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Gauged Linear Sigma Models for Noncompact Calabi-Yau Varieties

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Two-dimensional field theory is a powerful framework

(toy model of χSB, mass gap, confinement, CFT, etc..)

– 1986: Nonlinear Realization in SUSY theories

by Bando, Kuramoto, Maskawa and Uehara; Itoh, Kugo and Kunitomo

– 1999: SUSY nonlinear sigma models on hermitian symmetric spaces

introducing an auxiliary gauge field

by Higashijima and Nitta

2001 – 2002: Ricci-flat metrics on noncompact Kähler manifolds ($\equiv$ noncompact Calabi-Yau's)

by Higashijima, Nitta and TK:

hep-th/0104184, 0107100, 0108084, 0110216, 0202064

Metrics on Noncompact Calabi-Yau

(K.Higashijima, M.Nitta and TK, 2001, 2002)

“Complex Line Bundle on Einstein-Kähler Coset G/H”

$$\frac{\mathrm{d}}{\mathrm{d}X}\mathcal{K}_{\mathrm{CY}}(\rho,\varphi) \;=\; \left(\mathrm{e}^{CX}+b\right)^{1/D} \qquad X \;=\; \log|\rho^{1/\ell}|^2 + K_{G/H}(\varphi)$$

C = dual Coxeter number of G $\qquad$ ℓ = some specific number

(example)

$$G/H : \frac{SU(N)}{SU(N-1)\times U(1)} = \mathbb{C}\mathrm{P}^{N-1}, \quad \frac{U(N)}{U(M)\times U(N-M)} = \mathrm{G}_{N,M}(\mathbb{C})$$

$$\frac{SO(N)}{SO(N-2)\times U(1)}, \quad \frac{SO(2N)}{U(N)}, \quad \frac{Sp(N)}{U(N)}, \quad \text{etc.}$$

These CYs can be candidates of string background

when we study gauge/gravity duality in type II string.

typical example: deformed/resolved conifold

CFT descriptions (Virasoro- and current-algebras)?

global aspects of noncompact geometries?

and

mirror geometries?

Gauged Linear Sigma Model and its T-duality

including

NLSM (geometry)

Landau-Ginzburg theory (chiral rings)

Gaugeded linear sigma model

E. Witten (1993), K. Hori and C. Vafa (2000)

$\mathcal{N} = (2,2)$ SUSY gauge theory with matters $(\mathrm{FI}: \ t \equiv r - i\theta)$

$$\mathscr{L} \ = \int \mathrm{d}^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_a \overline{\Phi}_a \, \mathrm{e}^{2Q_aV} \Phi_a \Big\} + \Big(\frac{1}{\sqrt{2}} \int \mathrm{d}^2\widetilde{\theta} \, (-\Sigma \, t) + c.c. \Big) + \Big(\int \mathrm{d}^2\theta \, W_{\mathrm{GLSM}}(\Phi_a) + c.c. \Big)$$

- Φ_a : charged chiral superfield, $\overline{D}_{\pm}\Phi_a = 0$
  Σ : twisted chiral superfield, $\overline{D}_+\Sigma = D_-\Sigma = 0$, $\Sigma = \frac{1}{\sqrt{2}}\overline{D}_+ D_- V$
- There exist at least two phases:

  $\mathrm{FI} \gg 0$: geometric phase $\longrightarrow$ SUSY NLSM

  $\mathrm{FI} \ll 0$: non-geometric phase $\longrightarrow$ LG, orbifold, SCFT

- CY/LG correspondence

  harmonic forms $\longleftrightarrow$ NS-NS chiral primary states

- "Mirror geometry" appears in the T-dual theory in terms of twisted chiral superfields Y_a

$$Y_a + \overline{Y}_a \ \equiv \ 2 \, \overline{\Phi}_a \, \mathrm{e}^{2Q_aV} \Phi_a$$

Effective theories

The potential energy density is given by

$$\mathcal{U}(\phi,\sigma) \;=\; \frac{e^2}{2}\mathcal{D}^2 + \sum_a |F_a|^2 + \mathcal{U}_\sigma(\phi,\sigma)$$

$$\mathcal{D} \;=\; \frac{1}{e^2}D \;=\; r - \sum_a Q_a|\phi_a|^2 \;, \quad \overline{F}_a \;=\; -\frac{\partial}{\partial\phi_a}W_{\mathrm{GLSM}}(\phi) \;, \quad \mathcal{U}_\sigma(\phi,\sigma) \;=\; 2|\sigma|^2\sum_a Q_a^2|\phi_a|^2$$

The supersymmetric vacuum manifold $\mathcal{M}$ is defined by

$$\boxed{\mathcal{M} \;=\; \Big\{(\phi_a,\sigma)\in\mathbb{C}^n \,\Big|\, \mathcal{D}=F_a=\mathcal{U}_\sigma=0\Big\}\Big/U(1)}$$

In the IR limit $e\to\infty$, there appears the SUSY NLSM on $\mathcal{M}$ whose coupling is

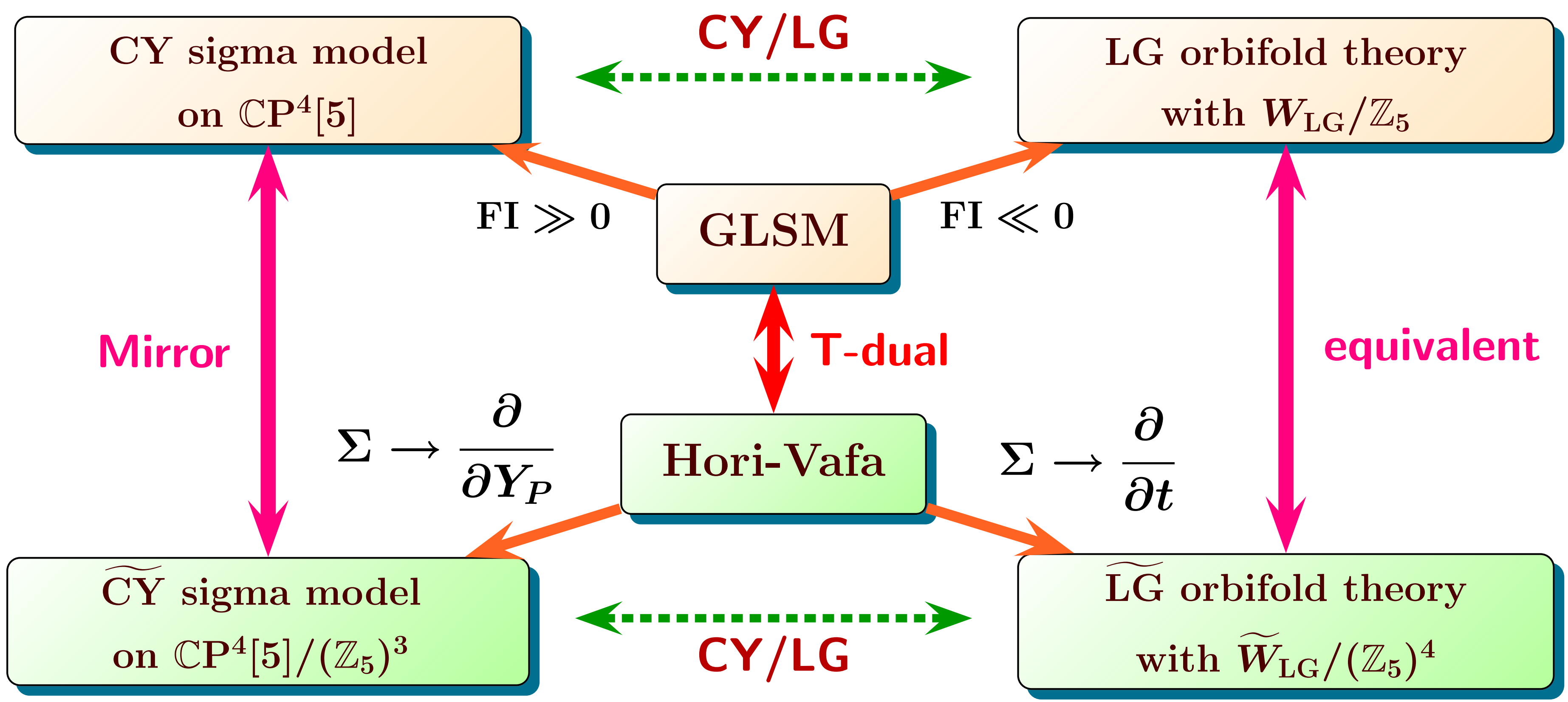
radius of Kähler manifold $\longrightarrow$ $\quad r \;=\; \dfrac{1}{g^2} \quad$ $\longleftarrow$ coupling constant of NLSM

Renormalization of the FI parameter is $\quad r_0 \;=\; r_R + s\cdot\log\left(\dfrac{\Lambda_{\mathrm{UV}}}{\mu}\right), \quad s \;=\; \sum_a Q_a$

Thus we find that

$s>0$	$\longrightarrow$	the theory is asymptotic free
$s=0$	$\longrightarrow$	the theory is conformal
$s<0$	$\longrightarrow$	the theory is infrared free

Example: quintic hypersurface



— configuration —

$h_{21}(\mathbb{C}P^4[5]) = h_{11}(\mathbb{C}P^4[5]/(\mathbb{Z}_5)^3) = 101$ $\qquad$ $h_{11}(\mathbb{C}P^4[5]) = h_{21}(\mathbb{C}P^4[5]/(\mathbb{Z}_5)^3) = 1$

Gauged Linear Sigma Model for $\mathcal{O}(-N+\ell)$ bundle on $\mathbb{CP}^{N-1}[\ell]$

chiral superfield	S_1	$\cdots$	S_N	P_1	P_2
$U(1)$ charge	1	$\cdots$	1	$-\ell$	$-N+\ell$

$$W_{\rm GLSM} = P_1 \cdot G_\ell(S_i)$$

$G_\ell(S_i)$: homogeneous polynomial of degree ℓ

Transverse condition:

if $G_\ell(s) = \partial_1 G_\ell(s) = \cdots = \partial_N G_\ell(s) = 0$, then ${}^\forall s_i = 0$

Potential energy density:

$$\mathcal{U} = \frac{e^2}{2}\mathcal{D}^2 + \big|G_\ell(s)\big|^2 + \sum_{i=1}^N \big|p_1 \partial_i G_\ell(s)\big|^2 + \mathcal{U}_\sigma$$

$$\mathcal{D} = r - \sum_{i=1}^N |s_i|^2 + \ell|p_1|^2 + (N-\ell)|p_2|^2 \,, \quad \mathcal{U}_\sigma = 2|\sigma|^2\Big\{ \sum_{i=1}^N |s_i|^2 + \ell^2|p_1|^2 + (N-\ell)^2|p_2|^2 \Big\}$$

Let us analyze SUSY vacuum manifold $\mathcal{U} = 0$ and massless effective theories

$r \gg 0$ region: CY phase appears

$$\mathcal{M}_{\rm CY} = \Big\{ (s_i; p_2) \in \mathbb{C}^{N+1} \, \Big| \, r = \sum_{i=1}^{N} |s_i|^2 - (N-\ell)|p_2|^2 > 0, \quad G_\ell(s_i) = 0 \Big\} \Big/ U(1)$$

➡ $\mathcal{O}(-N+\ell)$ **bundle on** $\mathbb{C}\mathrm{P}^{N-1}[\ell]$

$r \ll 0$ region: orbifold phase and "new" phase appear

$$\begin{aligned} \mathcal{M}_{r<0} &= \Big\{ (p_1, p_2; s_i) \in \mathbb{C}^{N+2} \, \Big| \, \mathcal{D} = G_\ell(S_i) = p_1 \partial_i G_\ell = 0, \ r < 0 \Big\} \Big/ U(1) \\ &= \mathcal{M}^1_{r<0} \cup \mathcal{M}^2_{r<0} \end{aligned}$$

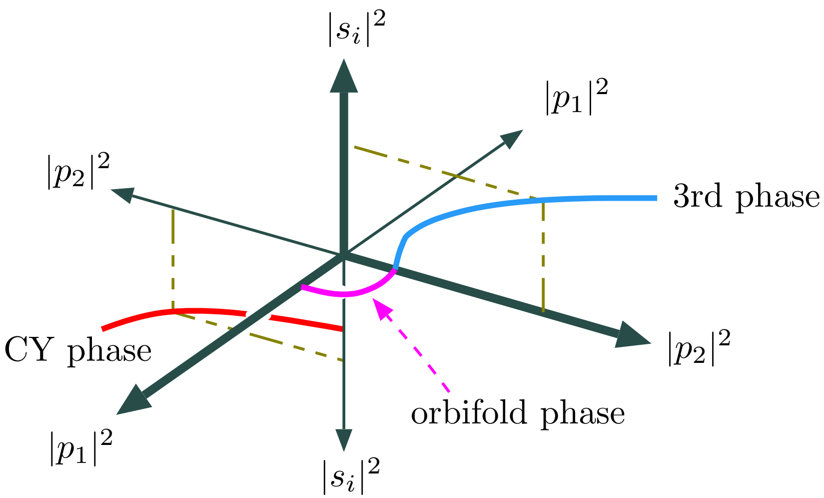
$$\mathcal{M}^1_{r<0} := \Big\{ (p_1, p_2) \in \mathbb{C}^2 \, \Big| \, r = -\ell |p_1|^2 - (N-\ell)|p_2|^2 < 0 \Big\} \Big/ U(1) = \mathrm{W}\mathbb{C}\mathrm{P}^1_{\ell, N-\ell}$$

$$\mathcal{M}^2_{r<0} := \Big\{ (p_2; s_i) \in \mathbb{C}^{N+1} \, \Big| \, r = \sum_i |s_i|^2 - (N-\ell)|p_2|^2 < 0, \ G_\ell = 0 \Big\} \Big/ U(1)$$

The property of $G_\ell(s)$ decomposes $\mathcal{M}_{r<0}$ into two parts!!

$$\left(\begin{array}{l} \mathcal{M}^1_{r<0} : \ G_\ell(s) \text{ is trivially zero} \\ \mathcal{M}^2_{r<0} : \ \text{the non-zero value } \partial^{(\ell)} G_\ell(s)\big|_{\rm VEV} \neq 0 \text{ exists} \end{array} \right)$$

Schematic illustration on effective theories



- **CY phase on $\mathcal{M}_{\rm CY}$**

  conformal sigma model on $\mathcal{M}_{\rm CY}$

- **orbifold phase on $\mathcal{M}^1_{r<0}$ (two "LG"s appear)**

  $\{$CFT on $\mathbb{C}^1 \otimes$ LG with $W_{\rm LG} = \langle p_1 \rangle G_\ell(S)\}/\mathbb{Z}_\ell$

  $\{$"LG" with $W_{\rm LG} = P_1 \cdot G_\ell(S)\}/\mathbb{Z}_{N-\ell}$

- **3rd phase on $\mathcal{M}^2_{r<0}$ NEW!**

  conformal sigma model on $\mathcal{M}^2_{r<0}$

$$\mathcal{M}_{\rm CY} = \Big\{(s_i; p_2) \in \mathbb{C}^{N+2} \Big| \mathcal{D} = G_\ell = 0,\ r > 0\Big\}\Big/U(1) \equiv \mathcal{O}(-N+\ell) \text{ bundle on } \mathbb{CP}^{N-1}[\ell]$$

$$\mathcal{M}^1_{r<0} = \Big\{(p_1, p_2) \in \mathbb{C}^2 \Big| \mathcal{D} = 0, r < 0\Big\}\Big/U(1) \equiv \mathrm{W}\mathbb{CP}^1_{\ell, N-\ell}$$

$$\mathcal{M}^2_{r<0} = \Big\{(s_i; p_2) \in \mathbb{C}^{N+2} \Big| \mathcal{D} = G_\ell = 0,\ r < 0\Big\}\Big/U(1)$$

Caution!: These effective theories are just approximately described because we do not integrate out but ignore all massive mode.

T-dual description of the GLSM is also powerful

to investigate low energy theories.

Analyzing them

we will re-investigate the massless effective theories in the original GLSM.

T-dual Theory

$$\mathscr{L} = \int d^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma - \sum_a \Big(\frac{1}{2}(Y_a + \overline{Y}_a)\log(Y_a + \overline{Y}_a)\Big)\Big\} + \Big(\frac{1}{\sqrt{2}}\int d^2\widetilde{\theta}\,\widetilde{W} + c.c.\Big)$$

$$\widetilde{W} = \Sigma\Big(\sum_{i=1}^N Y_i - \ell Y_{P_1} - (N-\ell)Y_{P_2} - t\Big) + \sum_{i=1}^N e^{-Y_i} + e^{-Y_{P_1}} + e^{-Y_{P_2}}$$

We often use the following functional integral:

Period integral : $\widehat{\Pi} \equiv \int d\Sigma \prod_{i=1}^N dY_i\, dY_{P_1}\, dY_{P_2}\, (\ell\Sigma)\, \exp\big(-\widetilde{W}\big)$

roughly, this functional is a "partition function of topological theory"

chiral superfield	S_1	S_2	$\cdots$	S_N	P_1	P_2
$U(1)$ charge	1	1	$\cdots$	1	$-\ell$	$-N+\ell$
twisted chiral	Y_1	Y_2	$\cdots$	Y_N	Y_{P_1}	Y_{P_2}

where, $2\,\overline{\Phi}_a\, e^{2Q_aV}\Phi_a = Y_a + \overline{Y}_a$

$U(1)$ **phase rotation symmetry on** Φ_a ➡ **shift symmetry on** Y_a: $Y_a \equiv Y_a + 2\pi i$

▼ Twisted Landau-Ginzburg theory: $\Sigma \to \dfrac{\partial}{\partial t}$

$$\widehat{\Pi} \;=\; \ell\frac{\partial}{\partial t}\int \prod_{i=1}^{N} \mathrm{d}Y_i \mathrm{d}Y_{P_1} \mathrm{d}Y_{P_2}\, \delta\Big(\sum_i Y_i - \ell Y_{P_1} - (N-\ell) Y_{P_2} - t\Big) \exp\Big(-\sum_i \mathrm{e}^{-Y_i} - \mathrm{e}^{-Y_{P_1}} - \mathrm{e}^{-Y_{P_2}}\Big)$$

Solve the δ-functional in terms of Y_{P_1}:

$$Y_{P_1} \;=\; \frac{1}{\ell}\Big\{t - \sum_{i=1}^{N} Y_i + (N-\ell) Y_{P_2}\Big\}$$

Field re-definition preserving canonical measure in $\widehat{\Pi}$ (to avoid anomaly):

$$X_i \;\equiv\; \mathrm{e}^{-\frac{1}{\ell}Y_i}\;, \quad X_{P_2} \;\equiv\; \mathrm{e}^{\frac{N-\ell}{\ell}Y_{P_2}}\;, \quad X_i \;\to\; \omega_i X_i\;, \quad X_{P_2} \;\to\; \omega_{P_2} X_{P_2}\;, \quad (\mathbb{Z}_\ell)^N \text{ symmetry}$$

Thus we obtain the twisted LG superpotential:

$$\boxed{\Big\{\widetilde{W}_\ell \;=\; X_1^\ell + \cdots + X_N^\ell + X_{P_2}^{-\frac{\ell}{N-\ell}} + \mathrm{e}^{t/\ell} X_1 \cdots X_N X_{P_2}\Big\}\Big/(\mathbb{Z}_\ell)^N}$$

The negative power term describes $\mathcal{N}=2$ Kazama-Suzuki model on $SL(2,\mathbb{R})_k/U(1)$:

$$\frac{\ell}{N-\ell} \;=\; k$$

Thus we argue that

this effective theory is the LG minimal model coupled to the KS model with $(\mathbb{Z}_\ell)^N$ orbifold symmetry

Return to the original GLSM

Recall the following two arguments:

- $\mathcal{N}=2$ SCFT on $SL(2,\mathbb{R})_k/U(1)$ is equivalent to $\mathcal{N}=2$ Liouville theory via T-duality: $k=\dfrac{2}{Q^2}$
- If a CFT $\mathcal{C}$ has an abelian discrete symmetry group Γ, the orbifold CFT $\mathcal{C}'=\mathcal{C}/\Gamma$ has a symmetry group Γ' which is isomorphic to Γ. Furthermore a new orbifold CFT $\mathcal{C}'/\Gamma'$ is identical to the original CFT $\mathcal{C}$.

Thus we insist that

> "$\{$CFT on $\mathbb{C}^1\otimes$ LG with $W_{\rm LG}=\langle p_1\rangle G_\ell(S)\}/\mathbb{Z}_\ell$" in the original GLSM
>
> is described by
>
> $\{\mathcal{N}=2$ Liouville theory coupled to the LG minimal model with $W_{\rm LG}\}/\mathbb{Z}_\ell$
>
> as an exact effective theory

Summary

- We found three non-trivial phases and effective theories in the GLSM

  CY sigma models

  orbifolded LG theories coulped to 1-dim. SCFT

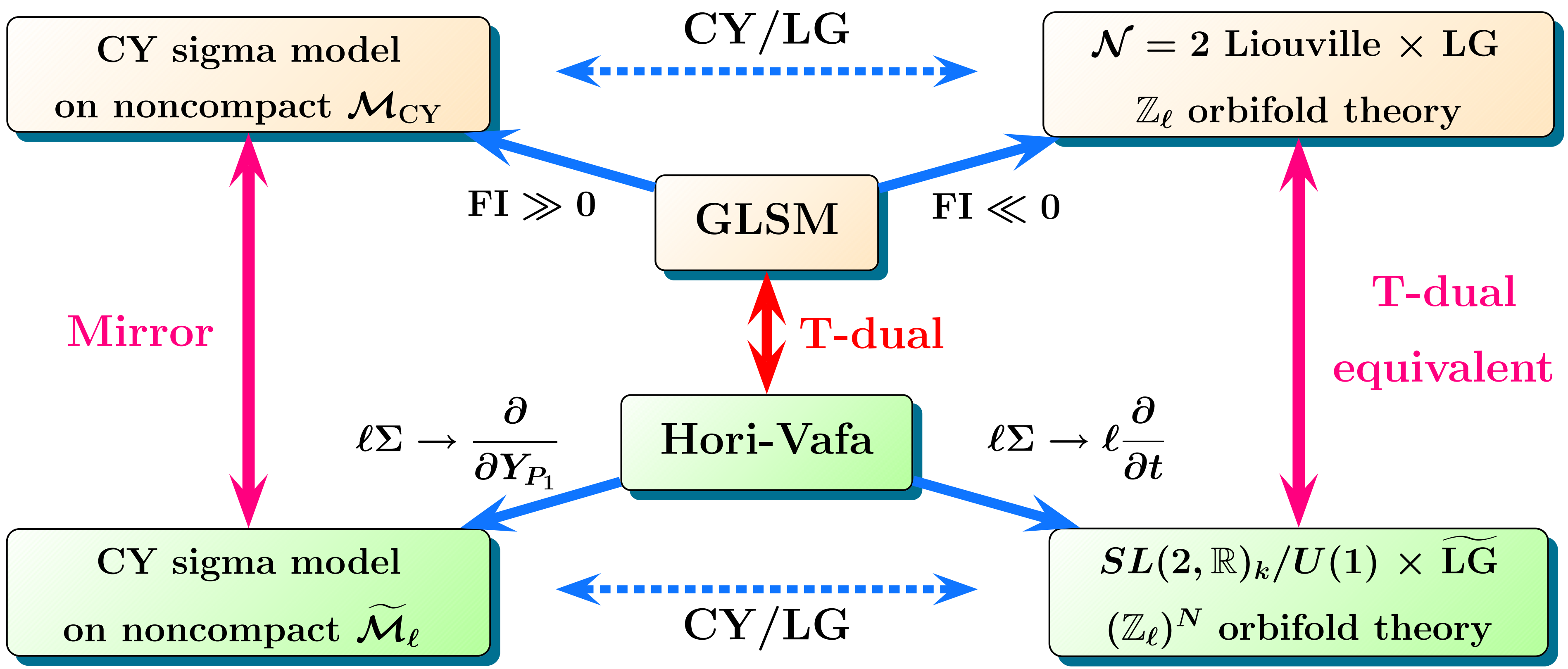
- We constructed exact effective theories in the T-dual theory

  orbifolded LG theories including a term with negative power $-k$

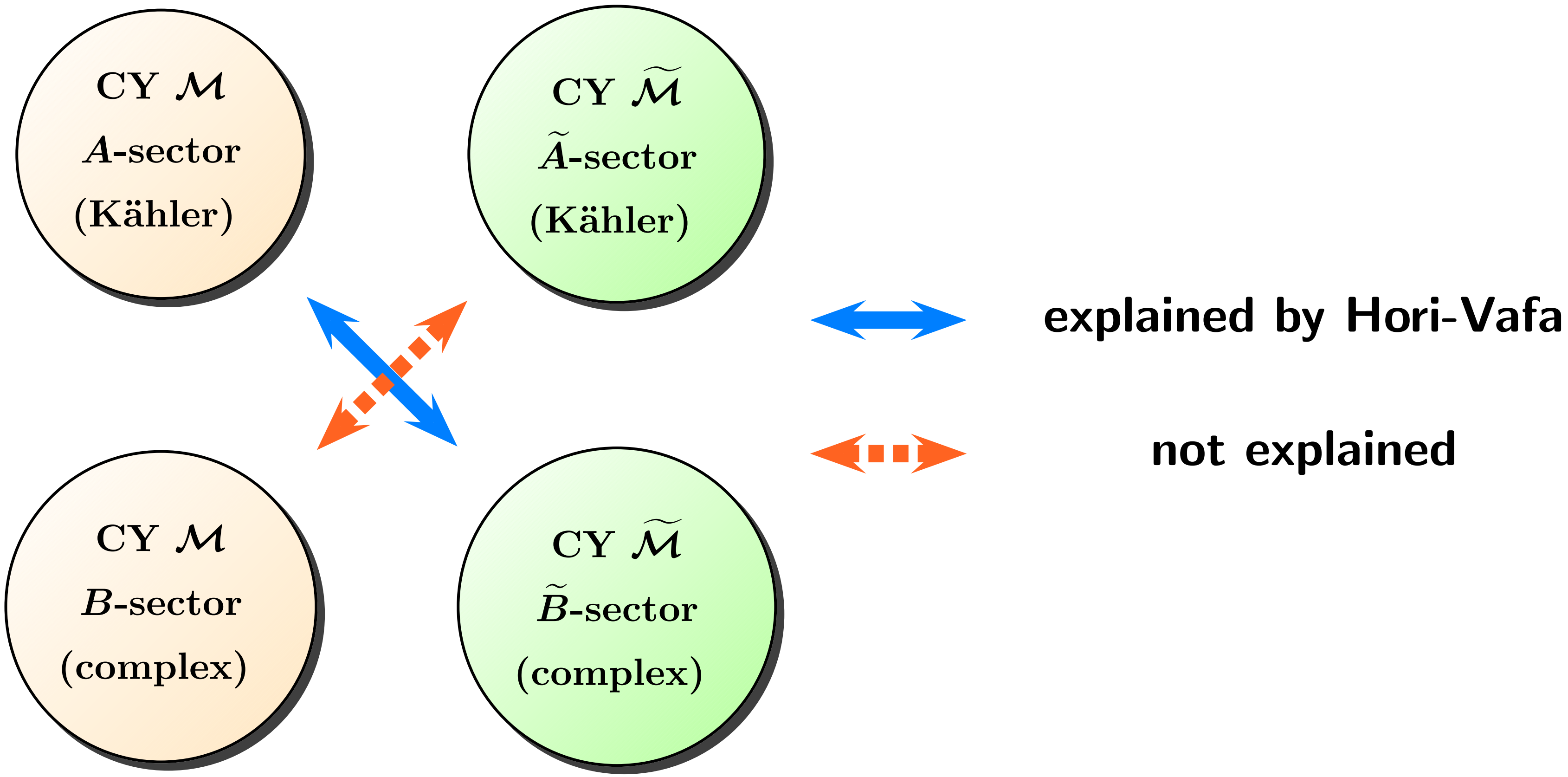
  This term represents a gauged WZW model on $SL(2,\mathbb{R})_k/U(1)$ at level k

  NLSMs on mirror CY geometries

- We argue that the LG theories in the original GLSM can be interpreted as $\mathcal{N}=2$ Liouville theories coupled to LG minimal models

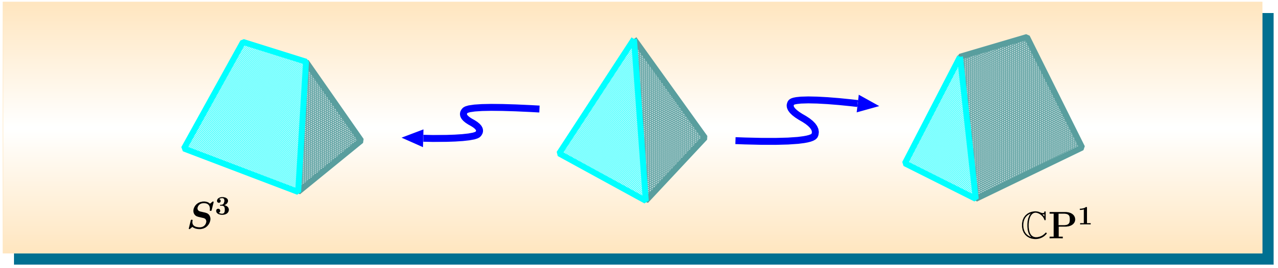

CY sigma model on noncompact $\mathcal{M}_{\rm CY}$
CY/LG
$\mathcal{N}=2$ Liouville $\times$ LG $\mathbb{Z}_\ell$ orbifold theory
FI $\gg 0$
GLSM
FI $\ll 0$
Mirror
T-dual
T-dual equivalent
$\ell\Sigma \to \frac{\partial}{\partial Y_{P_1}}$
Hori-Vafa
$\ell\Sigma \to \ell\frac{\partial}{\partial t}$
CY sigma model on noncompact $\widetilde{\mathcal{M}}_\ell$
$SL(2,\mathbb{R})_k/U(1) \times \widetilde{\rm LG}$ $(\mathbb{Z}_\ell)^N$ orbifold theory
CY/LG

Discussions



- Hori-Vafa's T-dual theory is only valid when we consider the GLSM without a superpotential or with a superpotential given simply by a homogeneous polynomial such as $W_{\rm GLSM} = P \cdot G_{\ell}(S)$. Even though the polynomial $G_{\ell}(S)$ has an additional symmetry, the period integral $\widehat{\Pi}$ cannot recognize the existence of this additional symmetry. Thus the T-dual theory does not map all structures of the CY $\mathcal{M}$ to the mirror geometry completely.

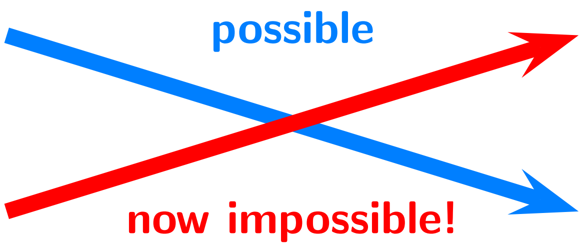
Example: resolved/deformed conifold



- deformed conifold: deformation of complex moduli
- resolved conifold: deformation of Kähler moduli

GLSM for resolved conifold

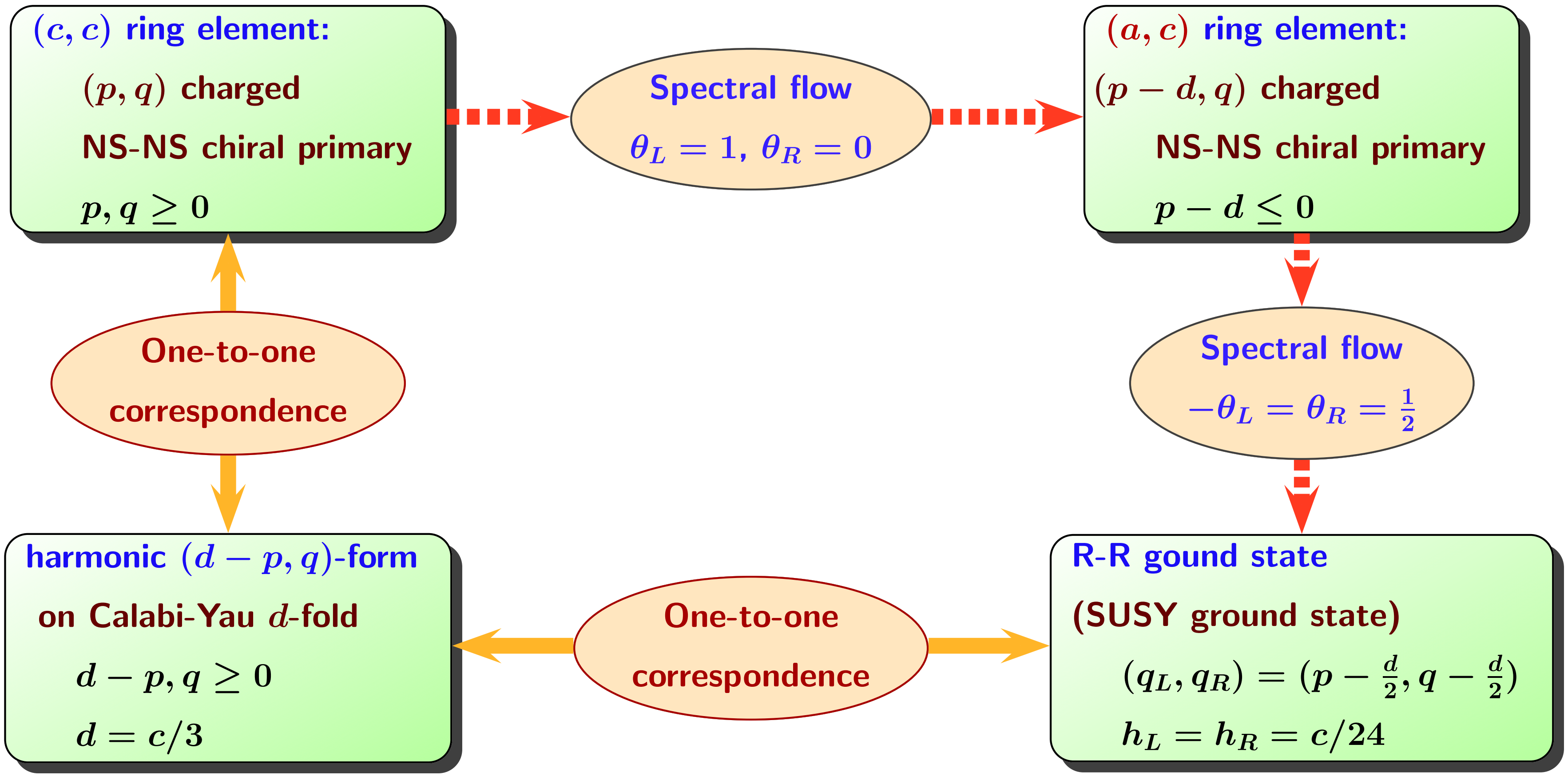
GLSM for deformed conifold



Hori-Vafa's theory for resolved conifold

Hori-Vafa's theory for deformed conifold

$\mathcal{N} = (2,2)$ "LG" SCFT and (compact) Calabi-Yau geometry



I would like to apply the above strategy to "noncompact" CY varieties

Now we ignore precise definitions of "topological charges" (normalizability etc.)

Massless effective theory on $\mathcal{M}^1_{r<0}$:

$$\mathcal{N} = (2,2) \text{ supersymmetric NLSM on } \mathrm{W}\mathbb{C}\mathrm{P}^1_{\ell,N-\ell}$$

$$\text{coupled to ``LG'' theory with } \Big\{W_{\rm LG} \;=\; (\langle p_1\rangle + P_1)G_\ell(S)\Big\}\Big/\mathbb{Z}_\alpha \qquad ([1])$$

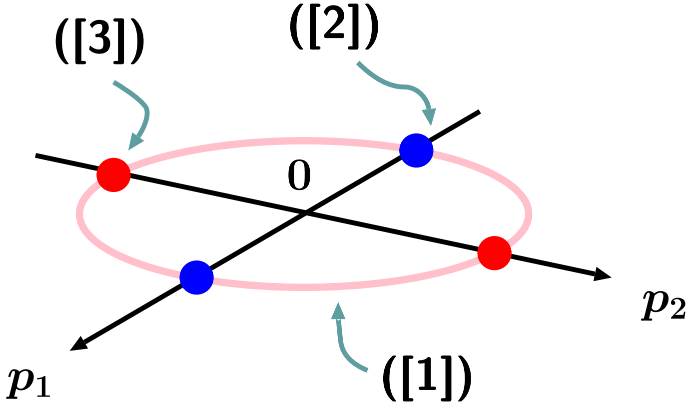
$$(\alpha \;=\; \mathrm{GCM}\{\ell, N-\ell\})$$

Especially on the point $(p_1, p_2) = (*, 0) \in \mathcal{M}^1_{r<0}$, the theory looks like

$$\boxed{\Big\{\text{CFT on } \mathbb{C}^1 \;\otimes\; \text{LG theory with } W_{\rm LG} = \langle p_1\rangle G_\ell(S)\Big\}\Big/\mathbb{Z}_\ell} \;\; \textbf{(?)} \qquad ([2])$$

and on the point $(p_1, p_2) = (0, *) \in \mathcal{M}^1_{r<0}$, the theory looks like

$$\boxed{\Big\{\text{LG theory with } W_{\rm LG} = P_1 \cdot G_\ell(S) \;\text{ on } \mathbb{C}^{N+1}\Big\}\Big/\mathbb{Z}_{N-\ell}} \;\; \textbf{(?)} \qquad ([3])$$



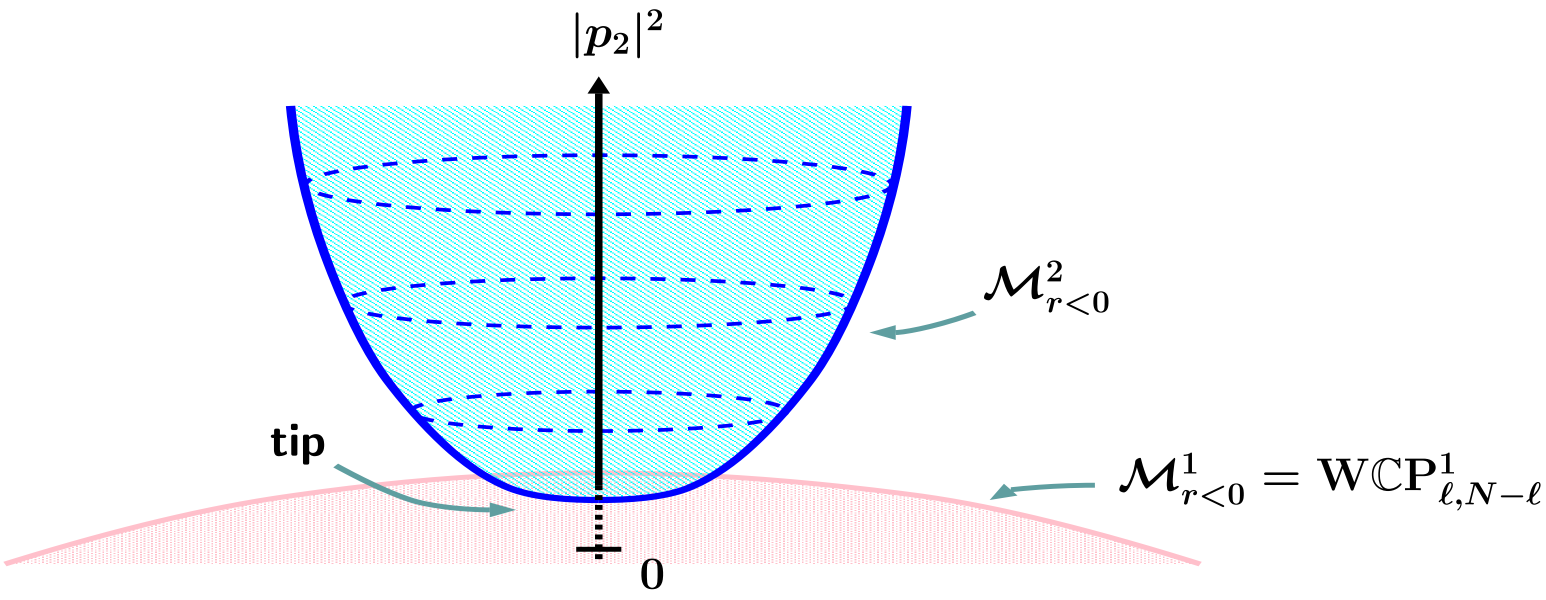
Massless effective theory on $\mathcal{M}^2_{r<0}$:

Conformal $\mathcal{N} = (2,2)$ SNLSM on $\mathcal{M}^2_{r<0}$

3rd phase!

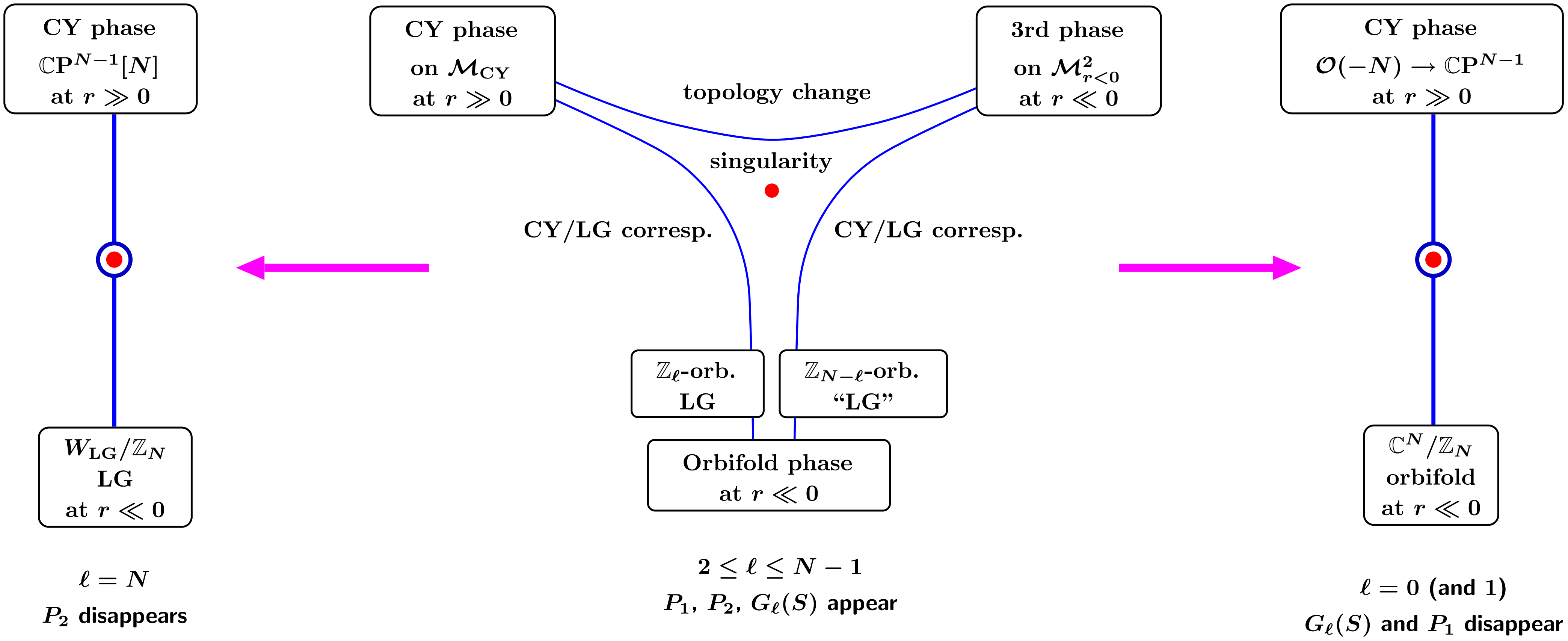
- massless fluctuation modes $(\widetilde{S}_1, \cdots, \widetilde{S}_N; \widetilde{P}_2)$ live in $\mathcal{M}^2_{r<0}$
- $U(1)$ gauge symmetry is completely broken,

  except for the tip $(\langle s_1 \rangle, \cdots, \langle s_N \rangle; \langle p_2 \rangle) = (0, \cdots, 0; *)$ in $\mathcal{M}^2_{r<0}$: partially restored to $\mathbb{Z}_{N-\ell}$
- On the tip, $\mathcal{M}^2_{r<0}$ connects to $\mathcal{M}^1_{r<0} = \mathrm{W}\mathbb{C}\mathrm{P}^1_{\ell,N-\ell}$ with phase transition



Relations: CY/LG correspondence and topology change

The four theories are related to each other via CY/LG correspondence and topology change:



We also notice that we have obtained various massless effective theories by decomposing (not by integrating out) all massive modes. Thus they are just approximate descriptions.

Derivation of the T-dual Lagrangian

Here we briefly review the T-duality of a generic GLSM without any superpotentials. We start from

$$\mathscr{L}' = \int d^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_a \Big(e^{2Q_a V + B_a} - \frac{1}{2}(Y_a + \overline{Y}_a) B_a \Big) \Big\} + \Big(\frac{1}{\sqrt{2}} \int d^2\widetilde{\theta}\, (-\Sigma\, t) + (c.c.) \Big) , \tag{1}$$

where Y_a and B_a are twisted chiral superfields and a real superfields B_a.

Integrating out twisted chiral superfields Y_a, we obtain $\overline{D}_+ D_- B_a = D_+ \overline{D}_- B_a = 0$, whose solutions are written in terms of chiral superfields Ψ_a and $\overline{\Psi}_a$ such as $B_a = \Psi_a + \overline{\Psi}_a$. When we substitute them into (1), $\mathscr{L}_{\rm GLSM}$ appears:

$$\mathscr{L}'\Big|_{B_a = \Psi_a + \overline{\Psi}_a} = \int d^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_a \overline{\Phi}_a\, e^{2Q_a V} \Phi_a \Big\} + \Big(\frac{1}{\sqrt{2}} \int d^2\widetilde{\theta}\, (-\Sigma\, t) + (c.c.) \Big) \equiv \mathscr{L}_{\rm GLSM} , \tag{2}$$

where we re-wrote $\Phi_a = e^{\Psi_a}$.

On the other hand, when we first integrate out B_a in $\mathscr{L}'$, we obtain $B_a = -2Q_a V + \log\Big(\dfrac{Y_a + \overline{Y}_a}{2}\Big)$. Let us insert these solutions into (1). By using a deformation $\int d^4\theta\, Q_a V Y_a = -\dfrac{Q_a}{2} \int d^2\widetilde{\theta}\, \overline{D}_+ D_- V Y_a = -\dfrac{Q_a}{\sqrt{2}} \int d^2\widetilde{\theta}\, \Sigma Y_a$, we find that a Lagrangian of twisted chiral superfields appears:

$$\begin{aligned} \mathscr{L}_{\rm T} &= \int d^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma - \sum_a \Big(\frac{1}{2}(Y_a + \overline{Y}_a) \log(Y_a + \overline{Y}_a) \Big) \Big\} + \Big(\frac{1}{\sqrt{2}} \int d^2\widetilde{\theta}\, \widetilde{W} + (c.c.) \Big) , \\ \widetilde{W} &= \Sigma\Big(\sum_a Q_a Y_a - t \Big) + \mu \sum_a e^{-Y_a} . \end{aligned}$$

Notice that the twisted superpotential $\widetilde{W}$ is corrected by instanton effects where the instantons are the vortices of the gauge theory. In attempt to analyze a model satisfying $\sum_a Q_a = 0$, the scale parameter μ is omitted by field re-definitions.

Derivation of the twisted geometry

Let us study how to obtain the geometry with $\mathbb{Z}_\ell$-type orbifold symmetry. Replacing $\ell\Sigma$ in $\widehat{\Pi}$ to

$$\ell\Sigma \;\to\; \frac{\partial}{\partial Y_{P_1}} \,,$$

we can perform the integration of Σ and obtain

$$\widehat{\Pi} \;=\; \int \prod_{i=1}^{N} dY_i \, (e^{-Y_{P_1}} dY_{P_1}) \, dY_{P_2} \, \delta\Big(\sum_i Y_i - \ell Y_{P_1} - (N-\ell) Y_{P_2} - t \Big) \exp\Big(- \sum_i e^{-Y_i} - e^{-Y_{P_1}} - e^{-Y_{P_2}} \Big) \,. \tag{3}$$

We perform the re-definitions of the variables Y_i, Y_{P_1} and Y_{P_2}:

$$e^{-Y_{P_1}} \;=\; \widetilde{P}_1 \,, \quad e^{-Y_a} \;=\; \widetilde{P}_1 U_a \quad \text{for } a = 1, \ldots, \ell \,, \qquad e^{-Y_{P_2}} \;=\; \widetilde{P}_2 \,, \quad e^{-Y_b} \;=\; \widetilde{P}_2 U_b \quad \text{for } b = \ell+1, \ldots, N \,.$$

Substituting these re-defined variables into (3), we continue the calculation:

$$\begin{aligned}
\widehat{\Pi} \;&=\; \int \prod_{i=1}^{N} \Big(\frac{dU_i}{U_i} \Big) \, d\widetilde{P}_1 \Big(\frac{d\widetilde{P}_2}{\widetilde{P}_2} \Big) \, \delta\Big(\log \Big(\prod_i U_i \Big) + t \Big) \, \exp\Big\{ - \widetilde{P}_1 \Big(\sum_{a=1}^{\ell} U_a + 1 \Big) - \widetilde{P}_2 \Big(\sum_{b=\ell+1}^{N} U_b + 1 \Big) \Big\} \\
&=\; \int \prod_i \Big(\frac{dU_i}{U_i} \Big) \, d\widetilde{P}_2 \, du \, dv \, \delta\Big(\log \Big(\prod_i U_i \Big) + t \Big) \, \delta\Big(\sum_a U_a + 1 \Big) \, \exp\Big\{ - \widetilde{P}_2 \Big(\sum_b U_b + 1 - uv \Big) \Big\} \\
&=\; \int \prod_i \Big(\frac{dU_i}{U_i} \Big) \, du \, dv \, \delta\Big(\log \Big(\prod_i U_i \Big) + t \Big) \, \delta\Big(\sum_a U_a + 1 \Big) \, \delta\Big(\sum_b U_b + 1 - uv \Big) \,,
\end{aligned} \tag{4}$$

where we introduced new variables u and v taking values in $\mathbb{C}$ and used a following equation

$$\frac{1}{\widetilde{P}_2} \;=\; \int du \, dv \, \exp\big(\widetilde{P}_2 \, uv \big) \,.$$

It is obvious that (4) still includes a non-canonical integral measure. Thus we perform further re-definitions such as

$$U_a \;=\; \mathrm{e}^{-t/\ell} \frac{Z_a^\ell}{Z_1 \cdots Z_N} \;, \qquad U_b \;=\; Z_b^\ell \;.$$

Note that the period integral (4) is invariant under the following transformations acting on the new variables Z_i:

$$Z_a \;\mapsto\; \lambda\, \omega_a\, Z_a \;, \quad Z_b \;\mapsto\; \omega_b\, Z_b \;, \qquad \omega_a^\ell \;=\; \omega_b^\ell \;=\; \omega_1 \cdots \omega_N \;=\; 1 \;,$$

where λ is an arbitrary number taking in $\mathbb{C}^*$. The ω_i come from the shift symmetry of the original variables $Y_i \equiv Y_i + 2\pi i$. Combining these transformations we find that $\widehat{\Pi}$ has $\mathbb{C}^* \times (\mathbb{Z}_\ell)^{N-2}$ symmetries. Substituting Z_i into (4), we obtain

$$\widehat{\Pi} \;=\; \int \frac{1}{\text{vol.}(\mathbb{C}^*)} \prod_{i=1}^N \mathrm{d}Z_i \, \mathrm{d}u \, \mathrm{d}v \, \delta\Big(\sum_{a=1}^\ell Z_a^\ell + \mathrm{e}^{t/\ell} Z_1 \cdots Z_N \Big) \, \delta\Big(\sum_{b=\ell+1}^N Z_b^\ell + 1 - uv \Big) \;,$$

which indicates that the resulting mirror geometry is described by

$$\widetilde{\mathcal{M}}_\ell \;=\; \Big\{ (Z_i; u, v) \in \mathbb{C}^{N+2} \Big| \, \big\{ \mathcal{F}(Z_i) \;=\; 0 \big\} / \mathbb{C}^* \;, \;\; \mathcal{G}(Z_b; u, v) \;=\; 0 \Big\} \Big/ (\mathbb{Z}_\ell)^{N-2} \;,$$

$$\mathcal{F}(Z_i) \;=\; \sum_{a=1}^\ell Z_a^\ell + \psi Z_1 \cdots Z_\ell \;, \qquad \mathcal{G}(Z_b; u, v) \;=\; \sum_{b=\ell+1}^N Z_b^\ell + 1 - uv \;, \qquad \psi \;=\; \mathrm{e}^{t/\ell} Z_{\ell+1} \cdots Z_N \;.$$

This is an $(N-1)$-dimensional complex manifold.

The equation $\mathcal{F}(Z_i) = 0$ denotes that the complex variables Z_a consist of the degree ℓ hypersurface in the projective space: $\mathbb{C}\mathrm{P}^{\ell-1}[\ell]$. This subspace itself is a compact CY manifold, which is parametrized by a parameter ψ which is subject to the equation $\mathcal{G}(Z_b; u, v) = 0$. Moreover we can also interpret that the total space is a noncompact CY manifold whose compact directions are described by Z_i, while the variables u and v run in the noncompact directions under the equations.

Twisted mirror geometry:

We replace $\ell\Sigma$ to $\dfrac{\partial}{\partial Y_{P_1}}$ and obtain

$$\widehat{\Pi} = \int \prod_{i=1}^{N} \mathrm{d}Y_i \, (\mathrm{e}^{-Y_{P_1}} \mathrm{d}Y_{P_1}) \, \mathrm{d}Y_{P_2} \, \delta\Big(\sum_i Y_i - \ell Y_{P_1} - (N-\ell) Y_{P_2} - t \Big) \exp\Big(- \sum_i \mathrm{e}^{-Y_i} - \mathrm{e}^{-Y_{P_1}} - \mathrm{e}^{-Y_{P_2}} \Big)$$

Re-defining the variables in order to obtain the canonical measure (to avoid anomaly):

$$\mathrm{e}^{-Y_{P_1}} = \widetilde{P}_1 \, , \quad \mathrm{e}^{-Y_a} = \mathrm{e}^{-t/\ell} \frac{\widetilde{P}_1 Z_a^\ell}{Z_1 \cdots Z_N} \, , \qquad \mathrm{e}^{-Y_{P_2}} = \widetilde{P}_2 \, , \quad \mathrm{e}^{-Y_b} = \widetilde{P}_2 Z_b^\ell$$

We obtain

$$\widetilde{\mathcal{M}}_\ell = \Big\{ \{ \mathcal{F}(Z_i) = 0 \} / \mathbb{C}^* \, , \, \mathcal{G}(Z_b; u, v) = 0 \Big\} \Big/ (\mathbb{Z}_\ell)^{N-2}$$

$$\mathcal{F}(Z_i) = Z_1^\ell + \cdots + Z_\ell^\ell + \psi Z_1 \cdots Z_\ell \, , \quad \psi = \mathrm{e}^{t/\ell} Z_{\ell+1} \cdots Z_N$$

$$\mathcal{G}(Z_b; u, v) = Z_{\ell+1}^\ell + \cdots + Z_N^\ell + 1 - uv$$

$Z_a \mapsto \lambda \, \omega_a \, Z_a$ for $a = 1, \cdots, \ell$ (homogeneous coordinates of $\mathbb{C}\mathrm{P}^{\ell-1}[\ell]$)

$Z_b \mapsto \omega_b \, Z_b$ for $b = \ell+1, \cdots, N$ (homogeneous coordinates of $\mathbb{C}^{N-\ell}$)

$$\omega_a^\ell = \omega_b^\ell = \omega_1 \cdots \omega_N = 1 \, , \qquad \lambda : \mathbb{C}^*\text{-value}$$

$(\mathbb{Z}_{N-\ell})^*$-orbifolded LG theory

Solve Y_{P_2} by using the constraint derived from integrating out Σ:

$$Y_{P_2} = \frac{1}{N-\ell}\Big\{t - \sum_{i=1}^{N} Y_i + \ell Y_{P_1}\Big\}$$

Field re-definition preserving canonical measure in $\widehat{\Pi}$:

$$X_i \equiv e^{-\frac{1}{N-\ell}Y_i} \,, \quad X_{P_1} \equiv e^{\frac{\ell}{N-\ell}Y_{P_1}} \,, \quad X_i \to \omega_i X_i \,, \quad X_{P_1} \to \omega_{P_1} X_{P_1} \,, \quad (\mathbb{Z}_{N-\ell})^N \text{ symmetry}$$

Thus we obtain the twisted LG superpotential:

$$\Big\{\widetilde{W}_{N-\ell} = X_1^{N-\ell} + \cdots + X_N^{N-\ell} + X_{P_1}^{-\frac{N-\ell}{\ell}} + e^{t/\ell} X_1 \cdots X_N X_{P_1}\Big\} \Big/ (\mathbb{Z}_{N-\ell})^N$$

negative power term = interpreted as $\mathcal{N}=2$ Kazama-Suzuki model on $SL(2,\mathbb{R})_k/U(1)$:

$$\frac{N-\ell}{\ell} = k = \frac{2}{Q^2}$$

Thus we argue that

this effective theory is the LG minimal model coupled to the KS model with $(\mathbb{Z}_{N-\ell})^N$ orbifold symmetry

$(\mathbb{Z}_{N-\ell})^*$-orbifolded algebraic geometry

We replace $\ell\Sigma$ to $\dfrac{\partial}{\partial Y_{P_2}}$ and obtain

$$\widehat{\Pi} \;=\; \int \prod_{i=1}^{N} \mathrm{d}Y_i \, \mathrm{d}Y_{P_1} \, (\mathrm{e}^{-Y_{P_2}} \mathrm{d}Y_{P_2}) \, \delta\Big(\sum_i Y_i - \ell Y_{P_1} - (N-\ell) Y_{P_2} - t \Big) \exp\Big(- \sum_i \mathrm{e}^{-Y_i} - \mathrm{e}^{-Y_{P_1}} - \mathrm{e}^{-Y_{P_2}} \Big)$$

Re-defining the variables in order to obtain the canonical measure, we obtain

$$\widetilde{\mathcal{M}}_{N-\ell} \;=\; \Big\{ F(Z_a; u, v) \;=\; 0 \,,\; \{ G(Z_i) \;=\; 0 \} / \mathbb{C}^* \Big\} \Big/ (\mathbb{Z}_{N-\ell})^{N-2}$$

$$F(Z_a; u, v) \;=\; Z_1^{N-\ell} + \cdots + Z_\ell^{N-\ell} + 1 - uv$$

$$G(Z_i) \;=\; Z_{\ell+1}^{N-\ell} + \cdots + Z_N^{N-\ell} + \psi Z_{\ell+1} \cdots Z_N \,, \qquad \psi = \mathrm{e}^{t/(N-\ell)} Z_1 \cdots Z_\ell$$

$Z_a \;\mapsto\; \omega_a Z_a$ for $a \;=\; 1, \cdots, \ell$ (homogeneous coordinates of $\mathbb{C}^\ell$)

$Z_b \;\mapsto\; \lambda\, \omega_b Z_b$ for $b \;=\; \ell+1, \cdots, N$ (homogeneous coordinates of $\mathbb{CP}^{N-\ell-1}[N-\ell]$)

$$\omega_a^{N-\ell} \;=\; \omega_b^{N-\ell} \;=\; \omega_1 \cdots \omega_N \;=\; 1 \,, \qquad \lambda \;:\; \mathbb{C}^*\text{-value}$$

linear dilaton CFT and Liouville theory

$$\mathbb{R}^{9,1} \;\to\; \underbrace{\mathbb{R}^{d-1,1}}_{\text{free SCFT}} \times \underbrace{X^{2n}}_{\text{singular CY}} \;\sim\; \mathbb{R}^{d-1,1} \times \underbrace{\mathbb{R}_\phi \times S^1}_{\text{linear dilaton SCFT}} \times \underbrace{\mathcal{M}/U(1)}_{\mathcal{N}=2\ \text{Landau-Ginzburg}}$$

linear dilaton: $\Phi = -\frac{Q}{2}\phi$

Landau-Ginzburg: $W_{\rm LG} = F(Z_a), \;\; F(\lambda^{r_a} Z_a) = \lambda F(Z_a)$

$$c_{\rm total} = c_d + c_{\rm dilaton} + c_{\rm LG} \quad\to\quad 15 = \frac{3}{2}d + \left(\frac{3}{2} + 3Q^2\right) + 3\sum_{a=1}^{n+1}(1 - 2r_a)$$

$$\mathcal{N} = 2\ \text{“LG” on } \mathbb{R}_\phi \times S^1 \times \mathcal{M}/U(1):\; W = -\mu Z_0^{-k} + F(Z_a)$$

$$k = \frac{1}{r_\Omega} = \frac{2}{Q^2}\,, \qquad r_\Omega \equiv \sum_a r_a - 1$$

linear dilaton SCFT on $\mathbb{R}_\phi \times S^1$ $\equiv$ “LG” with $W = -\mu Z_0^{-k}$

$\equiv$ Kazama-Suzuki model on $SL(2,\mathbb{R})_k/U(1)$

$\overset{\text{T-dual}}{\equiv}$ Liouville theory of charge Q

Strictly, we consider the Euclidean black hole: $SL(2,\mathbb{R})_k/U(1) \to [SL(2,\mathbb{C})_k/SU(2)]/U(1)$