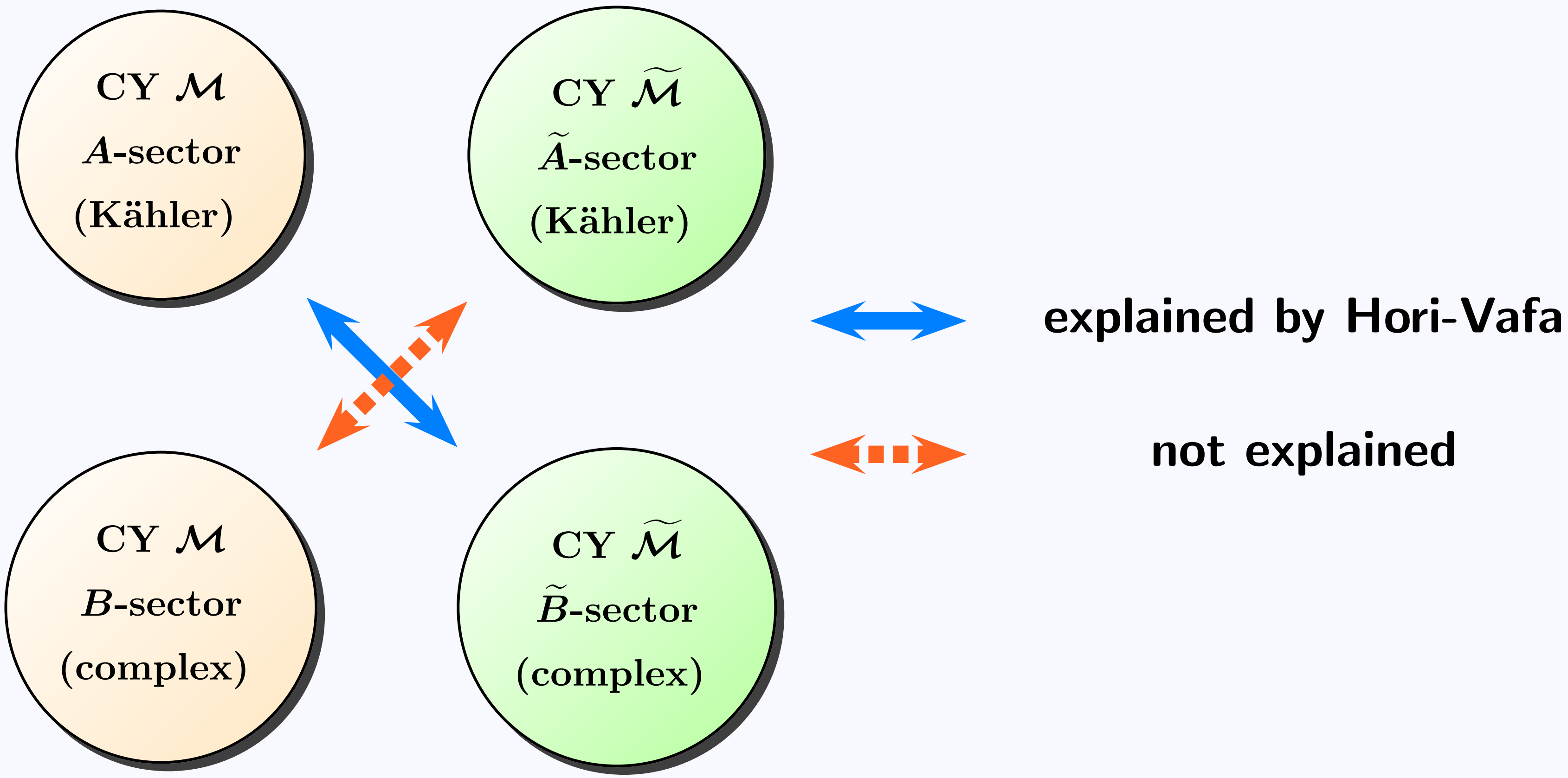
$$SL(2,\mathbb{R})_k/U(1)\times\widetilde{\rm LG}: \quad \Big\{\widetilde{W}_{\rm LG} = X_1^\ell + \cdots + X_N^\ell + X_{P_2}^{-\frac{\ell}{N-\ell}} + e^{t/\ell} X_1\cdots X_N X_{P_2}\Big\}\Big/(\mathbb{Z}_\ell)^N$$



- **Hori-Vafa's T-dual theory is only valid when we consider the GLSM without a superpotential or with a superpotential given simply by a homogeneous polynomial such as $W_{\text{gauge}} = P \cdot G_\ell(S)$.**

[Counter examples]

Sigma models on line bundles on HSS are also given as gauge theories, but...

$$\frac{SO(N+2)}{SO(N)\times U(1)} \ : \quad \mathbb{C}\mathrm{P}^{N+1} \text{ with } W_{\text{gauge}} = P_2 \cdot S_i S_j J_{ij} \ , \quad J_{ij} \ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1_N & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\frac{E_7}{E_6\times U(1)} \ : \quad \mathbb{C}\mathrm{P}^{55} \text{ with } W_{\text{gauge}} = d_{ijkl}\, P_i S_j S_k S_l$$

$$\text{where } S_i \in 56 = (27, -\tfrac{1}{3}) \oplus (\overline{27}, \tfrac{1}{3}) \oplus (1,-1) \oplus (1,1)$$

The superpotentials $W_{\text{LG}} = W_{\text{gauge}}\Big|_{\langle P\rangle}$ are no longer homogeneous.

$\longrightarrow$ NLSMs on the above line bundles are beyond the scope of GLSM,

$i.e.$, GLSM cannot trace the complex structure moduli very well.

Instead...

- **NLSM: We can trace complex structure moduli**
- **GLSM: We can trace Kähler moduli**

Is there a way to trace both of them in terms of one "Language"?

Generalization

Hitchin extended the almost complex structure to

generalized complex structure (GCS) and generalized complex geometry (GCG)

Hitchin [math.DG/0209099]

Gualtieri [math.DG/0401221]

- almost complex structure:

  a mapping tangent bundle $\mathcal{TM}$ $\longrightarrow$ tangent bundle $\mathcal{TM}$

- generalized complex structure:

  a mapping $\mathcal{TM} \oplus \mathcal{T}^*\mathcal{M} \longrightarrow \mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$

Mirror Symmetry is an exchanging rule bewteen 2 GCS: $\mathcal{J}_1 \leftrightarrow \mathcal{J}_2$

(ex.)
$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ * & -J^{\mathrm{T}} \end{pmatrix} , \qquad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & * \end{pmatrix}$$

complex strucuture   symplectic structure

Other motivations

Flux Compactifications

via NS-NS flux and R-R fluxes

Strominger [Nucl. Phys. B274 (1986) 253]

Giddings, Kachru and Polchinski [hep-th/0105097]

Kachru, Kallosh, Linde and Trivedi [hep-th/0301240]

Gauntlett, Martelli and Waldram [hep-th/0302158]

Modification of Mirror Symmetry

torsion as a back reaction of NS-NS flux

Gurrieri, Louis, Micu and Waldram [hep-th/0211102]

Cardoso, Curio, Dall'Agata, Lüst, Manousselis and Zoupanos [hep-th/0211118]

Fidanza, Minasian and Tomasiello [hep-th/0311122]

Sigma Models on GCG

Worldsheet Theories

The way how to generalize a string sigma model

to a sigma model on generalized complex geometries

- **introduce not only $\mathrm{d}X^A \in \mathcal{T}\mathcal{M}$ but also $\eta_A \in \mathcal{T}^*\mathcal{M}$ (1st order action)**
- **supersymmetrize**
- **extend the SUSY transformation rule**

Virtue

We can trace complex structure J and symplectic structure ω,

both of which are included in extended SUSY transformations as GCS

Extension of nonlinear sigma models

sigma model written by $\mathrm{d}X^\mu \in \mathcal{TM}$, $g_{\mu\nu}$ and $B_{\mu\nu}$ (2nd order action)

$$\boldsymbol{S} \;=\; \frac{1}{2}\int \Big\{ \boldsymbol{g}_{\mu\nu}(\boldsymbol{X})\,\mathrm{d}\boldsymbol{X}^\mu \wedge *\mathrm{d}\boldsymbol{X}^\nu + \boldsymbol{B}_{\mu\nu}(\boldsymbol{X})\,\mathrm{d}\boldsymbol{X}^\mu \wedge \mathrm{d}\boldsymbol{X}^\nu \Big\}$$

Extension of nonlinear sigma models

sigma model written by $\mathrm{d}X^\mu \in \mathcal{TM}$, $g_{\mu\nu}$ and $B_{\mu\nu}$ (2nd order action)

$$S = \frac{1}{2}\int \left\{ g_{\mu\nu}(X)\,\mathrm{d}X^\mu \wedge *\mathrm{d}X^\nu + B_{\mu\nu}(X)\,\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu \right\}$$

introduce $\eta_\mu \in \mathcal{T}^*\mathcal{M}$ as Lagrange multipliers (1st order action)

$$S = \frac{1}{2}\int \left\{ \eta_\mu \wedge \mathrm{d}X^\mu + \frac{1}{2}\theta^{\mu\nu}\,\eta_\mu \wedge \eta_\nu + \frac{1}{2}G^{\mu\nu}\,\eta_\mu \wedge *\eta_\nu + \frac{1}{2}(B-b)_{\mu\nu}\,\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu \right\}$$

$$E_{\mu\nu} = g_{\mu\nu} + b_{\mu\nu}\,, \qquad E^{\mu\lambda}E_{\lambda\nu} = \delta^\mu{}_\nu$$

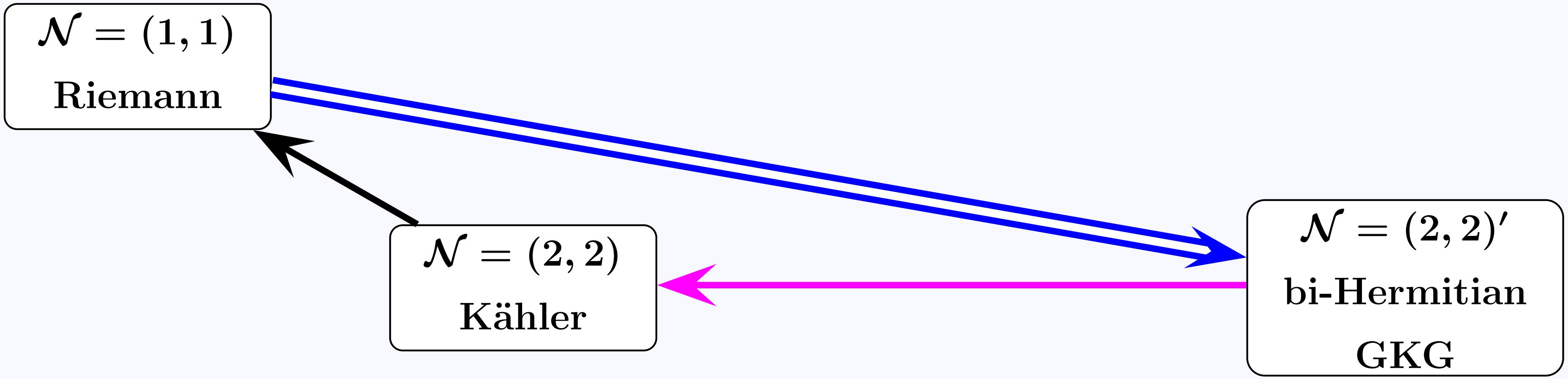
$$G^{\mu\nu} = \frac{1}{2}(E^{\mu\nu} + E^{\nu\mu})\,, \qquad \theta^{\mu\nu} = \frac{1}{2}(E^{\mu\nu} - E^{\nu\mu})$$

supersymmetrize this to a new sigma model on GCG

Lindström, Minasian, Tomasiello and Zabzine [hep-th/0405085]

Lindström, Roček, Unge and Zabzine [hep-th/0411186]

Various supersymmetric sigma models

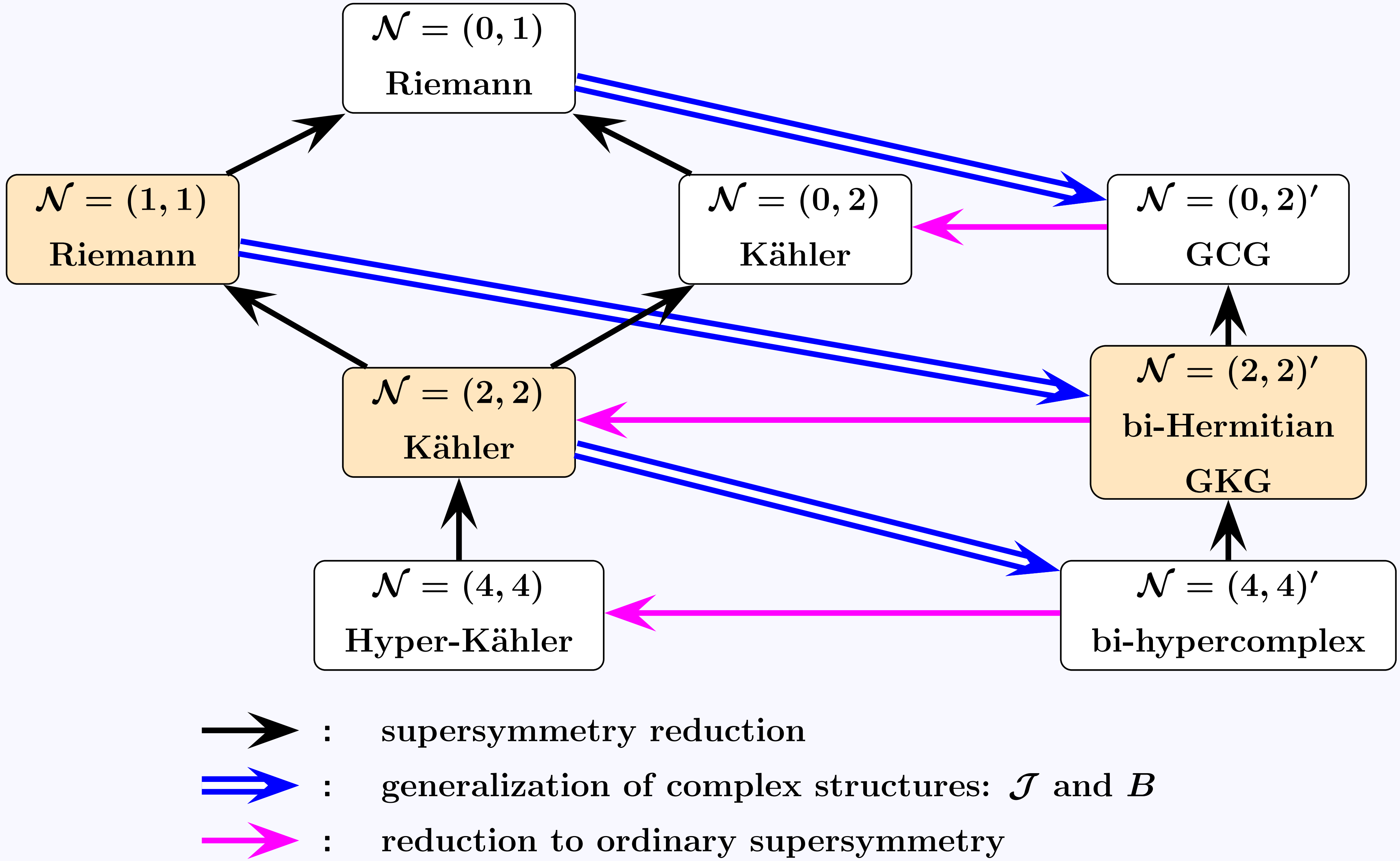


: supersymmetry reduction

: generalization of complex structures: $\mathcal{J}$ and B

: reduction to ordinary supersymmetry

Various supersymmetric sigma models



$$\mathcal{N} = (2,2)'$$

Sigma Models

$\mathcal{N} = (2,2)$ superfields

chiral superfield $\Phi^{(2,2)}$

$$\overline{D}_\pm \Phi^{(2,2)} = 0 \qquad \Phi^{(2,2)} = \overline{D}_+ \overline{D}_- \Theta$$

$$\Phi^{(2,2)} = \phi + i\sqrt{2}\theta^+ \psi_+ + i\sqrt{2}\theta^- \psi_- + 2i\theta^+\theta^- F$$
$$+\{(\partial_0 \pm \partial_1)\text{-derivative terms}\}$$

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chiral superfield $\Phi^{(2,2)}$ $\qquad \overline{D}_{\pm}\Phi^{(2,2)} = 0 \qquad \Phi^{(2,2)} = \overline{D}_{+}\overline{D}_{-}\Theta$

left semi-chiral superfield $\mathbb{X}^{(2,2)}$ $\qquad \overline{D}_{+}\mathbb{X}^{(2,2)} = 0 \qquad \mathbb{X}^{(2,2)} = \overline{D}_{+}\Theta$

$$\begin{aligned}\Phi^{(2,2)} &= \phi + i\sqrt{2}\theta^{+}\psi_{+} + i\sqrt{2}\theta^{-}\psi_{-} + 2i\theta^{+}\theta^{-}F \\ &\quad +\{(\partial_0 \pm \partial_1)\text{-derivative terms}\}\end{aligned}$$

$$\begin{aligned}\mathbb{X}^{(2,2)} &= \phi + i\sqrt{2}\theta^{+}\psi_{+} + i\sqrt{2}(\theta^{-}\psi_{-} + \overline{\theta}^{-}\chi_{-}) + 2i\theta^{+}(\theta^{-}F + \overline{\theta}^{-}G) \\ &\quad + \theta^{-}\overline{\theta}^{-}A_{=} + 2\theta^{+}\theta^{-}\overline{\theta}^{-}\zeta_{-} \\ &\quad +\{(\partial_0 + \partial_1)\text{-derivative terms}\}\end{aligned}$$

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$$\mathbb{X}^{(2,2)} = \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}(\theta^-\psi_- + \overline{\theta}^-\chi_-) + 2i\theta^+(\theta^- F + \overline{\theta}^- G)$$
$$+\theta^-\overline{\theta}^- A_= + 2\theta^+\theta^-\overline{\theta}^-\zeta_-$$
$$+\{(\partial_0 + \partial_1)\text{-derivative terms}\}$$

$\mathcal{N} = (1,1)$ scalar/spinor superfields from $\mathcal{N} = (2,2)$ semi-chiral superfields

$$\mathbb{X}^{(1,1)} = \phi + i\sqrt{2}\theta_1^+\widehat{\psi}_+ + i\sqrt{2}\theta_1^-(\widehat{\psi}_- + \widehat{\chi}_-) + 2i\theta_1^+\theta_1^-(\widehat{F} + \widehat{G})$$
$$\Psi_-^{(1,1)} = i(\widehat{\psi}_- - \widehat{\chi}_-) - i\sqrt{2}\theta_1^+(\widehat{F} - \widehat{G}) + \sqrt{2}\theta_1^- A_= + 2\sqrt{2}\theta_1^+\theta_1^-\widehat{\zeta}_-$$

Sigma Models on Generalized Complex Geometries

$\mathcal{N} = (2,2)$ supersymmetric sigma models of semi-chiral superfields

We use semi-chiral superfields $\mathbb{X}$, $\mathbb{Y}$ defined by

$$\overline{D}_+ \mathbb{X} = 0 \, , \qquad \overline{D}_- \mathbb{Y} = 0$$

$\mathcal{N} = (2,2)$ Lagrangian

$$\mathscr{L} = \int d^4\theta \, K(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}})$$

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reduce $\mathcal{N} = (2,2)$ to $\mathcal{N} = (1,1)$ (i.e., reduce complex $\theta^{\pm}$ to real $\theta_1^{\pm}$):

$$\theta_1^{\pm} \;\equiv\; -i\mathrm{e}^{-i\nu_{\pm}}\theta^{\pm} \;=\; i\mathrm{e}^{+i\nu_{\pm}}\overline{\theta}^{\pm}$$

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$\mathcal{N}=(2,2)$ supersymmetric sigma models of semi-chiral superfields

We use semi-chiral superfields $\mathbb{X}$, $\mathbb{Y}$ defined by

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reduce $\mathcal{N}=(2,2)$ Lagrangian to $\mathcal{N}=(1,1)$ Lagrangian:

$$\mathscr{L} \;=\; \int \mathrm{d}^4\theta \, K(\mathbb{X},\overline{\mathbb{X}},\mathbb{Y},\overline{\mathbb{Y}}) \;=\; -\frac{1}{8}\int \mathrm{d}\theta_1^+ \mathrm{d}\theta_1^- \, \widetilde{Q}{}^1_+ \widetilde{Q}{}^1_- K(\mathbb{X},\overline{\mathbb{X}},\mathbb{Y},\overline{\mathbb{Y}})$$

reduce $\mathcal{N}=(2,2)$ to $\mathcal{N}=(1,1)$ (i.e., reduce complex $\theta^{\pm}$ to real $\theta_1^{\pm}$):

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Semi-chiral superfields are decomposed into 2 independent $\mathcal{N}=(1,1)$ superfields:

$$\begin{aligned} \mathbb{X}^{(2,2)} \;&\longrightarrow\; \big\{\mathbb{X}^{(1,1)}\,,\; \Psi_-^{(1,1)}\big\} & \qquad \Psi_-^{(1,1)} \;&\equiv\; \widetilde{Q}{}^1_- \mathbb{X}^{(2,2)}\big| \\ \mathbb{Y}^{(2,2)} \;&\longrightarrow\; \big\{\mathbb{Y}^{(1,1)}\,,\; \Upsilon_+^{(1,1)}\big\} & \qquad \Upsilon_+^{(1,1)} \;&\equiv\; \widetilde{Q}{}^1_+ \mathbb{Y}^{(2,2)}\big| \end{aligned}$$

Buscher, Lindström and Roček [Phys. Lett. B202 (1988) 94]

"Topological" sigma models (useful to look at the generalized complex structures)

Consider an $\mathcal{N}=(1,1)$ Lagrangian including only $\mathbb{X}^A = \{\mathbb{X}^a, \overline{\mathbb{X}^{\overline{a}}}\}$:

$$\mathscr{L}_{\mathbb{X}} = \int \mathrm{d}^4\theta\, K(\mathbb{X},\overline{\mathbb{X}}) = -\frac{1}{4}\int \mathrm{d}\theta_1^+ \mathrm{d}\theta_1^- \left\{ S_{A-} D_+^1 \mathbb{X}^A \right\}$$

re-definition of superfields (from $\mathcal{TM}$ to $\mathcal{T}^*\mathcal{M}$):

$$S_{A-} = \Psi_-^B\, \omega_{BA}\,, \qquad 2\omega_{AB} \equiv J_A{}^C K_{CB} - K_{AC} J^C{}_B\,, \qquad K_{AB} \equiv \frac{\partial^2 K}{\partial\mathbb{X}^A \partial\mathbb{X}^B}$$

$$D_+^1 \mathbb{X}^A \in \mathcal{TM}\,, \qquad S_{A-} \in \mathcal{T}^*\mathcal{M}$$

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$$D_+^1 \mathbb{X}^A \in \mathcal{TM}\,, \qquad S_{A-} \in \mathcal{T}^*\mathcal{M}$$

introduce another $\mathcal{N}=(1,1)$ SUSY to extend the model to a new $\mathcal{N}=(2,2)$:

$$\widetilde{\delta}^{(+)} \mathbb{X}^A = \widetilde{\varepsilon}^+ J^A{}_B\, D_+^1 \mathbb{X}^B\,, \qquad \widetilde{\delta}^{(-)} \mathbb{X}^A = -\widetilde{\varepsilon}^- \omega^{AB} (S_{B-})^{\mathrm{T}}$$

$$\widetilde{\delta}^{(+)} S_{A-} = -\widetilde{\varepsilon}^+ D_+^1 S_{B-} J^B{}_A$$

$$\widetilde{\delta}^{(-)} S_{A-} = -i\widetilde{\varepsilon}^- \left\{ \omega_{AC} (\partial_0 - \partial_1) \mathbb{X}^C \right\}^{\mathrm{T}}$$

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introduce another $\mathcal{N}=(1,1)$ SUSY to extend the model to a new $\mathcal{N}=(2,2)'$:

$$\begin{aligned}
\widetilde{\delta}^{(+)}\mathbb{X}^A &= \widetilde{\varepsilon}^+ J^A{}_B D_+^1 \mathbb{X}^B \, , \qquad \widetilde{\delta}^{(-)}\mathbb{X}^A = -\widetilde{\varepsilon}^- \omega^{AB} (S_{B-})^{\mathrm{T}} \\
\widetilde{\delta}^{(+)} S_{A-} &= -\widetilde{\varepsilon}^+ D_+^1 S_{B-} J^B{}_A \\
&\quad + (\omega_{AB})^{\mathrm{T}} \Big\{ (\widetilde{\delta}^{(+)}\mathbb{X}^E)^{\mathrm{T}} \delta_E^G - \widetilde{\varepsilon}^+ (D_+^1 \mathbb{X}^E)^{\mathrm{T}} J^G{}_E \Big\} \partial_G (\omega^{BC})^{\mathrm{T}} S_{C-} \\
\widetilde{\delta}^{(-)} S_{A-} &= -i\widetilde{\varepsilon}^- \left\{ \omega_{AC} (\partial_0 - \partial_1) \mathbb{X}^C \right\}^{\mathrm{T}} \\
&\quad + S_{C-} \omega^{CB} \partial_E(\omega_{BA}) (\widetilde{\delta}^{(-)}\mathbb{X}^E)
\end{aligned}$$

Embedded generalized complex structures on $\mathcal{N} = (2,2)$

standard complex structure $\mathcal{J}_1$ and symplectic form $\mathcal{J}_2$

$$\left\{ \begin{aligned} \widetilde{\delta}^{(+)} \mathbb{X}^A &= \widetilde{\varepsilon}^+ J^A{}_B \, D^1_+ \mathbb{X}^B \\ \widetilde{\delta}^{(+)} (S_{A-})^{\mathrm{T}} &= -\widetilde{\varepsilon}^+ J_A{}^B \, (D^1_+ S_{B-})^{\mathrm{T}} \end{aligned} \right. \qquad \longleftrightarrow \qquad \mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^{\mathrm{T}} \end{pmatrix}$$

$$\left\{ \begin{aligned} \widetilde{\delta}^{(-)} \mathbb{X}^A &= -\widetilde{\varepsilon}^- \omega^{AB} (S_{B-})^{\mathrm{T}} \\ \widetilde{\delta}^{(-)} (S_{A-})^{\mathrm{T}} &= -i \widetilde{\varepsilon}^- \omega_{AB} (\partial_0 - \partial_1) \mathbb{X}^B \end{aligned} \right. \qquad \longleftrightarrow \qquad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

$$\left\{ \begin{aligned} \widetilde{\delta}^{(+)} \mathbb{Y}^{A'} &= -\widetilde{\varepsilon}^+ \omega^{A'B'} (S_{B'+})^{\mathrm{T}} \\ \widetilde{\delta}^{(+)} (S_{A'+})^{\mathrm{T}} &= -i \widetilde{\varepsilon}^+ \omega_{A'B'} (\partial_0 + \partial_1) \mathbb{Y}^{B'} \end{aligned} \right. \qquad \longleftrightarrow \qquad \mathcal{J}'_2 = \begin{pmatrix} 0 & -\omega'^{-1} \\ \omega' & 0 \end{pmatrix}$$

$$\left\{ \begin{aligned} \widetilde{\delta}^{(-)} \mathbb{Y}^{A'} &= \widetilde{\varepsilon}^- J^{A'}{}_{B'} \, D^1_- \mathbb{Y}^{B'} \\ \widetilde{\delta}^{(-)} (S_{A'+})^{\mathrm{T}} &= -\widetilde{\varepsilon}^- J_{A'}{}^{B'} \, (D^1_- S_{B'+})^{\mathrm{T}} \end{aligned} \right. \qquad \longleftrightarrow \qquad \mathcal{J}'_1 = \begin{pmatrix} J' & 0 \\ 0 & -J'^{\mathrm{T}} \end{pmatrix}$$

Lindström, Roček, Unge and Zabzine [hep-th/0411186]

Embedded generalized complex structures on $\mathcal{N} = (2,2)'$

standard complex structure $\mathcal{J}_1$ and symplectic form $\mathcal{J}_2$ are extended:

$$\begin{cases} \widetilde{\delta}^{(+)}\mathbb{X}^{A} = \widetilde{\varepsilon}^{+} J^{A}{}_{B}\, D^{1}_{+}\mathbb{X}^{B} \\ \widetilde{\delta}^{(+)}(S_{A-})^{\mathrm{T}} = -\widetilde{\varepsilon}^{+} J_{A}{}^{B}\, (D^{1}_{+}S_{B-})^{\mathrm{T}} + \cdots \end{cases} \longleftrightarrow \quad \mathcal{J}_1 = \begin{pmatrix} J & 0 \\ * & -J^{\mathrm{T}} \end{pmatrix}$$

$$\begin{cases} \widetilde{\delta}^{(-)}\mathbb{X}^{A} = -\widetilde{\varepsilon}^{-} \omega^{AB}(S_{B-})^{\mathrm{T}} \\ \widetilde{\delta}^{(-)}(S_{A-})^{\mathrm{T}} = -i\widetilde{\varepsilon}^{-} \omega_{AB}(\partial_0 - \partial_1)\mathbb{X}^{B} + \cdots \end{cases} \longleftrightarrow \quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & * \end{pmatrix}$$

$$\begin{cases} \widetilde{\delta}^{(+)}\mathbb{Y}^{A'} = -\widetilde{\varepsilon}^{+} \omega^{A'B'}(S_{B'+})^{\mathrm{T}} \\ \widetilde{\delta}^{(+)}(S_{A'+})^{\mathrm{T}} = -i\widetilde{\varepsilon}^{+} \omega_{A'B'}(\partial_0 + \partial_1)\mathbb{Y}^{B'} + \cdots \end{cases} \longleftrightarrow \quad \mathcal{J}'_2 = \begin{pmatrix} 0 & -\omega'^{-1} \\ \omega' & * \end{pmatrix}$$

$$\begin{cases} \widetilde{\delta}^{(-)}\mathbb{Y}^{A'} = \widetilde{\varepsilon}^{-} J^{A'}{}_{B'}\, D^{1}_{-}\mathbb{Y}^{B'} \\ \widetilde{\delta}^{(-)}(S_{A'+})^{\mathrm{T}} = -\widetilde{\varepsilon}^{-} J_{A'}{}^{B'}\, (D^{1}_{-}S_{B'+})^{\mathrm{T}} + \cdots \end{cases} \longleftrightarrow \quad \mathcal{J}'_1 = \begin{pmatrix} J' & 0 \\ * & -J'^{\mathrm{T}} \end{pmatrix}$$

Lindström, Roček, Unge and Zabzine [hep-th/0411186]

A conjecture on the mirror symmetry

As mentioned, the mirror symmetry should be interpreted as

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ * & -J^{\mathrm{T}} \end{pmatrix} \longleftrightarrow \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & * \end{pmatrix}$$

This can be regarded as exchange between $\mathbb{X}^A$ and $\mathbb{Y}^{A'}$(?)

If the mirror dual transformation means the mapping from $(\mathbb{X}^A, \mathbb{Y}^{A'})$ to $(\widehat{\mathbb{Y}}^{B'}, \widehat{\mathbb{X}}^B)$, we can insist that

> A theory $\mathscr{L}(\mathbb{X}^A, \mathbb{Y}^{A'})$ should be mapped
> to another theory $\mathscr{L}(\widehat{\mathbb{Y}}^{B'}, \widehat{\mathbb{X}}^B)$, and vice versa.

We wish to consider the duality transformation procedure.

[Note] The mirror dual in two-dimensional worldsheet:

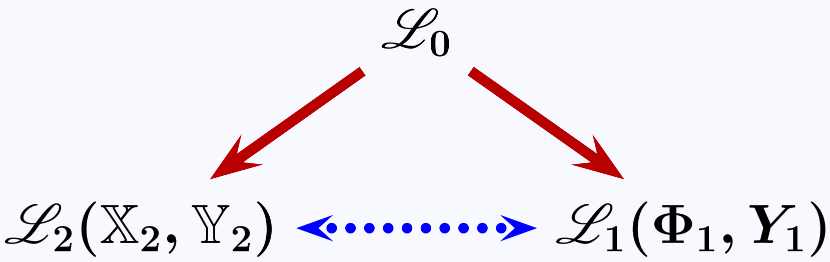
$$\begin{array}{ccc} (\Phi, \overline{\Phi}) & & (Y, \overline{Y}) \\ & \longleftrightarrow & \\ (c,c)\text{-ring} & & (a,c)\text{-ring} \end{array}$$

Duality Transformation

Exchanging between $(\mathbb{X}^{A}, \mathbb{Y}^{A'})$ and $(\widehat{\mathbb{Y}}^{B'}, \widehat{\mathbb{X}}^{B})$

Duality transformation

— IDEA —



Grisaru, Massar, Sevrin and Troost [hep-th/9801080]

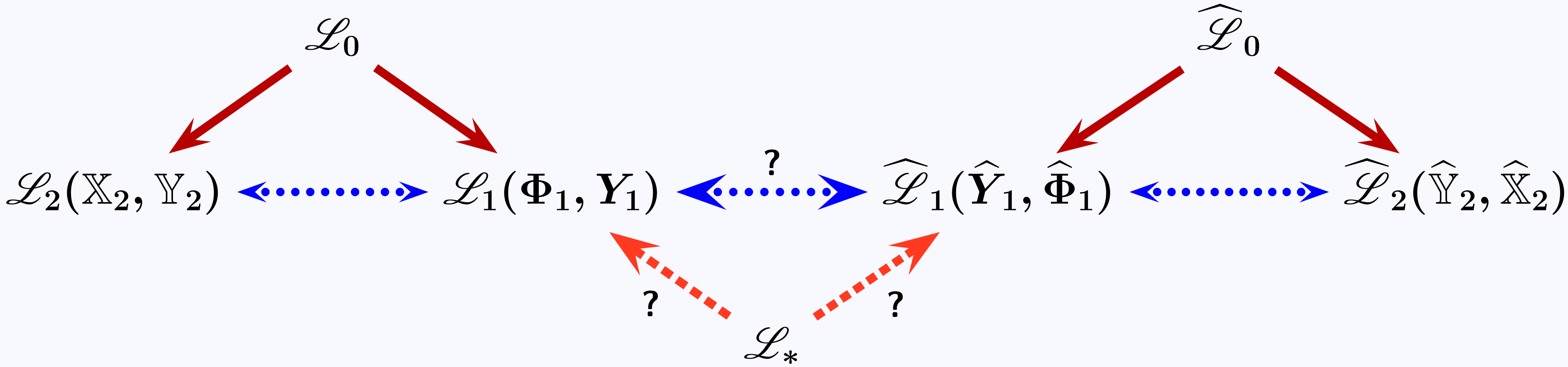
Duality transformation

— IDEA —

Grisaru, Massar, Sevrin and Troost [hep-th/9801080]

Duality transformation

— IDEA —



Grisaru, Massar, Sevrin and Troost [hep-th/9801080]

Roček and Verlinde [hep-th/9110053]

Let's find unknown Lagrangian $\mathscr{L}_*$ and new duality transformations!!

$$\mathscr{L}_2(\mathbb{X}_2, \mathbb{Y}_2) \longleftrightarrow \widehat{\mathscr{L}}_2(\widehat{\mathbb{Y}}_2, \widehat{\mathbb{X}}_2)$$

Discussions

PROBLEMS

- Doubling problem of degrees of freedom

  Which DOF corresponds to "coordinates" of geometry? (combination of $\mathbb{X}^{A}$ and $\mathbb{Y}^{A'}$?)

- How to relax the integrability condition? (to realize the half-flat manifold)

CHECK

- Relation between GCG and CY with H_3-flux (or G-structure manifolds)

- Consistency check of T-duality on $\mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$

  Dabholkar and Hull [hep-th/0512005]

APPLICATIONS

- Sigma models of heterotic strings

  $\mathcal{N} = (2,2)' \longrightarrow \mathcal{N} = (0,2)'$?

- Topological strings on generalized complex manifolds

  Kapustin and Li [hep-th/0407249]

Appendix

G-structure Manifolds

Appendix

G-structure manifolds

Now suppose $\mathcal{M}$ has a G-structure. In general the Levi-Civita connection does not preserve the G-invariant tensors (or spinor) ξ (i.e., $\nabla\xi \neq 0$). However, one can show that there always exist some other connection $\nabla^{(T)}$ which is compatible with the G-structure so that

$$\nabla^{(T)}\xi \; = \; 0 \; .$$

Thus for instance, on an almost Hermitian manifold one can always find $\nabla^{(T)}$ such that $\nabla^{(T)}J = 0$. On a manifold with $SU(3)$-structure, it means we can always find $\nabla^{(T)}$ such that both $\nabla^{(T)}J = 0$ and $\nabla^{(T)}\Omega = 0$ and the solutions are

$$\begin{aligned} 0 &= \nabla^{(T)}J & \rightarrow \quad (\mathrm{d}J)_{mnp} &= -6T^{r}{}_{[mn}J_{|r|p]} \; , \\ 0 &= \nabla^{(T)}\Omega & \rightarrow \quad (\mathrm{d}\Omega)_{mnpq} &= -12T^{r}{}_{[mn}\Omega_{|r|pq]} \; . \end{aligned}$$

Since the existence of an $SU(3)$-structure is also equivalent to the existence of an invariant spinor η, this is equivalent to the condition $\nabla^{(T)}\eta = 0$ and then

$$\nabla^{(T)}\eta \; = \; 0 \; , \qquad iJ_{mn} \; = \; \eta^{\dagger}\gamma_{mn}\gamma\eta \; , \qquad i\Omega_{mnp} \; = \; \eta^{\dagger}\gamma_{mnp}(1+\gamma)\eta \; .$$

Let κ be the contorsion tensor corresponding to $\nabla^{(T)}$. We see that κ is an element of $\Lambda^1 \otimes \Lambda^2$ where Λ^n is the space of n-forms. Alternatively, since $\Lambda^2 \cong \mathfrak{so}(d)$, it is more natural to think of $\kappa^p{}_{mn}$ as one-form with values in the Lie-algebra $\mathfrak{so}(d)$ that is $\Lambda^1 \otimes \mathfrak{so}(d)$. Given the existence of a G-structure, we can decompose $\mathfrak{so}(d)$ into a part in the Lie algebra $\mathfrak{g}$ of $G \subset SO(d)$ and an orthogonal piece $\mathfrak{g}^\perp = \mathfrak{so}(d)/\mathfrak{g}$. The contorsion κ splits accordingly into

$$\boldsymbol{\kappa} \;=\; \boldsymbol{\kappa}^0 + \boldsymbol{\kappa}^{\mathfrak{g}} \,,$$

where $\boldsymbol{\kappa}^0$ is the part in $\Lambda^1 \otimes \mathfrak{g}^\perp$. Since an invariant tensor (or spinor) $\boldsymbol{\xi}$ is fixed under G rotations, that action of $\mathfrak{g}$ on $\boldsymbol{\xi}$ vanishes and we have, by definition,

$$\boldsymbol{\nabla}^{(T)}\boldsymbol{\xi} \;=\; \left(\boldsymbol{\nabla} + \boldsymbol{\kappa}^0 + \boldsymbol{\kappa}^{\mathfrak{g}}\right)\boldsymbol{\xi} \;=\; \left(\boldsymbol{\nabla} + \boldsymbol{\kappa}^0\right)\boldsymbol{\xi} \;=\; 0 \,.$$

Thus, any two G-compatible connections must differ by a piece proportional to $\boldsymbol{\kappa}^{\mathfrak{g}}$ and they have a common term $\boldsymbol{\kappa}^0$ in $\Lambda^1 \otimes \mathfrak{g}^\perp$ called the "intrinsic contorsion". It is more conventional in the mathematics literature to define the corresponding torsion

$$(\boldsymbol{T}^0)^p{}_{mn} \;=\; (\boldsymbol{\kappa}^0)^p{}_{[mn]} \in \boldsymbol{\Lambda}^1 \otimes \mathfrak{g}^\perp \,,$$

known as the intrinsic torsion.

Let us consider the decomposition of T^0 in the case of $SU(3)$-structure. The relevant representations are

$$\Lambda^1 \sim \mathbf{3} \oplus \overline{\mathbf{3}} \,, \qquad \mathfrak{su}(3) \sim \mathbf{8} \,, \qquad \mathfrak{su}(3)^\perp \sim \mathbf{1} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}$$

Thus the intrinsic torsion $T^0 \in \Lambda^1 \otimes \mathfrak{su}(3)^\perp$ can be decomposed into the following $SU(3)$ representation

$$\begin{aligned} (T^0)^p{}_{mn} \in \Lambda^1 \otimes \mathfrak{su}(3)^\perp &= (\mathbf{3} \oplus \overline{\mathbf{3}}) \otimes (\mathbf{1} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}) \\ &= \underset{W_1}{(\mathbf{1} \oplus \mathbf{1})} \oplus \underset{W_2}{(\mathbf{8} \oplus \mathbf{8})} \oplus \underset{W_3}{(\mathbf{6} \oplus \overline{\mathbf{6}})} \oplus \underset{W_4}{(\mathbf{3} \oplus \overline{\mathbf{3}})} \oplus \underset{W_5}{(\mathbf{3} \oplus \overline{\mathbf{3}})'} \end{aligned}$$

where

W_1 : complex scalar in $(\mathbf{1} \oplus \mathbf{1})$

W_2 : complex primitive 2-form in $(\mathbf{8} \oplus \mathbf{8})$

W_3 : real primitive $(2,1) \oplus (1,2)$-form in $(\mathbf{6} \oplus \overline{\mathbf{6}})$

W_4 : real 1-form in $(\mathbf{3} \oplus \overline{\mathbf{3}})$

W_5 : complex $(1,0)$-form in $(\mathbf{3} \oplus \overline{\mathbf{3}})'$

Chiossi and Salamon [math.DG/0202282]

Calculation rules

$(2,1)$-form $\beta^{(2,1)}$ includes a primitive $(2,1)$-form $\beta_0^{(2,1)}$ and a (primitive) $(1,0)$-form $\beta_0^{(1,0)}$ such as

$$\beta^{(2,1)} = \beta_0^{(2,1)} \oplus \beta_0^{(1,0)} \wedge J \ .$$

Now let us express W_a explicitly:

$$\begin{aligned}
\mathrm{d}\psi_+ \wedge J &= \psi_+ \wedge \mathrm{d}J \equiv W_1^+ J \wedge J \wedge J \ , \qquad \mathrm{d}\psi_- \wedge J = \psi_- \wedge \mathrm{d}J \equiv W_1^- J \wedge J \wedge J \\
(\mathrm{d}\psi_+)^{(2,2)} &\equiv W_1^+ J \wedge J + W_2^+ \wedge J \ , \qquad (\mathrm{d}\psi_-)^{(2,2)} \equiv W_1^- J \wedge J + W_2^- \wedge J \\
W_4 &= \frac{1}{2} J \lrcorner \mathrm{d}J \ , \qquad W_5 = \frac{1}{2} \psi_+ \lrcorner \mathrm{d}\psi_+ \\
(\mathrm{d}J)^{(2,1)} &= (J \wedge W_4)^{(2,1)} + W_3
\end{aligned}$$

where

$$\begin{aligned}
J \cdot J &= -1 \ , \qquad \Psi \equiv \frac{\Omega}{||\Omega||} = \psi_+ + i\psi_- \ , \qquad W_a = W_a^+ + W_a^- \\
J \wedge \psi_\pm &= 0 \ , \qquad \psi_+ \wedge \psi_- = \frac{2}{3} J \wedge J \wedge J \\
\lrcorner : \ \bigwedge^k T^* \otimes \bigwedge^n T^* &\to \bigwedge^{n-k} T^* \ , \qquad (L_{(k)}, M_{(n)}) \mapsto \frac{1}{n!} \begin{pmatrix} n \\ k \end{pmatrix} L^{a_1 \cdots a_k} M_{a_1 \cdots a_n} \, e^{a_{k+1}} \cdots e^{a_n} \ .
\end{aligned}$$

Note that e^i is a vielbein. For instance, $(e^1 \wedge e^2) \lrcorner (e^1 \wedge e^2 \wedge e^3 \wedge e^4) = e^3 \wedge e^4$.

- **complex manifolds**

$W_1 = W_2 = 0$	$T^0 \in W_3 \oplus W_4 \oplus W_5$	hermitian
$W_1 = W_2 = W_4 = 0$	$T^0 \in W_3 \oplus W_5$	balanced
$W_1 = W_2 = W_4 = W_5 = 0$	$T^0 \in W_3$	special-hermitian
$W_1 = W_2 = W_3 = W_4 = 0$	$T^0 \in W_5$	Kähler
$W_1 = W_2 = W_3 = W_4 = W_5 = 0$	$T^0 = 0$	Calabi-Yau
$W_1 = W_2 = W_3 = 3W_4 + 2W_5 = 0$	$T^0 \in W_4 \oplus W_5$	conformally rescaled CY

- **non-complex manifolds**

$W_1 = W_3 = W_4 = 0$	$T^0 \in W_2 \oplus W_5$	symplectic
$W_2 = W_3 = W_4 = W_5 = 0$	$T^0 \in W_1$	nearly-Kähler
$W_1 = W_3 = W_4 = W_5 = 0$	$T^0 \in W_2$	almost-Kähler
$W_3 = W_4 = W_5 = 0$	$T^0 \in W_1 \oplus W_2$	quasi-Kähler
$W_4 = W_5 = 0$	$T^0 \in W_1 \oplus W_2 \oplus W_3$	semi-Kähler
$W_1^- = W_2^- = W_4 = W_5 = 0$	$T^0 \in W_1^+ \oplus W_2^+ \oplus W_3$	half-flat
$W_2 = 0$	$T^0 \in W_1 \oplus W_3 \oplus W_4 \oplus W_5$	$\mathcal{G}_1$
$W_1 = 0$	$T^0 \in W_2 \oplus W_3 \oplus W_4 \oplus W_5$	$\mathcal{G}_2$

Gray and Hervella [Ann. Math. Pura. Appl. 123 (1980) 35]

Cardoso, Curio, Dall'Agata, Lüst, Manousselis and Zoupanos [hep-th/0211118]

T-duality

R-R $(2n+1)$-form	$\longleftrightarrow$	R-R $2n$- or $(2n+2)$-form
NS-NS 3-form	$\longleftrightarrow$	NS-NS ??-form
		$\downarrow$
		torsion in 6-dim. space

torsion is forbiddon in Kähler manifolds

$\longrightarrow$ the 6-dim. space is no longer a Kähler

(Example)

T^6 compactified type IIB supergravity with NS-NS 3-form flux

the mirror manifold is not a complex 3-torus

but a half-flat manifold with torsion

Appendix

Generalized Complex Structures

Generalized complex structures

Usual complex geometry deals with the tangent bundle of a manifold $\mathcal{TM}$, whose sections are vectors X, and separately, with the cotangent bundle $\mathcal{T}^*\mathcal{M}$, whose sections are 1-forms ζ. In generalized complex geometry the tangent and cotangent bundle are joined as a single bundle, $\mathcal{TM}\oplus\mathcal{T}^*\mathcal{M}$. Its sections are the sum of a vector field plus a 1-form $X+\zeta$. The standard machinery of complex geometry can be generalized to this bundle. On this even-dimensional bundle, one can construct a generalized almost complex structure $\mathcal{J}$, which is a map of $\mathcal{TM}\oplus\mathcal{T}^*\mathcal{M}$ to itself that squares to $-\mathbb{I}_{2d}$ (d is real the dimension of the manifold). This is analogous to an almost complex structure $I^m{}_n$ which is a bundle map from $\mathcal{TM}$ to itself that squares to $-\mathbb{I}_d$. As for an almost complex structure, $\mathcal{J}$ must also satisfy the hermiticity condition $\mathcal{J}^{\mathrm{T}}\mathcal{G}\mathcal{J}=\mathcal{G}$, with the respect to the natural metric on $\mathcal{TM}\oplus\mathcal{T}^*\mathcal{M}$, $\mathcal{G}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

Usual complex structures I are naturally embedded into generalized ones $\mathcal{J}$: take $\mathcal{J}$ to be

$$\mathcal{J}_1\equiv\begin{pmatrix}I&0\\0&-I^{\mathrm{T}}\end{pmatrix}\tag{1}$$

with $I^m{}_n$ a regular almost complex structure (i.e. $I^2=-\mathbb{I}_d$). This $\mathcal{J}$ satisfies the desired properties, namely $\mathcal{J}^2=-\mathbb{I}_{2d}$, $\mathcal{J}^{\mathrm{T}}\mathcal{G}\mathcal{J}=\mathcal{G}$. Another example of generalized almost complex structure can be built using a non degenerate 2-form J_{mn},

$$\mathcal{J}_2 \equiv \begin{pmatrix} 0 & -J^{-1} \\ J & 0 \end{pmatrix} . \tag{2}$$

Given an almost complex structure $I^m{}_n$, one can build holomorphic and antiholomorphic projectors $\pi_\pm = \frac{1}{2}(\mathbb{I}_d \pm iI)$. Correspondingly, projectors can be build out of a generalized almost complex structure, $\Pi_\pm = \frac{1}{2}(\mathbb{I}_{2d} \pm i\mathcal{J})$. There is an integrability condition for generalized almost complex structures, analogous to the integrability condition for usual almost complex structures. For the usual complex structures, integrability, namely the vanishing of the Nijenhuis tensor, can be written as the condition $\pi_\mp[\pi_\pm X, \pi_\pm Y] = 0$, i.e. the Lie bracket of two holomorphic vectors should again be holomorphic. For generalized almost complex structures, integrability condition reads exactly the same, with π and X replaced respectively by Π and $X + \zeta$, and the Lie bracket replaced by the Courant bracket[1] on $\mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$. The Courant bracket does not satisfy Jacobi identity in general, but it does on the i–eigenspaces of $\mathcal{J}$. In case these conditions are fulfilled, we can drop the "almost" and speak of generalized complex structures.

[1]The Courant bracket is defined as follows: $[X + \zeta, Y + \eta]_C = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\zeta - \frac{1}{2}\mathrm{d}(\iota_X\eta - \iota_Y\zeta)$.

For the two examples of generalized almost complex structure given above, $\mathcal{J}_1$ and $\mathcal{J}_2$, integrability condition turns into a condition on their building blocks, $I^m{}_n$ and J_{mn}. Integrability of $\mathcal{J}_1$ enforces I to be an integrable almost complex structure on $\mathcal{TM}$, and hence I is a complex structure, or equivalently the manifold is complex. For $\mathcal{J}_2$, which was built from a 2-form J_{mn}, integrability imposes $\mathrm{d}J = 0$, thus making J into a symplectic form, and the manifold a symplectic one.

Clifford$(6,6)$ algebra

Spinors on $\mathcal{TM}$ transform under *Clifford*(6), whose algebra is $\{\gamma^m, \gamma^n\} = 2g^{mn}$. There is a representation of this algebra in terms of forms. We can take[2] $\gamma^m = \mathrm{d}x^m \wedge + g^{mn}\iota_n$. These satisfy the *Clifford*(d) algebra. The algebra of *Clifford*(d,d) is instead

$$\{\Gamma^m, \Gamma^n\} = 0 \ , \qquad \{\Gamma^m, \Gamma_n\} = \delta^m_n \ , \qquad \{\Gamma_m, \Gamma_n\} = 0 \ .$$

Γ^m and Γ_m are independent, they cannot be obtained from one another by raising or lowering indices with the metric. There is also a representation of this algebra in terms of forms, namely

$$\Gamma^m = \mathrm{d}x^m \wedge \ , \qquad \Gamma_n = \iota_n \ . \tag{3}$$

The holomorphic 3-form Ω is a good vacuum of *Clifford*$(6,6)$, as it is annihilated by the holomorphic Γ^i and the antiholomorphic $\Gamma_{\bar{\imath}}$. These are half of the total gamma matrices, which implies that Ω is a pure *Clifford*$(6,6)$ spinor. Acting with the other half, $\Gamma^{\bar{\imath}}$ and Γ_i we get forms of all possible degrees. *Clifford*$(6,6)$ spinors are therefore equivalent to (p,q)-forms.

Using the Clifford map, a *Clifford*$(6,6)$ spinor can also be mapped to a bispinor:

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1\cdots i_k} \mathrm{d}x^{i_i} \wedge \cdots \wedge \mathrm{d}x^{i_k} \qquad \longleftrightarrow \qquad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1\cdots i_k} \gamma^{i_i\cdots i_k}_{\alpha\beta} \ . \tag{4}$$

[2] ι_n: $\Lambda^p T^* \to \Lambda^{p-1} T^*$, $\iota_n \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} = p\delta^{[i_1}_n \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_p]}$.

On a space of $SU(3)$ structure, there is a nowhere vanishing $SU(3)$ invariant $\mathit{Clifford}(6)$ spinor η. Out of it, we can construct two nowhere vanishing $SU(3,3)$ invariant bispinors by tensoring η with its dagger, namely

$$\Phi_+ = \eta_+ \otimes \eta_+^\dagger \ , \qquad \Phi_- = \eta_+ \otimes \eta_-^\dagger \ . \tag{5}$$

(and its complex conjugates). Using Fierz identities, this tensor product can be written in terms of the bilinears by

$$\eta_+ \otimes \eta_\pm^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_\pm^\dagger \gamma_{i_1 \cdots i_k} \eta_+ \gamma^{i_k \cdots i_1} \tag{6}$$

Using the Clifford map (4) backwards, the tensor products in (5) are identified with regular forms. From now on, we will use $\Phi_\pm$ to denote just the forms.

The subindices plus and minus in $\Phi_\pm$ denote the $Spin(6,6)$ chirality: positive corresponds to an even form, and negative to an odd form. Irreducible $Spin(6,6)$ representations are actually "Majorana-Weyl", namely they are of definite parity –"Weyl"– and real –"Majorana"–.

The explicit expression for the $\mathit{Clifford}(6,6)$ spinors in (5) in terms of the defining forms for the $SU(3)$ structure is

$$\Phi_+ = \eta_+ \otimes \eta_+^\dagger = \frac{1}{8} \mathrm{e}^{-iJ} \ , \qquad \Phi_- = \eta_+ \otimes \eta_-^\dagger = -\frac{i}{8} \Omega \ . \tag{7}$$

The forms in (5), (7) are pure. This can be seen from writing the usual gamma matrices acting on the left of Φ (denoted as $\overrightarrow{\gamma}^m$) and on the right (denoted as $\overleftarrow{\gamma}^m$) in terms of the $Clifford(6,6)$ gamma matrices (3)

$$\overrightarrow{\gamma}^m = \frac{1}{2}(\mathrm{d}x^m \wedge + g^{mn}\iota_n) \ , \qquad \overleftarrow{\gamma}^m = \frac{1}{2}(\mathrm{d}x^m \wedge \pm g^{mn}\iota_n) \ , \tag{8}$$

where the $\pm$ sign depends on the parity of the spinor on which $\overleftarrow{\gamma}^m$ acts. We can check now that the forms (5) are indeed pure: the six gamma matrices that annihilate them are

$$(\delta + iI)_m{}^n \gamma_n \, \eta_+ \otimes \eta_\pm^\dagger = 0 \ , \qquad \eta_+ \otimes \eta_\pm^\dagger \, \gamma_n (\delta \mp iI)_m{}^n = 0 \ . \tag{9}$$

where I is the almost complex structure on the tangent bundle.

On a space of $SU(3)$ structure on $\mathcal{TM}$, there are therefore two $SU(3,3)$ invariant pure forms (and their complex conjugates), e^{-iJ} and Ω. This implies that the structure group on $\mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$, which is generically $SO(d,d)$, is reduced in this case to $SU(3) \times SU(3)$.

There is a one to one correspondence between a a pure spinor Φ and a generalized almost complex structure $\mathcal{J}$. It maps the $+i$ eigenspace of $\mathcal{J}$ to the annihilator space of the spinor Φ. Under this correspondence

$$\Phi_- = -\frac{i}{8}\Omega \ \leftrightarrow \mathcal{J}_1 \ , \qquad \Phi_+ = \frac{1}{8}\mathrm{e}^{-iJ} \leftrightarrow \mathcal{J}_2 \tag{10}$$

where $\mathcal{J}_1$ and $\mathcal{J}_2$ are defined in (1) and (2).

Integrability condition for the generalized complex structure corresponds on the pure spinor side to the condition

$$\mathcal{J} \text{ is integrable} \Leftrightarrow \exists \text{ vector } v \text{ and } 1\text{-form } \zeta \text{ such that } d\Phi = (v\llcorner + \zeta\wedge)\Phi$$

A generalized Calabi-Yau is a manifold on which a closed pure spinor exists:

$$\text{Generalized Calabi-Yau} \Leftrightarrow \exists\ \Phi \text{ pure such that } \mathrm{d}\Phi = 0$$

There is also the possibility of twisting by a closed 3-form H. Using a 3-form, the Courant bracket can be modified[3], and with it the integrability condition. In terms of "integrability" of the pure spinors Φ, adding H amounts to twisting the differential conditition for integrability and for Generalized Calabi-Yau. More precisely,

$$\text{“Twisted” Generalized Calabi-Yau} \Leftrightarrow \exists\ \Phi \text{ pure, and } H \text{ closed such that } (\mathrm{d} - H\wedge)\Phi = 0$$

[3] $[X+\zeta, Y+\eta]_H = [X+\zeta, Y+\eta]_C + \iota_X \iota_Y H$.

Mirror symmetry

Minasian et al suggested a new mirror symmetry on generalized complex geometries such as

$$\mathrm{e}^{-(B+iJ)} \ \leftrightarrow \ \Omega$$

Fidanza, Minasian and Tomasiello [hep-th/0311122]

This means that this mirror symmetry is an exchange of two $Clifford(6,6)$ pure spinors.

This also means the exchange of two generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$:

$SU(3)$-invariant tensors	$Clifford(6,6)$ pure spinors	GCS
$-\frac{i}{8}\Omega$	$\Phi_- = \eta_+ \otimes \eta_-^\dagger$	$\mathcal{J}_1 = \begin{pmatrix} I & 0 \\ * & -I^{\mathrm{T}} \end{pmatrix}$
$\frac{1}{8}\mathrm{e}^{-(B+iJ)}$	$\Phi_+ = \eta_+ \otimes \eta_+^\dagger$	$\mathcal{J}_2 = \begin{pmatrix} 0 & -J^{-1} \\ J & * \end{pmatrix}$

Appendix

$\mathcal{N} = (2, 2), (1, 1)$ SUSY

Appendix

$\mathcal{N} = (2,2)$ supersymmetry

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm(\partial_0 \pm \partial_1) , \qquad \overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm(\partial_0 \pm \partial_1) ,$$

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm(\partial_0 \pm \partial_1) , \qquad \overline{Q}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm(\partial_0 \pm \partial_1) .$$

$$Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0 ,$$

$$\{Q_\pm, \overline{Q}_\pm\} = -2i(\partial_0 \pm \partial_1) = 2(H \mp P) ,$$

$$\{\overline{Q}_+, \overline{Q}_-\} = 0 , \quad \{Q_+, Q_-\} = 0 , \quad \{Q_-, \overline{Q}_+\} = 0 , \quad \{Q_+, \overline{Q}_-\} = 0 ,$$

$$\{D_\pm, \overline{D}_\pm\} = 2i(\partial_0 \pm \partial_1) ,$$

$$\{\overline{D}_\alpha, \overline{D}_\beta\} = \{D_\alpha, D_\beta\} = \{D_\pm, \overline{D}_\mp\} = 0 ,$$

$$\{D_\alpha, Q_\beta\} = \{\overline{D}_\alpha, Q_\beta\} = \{D_\alpha, \overline{Q}_\beta\} = \{\overline{D}_\alpha, \overline{Q}_\beta\} = 0 ,$$

$$[M, Q_\pm] = \mp Q_\pm , \qquad [M, \overline{Q}_\pm] = \mp\overline{Q}_\pm ,$$

$$[F_V, Q_\pm] = -Q_\pm , \qquad [F_V, \overline{Q}_\pm] = \overline{Q}_\pm ,$$

$$[F_A, Q_\pm] = \mp Q_\pm , \qquad [F_A, \overline{Q}_\pm] = \pm\overline{Q}_\pm .$$

$\mathcal{N} = (2,2)$ superfields

Here we summarize $\mathcal{N} = (2,2)$ superfields:

chiral superfield Φ	$\overline{D}_\pm \Phi \;=\; 0$	$\Phi \;=\; \overline{D}_+ \overline{D}_- \Theta$
twisted chiral superfield Y	$\overline{D}_+ Y \;=\; D_- Y \;=\; 0$	$Y \;=\; \overline{D}_+ D_- \Theta$
real linear superfield G	$\overline{D}_+ \overline{D}_- G \;=\; D_+ D_- G \;=\; 0$	$G \;=\; Y + \overline{Y}$
real twisted linear superfield H	$\overline{D}_+ D_- H \;=\; D_+ \overline{D}_- H \;=\; 0$	$H \;=\; \Phi + \overline{\Phi}$
left semi-chiral superfield $\mathbb{X}$	$\overline{D}_+ \mathbb{X} \;=\; 0$	$\mathbb{X} \;=\; \overline{D}_+ \Theta$
right semi-chiral superfield $\mathbb{Y}$	$\overline{D}_- \mathbb{Y} \;=\; 0$	$\mathbb{Y} \;=\; \overline{D}_- \Theta$
complex linear superfield Σ	$\overline{D}_+ \overline{D}_- \Sigma \;=\; 0$	$\Sigma \;=\; a\mathbb{X} + b\mathbb{Y}$
complex twisted linear superfield $\widetilde{\Sigma}$	$\overline{D}_+ D_- \widetilde{\Sigma} \;=\; 0$	$\widetilde{\Sigma} \;=\; a\mathbb{X} + b\overline{\mathbb{Y}}$

where Θ is an unconstrained superfield; a and b are complex constants.

$$
\begin{aligned}
\Phi &= \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}\theta^-\psi_- + 2i\theta^+\theta^- F \\
&\quad - i\theta^+\overline{\theta}^+(\partial_0+\partial_1)\phi - i\theta^-\overline{\theta}^-(\partial_0-\partial_1)\phi + \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0^2-\partial_1^2)\phi \\
&\quad + \sqrt{2}\theta^+\overline{\theta}^+\theta^-(\partial_0+\partial_1)\psi_- + \sqrt{2}\theta^-\overline{\theta}^-\theta^+(\partial_0-\partial_1)\psi_+ \ ,
\end{aligned}
$$

$$
\begin{aligned}
Y &= y + i\sqrt{2}\theta^+\overline{\chi}_+ + i\sqrt{2}\overline{\theta}^-\chi_- + 2i\theta^+\overline{\theta}^- G \\
&\quad - i\theta^+\overline{\theta}^+(\partial_0+\partial_1)y + i\theta^-\overline{\theta}^-(\partial_0-\partial_1)y - \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0^2-\partial_1^2)y \\
&\quad - \sqrt{2}\theta^-\overline{\theta}^-\theta^+(\partial_0-\partial_1)\overline{\chi}_+ + \sqrt{2}\theta^+\overline{\theta}^+\overline{\theta}^-(\partial_0+\partial_1)\chi_- \ ,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{X} &= \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}(\theta^-\psi_- + \overline{\theta}^-\chi_-) + 2i\theta^+(\theta^- F + \overline{\theta}^- G) \\
&\quad + \theta^-\overline{\theta}^- A_= + 2\theta^+\theta^-\overline{\theta}^-\zeta_- - i\theta^+\overline{\theta}^+(\partial_0+\partial_1)\phi \\
&\quad + \sqrt{2}\theta^+\overline{\theta}^+(\partial_0+\partial_1)(\theta^-\psi_- + \overline{\theta}^-\chi_-) + i\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0+\partial_1)A_= \ ,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{Y} &= \phi + i\sqrt{2}(\theta^+\psi_+ + \overline{\theta}^+\chi_+) + i\sqrt{2}\theta^-\psi_- + 2i\theta^+\theta^- F + 2i\overline{\theta}^+\theta^- N \\
&\quad - i\theta^-\overline{\theta}^-(\partial_0-\partial_1)\phi + \theta^+\overline{\theta}^+ B_{\#} - 2\theta^-\theta^+\overline{\theta}^+\zeta_+ \\
&\quad + \sqrt{2}\theta^-\overline{\theta}^-(\partial_0-\partial_1)(\theta^+\psi_+ + \overline{\theta}^+\chi_+) + i\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0-\partial_1)B_{\#} \ .
\end{aligned}
$$

$\mathcal{N} = (1,1)$ supersymmetry

Here we consider $\mathcal{N} = (1,1)$ supersymmetry which has two real supercharges, one with positive chirality and the other with negative chirality:

$$\theta_1^\pm \;\equiv\; -i\mathrm{e}^{-i\nu_\pm}\theta^\pm \;=\; i\mathrm{e}^{+i\nu_\pm}\overline{\theta}^\pm \qquad \text{where } \theta_1^\pm \text{ is real} \tag{11}$$

We introduce the following differential operators

$$\begin{aligned} Q_\pm^1 \;&\equiv\; \frac{1}{\sqrt{2}}\Big\{\mathrm{e}^{i\nu_\pm}Q_\pm + \mathrm{e}^{-i\nu_\pm}\overline{Q}_\pm\Big\}\Big|_{\text{eq.(11)}} \;=\; -\frac{i}{\sqrt{2}}\frac{\partial}{\partial\theta_1^\pm} + \sqrt{2}\theta_1^\pm\big(\partial_0 \pm \partial_1\big) \;, \\ D_\pm^1 \;&\equiv\; \frac{1}{\sqrt{2}}\Big\{\mathrm{e}^{i\nu_\pm}D_\pm + \mathrm{e}^{-i\nu_\pm}\overline{D}_\pm\Big\}\Big|_{\text{eq.(11)}} \;=\; -\frac{i}{\sqrt{2}}\frac{\partial}{\partial\theta_1^\pm} - \sqrt{2}\theta_1^\pm\big(\partial_0 \pm \partial_1\big) \;. \end{aligned}$$

These operators obey the anti-commutation relations

$$\begin{aligned} \{Q_\pm^1, Q_\pm^1\} \;&=\; -2i\big(\partial_0 \pm \partial_1\big) \;=\; 2(H \mp P) \;, \qquad \{Q_+^1, Q_-^1\} \;=\; 0 \;, \\ \{D_\pm^1, D_\pm^1\} \;&=\; +2i\big(\partial_0 \pm \partial_1\big) \;, \qquad \{D_+^1, D_-^1\} \;=\; 0 \;, \qquad \{Q_\alpha^1, D_\beta^1\} \;=\; 0 \;. \end{aligned}$$

We also define the following "differential operators":

$$\begin{aligned} \widetilde{Q}_\pm^1 \;&\equiv\; \frac{i}{\sqrt{2}}\Big\{\mathrm{e}^{i\nu_\pm}Q_\pm - \mathrm{e}^{-i\nu_\pm}\overline{Q}_\pm\Big\} \;, \qquad \text{under the constraint (11):} \quad \widetilde{Q}_\pm^1\Big| \;=\; 0 \;, \\ \widetilde{D}_\pm^1 \;&\equiv\; \frac{i}{\sqrt{2}}\Big\{\mathrm{e}^{i\nu_\pm}D_\pm - \mathrm{e}^{-i\nu_\pm}\overline{D}_\pm\Big\} \;, \qquad \text{under the constraint (11):} \quad \widetilde{D}_\pm^1\Big| \;=\; 0 \;. \end{aligned}$$

Under the constraint (11) these operators are trivially zero. However, they are another two supercharges and two covariant derivatives in the original $\mathcal{N} = (2,2)$ supersymmetry. We can easily find this "second" $(1,1)$ supersymmetry operators satisfy the following anti-commutation relations in the $(2,2)$ supersymmetry level:

$$\begin{aligned} \{\widetilde{Q}^1_{\pm}, \widetilde{Q}^1_{\pm}\} &= -2i(\partial_0 \pm \partial_1) = 2(H \mp P) \,, & \{\widetilde{Q}^1_{+}, \widetilde{Q}^1_{-}\} &= 0 \,, \\ \{\widetilde{D}^1_{\pm}, \widetilde{D}^1_{\pm}\} &= +2i(\partial_0 \pm \partial_1) \,, & \{\widetilde{D}^1_{+}, \widetilde{D}^1_{-}\} &= 0 \,, \\ \{\widetilde{Q}^1_{\alpha}, \widetilde{D}^1_{\beta}\} &= 0 \,. \end{aligned}$$

Furthermore we can check that the first $(1,1)$ supersymmetry and the second $(1,1)$ supersymmetry commute with each other:

$$\{Q^1_{\alpha}, \widetilde{Q}^1_{\beta}\} = \{D^1_{\alpha}, \widetilde{D}^1_{\beta}\} = \{Q^1_{\alpha}, \widetilde{D}^1_{\beta}\} = \{D^1_{\alpha}, \widetilde{Q}^1_{\beta}\} = 0 \,.$$

This result is consistent with the original $\mathcal{N} = (2,2)$ supersymmetry.

$\mathcal{N}=(1,1)$ scalar/spinor superfields from $\mathcal{N}=(2,2)$ semi-chiral superfields

$$
\begin{aligned}
\mathbb{X}^{(1,1)} &= \mathbb{X}^{(2,2)}\Big| \\
&= \phi + i\sqrt{2}\theta_1^{+}\widehat{\psi}_{+} + i\sqrt{2}\theta_1^{-}(\widehat{\psi}_{-} + \widehat{\chi}_{-}) + 2i\theta_1^{+}\theta_1^{-}(\widehat{F} + \widehat{G}) \ , \\
\boldsymbol{\Psi}_{-}^{(1,1)} &= \widetilde{Q}_{-}^{1}\mathbb{X}^{(2,2)}\Big| \\
&= i(\widehat{\psi}_{-} - \widehat{\chi}_{-}) - i\sqrt{2}\theta_1^{+}(\widehat{F} - \widehat{G}) + \sqrt{2}\theta_1^{-}A_{=} + 2\sqrt{2}\theta_1^{+}\theta_1^{-}\widehat{\zeta}_{-} \ ,
\end{aligned}
$$

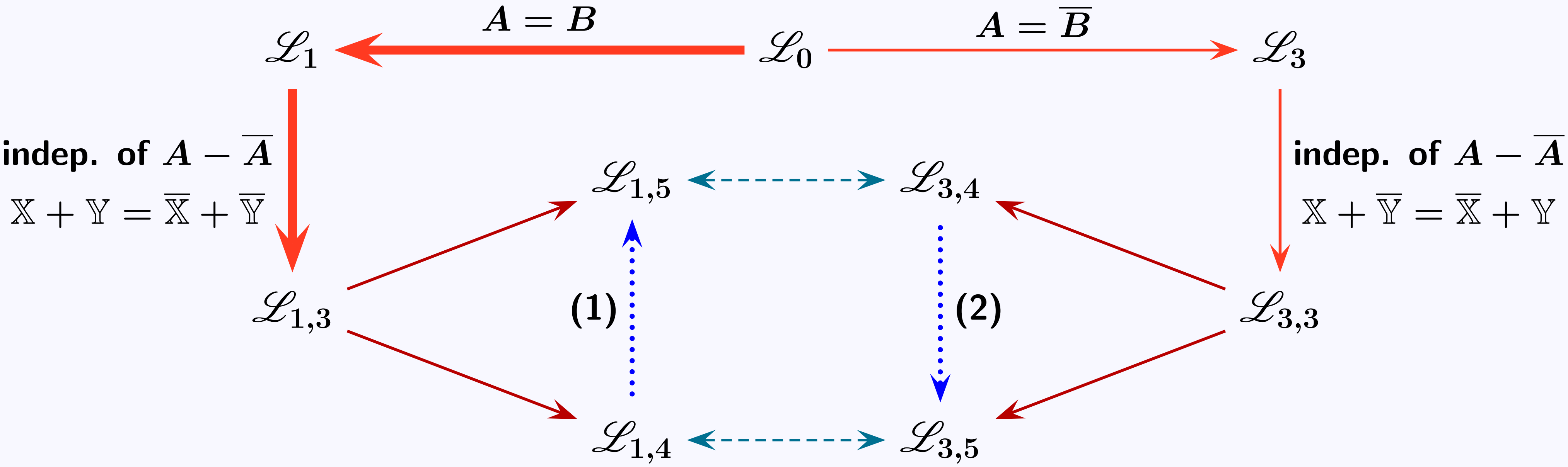
$$
\begin{aligned}
\mathbb{Y}^{(1,1)} &= \mathbb{Y}^{(2,2)}\Big| \\
&= \phi + i\sqrt{2}\theta_1^{+}(\widehat{\psi}_{+} + \widehat{\chi}_{+}) + i\sqrt{2}\theta_1^{-}\widehat{\psi}_{-} + 2i\theta_1^{+}\theta_1^{-}(\widehat{F} + \widehat{N}) \ , \\
\boldsymbol{\Upsilon}_{+}^{(1,1)} &= \widetilde{Q}_{+}^{1}\mathbb{Y}^{(2,2)}\Big| \\
&= i(\widehat{\psi}_{+} - \widehat{\chi}_{+}) + \sqrt{2}\theta_1^{+}B_{\#} + i\sqrt{2}\theta_1^{-}(\widehat{F} - \widehat{N}) - 2\sqrt{2}\theta_1^{+}\theta_1^{-}\widehat{\zeta}_{+} \ .
\end{aligned}
$$

Appendix
Duality transformation case 1

$$\mathscr{L}(\mathbf{\Phi}, \overline{\mathbf{\Phi}}) \leftrightarrow \mathscr{L}(\boldsymbol{Y}, \overline{\boldsymbol{Y}})$$

Duality transformation 1 (generic)

$$\mathscr{L}_0 \;=\; \int \mathrm{d}^4\boldsymbol{\theta}\,\Big\{\boldsymbol{K}_0(\boldsymbol{A},\overline{\boldsymbol{A}},\boldsymbol{B},\overline{\boldsymbol{B}};\cdots) - \mathbb{X}\boldsymbol{A} - \overline{\mathbb{X}\boldsymbol{A}} - \mathbb{Y}\boldsymbol{B} - \overline{\mathbb{Y}\boldsymbol{B}}\Big\}$$



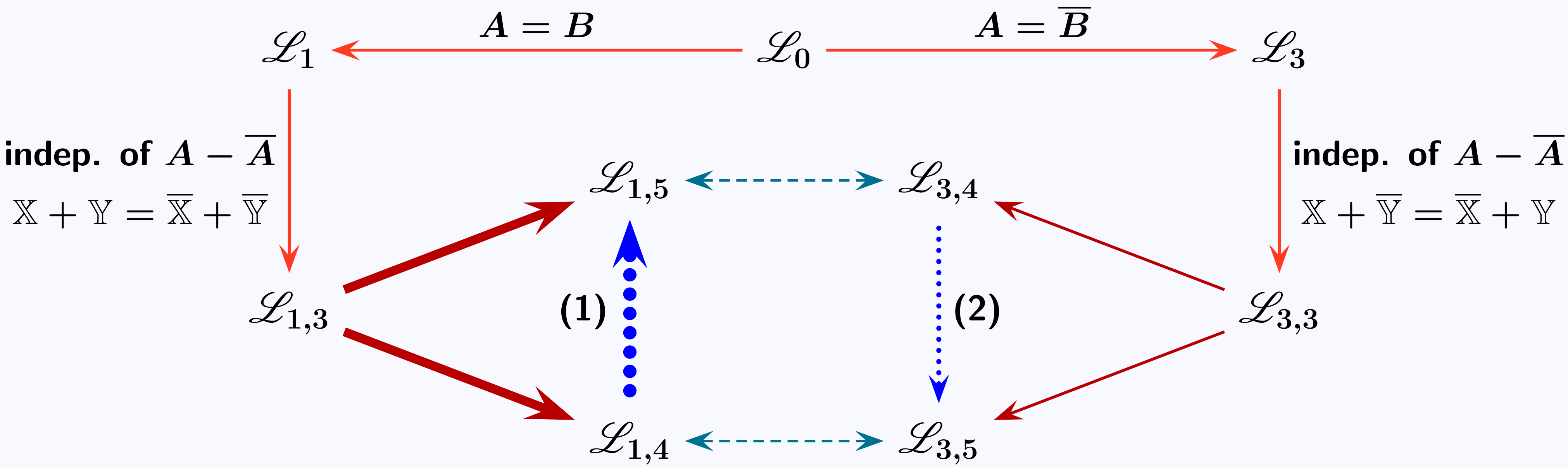
$$\mathscr{L}_1 \;\equiv\; \int \mathrm{d}^4\boldsymbol{\theta}\,\Big\{\boldsymbol{K}_{1,3}(\boldsymbol{A}+\overline{\boldsymbol{A}};\cdots) - \tfrac{1}{2}(\mathbb{X}+\mathbb{Y}+\overline{\mathbb{X}}+\overline{\mathbb{Y}})(\boldsymbol{A}+\overline{\boldsymbol{A}}) - \tfrac{1}{2}(\mathbb{X}+\mathbb{Y}-\overline{\mathbb{X}}-\overline{\mathbb{Y}})(\boldsymbol{A}-\overline{\boldsymbol{A}})\Big\}$$

$$\therefore \quad \mathbb{X}+\mathbb{Y} \;=\; \boldsymbol{Y}_1 + \overline{\boldsymbol{Y}}_1$$

$$\mathscr{L}_{1,3} \;\equiv\; \int \mathrm{d}^4\boldsymbol{\theta}\,\Big\{\boldsymbol{K}_{1,3}(\boldsymbol{A}+\overline{\boldsymbol{A}};\cdots) - (\boldsymbol{Y}_1+\overline{\boldsymbol{Y}}_1)(\boldsymbol{A}+\overline{\boldsymbol{A}})\Big\}$$

Duality transformation 1 (generic)

$$\mathscr{L}_0 \;=\; \int \mathrm{d}^4\theta \left\{ \boldsymbol{K}_0(\boldsymbol{A}, \overline{\boldsymbol{A}}, \boldsymbol{B}, \overline{\boldsymbol{B}}; \cdots) - \mathbb{X}\boldsymbol{A} - \overline{\mathbb{X}\boldsymbol{A}} - \mathbb{Y}\boldsymbol{B} - \overline{\mathbb{Y}\boldsymbol{B}} \right\}$$



$$\boldsymbol{\Phi}_1 + \overline{\boldsymbol{\Phi}}_1 \;=\; \boldsymbol{A} + \overline{\boldsymbol{A}} \;=\; \widetilde{\boldsymbol{A}}(\boldsymbol{Y}_1, \overline{\boldsymbol{Y}}_1; \cdots)\,, \qquad \mathbb{X} + \mathbb{Y} \;=\; \boldsymbol{Y}_1 + \overline{\boldsymbol{Y}}_1$$

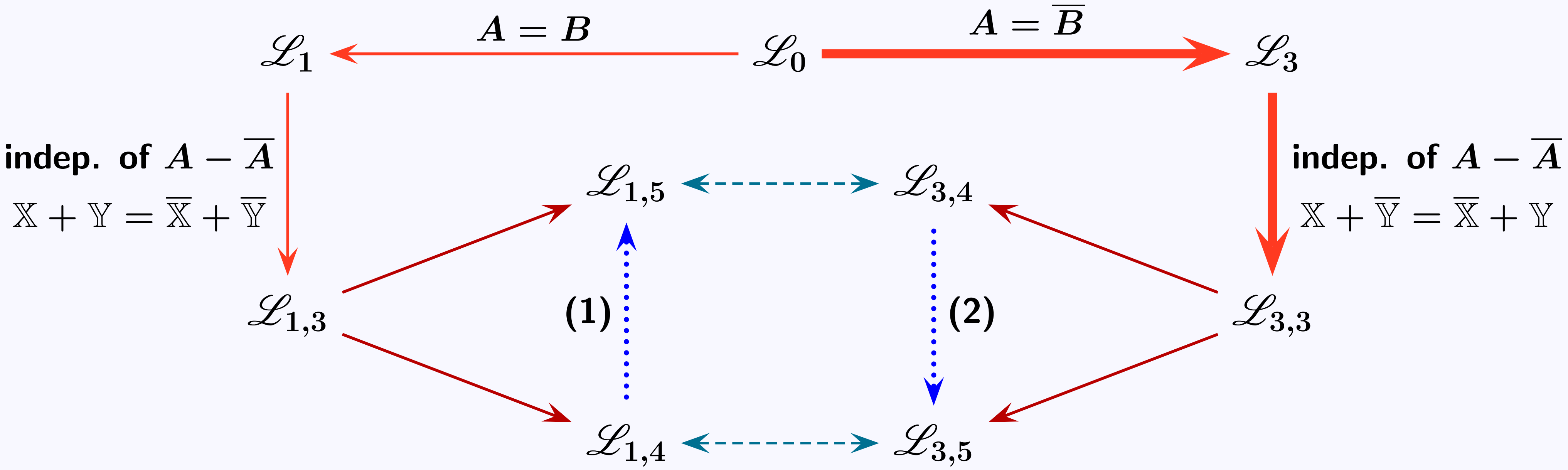
(1) :

$$\mathscr{L}_{1,4} \;\equiv\; \mathscr{L}_{1,3}\Big|_{\text{EOM of } Y_1 + \overline{Y}_1} \;\equiv\; \int \mathrm{d}^4\theta\, \boldsymbol{K}_{1,4}(\boldsymbol{\Phi}_1, \overline{\boldsymbol{\Phi}}_1; \cdots)$$

$$\mathscr{L}_{1,5} \;\equiv\; \mathscr{L}_{1,3}\Big|_{\text{EOM of } A + \overline{A}} \;\equiv\; \int \mathrm{d}^4\theta\, \boldsymbol{K}_{1,5}(\boldsymbol{Y}_1, \overline{\boldsymbol{Y}}_1; \cdots)$$

Duality transformation 1 (generic)

$$\mathscr{L}_0 \;=\; \int \mathrm{d}^4\boldsymbol{\theta}\left\{\boldsymbol{K}_0(\boldsymbol{A},\overline{\boldsymbol{A}},\boldsymbol{B},\overline{\boldsymbol{B}};\cdots) - \mathbb{X}\boldsymbol{A} - \overline{\mathbb{X}\boldsymbol{A}} - \mathbb{Y}\boldsymbol{B} - \overline{\mathbb{Y}\boldsymbol{B}}\right\}$$



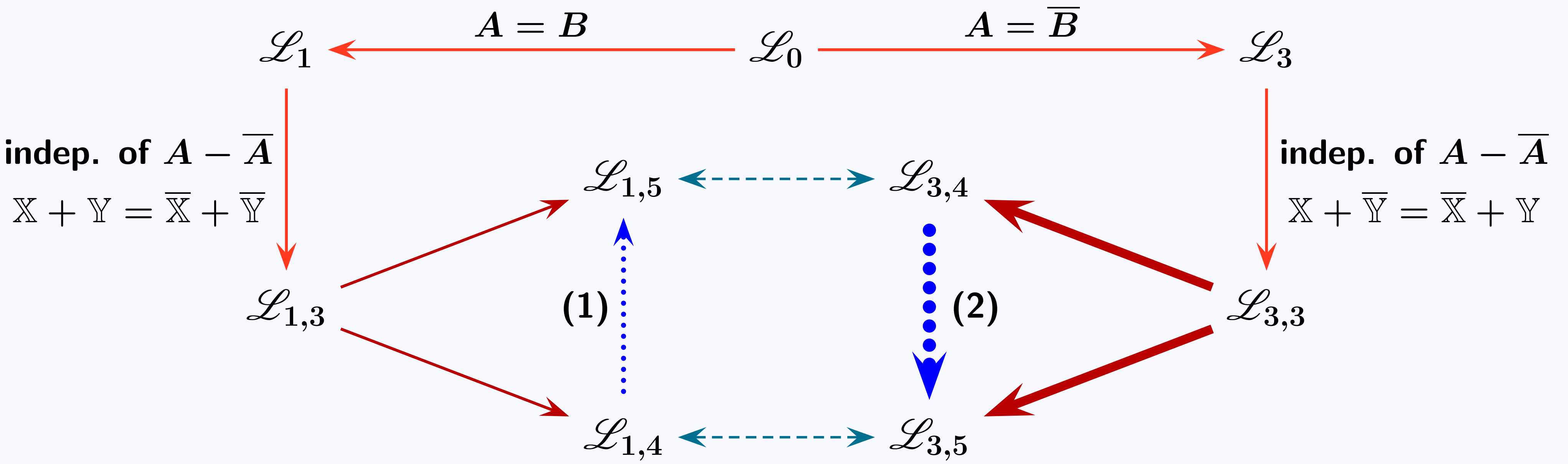
$$\mathscr{L}_3 \;\equiv\; \int \mathrm{d}^4\boldsymbol{\theta}\left\{\boldsymbol{K}_{3,3}(\boldsymbol{A}+\overline{\boldsymbol{A}};\cdots) - \tfrac{1}{2}(\mathbb{X}+\overline{\mathbb{Y}}+\overline{\mathbb{X}}+\mathbb{Y})(\boldsymbol{A}+\overline{\boldsymbol{A}}) - \tfrac{1}{2}(\mathbb{X}+\overline{\mathbb{Y}}-\overline{\mathbb{X}}-\mathbb{Y})(\boldsymbol{A}-\overline{\boldsymbol{A}})\right\}$$

$$\therefore \quad \mathbb{X}+\overline{\mathbb{Y}} \;=\; \boldsymbol{\Phi}_3+\overline{\boldsymbol{\Phi}}_3$$

$$\mathscr{L}_{3,3} \;\equiv\; \int \mathrm{d}^4\boldsymbol{\theta}\left\{\boldsymbol{K}_{3,3}(\boldsymbol{A}+\overline{\boldsymbol{A}};\cdots) - (\boldsymbol{\Phi}_3+\overline{\boldsymbol{\Phi}}_3)(\boldsymbol{A}+\overline{\boldsymbol{A}})\right\}$$

Duality transformation 1 (generic)

$$\mathscr{L}_0 \;=\; \int \mathrm{d}^4\theta \left\{ \boldsymbol{K}_0(\boldsymbol{A}, \overline{\boldsymbol{A}}, \boldsymbol{B}, \overline{\boldsymbol{B}}; \cdots) - \mathbb{X}\boldsymbol{A} - \overline{\mathbb{X}\boldsymbol{A}} - \mathbb{Y}\boldsymbol{B} - \overline{\mathbb{Y}\boldsymbol{B}} \right\}$$



$$\boldsymbol{Y}_3 + \overline{\boldsymbol{Y}}_3 \;=\; \boldsymbol{A} + \overline{\boldsymbol{A}} \;=\; \widetilde{\boldsymbol{A}}(\boldsymbol{\Phi}_3, \overline{\boldsymbol{\Phi}}_3; \cdots)\,, \qquad \mathbb{X} + \overline{\mathbb{Y}} \;=\; \boldsymbol{\Phi}_3 + \overline{\boldsymbol{\Phi}}_3$$

(2) :

$$\mathscr{L}_{3,4} \;\equiv\; \mathscr{L}_{3,3}\Big|_{\text{EOM of } \Phi_3 + \overline{\Phi}_3} \;\equiv\; \int \mathrm{d}^4\theta\, \boldsymbol{K}_{3,4}(\boldsymbol{Y}_3, \overline{\boldsymbol{Y}}_3; \cdots)$$

$$\mathscr{L}_{3,5} \;\equiv\; \mathscr{L}_{3,3}\Big|_{\text{EOM of } A + \overline{A}} \;\equiv\; \int \mathrm{d}^4\theta\, \boldsymbol{K}_{3,5}(\boldsymbol{\Phi}_3, \overline{\boldsymbol{\Phi}}_3; \cdots)$$

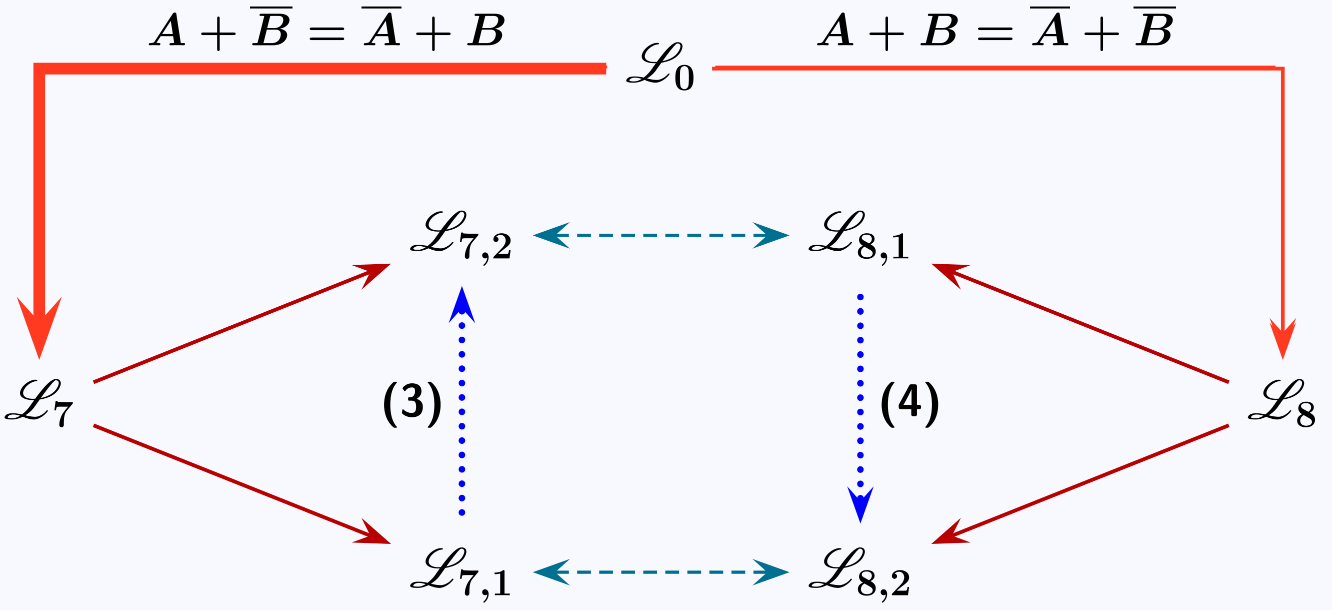
Appendix

Duality transformation case 2

$$\mathscr{L}(\mathbf{\Phi}, \overline{\mathbf{\Phi}}, \boldsymbol{Y}, \overline{\boldsymbol{Y}}) \leftrightarrow \mathscr{L}(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}})$$

Duality transformation 2 (generic)

$$\mathscr{L}_0 \;=\; \int \mathrm{d}^4\theta \left\{ \boldsymbol{K}_0(\boldsymbol{A}, \overline{\boldsymbol{A}}, \boldsymbol{B}, \overline{\boldsymbol{B}}; \cdots) - \mathbb{X}\boldsymbol{A} - \overline{\mathbb{X}\boldsymbol{A}} - \mathbb{Y}\boldsymbol{B} - \overline{\mathbb{Y}\boldsymbol{B}} \right\}$$



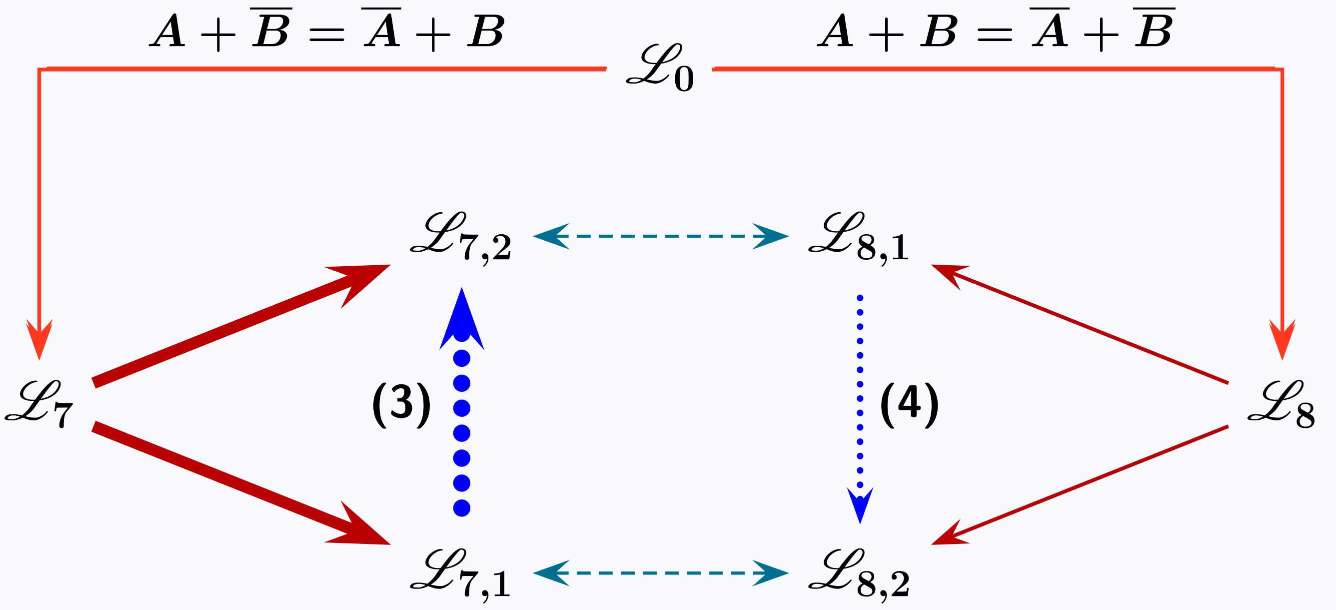
$$\boldsymbol{A} \;=\; \boldsymbol{A}' + \boldsymbol{B}' \,, \qquad \boldsymbol{B} \;=\; \boldsymbol{A}' - \overline{\boldsymbol{B}}' + \boldsymbol{C}'$$

$$\overline{\boldsymbol{C}}' \;=\; \boldsymbol{C}'$$

$$\mathscr{L}_7 \;=\; \int \mathrm{d}^4\theta \left\{ \boldsymbol{K}_7(\boldsymbol{B}' + \overline{\boldsymbol{B}}' - \boldsymbol{C}', \boldsymbol{A}' - \overline{\boldsymbol{B}}' + \boldsymbol{C}', \overline{\boldsymbol{A}}' - \boldsymbol{B}' + \boldsymbol{C}'; \cdots) \right.$$
$$\left. - (\boldsymbol{B}' + \overline{\boldsymbol{B}}' - \boldsymbol{C}')(\mathbb{X} + \overline{\mathbb{X}}) - (\boldsymbol{A}' - \overline{\boldsymbol{B}}' + \boldsymbol{C}')(\mathbb{X} + \mathbb{Y}) - (\overline{\boldsymbol{A}}' - \boldsymbol{B}' + \boldsymbol{C}')(\overline{\mathbb{X}} + \overline{\mathbb{Y}}) \right\}$$

Duality transformation 2 (generic)

$$\mathscr{L}_0 \;=\; \int \mathrm{d}^4\theta \left\{ \boldsymbol{K}_0(\boldsymbol{A}, \overline{\boldsymbol{A}}, \boldsymbol{B}, \overline{\boldsymbol{B}}; \cdots) - \mathbb{X}\boldsymbol{A} - \overline{\mathbb{X}\boldsymbol{A}} - \mathbb{Y}\boldsymbol{B} - \overline{\mathbb{Y}\boldsymbol{B}} \right\}$$



$$\boldsymbol{\Phi}_7 \;=\; \boldsymbol{A}' \;=\; \boldsymbol{A}'(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}; \cdots)\,, \qquad \boldsymbol{Y}_7 \;=\; \boldsymbol{B}' \;=\; \boldsymbol{B}'(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}; \cdots)$$

$$\boldsymbol{0} \;=\; \boldsymbol{C}' \;=\; \boldsymbol{C}'(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}; \cdots)$$

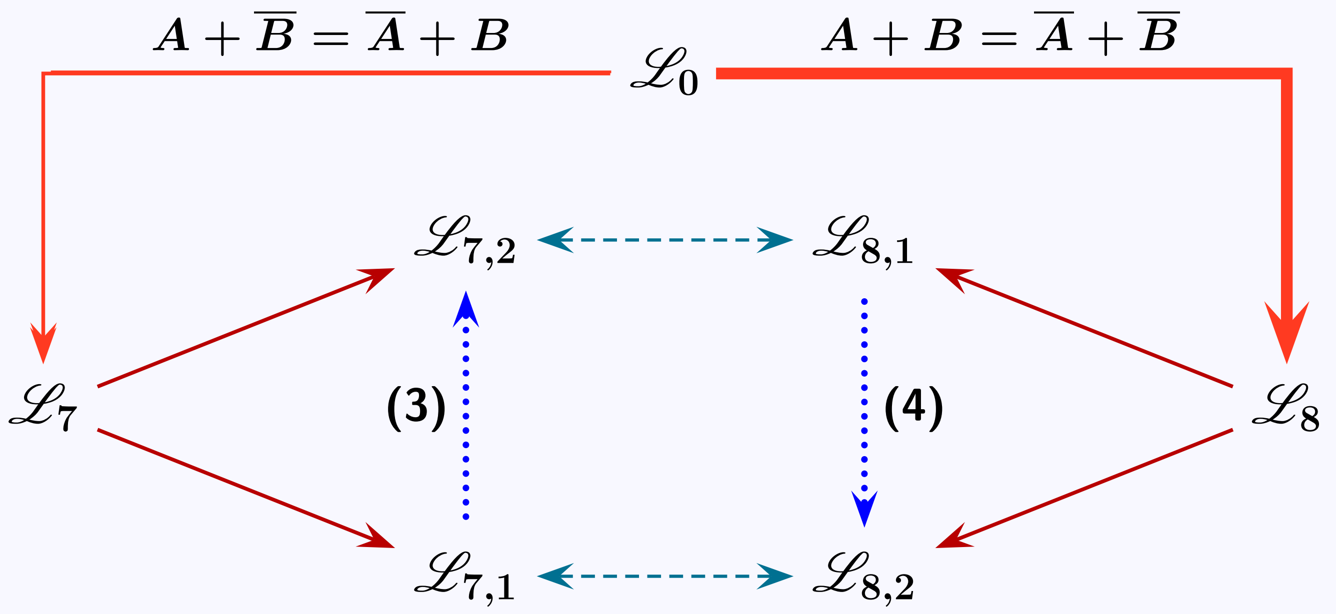
(3) :

$$\mathscr{L}_{7,1} \;\equiv\; \mathscr{L}_7\Big|_{\text{EOM of } \mathbb{X},\, \mathbb{Y}} \;\equiv\; \int \mathrm{d}^4\theta\, \boldsymbol{K}_{7,1}(\boldsymbol{\Phi}_7, \overline{\boldsymbol{\Phi}}_7, \boldsymbol{Y}_7, \overline{\boldsymbol{Y}}_7; \cdots)$$

$$\mathscr{L}_{7,2} \;\equiv\; \mathscr{L}_7\Big|_{\text{EOM of } A',\, B',\, C'} \;\equiv\; \int \mathrm{d}^4\theta\, \boldsymbol{K}_{7,2}(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}; \cdots)$$

Duality transformation 2 (generic)

$$\mathscr{L}_0 = \int \mathrm{d}^4\theta \left\{ \boldsymbol{K}_0(\boldsymbol{A}, \overline{\boldsymbol{A}}, \boldsymbol{B}, \overline{\boldsymbol{B}}; \cdots) - \mathbb{X}\boldsymbol{A} - \overline{\mathbb{X}\boldsymbol{A}} - \mathbb{Y}\boldsymbol{B} - \overline{\mathbb{Y}\boldsymbol{B}} \right\}$$



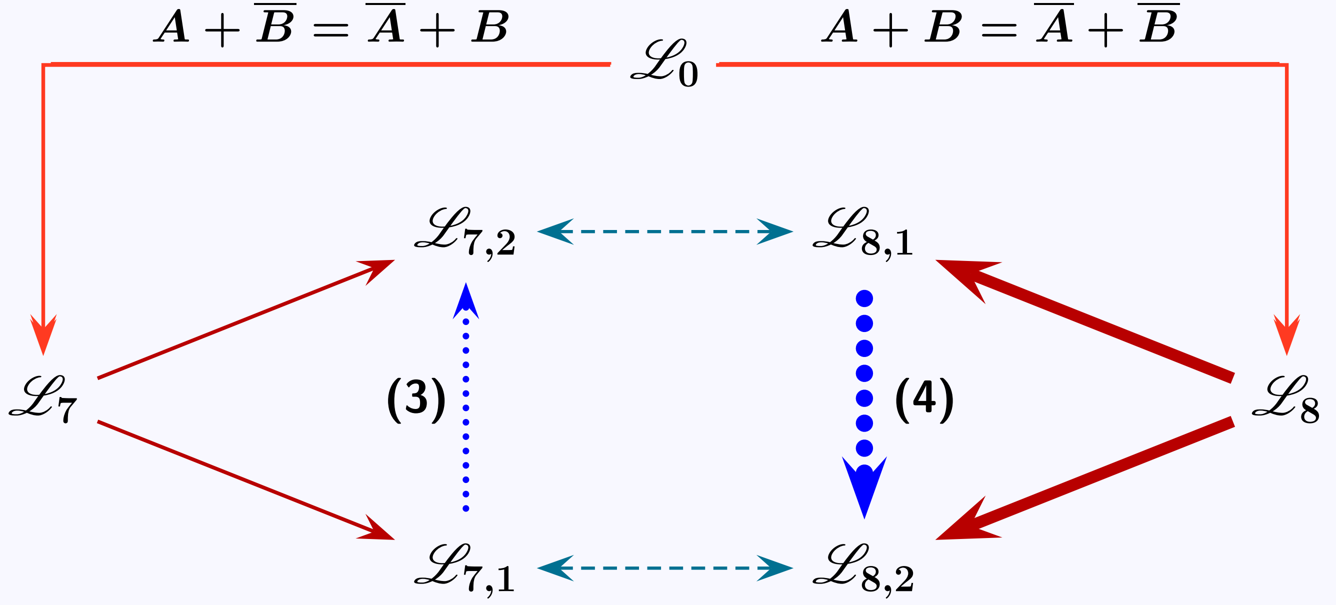
$$\boldsymbol{A} = \mathbb{X}' , \quad \overline{\boldsymbol{A}} = \overline{\mathbb{X}}' , \quad \boldsymbol{B} = \mathbb{Y}' , \quad \overline{\boldsymbol{B}} = \overline{\mathbb{Y}}'$$

$$\boldsymbol{M}_8 \equiv \boldsymbol{A} + \overline{\boldsymbol{A}} , \quad \boldsymbol{N}_8 \equiv -i(\boldsymbol{A} - \overline{\boldsymbol{A}}) , \quad \boldsymbol{P}_8 \equiv \boldsymbol{A} + \overline{\boldsymbol{B}} , \quad \overline{\boldsymbol{P}}_8 \equiv \overline{\boldsymbol{A}} + \boldsymbol{B}$$

$$\mathscr{L}_8 = \int \mathrm{d}^4\theta \left\{ \boldsymbol{K}_8(\boldsymbol{M}_8, \boldsymbol{P}_8, \overline{\boldsymbol{P}}_8; \cdots) - \tfrac{i}{2}\boldsymbol{N}_8(\mathbb{X} - \overline{\mathbb{X}} + \mathbb{Y} - \overline{\mathbb{Y}}) \right.$$
$$\left. - \tfrac{1}{2}\boldsymbol{M}_8(\mathbb{X} + \overline{\mathbb{X}} - \mathbb{Y} - \overline{\mathbb{Y}}) - \overline{\boldsymbol{P}}_8\mathbb{Y} - \boldsymbol{P}_8\overline{\mathbb{Y}} \right\}$$

Duality transformation 2 (generic)

$$\mathscr{L}_0 \;=\; \int \mathrm{d}^4\theta \left\{ \boldsymbol{K}_0(\boldsymbol{A}, \overline{\boldsymbol{A}}, \boldsymbol{B}, \overline{\boldsymbol{B}}; \cdots) - \mathbb{X}\boldsymbol{A} - \overline{\mathbb{X}\boldsymbol{A}} - \mathbb{Y}\boldsymbol{B} - \overline{\mathbb{Y}\boldsymbol{B}} \right\}$$



$$\mathbb{X}' + \overline{\mathbb{X}}' \;=\; \boldsymbol{A} + \overline{\boldsymbol{A}} \;=\; \boldsymbol{M}_8(\boldsymbol{\Phi}_8, \overline{\boldsymbol{\Phi}}_8, \boldsymbol{Y}_8, \overline{\boldsymbol{Y}}_8; \cdots)$$

$$\mathbb{X}' + \overline{\mathbb{Y}}' \;=\; \boldsymbol{A} + \overline{\boldsymbol{B}} \;=\; \boldsymbol{P}_8(\boldsymbol{\Phi}_8, \overline{\boldsymbol{\Phi}}_8, \boldsymbol{Y}_8, \overline{\boldsymbol{Y}}_8; \cdots)$$

(4) :

$$\mathscr{L}_{8,1} \;\equiv\; \mathscr{L}_8\Big|_{\textsf{EOM of } \mathbb{X},\, \mathbb{Y}} \;\equiv\; \int \mathrm{d}^4\theta\, \boldsymbol{K}_{8,1}(\mathbb{X}', \overline{\mathbb{X}}', \mathbb{Y}', \overline{\mathbb{Y}}'; \cdots)$$

$$\mathscr{L}_{8,2} \;\equiv\; \mathscr{L}_8\Big|_{\textsf{EOM of } M_8,\, P_8} \;\equiv\; \int \mathrm{d}^4\theta\, \boldsymbol{K}_{8,2}(\boldsymbol{\Phi}_8, \overline{\boldsymbol{\Phi}}_8, \boldsymbol{Y}_8, \overline{\boldsymbol{Y}}_8; \cdots)$$

Appendix
Line Bundles on EK

SUSY sigma models on Hermitian symmetric spaces

Here we construct SUSY sigma models.

$$\Phi : M \times N \text{ matrix-valued chiral superfield}$$

is imposed $U(M) \times U(N)$ global symmetry:

$$\Phi \to \Phi' = g_{\rm L} \Phi g_{\rm R} \,, \qquad (g_{\rm L}, g_{\rm R}) \in (U(M), U(N))$$

introduce $U(N)$ gauge transformation (V : auxiliary vector superfield):

$$\Phi \to \Phi' = \Phi \mathrm{e}^{-i\Lambda} \,, \qquad \mathrm{e}^{V} \to \mathrm{e}^{V'} = \mathrm{e}^{i\Lambda} \mathrm{e}^{V} \mathrm{e}^{-i\overline{\Lambda}}$$

We consider the Lagrangian:

$$\mathcal{L} = \int \mathrm{d}^4\theta \left\{ \mathrm{tr}(\overline{\Phi} \Phi \, \mathrm{e}^{V}) - c \, \mathrm{tr} V \right\} + \left(\int \mathrm{d}^2\theta \, W + (c.c.) \right)$$

Integrating out V: $\Phi = \begin{pmatrix} 1_N \\ \varphi_{Aa} \end{pmatrix}$, φ_{Aa} **is $(M-N) \times N$ matrix-valued**

- $W = 0$:

$$K_{\text{compact}} = c \log\det\left(1_N + \overline{\varphi}\varphi\right), \qquad G_{M,N} = \frac{U(M)}{U(M-N)\times U(N)}$$

- $W = \mathrm{tr}(\Phi_0 \Phi^T J' \Phi)$ on $G_{2N,N}$; Φ_0 is $N \times N$ matrix-valued and

$$J' = \begin{pmatrix} 0 & 1_N \\ \epsilon 1_N & 0 \end{pmatrix}, \qquad \epsilon = \pm 1$$

Integrating out V and Φ_0, we obtain sigma models on

$$K_{\text{compact}} = c \log\det\left(1_N + \overline{\varphi}\varphi\right)$$

$$\frac{Sp(N)}{U(N)} \quad (\epsilon = -1), \qquad \frac{SO(2N)}{U(N)} \quad (\epsilon = +1)$$

Apply this method to construct sigma models on noncompact CYs

$$\mathbb{R}_+ \times \frac{\boldsymbol{SU(N+1) \times U(1)_{\mathrm{D}}}}{\boldsymbol{SU(N) \times U(1)'}} \xrightarrow{\boldsymbol{U(1)_{\mathrm{D}}}} \frac{\boldsymbol{SU(N+1)}}{\boldsymbol{S[U(N) \times U(1)]}}$$

$$\mathbb{R}_+ \times \frac{\boldsymbol{SO(N+2) \times U(1)_{\mathrm{D}}}}{\boldsymbol{SO(N) \times U(1)'}} \xrightarrow{\boldsymbol{U(1)_{\mathrm{D}}}} \frac{\boldsymbol{SO(N+2)}}{\boldsymbol{SO(N) \times U(1)}}$$

$$\mathbb{R}_+ \times \frac{\boldsymbol{E_6 \times U(1)_{\mathrm{D}}}}{\boldsymbol{SO(10) \times U(1)'}} \xrightarrow{\boldsymbol{U(1)_{\mathrm{D}}}} \frac{\boldsymbol{E_6}}{\boldsymbol{SO(10) \times U(1)}}$$

$$\mathbb{R}_+ \times \frac{\boldsymbol{E_7 \times U(1)_{\mathrm{D}}}}{\boldsymbol{E_6 \times U(1)'}} \xrightarrow{\boldsymbol{U(1)_{\mathrm{D}}}} \frac{\boldsymbol{E_7}}{\boldsymbol{E_6 \times U(1)}}$$

$$(\mathbb{R}_+)^{\boldsymbol{N^2}} \times \frac{\boldsymbol{U(M)_{\mathrm{L}} \times U(N)_{\mathrm{R}}}}{\boldsymbol{U(M-N)_{\mathrm{L}} \times U(N)_{\mathrm{V}}}} \xrightarrow{\boldsymbol{U(N)_{\mathrm{R}}}} \frac{\boldsymbol{U(M)_{\mathrm{L}}}}{\boldsymbol{U(M-N)_{\mathrm{L}} \times U(N)_{\mathrm{L}}}}$$

$$(\mathbb{R}_+)^{\boldsymbol{N^2}} \times \frac{\boldsymbol{SO(2N)_{\mathrm{L}} \times U(N)_{\mathrm{R}}}}{\boldsymbol{U(N)_{\mathrm{V}}}} \xrightarrow{\boldsymbol{U(N)_{\mathrm{R}}}} \frac{\boldsymbol{SO(2N)_{\mathrm{L}}}}{\boldsymbol{U(N)_{\mathrm{L}}}}$$

$$(\mathbb{R}_+)^{\boldsymbol{N^2}} \times \frac{\boldsymbol{Sp(N)_{\mathrm{L}} \times U(N)_{\mathrm{R}}}}{\boldsymbol{U(N)_{\mathrm{V}}}} \xrightarrow{\boldsymbol{U(N)_{\mathrm{R}}}} \frac{\boldsymbol{Sp(N)_{\mathrm{L}}}}{\boldsymbol{U(N)_{\mathrm{L}}}}$$

SUSY sigma models on noncompact CYs

Here we construct $\mathcal{N} = (2, 2)$ SUSY sigma models on

$$\mathcal{M}_{\text{CY}} = \mathbb{C} \ltimes \bigotimes_{a=1}^{N} (G_a/H_a) \ , \qquad \text{each } G_a/H_a \text{ is an compact Einstein-Kähler}$$

The Kähler potential follows $\frac{\mathrm{d}}{\mathrm{d}X} K_{\text{CY}} = (\mathrm{e}^X + b)^{1/D}$

where D is the number of complex dimensions of $\mathcal{M}_{\text{CY}}$ and

$$X = \log|\sigma|^2 + \sum_{a=1}^{N} h_a \, K_{G_a/H_a}(\varphi)$$

$$z^\mu = \{\sigma, \varphi_a^i\} \ : \ \text{chiral superfields, coordinates on } (\mathbb{C}, \otimes_a (G_a/H_a))$$

$$h_a \ : \ \text{a real number, } \mathcal{R}_{i\bar{j}} = h_a \, g^a_{i\bar{j}}$$

We have already imposed the Ricci-flatness condition exressed by

$$\mathcal{R}_{\mu\bar{\nu}} = -\partial_\mu \partial_{\bar{\nu}} \log\det g_{\rho\bar{\lambda}} \equiv 0$$

(Notice that $\mathcal{R}_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log\det g_{k\bar{l}} = h \partial_i \partial_{\bar{j}} K_{\text{EK}} \to \det g_{k\bar{l}} = \mathrm{e}^{-hK_{\text{EK}}} |\text{hol.}|^2$)

Examples of G_a/H_a

- **Hermitian symmetric spaces:**

$$h_a = \frac{1}{2v_a} C_2(G_a) \,, \qquad v_a = r_a^{-2} : \quad r_a \text{ is a radius of } G_a/H_a$$

- a non-symmetric space: $G_a/H_a = \dfrac{SU(\ell+m+n)}{S[U(\ell)\times U(m)\times U(n)]} \left(\ni \dfrac{SU(3)}{U(1)^2} \right)$

$$\text{(I)} \qquad h_a = \frac{m+n}{v_\ell} = \frac{\ell+m}{v_m} = \frac{\ell+2m+n}{v_n} \qquad v_n = v_\ell + v_m$$

$$\text{(II)} \qquad h_a = \frac{\ell+n}{v_\ell} = \frac{\ell+m}{v_m} = \frac{2\ell+m+n}{v_n} \qquad v_n = v_\ell + v_m$$

(v_ℓ, v_m, v_n) : sizes of subspaces related to $(U(\ell), U(m), U(n))$

(There exist two inequivalent complex structures.)

Metrics on Noncompact CYs

line bundles	total dim. D	$h = \frac{1}{2v} C_2(G)$
$\mathbb{C} \ltimes \left(\mathbb{CP}^{N-1} = \frac{SU(N)}{SU(N-1)\times U(1)} \right)$	$1 + (N-1)$	N
$\mathbb{C} \ltimes \left(Q^{N-2} = \frac{SO(N)}{SO(N-2)\times U(1)} \right)$	$1 + (N-2)$	$N-2$
$\mathbb{C} \ltimes E_6/[SO(10)\times U(1)]$	$1+16$	12
$\mathbb{C} \ltimes E_7/[E_6\times U(1)]$	$1+27$	18
$\mathbb{C} \ltimes \left(\mathrm{G}_{M,N} = \frac{U(M)}{U(M-N)\times U(N)} \right)$	$1 + N(M-N)$	MN
$\mathbb{C} \ltimes SO(2N)/U(N)$	$1 + \frac{1}{2}N(N-1)$	$N(N-1)$
$\mathbb{C} \ltimes Sp(N)/U(N)$	$1 + \frac{1}{2}N(N+1)$	$N(N+1)$

$$\frac{\mathrm{d}}{\mathrm{d}X} K_{\text{CY}}(\sigma,\varphi) \;=\; \left(\mathrm{e}^X + b\right)^{1/D}, \qquad X \;=\; \log|\sigma|^2 + hK_{\text{compact}}(\varphi)$$

(ex.) $\quad K_{\text{compact}} = K_{G_{M,N}}(\varphi) = v \log\det\left(1_N + \overline{\varphi}\varphi\right)$