「素粒子論と幾何学の最前線」 於 大阪大学大学院理学研究科
2005 年 11 月 7 日

Generalized Complex Geometries and Supersymmetric Theories

— *inspired by flux compactification* —

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work in progress with Prof. Piljin Yi (KIAS)

弦理論における動機

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Flux Compactifications and Moduli Stabilization

NS-NS flux や R-R flux の期待値による真空構造の変化

Strominger [Nucl. Phys. B274 (1986) 253]

Giddings, Kachru and Polchinski [hep-th/0105097]

Kachru, Kallosh, Linde and Trivedi [hep-th/0301240]

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Modification of Mirror Symmetry

NS-NS flux の T-双対の行き場としての torsion

Gurrieri, Louis, Micu and Waldram [hep-th/0211102]

Cardoso, Curio, Dall'Agata, Lüst, Manousselis and Zoupanos [hep-th/0211118]

Fidanza, Minasian and Tomasiello [hep-th/0311122]

Supergravities, string, CY moduli spaces, mirror symmetry, moduli stabilizatin などの review:

Graña [hep-th/0509003]

T-duality

R-R $(2n+1)$-form $\longleftrightarrow$ R-R $2n$- or $(2n+2)$-form

NS-NS 3-form $\longleftrightarrow$ NS-NS ??-form

$\downarrow$

6-dim. space の torsion になっている

Kähler manifold には torsion を入れられない

$\longrightarrow$ これはもはや Kähler ではない

(Example)

T^6 compactified type IIB supergravity with NS-NS 3-form flux

この mirror は complex 3-torus ではなく、

half-flat manifold と呼ばれる、(もはや CY ではない) torsion を持った geometry になる

$$
\begin{aligned}
(T^0)^p{}_{mn} \;=\; \Lambda^1 \otimes \mathfrak{su}(3)^{\perp} &= (\mathbf{3} \oplus \overline{\mathbf{3}}) \otimes (\mathbf{1} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}) \\
&= \underset{W_1}{(\mathbf{1} \oplus \mathbf{1})} \oplus \underset{W_2}{(\mathbf{8} \oplus \mathbf{8})} \oplus \underset{W_3}{(\mathbf{6} \oplus \overline{\mathbf{6}})} \oplus \underset{W_4}{(\mathbf{3} \oplus \overline{\mathbf{3}})} \oplus \underset{W_5}{(\mathbf{3} \oplus \overline{\mathbf{3}})'}
\end{aligned}
$$

W_1 : complex scalar in $(\mathbf{1} \oplus \mathbf{1})$

W_2 : complex primitive 2-form in $(\mathbf{8} \oplus \mathbf{8})$

W_3 : real primitive $(2,1) \oplus (1,2)$-form in $(\mathbf{6} \oplus \overline{\mathbf{6}})$

W_4 : real 1-form in $(\mathbf{3} \oplus \overline{\mathbf{3}})$

W_5 : complex $(1,0)$-form in $(\mathbf{3} \oplus \overline{\mathbf{3}})'$

$SU(3)$-structure manifolds はこの W_a を用いて分類される

($SU(3)$-holonomy manifold = CY 3-fold も含まれる)

Chiossi and Salamon [math.DG/0202282]

- **complex manifolds**

  $W_1 = W_2 = 0$ $\qquad T^0 \in W_3 \oplus W_4 \oplus W_5$ **complex**

- **non-complex manifolds**

  $W_1 = W_3 = W_4 = 0$ $\qquad T^0 \in W_2 \oplus W_5$ **symplectic**

- **complex manifolds**

$W_1 = W_2 = 0$ $\qquad T^0 \in W_3 \oplus W_4 \oplus W_5$ $\qquad$ **complex**

$W_1 = W_2 = W_3 = W_4 = 0$ $\qquad T^0 \in W_5$ $\qquad$ **Kähler**

$W_1 = W_2 = W_3 = W_4 = W_5 = 0$ $\qquad T^0 = 0$ $\qquad$ **Calabi-Yau**

- **non-complex manifolds**

$W_1 = W_3 = W_4 = 0$ $\qquad T^0 \in W_2 \oplus W_5$ $\qquad$ **symplectic**

$W_1^- = W_2^- = W_4 = W_5 = 0$ $\qquad T^0 \in W_1^+ \oplus W_2^+ \oplus W_3$ $\qquad$ **half-flat**

complex manifolds

$W_1 = W_2 = 0$	$T^0 \in W_3 \oplus W_4 \oplus W_5$	complex
$W_1 = W_2 = W_4 = 0$	$T^0 \in W_3 \oplus W_5$	balanced
$W_1 = W_2 = W_4 = W_5 = 0$	$T^0 \in W_3$	special-hermitian
$W_1 = W_2 = W_3 = W_4 = 0$	$T^0 \in W_5$	Kähler
$W_1 = W_2 = W_3 = W_4 = W_5 = 0$	$T^0 = 0$	Calabi-Yau

non-complex manifolds

$W_1 = W_3 = W_4 = 0$	$T^0 \in W_2 \oplus W_5$	symplectic
$W_2 = W_3 = W_4 = W_5 = 0$	$T^0 \in W_1$	nearly-Kähler
$W_1 = W_3 = W_4 = W_5 = 0$	$T^0 \in W_2$	almost-Kähler
$W_1^- = W_2^- = W_4 = W_5 = 0$	$T^0 \in W_1^+ \oplus W_2^+ \oplus W_3$	half-flat
$W_1 = W_2 = W_3 = 3W_4 + 2W_5 = 0$	$T^0 \in W_4 \oplus W_5$	conformally rescaled CY

complex manifolds

$W_1 = W_2 = 0$	$T^0 \in W_3 \oplus W_4 \oplus W_5$	**complex**
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$W_1 = W_2 = W_3 = 3W_4 + 2W_5 = 0$	$T^0 \in W_4 \oplus W_5$	**conformally rescaled CY**

	$\mathbf{1} \oplus \mathbf{1}$	$\mathbf{3} \oplus \overline{\mathbf{3}}$	$\mathbf{6} \oplus \overline{\mathbf{6}}$	$\mathbf{8} \oplus \mathbf{8}$
Torsion	$1\,(W_1)$	$2\,(W_4, W_5)$	$1\,(W_3)$	$1\,(W_2)$
H_3	1	1	1	0
IIA: F_{2n}	$2\,(F_0, F_2, F_4)$	$2\,(F_2, F_4)$	0	$1\,(F_2, F_4)$
IIB: F_{2n+1}	$1\,(F_3)$	$3\,(F_1, F_3, F_5)$	$1\,(F_3)$	0

非自明な **torsion (or flux)** が存在する **IIB supergravities** では、コンパクト化された空間は **complex**

一方、**IIA supergravities** では、**complex** だったり **symplectic** だったり

Hitchin は (概) 複素構造の定義を広げた **generalized complex structure (GCS)** と

それが乗る **generalized complex geometry (GCG)** を考案

Hitchin [math.DG/0209099]

Gualtieri [math.DG/0401221]

- 通常の **almost complex structure: tangent bundle** $\mathcal{TM} \to$ **tangent bundle** $\mathcal{TM}$ への写像
- **generalized almost complex structure:** $\mathcal{TM} \oplus \mathcal{T}^*\mathcal{M} \to \mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$ への写像

非自明な torsion (or flux) が存在するIIB supergravities では、コンパクト化された空間は complex

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- 通常の almost complex structure: tangent bundle $\mathcal{TM}$ $\to$ tangent bundle $\mathcal{TM}$ への写像
- generalized almost complex structure: $\mathcal{TM} \oplus \mathcal{T}^*\mathcal{M} \to \mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$ への写像

$\mathcal{TM}$ 上の spinors は $\mathit{Clifford}(d)$ の表現で変換

$\mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$ 上の spinors は $\mathit{Clifford}(d,d)$ の表現で変換

torsion がある時の invariant 2-form $\nabla^{(T)} J = 0$ と invariant 3-forms $\nabla^{(T)} \Omega = 0$ は

$\mathit{Clifford}(3,3)$ bispinors で記述できる

つまり、complex structure を広げることで、そこに torsion (flux) の情報を入れることが可能になる

Mirror symmetry

Minasian 達は GCG 上の mirror symmetry をこう考えた:

$$e^{-(B+iJ)} \;\leftrightarrow\; \Omega$$

Fidanza, Minasian and Tomasiello [hep-th/0311122]

つまり mirror symmetry は 2 つの $Clifford(6,6)$ bispinors の交換である

これはさらに 2 つの generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ 交換であると読み換えられる:

$SU(3)$-invariant tensors	$SU(3,3)$-invariant bispinors	$Clifford(6,6)$ (pure) spinors	GCS
$-\frac{i}{8}\Omega$	$\eta_+ \otimes \eta_+^\dagger$	Φ_-	$\mathcal{J}_1 = \begin{pmatrix} I & 0 \\ 0 & -I^T \end{pmatrix}$
$\frac{1}{8}e^{-(B+iJ)}$	$\eta_+ \otimes \eta_-^\dagger$	Φ_+	$\mathcal{J}_2 = \begin{pmatrix} 0 & -J^{-1} \\ J & 0 \end{pmatrix}$

目標

worldsheet theory (sigma model) 上で $\mathcal{J}_1 \leftrightarrow \mathcal{J}_2$ 交換に該当する変換則を確立する

Worldsheet Theories

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通常の **string sigma model** を

generalized complex geometry を伝播するストリング理論に拡張する方法

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generalized complex geometry を伝播するストリング理論に拡張する方法

- 通常の場 $(\mathrm{d}X^A \in \mathcal{T}\mathcal{M})$ で記述される sigma model に
  新たな場 $(\eta_A \in \mathcal{T}^*\mathcal{M})$ を導入して拡張 (1st order action)
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- 拡張した理論を超対称化
- 超対称性変換則に generalized complex structure の情報が入る様にさらに拡張

Extension of nonlinear sigma models

$\mathrm{d}X^\mu \in \mathcal{T}\mathcal{M}$ と $g_{\mu\nu}$, $B_{\mu\nu}$ が存在する通常の string sigma model (2nd order action)

$$S = \frac{1}{2}\int\Big\{g_{\mu\nu}(X)\,\mathrm{d}X^\mu\wedge *\mathrm{d}X^\nu + B_{\mu\nu}(X)\,\mathrm{d}X^\mu\wedge\mathrm{d}X^\nu\Big\}$$

$\eta_\mu \in \mathcal{T}^*\mathcal{M}$ を Lagrange multiplier として導入(1st order action)

$$S = \frac{1}{2}\int\Big\{\eta_\mu\wedge\mathrm{d}X^\mu + \frac{1}{2}\theta^{\mu\nu}\,\eta_\mu\wedge\eta_\nu + \frac{1}{2}G^{\mu\nu}\,\eta_\mu\wedge *\eta_\nu + \frac{1}{2}(B-b)_{\mu\nu}\,\mathrm{d}X^\mu\wedge\mathrm{d}X^\nu\Big\}$$

$$E_{\mu\nu} = g_{\mu\nu} + b_{\mu\nu}\,, \qquad E^{\mu\lambda}E_{\lambda\nu} = \delta^\mu{}_\nu$$

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これを超対称化して **generalized complex strcuture** の情報を持たせることで、

generalized almost complex geometry を持つ **sigma model** が構成できる

Lindström, Minasian, Tomasiello and Zabzine [hep-th/0405085]

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Lindström, Minasian, Tomasiello and Zabzine [hep-th/0405085]

逆に、supersymmetry を一度下げて、一般的な supersymmetry 変換を再度導入することで
generalized complex geometry を実現させる方法がある

Lindström [hep-th/0401100]

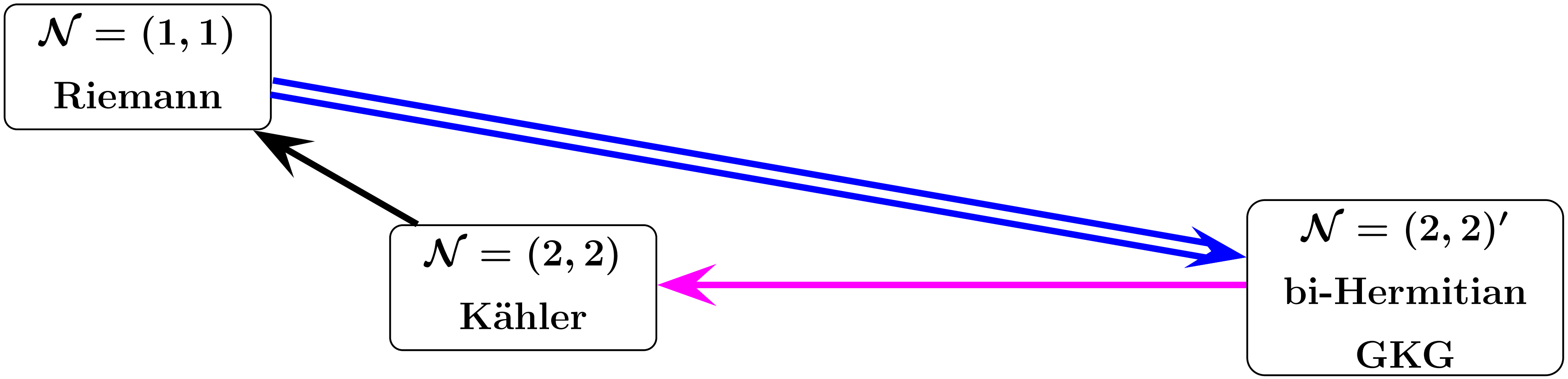
Lindström, Roček, Unge and Zabzine [hep-th/0411186]

いろいろな supersymmetric sigma models

$\boldsymbol{\mathcal{N} = (2,2)}$

Kähler

いろいろな supersymmetric sigma models

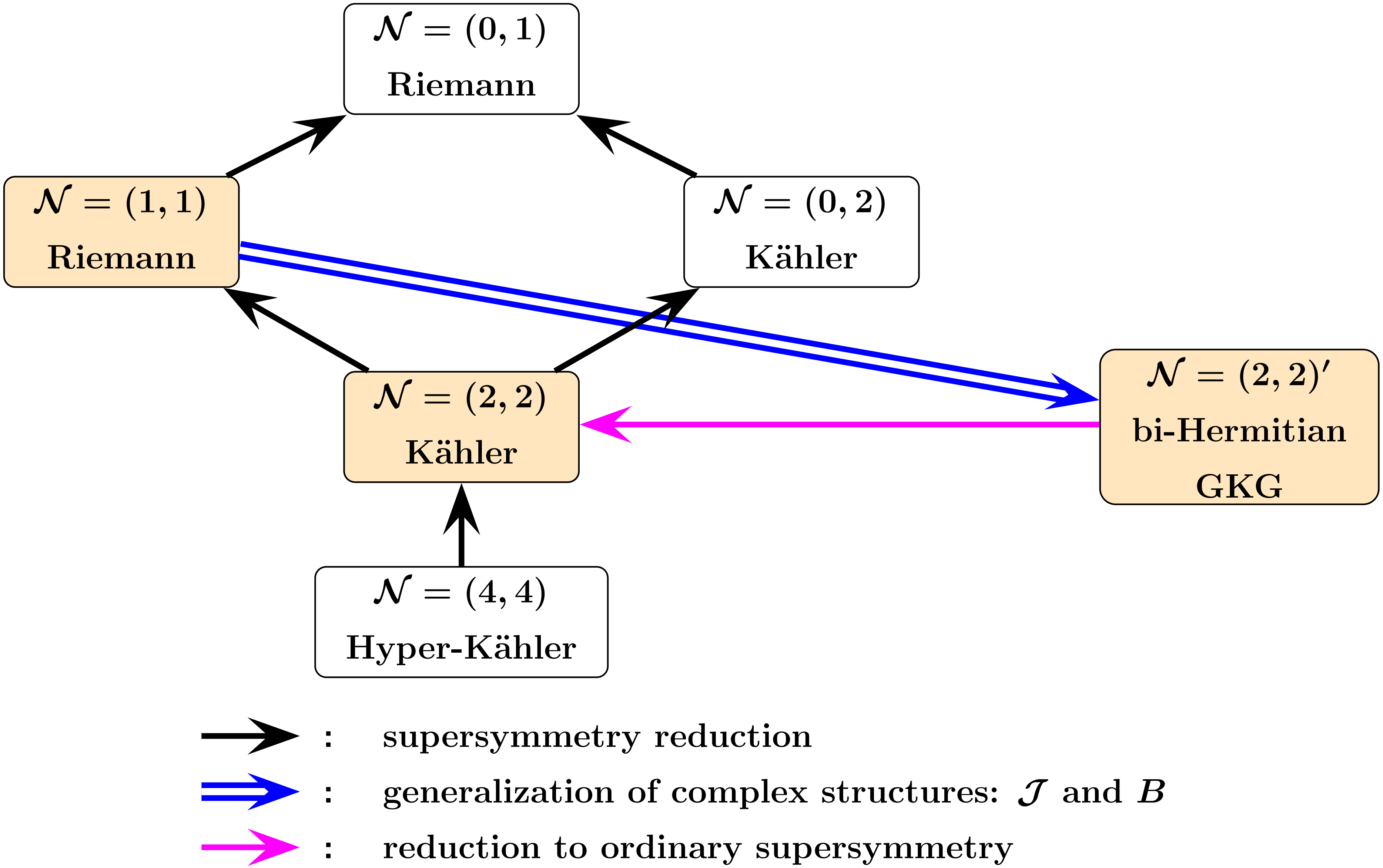


→ : supersymmetry reduction

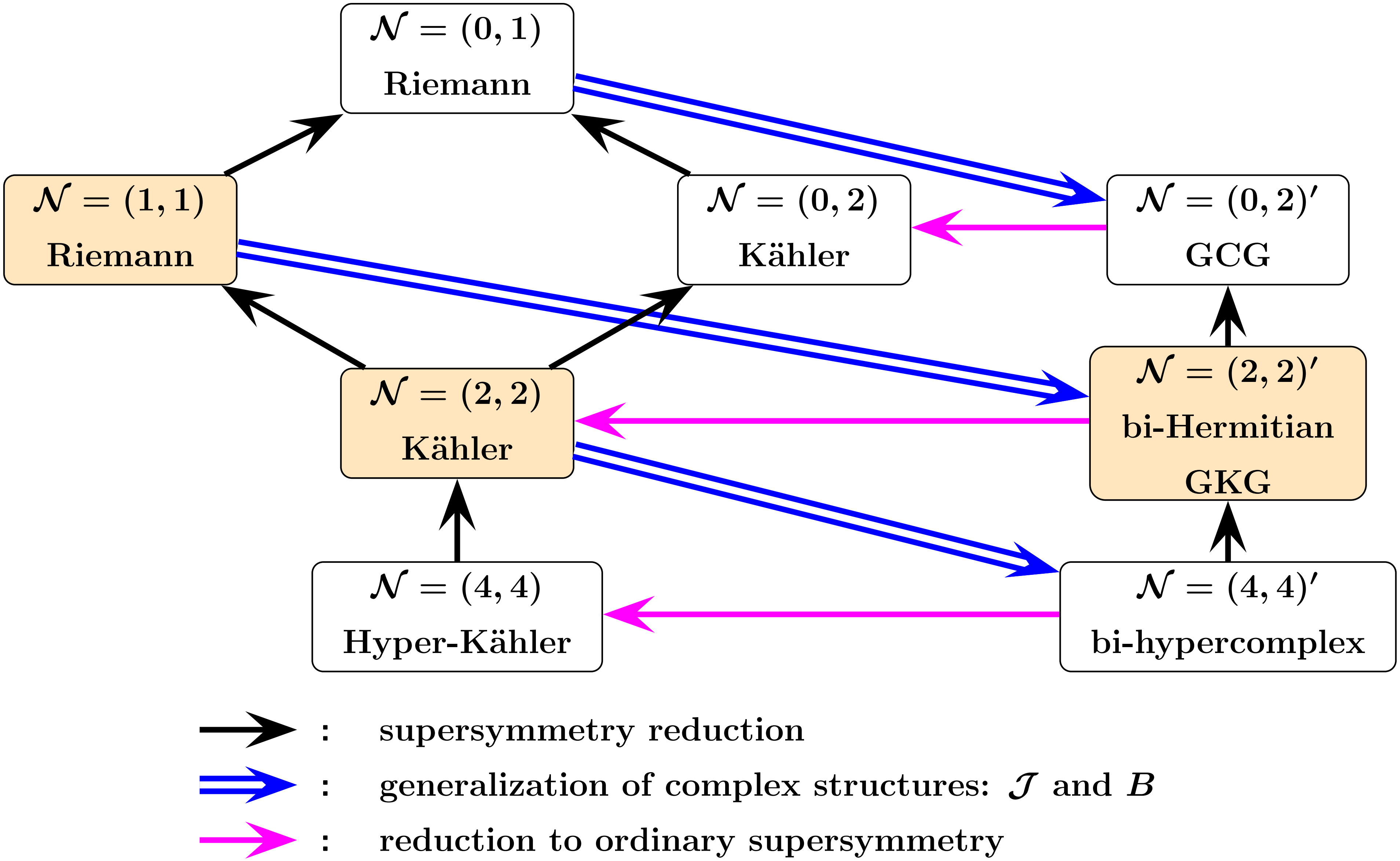
⇒ : generalization of complex structures: $\mathcal{J}$ and B

→ : reduction to ordinary supersymmetry

いろいろな supersymmetric sigma models



いろいろな supersymmetric sigma models



Sigma Models on GCG

Sigma Models on Generalized Complex Geometries

$\mathcal{N} = (2,2)$ supersymmetric sigma models of semi-chiral superfields

用いる超場: **semi-chiral superfields** $\mathbb{X}$, $\mathbb{Y}$

$$\overline{D}_+ \mathbb{X} \; = \; 0 \, , \qquad \overline{D}_- \mathbb{Y} \; = \; 0$$

$\mathcal{N} = (2,2)$ を $\mathcal{N} = (1,1)$ に簡約 (複素 $\theta^{\pm}$ を実$\theta_1^{\pm}$ に) する:

$$\theta_1^{\pm} \; \equiv \; -i \mathrm{e}^{-i\nu_{\pm}} \theta^{\pm} \; = \; i \mathrm{e}^{+i\nu_{\pm}} \overline{\theta}^{\pm}$$

この下で $\mathcal{N} = (1,1)$ **Lagrangian** に簡約する:

$$\mathscr{L} \; = \; \int \mathrm{d}^4\theta \, K(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}) \; = \; -\frac{1}{8} \int \mathrm{d}\theta_1^+ \mathrm{d}\theta_1^- \, \widetilde{Q}_+^1 \widetilde{Q}_-^1 K(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}})$$

Semi-chiral superfields は 2 種の互いに独立な $\mathcal{N} = (1,1)$ **superfields** に分解できる:

$$\begin{aligned} \mathbb{X}^{(2,2)} \; &\longrightarrow \; \big\{ \mathbb{X}^{(1,1)} \, , \; \Psi_-^{(1,1)} \big\} \qquad & \Psi_-^{(1,1)} \; &\equiv \; \widetilde{Q}_-^1 \mathbb{X}^{(2,2)} \big| \\ \mathbb{Y}^{(2,2)} \; &\longrightarrow \; \big\{ \mathbb{Y}^{(1,1)} \, , \; \Upsilon_+^{(1,1)} \big\} \qquad & \Upsilon_+^{(1,1)} \; &\equiv \; \widetilde{Q}_+^1 \mathbb{Y}^{(2,2)} \big| \end{aligned}$$

$\mathcal{N} = (2,2) \rightarrow (1,1)$ に簡約することで **generalized complex structure** を導入する余地を作り出す

Buscher, Lindström and Roček [Phys. Lett. B202 (1988) 94]

$\mathcal{N} = (2,2)$ superfields

chiral superfield $\boldsymbol{\Phi}$	$\overline{\boldsymbol{D}}_{\pm}\boldsymbol{\Phi} \;=\; 0$	$\boldsymbol{\Phi} \;=\; \overline{\boldsymbol{D}}_{+}\overline{\boldsymbol{D}}_{-}\boldsymbol{\Theta}$
left semi-chiral superfield $\mathbb{X}$	$\overline{\boldsymbol{D}}_{+}\mathbb{X} \;=\; 0$	$\mathbb{X} \;=\; \overline{\boldsymbol{D}}_{+}\boldsymbol{\Theta}$

$\mathcal{N} = (2,2)$ superfields

chiral superfield Φ	$\overline{D}_\pm \Phi = 0$	$\Phi = \overline{D}_+ \overline{D}_- \Theta$
left semi-chiral superfield $\mathbb{X}$	$\overline{D}_+ \mathbb{X} = 0$	$\mathbb{X} = \overline{D}_+ \Theta$

$$\begin{aligned}\Phi &= \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}\theta^-\psi_- + 2i\theta^+\theta^- F \\ &\quad - i\theta^+\overline{\theta}^+(\partial_0+\partial_1)\phi - i\theta^-\overline{\theta}^-(\partial_0-\partial_1)\phi + \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0^2-\partial_1^2)\phi \\ &\quad + \sqrt{2}\theta^+\overline{\theta}^+\theta^-(\partial_0+\partial_1)\psi_- + \sqrt{2}\theta^-\overline{\theta}^-\theta^+(\partial_0-\partial_1)\psi_+ \,,\end{aligned}$$

$$\begin{aligned}\mathbb{X} &= \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}(\theta^-\psi_- + \overline{\theta}^-\chi_-) + 2i\theta^+(\theta^- F + \overline{\theta}^- G) \\ &\quad - i\theta^+\overline{\theta}^+(\partial_0+\partial_1)\phi + \theta^-\overline{\theta}^- A_= + i\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0+\partial_1)A_= \\ &\quad + \sqrt{2}\theta^+\overline{\theta}^+(\partial_0+\partial_1)(\theta^-\psi_- + \overline{\theta}^-\chi_-) + 2\theta^+\theta^-\overline{\theta}^-\zeta_- \,.\end{aligned}$$

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$\mathcal{N} = (1,1)$ scalar/spinor superfields from $\mathcal{N} = (2,2)$ semi-chiral superfields

$$\begin{aligned}\mathbb{X}^{(1,1)} &= \phi + i\sqrt{2}\theta_1^+\widehat{\psi}_+ + i\sqrt{2}\theta_1^-(\widehat{\psi}_- + \widehat{\chi}_-) + 2i\theta_1^+\theta_1^-(\widehat{F} + \widehat{G}) \ , \\ \Psi_-^{(1,1)} &= \widetilde{Q}_-^1 \mathbb{X}^{(2,2)}\Big| \ = \ i(\widehat{\psi}_- - \widehat{\chi}_-) - i\sqrt{2}\theta_1^+(\widehat{F} - \widehat{G}) + \sqrt{2}\theta_1^- A_= + 2\sqrt{2}\theta_1^+\theta_1^-\widehat{\zeta}_- \ .\end{aligned}$$

Topological sigma models

$\mathbb{X}^a$ のみを含んだ $\mathcal{N} = (1,1)$ **Lagrangian を考えてみる:**

$$\mathscr{L}_{\mathbb{X}} \;=\; \int \mathrm{d}^4\boldsymbol{\theta}\, \boldsymbol{K}(\mathbb{X}, \overline{\mathbb{X}}) \;=\; -\frac{1}{4}\int \mathrm{d}\boldsymbol{\theta}_1^+ \mathrm{d}\boldsymbol{\theta}_1^- \Big\{ \boldsymbol{\Psi}_-^{\boldsymbol{A}}\, \boldsymbol{\omega}_{\boldsymbol{AB}}\, \boldsymbol{D}_+^1 \mathbb{X}^{\boldsymbol{B}} \Big\} \;=\; -\frac{1}{4}\int \mathrm{d}\boldsymbol{\theta}_1^+ \mathrm{d}\boldsymbol{\theta}_1^- \Big\{ \boldsymbol{S}_{\boldsymbol{A}-} \boldsymbol{D}_+^1 \mathbb{X}^{\boldsymbol{A}} \Big\}$$

superfields の再定義 ($\mathcal{T}\mathcal{M}$ 変数から $\mathcal{T}^*\mathcal{M}$ 変数へ):

$$\boldsymbol{S}_{\boldsymbol{A}-} \;=\; \boldsymbol{\Psi}_-^{\boldsymbol{B}}\, \boldsymbol{\omega}_{\boldsymbol{BA}}\,, \qquad 2\boldsymbol{\omega}_{\boldsymbol{AB}} \;\equiv\; \boldsymbol{J}_{\boldsymbol{A}}{}^{\boldsymbol{C}}\, \boldsymbol{K}_{\boldsymbol{CB}} - \boldsymbol{K}_{\boldsymbol{AC}}\, \boldsymbol{J}^{\boldsymbol{C}}{}_{\boldsymbol{B}}$$

$$\boldsymbol{D}_+^1 \mathbb{X}^{\boldsymbol{A}} \;\in\; \mathcal{T}\mathcal{M}\,, \qquad \boldsymbol{S}_{\boldsymbol{A}-} \;\in\; \mathcal{T}^*\mathcal{M}$$

Topological sigma models

$\mathbb{X}^a$ のみを含んだ $\mathcal{N} = (1,1)$ **Lagrangian** を考えてみる:

$$\mathscr{L}_{\mathbb{X}} \;=\; \int \mathrm{d}^4\boldsymbol{\theta}\, \boldsymbol{K}(\mathbb{X}, \overline{\mathbb{X}}) \;=\; -\frac{1}{4}\int \mathrm{d}\boldsymbol{\theta}_1^+ \mathrm{d}\boldsymbol{\theta}_1^- \Big\{ \Psi_-^A\, \boldsymbol{\omega}_{AB}\, \boldsymbol{D}_+^1 \mathbb{X}^B \Big\} \;=\; -\frac{1}{4}\int \mathrm{d}\boldsymbol{\theta}_1^+ \mathrm{d}\boldsymbol{\theta}_1^- \Big\{ \boldsymbol{S}_{A-} \boldsymbol{D}_+^1 \mathbb{X}^A \Big\}$$

superfields の再定義 ($\mathcal{TM}$ 変数から $\mathcal{T}^*\mathcal{M}$ 変数へ):

$$\boldsymbol{S}_{A-} \;=\; \Psi_-^B\, \boldsymbol{\omega}_{BA}\,, \qquad 2\boldsymbol{\omega}_{AB} \;\equiv\; \boldsymbol{J}_A{}^C\, \boldsymbol{K}_{CB} - \boldsymbol{K}_{AC}\, \boldsymbol{J}^C{}_B$$

$$\boldsymbol{D}_+^1 \mathbb{X}^A \;\in\; \mathcal{TM}\,, \qquad \boldsymbol{S}_{A-} \;\in\; \mathcal{T}^*\mathcal{M}$$

もう一度 $\mathcal{N} = (2,2)'$ に拡張するために、新たな $\mathcal{N} = (1,1)$ **supersymmetry** 変換則を

$$\begin{aligned}
\widetilde{\boldsymbol{\delta}}^{(+)} \mathbb{X}^A \;&=\; \widetilde{\varepsilon}^+\, \boldsymbol{J}^A{}_B\, \boldsymbol{D}_+^1 \mathbb{X}^B \\
\widetilde{\boldsymbol{\delta}}^{(-)} \mathbb{X}^A \;&=\; -\widetilde{\varepsilon}^-\, \boldsymbol{\omega}^{AB}\, (\boldsymbol{S}_{B-})^{\mathfrak{T}} \\
\widetilde{\boldsymbol{\delta}}^{(+)} \boldsymbol{S}_{A-} \;&=\; -\widetilde{\varepsilon}^+\, \boldsymbol{D}_+^1 \boldsymbol{S}_{B-}\, \boldsymbol{J}^B{}_A \\
\widetilde{\boldsymbol{\delta}}^{(-)} \boldsymbol{S}_{A-} \;&=\; -i\widetilde{\varepsilon}^-\, \big\{ \boldsymbol{\omega}_{AC}\, (\boldsymbol{\partial}_0 - \boldsymbol{\partial}_1) \mathbb{X}^C \big\}^{\mathfrak{T}}
\end{aligned}$$

Topological sigma models

$\mathbb{X}^a$ のみを含んだ $\mathcal{N} = (1,1)$ **Lagrangian** を考えてみる:

$$\mathscr{L}_{\mathbb{X}} \;=\; \int \mathrm{d}^4\theta\, K(\mathbb{X}, \overline{\mathbb{X}}) \;=\; -\frac{1}{4}\int \mathrm{d}\theta_1^+ \mathrm{d}\theta_1^- \Big\{ \Psi_-^A\, \omega_{AB}\, D_+^1 \mathbb{X}^B \Big\} \;=\; -\frac{1}{4}\int \mathrm{d}\theta_1^+ \mathrm{d}\theta_1^- \Big\{ S_{A-} D_+^1 \mathbb{X}^A \Big\}$$

superfields の再定義 ($\mathcal{T}\mathcal{M}$ 変数から $\mathcal{T}^*\mathcal{M}$ 変数へ):

$$S_{A-} \;=\; \Psi_-^B\, \omega_{BA}\,, \qquad 2\omega_{AB} \;\equiv\; J_A{}^C\, K_{CB} - K_{AC}\, J^C{}_B$$

$$D_+^1 \mathbb{X}^A \;\in\; \mathcal{T}\mathcal{M}\,, \qquad S_{A-} \;\in\; \mathcal{T}^*\mathcal{M}$$

もう一度 $\mathcal{N} = (2,2)'$ に拡張するために、新たな $\mathcal{N} = (1,1)$ **supersymmetry** 変換則を導入

$$\begin{aligned}
\widetilde{\delta}^{(+)} \mathbb{X}^A \;&=\; \widetilde{\varepsilon}^+\, J^A{}_B\, D_+^1 \mathbb{X}^B \\
\widetilde{\delta}^{(-)} \mathbb{X}^A \;&=\; -\widetilde{\varepsilon}^-\, \omega^{AB}\, (S_{B-})^{\mathfrak{T}} \\
\widetilde{\delta}^{(+)} S_{A-} \;&=\; -\widetilde{\varepsilon}^+\, D_+^1 S_{B-}\, J^B{}_A + (\omega_{AB})^{\mathfrak{T}} \Big\{ (\widetilde{\delta}^{(+)} \mathbb{X}^E)^{\mathfrak{T}}\, \delta_E^G - \widetilde{\varepsilon}^+\, (D_+^1 \mathbb{X}^E)^{\mathfrak{T}}\, J^G{}_E \Big\}\, \partial_G (\omega^{BC})^{\mathfrak{T}}\, S_{C-} \\
\widetilde{\delta}^{(-)} S_{A-} \;&=\; -i\widetilde{\varepsilon}^-\, \big\{ \omega_{AC}\, (\partial_0 - \partial_1) \mathbb{X}^C \big\}^{\mathfrak{T}} + S_{C-}\, \omega^{CB}\, \partial_E(\omega_{BA})\, (\widetilde{\delta}^{(-)} \mathbb{X}^E)
\end{aligned}$$

Embedded generalized complex structures on $\mathcal{N} = (2,2)$

通常の complex structure $\mathcal{J}_1$ と symplectic form $\mathcal{J}_2$ が

$$\left\{\begin{array}{rcl} \widetilde{\delta}^{(+)}\mathbb{X}^A &=& \widetilde{\varepsilon}^+ J^A{}_B D^1_+ \mathbb{X}^B \\ \widetilde{\delta}^{(+)}(S_{A-})^{\mathfrak{T}} &=& -\widetilde{\varepsilon}^+ J_A{}^B (D^1_+ S_{B-})^{\mathfrak{T}} \end{array}\right. \quad \longleftrightarrow \quad \mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^{\mathfrak{T}} \end{pmatrix}$$

$$\left\{\begin{array}{rcl} \widetilde{\delta}^{(-)}\mathbb{X}^A &=& -\widetilde{\varepsilon}^- \omega^{AB} (S_{B-})^{\mathfrak{T}} \\ \widetilde{\delta}^{(-)}(S_{A-})^{\mathfrak{T}} &=& -i\widetilde{\varepsilon}^- \omega_{AB} (\partial_0 - \partial_1)\mathbb{X}^B \end{array}\right. \quad \longleftrightarrow \quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

$$\left\{\begin{array}{rcl} \widetilde{\delta}^{(+)}\mathbb{Y}^{A'} &=& -\widetilde{\varepsilon}^+ \omega^{A'B'} (S_{B'+})^{\mathfrak{T}} \\ \widetilde{\delta}^{(+)}(S_{A'+})^{\mathfrak{T}} &=& -i\widetilde{\varepsilon}^+ \omega_{A'B'} (\partial_0 + \partial_1)\mathbb{Y}^{B'} \end{array}\right. \quad \longleftrightarrow \quad \widetilde{\mathcal{J}}_2 = \begin{pmatrix} 0 & -\omega'^{-1} \\ \omega' & 0 \end{pmatrix}$$

$$\left\{\begin{array}{rcl} \widetilde{\delta}^{(-)}\mathbb{Y}^{A'} &=& \widetilde{\varepsilon}^- J^{A'}{}_{B'} D^1_- \mathbb{Y}^{B'} \\ \widetilde{\delta}^{(-)}(S_{A'+})^{\mathfrak{T}} &=& -\widetilde{\varepsilon}^- J_{A'}{}^{B'} (D^1_- S_{B'+})^{\mathfrak{T}} \end{array}\right. \quad \longleftrightarrow \quad \widetilde{\mathcal{J}}_1 = \begin{pmatrix} J' & 0 \\ 0 & -J'^{\mathfrak{T}} \end{pmatrix}$$

$$\mathcal{J}_1 : \mathcal{TM} \to \mathcal{TM}, \ \mathcal{T}^*\mathcal{M} \to \mathcal{T}^*\mathcal{M}$$

$$\mathcal{J}_2 : \mathcal{TM} \to \mathcal{T}^*\mathcal{M}, \ \mathcal{T}^*\mathcal{M} \to \mathcal{TM}$$

Embedded generalized complex structures on $\mathcal{N} = (2,2)'$

通常の complex structure $\mathcal{J}_1$ と symplectic form $\mathcal{J}_2$ が拡張される

$$\begin{cases} \widetilde{\delta}^{(+)}\mathbb{X}^A = \widetilde{\varepsilon}^+ J^A{}_B\, D^1_+ \mathbb{X}^B \\ \widetilde{\delta}^{(+)}(S_{A-})^{\mathfrak{T}} = -\widetilde{\varepsilon}^+ J_A{}^B\, (D^1_+ S_{B-})^{\mathfrak{T}} + \cdots \end{cases} \longleftrightarrow \quad \mathcal{J}_1 = \begin{pmatrix} J & 0 \\ * & -J^{\mathfrak{T}} \end{pmatrix}$$

$$\begin{cases} \widetilde{\delta}^{(-)}\mathbb{X}^A = -\widetilde{\varepsilon}^- \omega^{AB} (S_{B-})^{\mathfrak{T}} \\ \widetilde{\delta}^{(-)}(S_{A-})^{\mathfrak{T}} = -i\widetilde{\varepsilon}^- \omega_{AB}(\partial_0 - \partial_1)\mathbb{X}^B + \cdots \end{cases} \longleftrightarrow \quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & * \end{pmatrix}$$

$$\begin{cases} \widetilde{\delta}^{(+)}\mathbb{Y}^{A'} = -\widetilde{\varepsilon}^+ \omega^{A'B'} (S_{B'+})^{\mathfrak{T}} \\ \widetilde{\delta}^{(+)}(S_{A'+})^{\mathfrak{T}} = -i\widetilde{\varepsilon}^+ \omega_{A'B'}(\partial_0 + \partial_1)\mathbb{Y}^{B'} + \cdots \end{cases} \longleftrightarrow \quad \widetilde{\mathcal{J}}_2 = \begin{pmatrix} 0 & -\omega'^{-1} \\ \omega' & * \end{pmatrix}$$

$$\begin{cases} \widetilde{\delta}^{(-)}\mathbb{Y}^{A'} = \widetilde{\varepsilon}^- J^{A'}{}_{B'}\, D^1_- \mathbb{Y}^{B'} \\ \widetilde{\delta}^{(-)}(S_{A'+})^{\mathfrak{T}} = -\widetilde{\varepsilon}^- J_{A'}{}^{B'}\, (D^1_- S_{B'+})^{\mathfrak{T}} + \cdots \end{cases} \longleftrightarrow \quad \widetilde{\mathcal{J}}_1 = \begin{pmatrix} J' & 0 \\ * & -J'^{\mathfrak{T}} \end{pmatrix}$$

$$\mathcal{J}_1 : \mathcal{TM} \to \mathcal{TM},\ \mathcal{T}^*\mathcal{M} \to \mathcal{T}^*\mathcal{M} \quad \Longrightarrow \quad \mathcal{TM} \oplus \mathcal{T}^*\mathcal{M} \to \mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$$

$$\mathcal{J}_2 : \mathcal{TM} \to \mathcal{T}^*\mathcal{M},\ \mathcal{T}^*\mathcal{M} \to \mathcal{TM} \quad \Longrightarrow \quad \mathcal{TM} \oplus \mathcal{T}^*\mathcal{M} \to \mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$$

A conjecture on the mirror symmetry

mirror symmetry は 2 つの generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ の交換である:

(pure) spinors	$SU(3)$-invariant tensors	$SU(3,3)$-invariant bispinors	generalized complex structures
Φ_-	$-\frac{i}{8}\Omega$	$\eta_+ \otimes \eta_+^\dagger$	$\mathcal{J}_1 = \begin{pmatrix} I & 0 \\ 0 & -I^{\mathfrak{T}} \end{pmatrix}$
Φ_+	$\frac{1}{8}\mathrm{e}^{-iJ}$	$\eta_+ \otimes \eta_-^\dagger$	$\mathcal{J}_2 = \begin{pmatrix} 0 & -J^{-1} \\ J & 0 \end{pmatrix}$

この予想を $\mathscr{L}'_{\mathbb{X}}$-theory と $\mathscr{L}'_{\mathbb{Y}}$-theory で読み取ろうとする:

$$\mathcal{J}_1 \longleftrightarrow \widetilde{\mathcal{J}}_2 \, , \qquad \mathcal{J}_2 \longleftrightarrow \widetilde{\mathcal{J}}_1$$

これは次の交換に読み換えられる: **exchange between $\mathbb{X}^A$ and $\mathbb{Y}^{A'}$**

> **The topological theory $\mathscr{L}'_{\mathbb{X}}$ should be mapped**
>
> **to the topological theory $\mathscr{L}'_{\mathbb{Y}}$, and vice versa.**

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> **The topological theory $\mathscr{L}'_{\mathbb{X}}$ should be mapped**
>
> **to the topological theory $\mathscr{L}'_{\mathbb{Y}}$, and vice versa.**

しかし $\mathcal{N} = (2,2)'$-theory でいきなりこの交換を確立するのは困難である

$\rightarrow \mathcal{N} = (2,2)$-theory での交換ルールを確認してみよう

$\mathcal{N} = (2,2)$ SUSY theories の範疇において$\cdots$

- 我々は、$(\Phi, \overline{\Phi})$ で書かれる **model** と $(Y, \overline{Y})$ で書かれる **model** の交換ルールを知っている
- 一方で、$(\mathbb{X}, \mathbb{Y}, \overline{\mathbb{X}}, \overline{\mathbb{Y}})$**-model** と $(\Phi, \overline{\Phi}, Y, \overline{Y})$**-model** の交換ルールも知っている

$\longrightarrow$ **Duality 変換 (例えば次の文献を参照):**

Roček and Verlinde [hep-th/9110053]

Ivanov, Kim and Roček [hep-th/9406063]

Penati and Zanon [hep-th/9712137]

Grisaru, Massar, Sevrin and Troost [hep-th/9801080]

しかし、$(\mathbb{X}, \overline{\mathbb{X}})$**-model** から $(\mathbb{Y}, \overline{\mathbb{Y}})$**-model** へ、

もしくは $(\mathbb{X}, \mathbb{Y}, \overline{\mathbb{X}}, \overline{\mathbb{Y}})$**-model** から $(\mathbb{X}, \mathbb{Y}, \overline{\mathbb{X}}, \overline{\mathbb{Y}})$**-model** への **Duality** 変換は知られていない

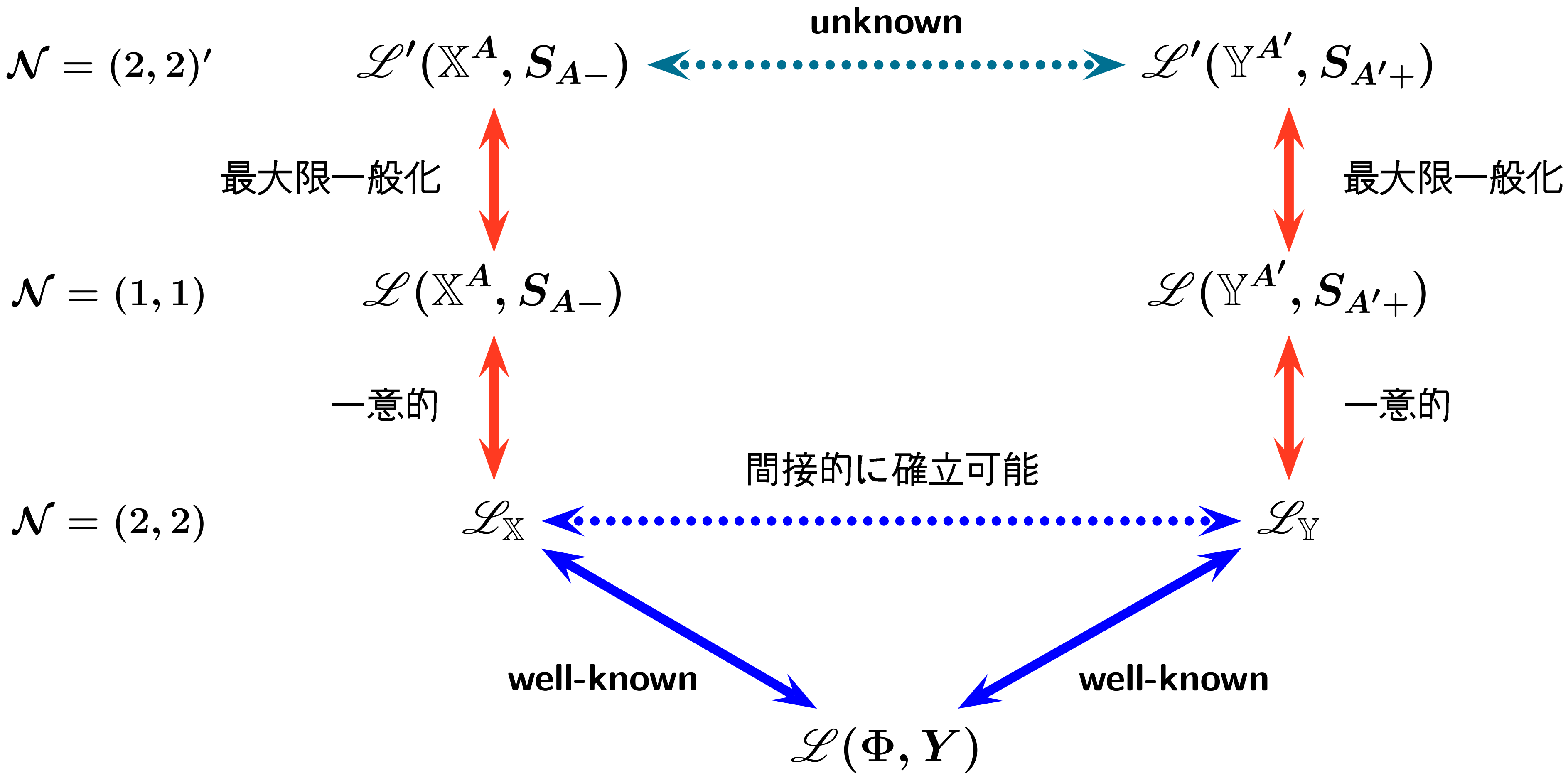
$\mathcal{N}=(2,2)'$-theory の交換ルールを間接的に確立する方法

$\boldsymbol{\mathcal{N}=(2,2)'}$ $\quad$ $\mathscr{L}'(\mathbb{X}^{A}, \boldsymbol{S}_{A-})$ ⟵ **unknown** ⟶ $\mathscr{L}'(\mathbb{Y}^{A'}, \boldsymbol{S}_{A'+})$

$\mathcal{N}=(2,2)'$-theory の交換ルールを間接的に確立する方法

$\boldsymbol{\mathcal{N}=(2,2)'}$	$\mathscr{L}'(\mathbb{X}^{A}, \boldsymbol{S}_{A-})$	$\longleftrightarrow$ unknown	$\mathscr{L}'(\mathbb{Y}^{A'}, \boldsymbol{S}_{A'+})$
	最大限一般化 $\updownarrow$		$\updownarrow$ 最大限一般化
$\boldsymbol{\mathcal{N}=(1,1)}$	$\mathscr{L}(\mathbb{X}^{A}, \boldsymbol{S}_{A-})$		$\mathscr{L}(\mathbb{Y}^{A'}, \boldsymbol{S}_{A'+})$
	一意的 $\updownarrow$		$\updownarrow$ 一意的
$\boldsymbol{\mathcal{N}=(2,2)}$	$\mathscr{L}_{\mathbb{X}}$		$\mathscr{L}_{\mathbb{Y}}$

$\mathcal{N}=(2,2)'$-theory の交換ルールを間接的に確立する方法



Duality 変換 その1

$$\mathscr{L}(\Phi, \overline{\Phi}) \leftrightarrow \mathscr{L}(\boldsymbol{Y}, \overline{\boldsymbol{Y}})$$

Duality 変換 1 (一般論)

$$\mathscr{L}_0 = \int \mathrm{d}^4\boldsymbol{\theta} \left\{ \boldsymbol{K}_0(\boldsymbol{A}, \overline{\boldsymbol{A}}, \boldsymbol{B}, \overline{\boldsymbol{B}}; \cdots) - \mathbb{X}\boldsymbol{A} - \overline{\mathbb{X}\boldsymbol{A}} - \mathbb{Y}\boldsymbol{B} - \overline{\mathbb{Y}\boldsymbol{B}} \right\}$$

$$\mathscr{L}_0$$

$$\downarrow \quad \boldsymbol{A} = \boldsymbol{B}$$

$$\mathscr{L}_1$$

$$\downarrow \quad \textbf{indep. of } \boldsymbol{A} - \overline{\boldsymbol{A}} \qquad \mathbb{X} + \mathbb{Y} = \overline{\mathbb{X}} + \overline{\mathbb{Y}}$$

$$\mathscr{L}_{1,3}$$

$$\swarrow \qquad \searrow$$

$$\mathscr{L}_{1,4} \longleftrightarrow \mathscr{L}_{1,5}$$

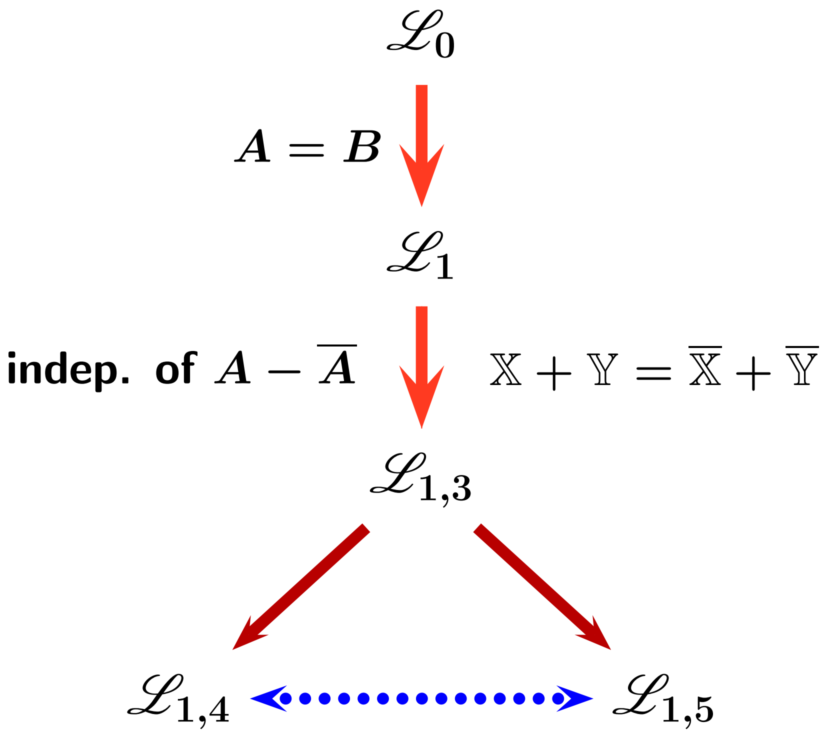
$$\mathscr{L}_1 \equiv \int \mathrm{d}^4\boldsymbol{\theta} \left\{ \boldsymbol{K}_{1,3}(\boldsymbol{A} + \overline{\boldsymbol{A}}; \cdots) - \tfrac{1}{2}(\mathbb{X} + \mathbb{Y} + \overline{\mathbb{X}} + \overline{\mathbb{Y}})(\boldsymbol{A} + \overline{\boldsymbol{A}}) - \tfrac{1}{2}(\mathbb{X} + \mathbb{Y} - \overline{\mathbb{X}} - \overline{\mathbb{Y}})(\boldsymbol{A} - \overline{\boldsymbol{A}}) \right\}$$

$$\therefore \quad \mathbb{X} + \mathbb{Y} = \boldsymbol{Y}_1 + \overline{\boldsymbol{Y}}_1$$

$$\mathscr{L}_{1,3} \equiv \int \mathrm{d}^4\boldsymbol{\theta} \left\{ \boldsymbol{K}_{1,3}(\boldsymbol{A} + \overline{\boldsymbol{A}}; \cdots) - (\boldsymbol{Y}_1 + \overline{\boldsymbol{Y}}_1)(\boldsymbol{A} + \overline{\boldsymbol{A}}) \right\}$$

Duality 変換 1 (一般論)

$$\mathscr{L}_0 = \int d^4\theta \left\{ \boldsymbol{K}_0(\boldsymbol{A}, \overline{\boldsymbol{A}}, \boldsymbol{B}, \overline{\boldsymbol{B}}; \cdots) - \mathbb{X}\boldsymbol{A} - \overline{\mathbb{X}\boldsymbol{A}} - \mathbb{Y}\boldsymbol{B} - \overline{\mathbb{Y}\boldsymbol{B}} \right\}$$



$$\boldsymbol{\Phi}_1 + \overline{\boldsymbol{\Phi}}_1 = \boldsymbol{A} + \overline{\boldsymbol{A}} = \widetilde{\boldsymbol{A}}(\boldsymbol{Y}_1, \overline{\boldsymbol{Y}}_1; \cdots)\,, \qquad \mathbb{X} + \mathbb{Y} = \boldsymbol{Y}_1 + \overline{\boldsymbol{Y}}_1$$

$$\mathscr{L}_{1,4} \equiv \int d^4\theta \left\{ \boldsymbol{K}_{1,3}(\boldsymbol{A} + \overline{\boldsymbol{A}}; \cdots) - (\boldsymbol{Y}_1 + \overline{\boldsymbol{Y}}_1)(\boldsymbol{A} + \overline{\boldsymbol{A}}) \right\}\Big|_{\text{EOM of } Y_1 + \overline{Y}_1} \equiv \int d^4\theta\, \boldsymbol{K}_{1,4}(\boldsymbol{\Phi}_1, \overline{\boldsymbol{\Phi}}_1; \cdots)$$

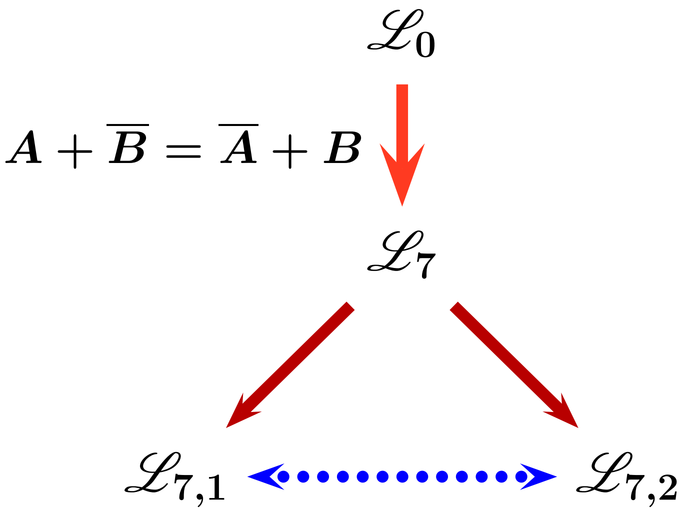
$$\mathscr{L}_{1,5} \equiv \int d^4\theta \left\{ \boldsymbol{K}_{1,3}(\boldsymbol{A} + \overline{\boldsymbol{A}}; \cdots) - (\boldsymbol{Y}_1 + \overline{\boldsymbol{Y}}_1)(\boldsymbol{A} + \overline{\boldsymbol{A}}) \right\}\Big|_{\text{EOM of } A + \overline{A}} \equiv \int d^4\theta\, \boldsymbol{K}_{1,5}(\boldsymbol{Y}_1, \overline{\boldsymbol{Y}}_1; \cdots)$$

Duality 変換 その2

$$\mathscr{L}(\boldsymbol{\Phi}, \overline{\boldsymbol{\Phi}}, \boldsymbol{Y}, \overline{\boldsymbol{Y}}) \leftrightarrow \mathscr{L}(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}})$$

Duality 変換 2 (一般論)

$$\mathscr{L}_0 \;=\; \int \mathrm{d}^4\boldsymbol{\theta} \left\{ \boldsymbol{K}_0(\boldsymbol{A}, \overline{\boldsymbol{A}}, \boldsymbol{B}, \overline{\boldsymbol{B}}; \cdots) - \mathbb{X}\boldsymbol{A} - \overline{\mathbb{X}\boldsymbol{A}} - \mathbb{Y}\boldsymbol{B} - \overline{\mathbb{Y}\boldsymbol{B}} \right\}$$



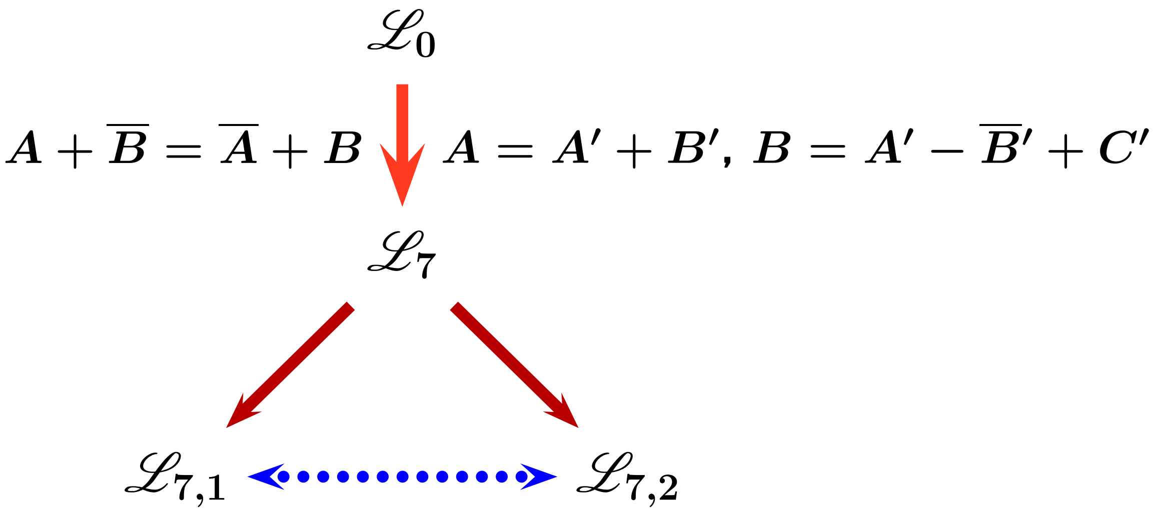
$$\boldsymbol{A} \;=\; \boldsymbol{A}' + \boldsymbol{B}' \,, \qquad \boldsymbol{B} \;=\; \boldsymbol{A}' - \overline{\boldsymbol{B}}' + \boldsymbol{C}'$$

$$\overline{\boldsymbol{C}}' \;=\; \boldsymbol{C}'$$

$$\begin{aligned} \mathscr{L}_7 \;=\; \int \mathrm{d}^4\boldsymbol{\theta} \Big\{ & \boldsymbol{K}_7(\boldsymbol{B}' + \overline{\boldsymbol{B}}' - \boldsymbol{C}', \boldsymbol{A}' - \overline{\boldsymbol{B}}' + \boldsymbol{C}', \overline{\boldsymbol{A}}' - \boldsymbol{B}' + \boldsymbol{C}'; \cdots) \\ & - (\boldsymbol{B}' + \overline{\boldsymbol{B}}' - \boldsymbol{C}')(\mathbb{X} + \overline{\mathbb{X}}) - (\boldsymbol{A}' - \overline{\boldsymbol{B}}' + \boldsymbol{C}')(\mathbb{X} + \mathbb{Y}) - (\overline{\boldsymbol{A}}' - \boldsymbol{B}' + \boldsymbol{C}')(\overline{\mathbb{X}} + \overline{\mathbb{Y}}) \Big\} \end{aligned}$$

Duality 変換 2 (一般論)

$$\mathscr{L}_0 = \int \mathrm{d}^4\theta \left\{ \boldsymbol{K}_0(\boldsymbol{A}, \overline{\boldsymbol{A}}, \boldsymbol{B}, \overline{\boldsymbol{B}}; \cdots) - \mathbb{X}\boldsymbol{A} - \overline{\mathbb{X}\boldsymbol{A}} - \mathbb{Y}\boldsymbol{B} - \overline{\mathbb{Y}\boldsymbol{B}} \right\}$$



$$\boldsymbol{\Phi}_7 = \boldsymbol{A}' = \boldsymbol{A}'(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}; \cdots), \qquad \boldsymbol{Y}_7 = \boldsymbol{B}' = \boldsymbol{B}'(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}; \cdots)$$

$$\boldsymbol{0} = \boldsymbol{C}' = \boldsymbol{C}'(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}; \cdots)$$

$$\mathscr{L}_{7,1} \equiv \mathscr{L}_7\Big|_{\text{EOM of } \mathbb{X},\ \mathbb{Y}} \equiv \int \mathrm{d}^4\theta\, \boldsymbol{K}_{7,1}(\boldsymbol{\Phi}_7, \overline{\boldsymbol{\Phi}}_7, \boldsymbol{Y}_7, \overline{\boldsymbol{Y}}_7; \cdots)$$

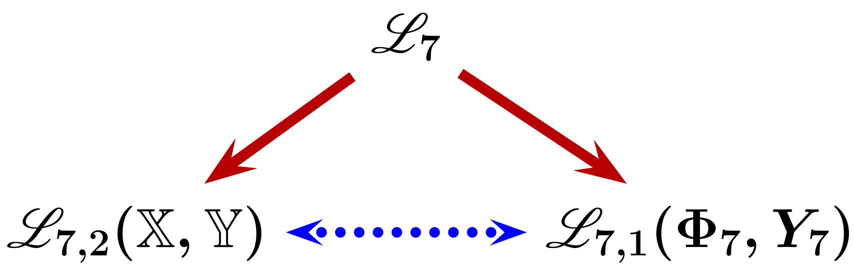
$$\mathscr{L}_{7,2} \equiv \mathscr{L}_7\Big|_{\text{EOM of } A',\ B',\ C'} \equiv \int \mathrm{d}^4\theta\, \boldsymbol{K}_{7,2}(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}; \cdots)$$

Duality 変換

$\mathscr{L}_{\mathbb{X}}$-theory から $\mathscr{L}_{\mathbb{Y}}$-theory へ

Duality 変換: $\mathscr{L}_{\mathbb{X}}$-theory から $\mathscr{L}_{\mathbb{Y}}$-theory へ

— アイデア —



Duality 変換 2

Duality 変換: $\mathscr{L}_{\mathbb{X}}$-theory から $\mathscr{L}_{\mathbb{Y}}$-theory へ

— アイデア —

$$\mathscr{L}_{\mathbf{7}}$$

$$\mathscr{L}_{\mathbf{7,2}}(\mathbb{X}, \mathbb{Y}) \longleftrightarrow \mathscr{L}_{\mathbf{7,1}}(\mathbf{\Phi}_7, \boldsymbol{Y}_7)$$

Duality 変換2

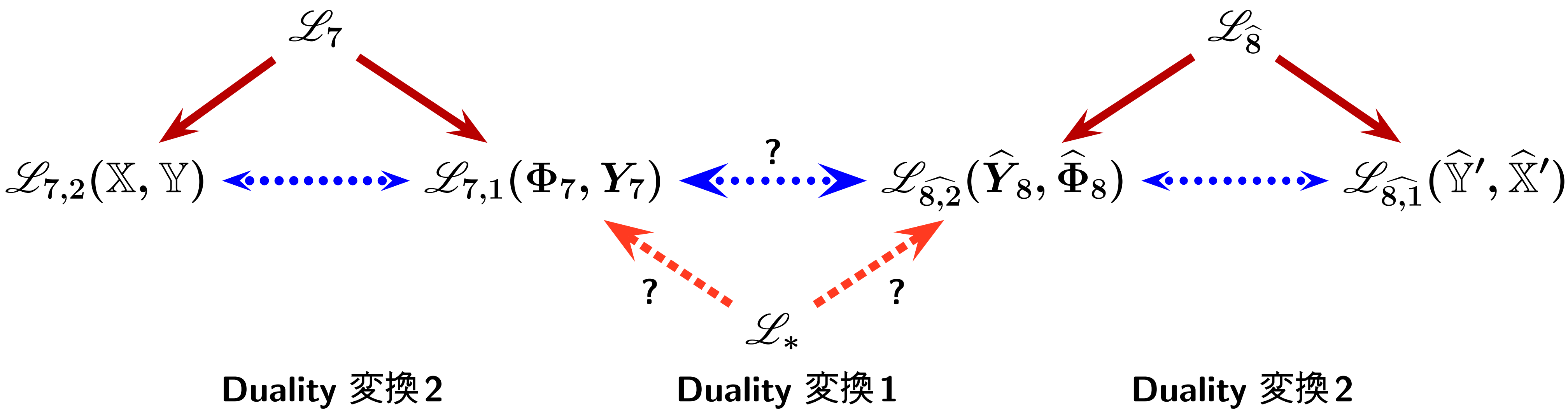
$$\mathscr{L}_{\widehat{\mathbf{8}}}$$

$$\mathscr{L}_{\widehat{\mathbf{8,2}}}(\widehat{\boldsymbol{Y}}_8, \widehat{\mathbf{\Phi}}_8) \longleftrightarrow \mathscr{L}_{\widehat{\mathbf{8,1}}}(\widehat{\mathbb{Y}}', \widehat{\mathbb{X}}')$$

Duality 変換2

Duality 変換: $\mathscr{L}_{\mathbb{X}}$-theory から $\mathscr{L}_{\mathbb{Y}}$-theory へ

— アイデア —



Discussions

確認したい点

- relation between CY with H_3-flux and GCG
- Consistency check of T-duality
  Hull and Reid-Edwards [hep-th/0503114]

応用点

- Gauge theory and a generalized moment map
  Lin and Tolman [math.DG/0509069]
- Topological strings on generalized complex manifolds
  Kapustin and Li [hep-th/0407249]

問題点・疑問点

- Doubling problem of degrees of freedom
  Bredthauer, Lindström and Persson [hep-th/0508228]
- How to break the complex structure? (to realize the half-flat manifold)

付録

G-structure Manifolds

Appendix

G-structure manifolds

CY 3-fold を理解するとき、supergravity の中にある fermion の SUSY 変換の期待値がゼロになれ ($\langle \delta\Psi_m \rangle = \langle \delta\chi \rangle = \langle \delta\lambda \rangle = 0$)、という非自明な条件を課すと、それは6-dim. manifold $\mathcal{M}$ 上の Killing spinor equation とみなすことができて、$SU(3)$ invariant spinor (covariantly constant spinor) $\nabla_m \eta = 0$ があることがわかります。これと同様にして、torsion ありの covariant derivative $\nabla_m^{(T)} = \nabla_m + \kappa_m$ について invariant な spinor が定義できます。この、torsion がある多様体はもはや $SU(3)$ holonomy manifold ではないのですが、frame bundle には $SU(3)$-structure group symmetry がまだ存在している、という事になります。この spinor が存在するとき、これらの多様体を

$SU(3)$-structure manifold

と呼びます。広い意味で、$SU(3)$ holonomy manifold も、$SU(3)$-structure manifold です。こうすると、例えば fundamental 2-form J や holomorphic 3-fom Ω が $SU(3)$ invariant spinor の直積で記述出来たりします:

$$J_{mn} \ = \ \mp 2i \, \eta_{\pm}^{\dagger} \gamma_{mn} \eta_{\pm} \ , \qquad \Omega_{mnp} \ = \ -2i \, \eta_{-}^{\dagger} \gamma_{mnp} \eta_{+} \ . \tag{1}$$

torsion 自身、この $SU(3)$-structure group を用いた表現で分解できます。この spinor の性質と、torsion の分解の仕方で、$SU(3)$-structure manifold を分類することができます。

Now suppose $\mathcal{M}$ has a G-structure. In general the Levi-Civita connection does not preserve the G-invariant tensors (or spinor) ξ (i.e., $\nabla\xi \neq 0$). However, one can show that there always exist some other connection $\nabla^{(T)}$ which is compatible with the G-structure so that

$$\nabla^{(T)}\xi \; = \; 0 \; .$$

Thus for instance, on an almost Hermitian manifold one can always find $\nabla^{(T)}$ such that $\nabla^{(T)}J = 0$. On a manifold with $SU(3)$-structure, it means we can always find $\nabla^{(T)}$ such that both $\nabla^{(T)}J = 0$ and $\nabla^{(T)}\Omega = 0$ and the solutions are

$$\begin{aligned} 0 \; &= \; \nabla^{(T)}J \qquad \rightarrow \qquad (\mathrm{d}J)_{mnp} \; = \; -6T^{r}{}_{[mn}J_{r|p]} \; , \\ 0 \; &= \; \nabla^{(T)}\Omega \qquad \rightarrow \qquad (\mathrm{d}\Omega)_{mnpq} \; = \; -12T^{r}{}_{[mn}\Omega_{r|pq]} \; . \end{aligned}$$

Since the existence of an $SU(3)$-structure is also equivalent to the existence of an invariant spinor η, this is equivalent to the condition $\nabla^{(T)}\eta = 0$ and then

$$\nabla^{(T)}\eta \; = \; 0 \; , \qquad iJ_{mn} \; = \; \eta^{\dagger}\gamma_{mn}\gamma\eta \; , \qquad i\Omega_{mnp} \; = \; \eta^{\dagger}\gamma_{mnp}(1+\gamma)\eta \; .$$

Let κ be the contorsion tensor corresponding to $\nabla^{(T)}$. We see that κ is an element of $\Lambda^1 \otimes \Lambda^2$ where Λ^n is the space of n-forms. Alternatively, since $\Lambda^2 \cong \mathfrak{so}(d)$, it is more natural to think of $\kappa^{p}{}_{mn}$ as one-form with values in the Lie-algebra $\mathfrak{so}(d)$ that is $\Lambda^1 \otimes \mathfrak{so}(d)$. Given the existence of a G-structure, we can decompose $\mathfrak{so}(d)$ into a part in the Lie algebra $\mathfrak{g}$ of $G \subset SO(d)$ and an orthogonal piece $\mathfrak{g}^\perp = \mathfrak{so}(d)/\mathfrak{g}$. The contorsion κ splits accordingly into

$$\kappa \;=\; \kappa^0 + \kappa^{\mathfrak{g}} \;,$$

where κ^0 is the part in $\Lambda^1 \otimes \mathfrak{g}^\perp$. Since an invariant tensor (or spinor) ξ is fixed under G rotations, that action of $\mathfrak{g}$ on ξ vanishes and we have, by definition,

$$\nabla^{(T)}\xi \;=\; \left(\nabla + \kappa^0 + \kappa^{\mathfrak{g}}\right)\xi \;=\; \left(\nabla + \kappa^0\right)\xi \;=\; 0 \;.$$

Thus, any two G-compatible connections must differ by a piece proportional to $\kappa^{\mathfrak{g}}$ and they have a common term κ^0 in $\Lambda^1 \otimes \mathfrak{g}^\perp$ called the "intrinsic contorsion". It is more conventional in the mathematics literature to define the corresponding torsion

$$(T^0)^p{}_{mn} \;=\; (\kappa^0)^p{}_{[mn]} \in \Lambda^1 \otimes \mathfrak{g}^\perp \;,$$

known as the intrinsic torsion.

Let us consider the decomposition of T^0 in the case of $SU(3)$-structure. The relevant representations are

$$\Lambda^1 \;\sim\; 3 \oplus \bar{3} \;, \qquad \mathfrak{su}(3) \;\sim\; 8 \;, \qquad \mathfrak{su}(3)^\perp \;\sim\; 1 \oplus 3 \oplus \bar{3}$$

Thus the intrinsic torsion $T^0 \in \Lambda^1 \otimes \mathfrak{su}(3)^\perp$ can be decomposed into the following $SU(3)$ representation

$$
\begin{aligned}
\Lambda^1 \otimes \mathfrak{su}(3)^\perp &= (\mathbf{3} \oplus \overline{\mathbf{3}}) \otimes (\mathbf{1} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}) \\
&= \underset{W_1}{(\mathbf{1} \oplus \mathbf{1})} \oplus \underset{W_2}{(\mathbf{8} \oplus \mathbf{8})} \oplus \underset{W_3}{(\mathbf{6} \oplus \overline{\mathbf{6}})} \oplus \underset{W_4}{(\mathbf{3} \oplus \overline{\mathbf{3}})} \oplus \underset{W_5}{(\mathbf{3} \oplus \overline{\mathbf{3}})'} \, .
\end{aligned}
$$

where

W_1 : complex scalar in $(\mathbf{1} \oplus \mathbf{1})$

W_2 : complex primitive 2-form in $(\mathbf{8} \oplus \mathbf{8})$

W_3 : real primitive $(2,1) \oplus (1,2)$-form in $(\mathbf{6} \oplus \overline{\mathbf{6}})$

W_4 : real 1-form in $(\mathbf{3} \oplus \overline{\mathbf{3}})$

W_5 : complex $(1,0)$-form in $(\mathbf{3} \oplus \overline{\mathbf{3}})'$

By using the above objects, we can represent the exterior derivatives of J and Ω such as

$$
\begin{aligned}
\mathrm{d}J &= -\frac{3}{2}\mathrm{Im}(W_1 \overline{\Omega}) + W_4 \wedge J + W_3 \, , \\
\mathrm{d}\Omega &= W_1 J \wedge J + W_2 \wedge J + \overline{W}_5 \wedge \Omega \, .
\end{aligned}
$$

If all W_a vanish, $\mathrm{d}J$ and $\mathrm{d}\Omega$ are also zero and the resulting geometry is a Kähler manifold, i.e., Calabi-Yau.

We can classify manifolds as follows:

- complex manifolds

$W_1 = W_2 = 0$	$T^0 \in W_3 \oplus W_4 \oplus W_5$	:	hermitian
$W_1 = W_2 = W_4 = 0$	$T^0 \in W_3 \oplus W_5$	:	balanced
$W_1 = W_2 = W_4 = W_5 = 0$	$T^0 \in W_3$	:	special-hermitian
$W_1 = W_2 = W_3 = W_4 = 0$	$T^0 \in W_5$	:	Kähler
$W_1 = W_2 = W_3 = W_4 = W_5 = 0$	$T^0 = 0$	:	Calabi-Yau

- non-complex manifolds

$W_2 = W_3 = W_4 = W_5 = 0$	$T^0 \in W_1$	:	nearly-Kähler
$W_1 = W_3 = W_4 = W_5 = 0$	$T^0 \in W_2$	:	almost-Kähler
$W_1^- = W_2^- = W_4 = W_5 = 0$	$T^0 \in W_1^+ \oplus W_2^+ \oplus W_3$	:	half-flat
$W_1 = W_2 = W_3 = 3W_4 + 2W_5 = 0$	$T^0 \in W_4 \oplus W_5$	:	conformally rescaled CY

Chiossi and Salamon [math.DG/0202282]

Cardoso, Curio, Dall'Agata, Lüst, Manousselis and Zoupanos [hep-th/0211118]

付録
Generalized Complex Structures

Generalized complex structures

Usual complex geometry deals with the tangent bundle of a manifold $\mathcal{T}\mathcal{M}$, whose sections are vectors X, and separately, with the cotangent bundle $\mathcal{T}^*\mathcal{M}$, whose sections are 1-forms ζ. In generalized complex geometry the tangent and cotangent bundle are joined as a single bundle, $\mathcal{T}\mathcal{M}\oplus\mathcal{T}^*\mathcal{M}$. Its sections are the sum of a vector field plus a one-form $X+\zeta$. The standard machinery of complex geometry can be generalized to this bundle. On this even-dimensional bundle, one can construct a generalized almost complex structure $\mathcal{J}$, which is a map of $\mathcal{T}\mathcal{M}\oplus\mathcal{T}^*\mathcal{M}$ to itself that squares to $-\mathbb{I}_{2d}$ (d is real the dimension of the manifold). This is analogous to an almost complex structure $I^m{}_n$ which is a bundle map from $\mathcal{T}\mathcal{M}$ to itself that squares to $-\mathbb{I}_d$. As for an almost complex structure, $\mathcal{J}$ must also satisfy the hermiticity condition $\mathcal{J}^{\mathfrak{T}}\mathcal{G}\mathcal{J}=\mathcal{G}$, with the respect to the natural metric on $\mathcal{T}\mathcal{M}\oplus\mathcal{T}^*\mathcal{M}$, $\mathcal{G}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

Usual complex structures I are naturally embedded into generalized ones $\mathcal{J}$: take $\mathcal{J}$ to be

$$\mathcal{J}_1\equiv\begin{pmatrix}I&0\\0&-I^{\mathfrak{T}}\end{pmatrix}, \tag{2}$$

with $I^m{}_n$ a regular almost complex structure (i.e. $I^2=-\mathbb{I}_d$). This $\mathcal{J}$ satisfies the desired properties, namely $\mathcal{J}^2=-\mathbb{I}_{2d}$, $\mathcal{J}^{\mathfrak{T}}\mathcal{G}\mathcal{J}=\mathcal{G}$. Another example of generalized almost complex structure can be built using a non degenerate two–form J_{mn},

$$\mathcal{J}_2 \equiv \begin{pmatrix} 0 & -J^{-1} \\ J & 0 \end{pmatrix} . \tag{3}$$

Given an almost complex structure $I^m{}_n$, one can build holomorphic and antiholomorphic projectors $\pi_\pm = \frac{1}{2}(\mathbb{I}_d \pm iI)$. Correspondingly, projectors can be build out of a generalized almost complex structure, $\Pi_\pm = \frac{1}{2}(\mathbb{I}_{2d} \pm i\mathcal{J})$. There is an integrability condition for generalized almost complex structures, analogous to the integrability condition for usual almost complex structures. For the usual complex structures, integrability, namely the vanishing of the Nijenhuis tensor, can be written as the condition $\pi_\mp[\pi_\pm X, \pi_\pm Y] = 0$, i.e. the Lie bracket of two holomorphic vectors should again be holomorphic. For generalized almost complex structures, integrability condition reads exactly the same, with π and X replaced respectively by Π and $X + \zeta$, and the Lie bracket replaced by the Courant bracket[1] on $\mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$. The Courant bracket does not satisfy Jacobi identity in general, but it does on the i–eigenspaces of $\mathcal{J}$. In case these conditions are fulfilled, we can drop the "almost" and speak of generalized complex structures.

For the two examples of generalized almost complex structure given above, $\mathcal{J}_1$ and $\mathcal{J}_2$, integrability condition turns into a condition on their building blocks, $I^m{}_n$ and J_{mn}. Integrability of $\mathcal{J}_1$ enforces

[1] The Courant bracket is defined as follows: $[X+\zeta, Y+\eta]_C = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\zeta - \frac{1}{2}\mathrm{d}(\iota_X\eta - \iota_Y\zeta)$.

I to be an integrable almost complex structure on $\mathcal{TM}$, and hence I is a complex structure, or equivalently the manifold is complex. For $\mathcal{J}_2$, which was built from a two-form J_{mn}, integrability imposes $\mathrm{d}J = 0$, thus making J into a symplectic form, and the manifold a symplectic one.

$\mathit{Clifford}(6,6)$ algebra

Spinors on $\mathcal{TM}$ transform under $\mathit{Clifford}(6)$, whose algebra is $\{\gamma^m, \gamma^n\} = 2g^{mn}$. There is a representation of this algebra in terms of forms. We can take[2] $\gamma^m = \mathrm{d}x^m \wedge + g^{mn}\iota_n$. These satisfy the $\mathit{Clifford}(d)$ algebra. The algebra of $\mathit{Clifford}(d,d)$ is instead

$$\{\Gamma^m, \Gamma^n\} = 0 \; , \qquad \{\Gamma^m, \Gamma_n\} = \delta^m_n \; , \qquad \{\Gamma_m, \Gamma_n\} = 0 \; .$$

Γ^m and Γ_m are independent, they cannot be obtained from one another by raising or lowering indices with the metric. There is also a representation of this algebra in terms of forms, namely

$$\Gamma^m = \mathrm{d}x^m \wedge \; , \qquad \Gamma_n = \iota_n \; . \tag{4}$$

The holomorphic 3-form Ω is a good vacuum of $\mathit{Clifford}(6,6)$, as it is annihilated by the holomorphic Γ^i and the antiholomorphic $\Gamma_{\bar{\imath}}$. These are half of the total gamma matrices, which implies that Ω is a pure $\mathit{Clifford}(6,6)$ spinor. Acting with the other half, $\Gamma^{\bar{\imath}}$ and Γ_i we get forms of all possible degrees. $\mathit{Clifford}(6,6)$ spinors are therefore equivalent to (p,q)-forms.

Using the Clifford map, a $\mathit{Clifford}(6,6)$ spinor can also be mapped to a bispinor:

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1 \cdots i_k} \mathrm{d}x^{i_i} \wedge \cdots \wedge \mathrm{d}x^{i_k} \qquad \longleftrightarrow \qquad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{i_1 \cdots i_k} \gamma^{i_i \cdots i_k}_{\alpha\beta} \; . \tag{5}$$

[2] ι_n: $\Lambda^p T^* \to \Lambda^{p-1} T^*$, $\iota_n \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} = p\delta_n^{[i_1} \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_p]}$.

On a space of $SU(3)$ structure, there is a nowhere vanishing $SU(3)$ invariant $\mathit{Clifford}(6)$ spinor η. Out of it, we can construct two nowhere vanishing $SU(3,3)$ invariant bispinors by tensoring η with its dagger, namely

$$\Phi_+ = \eta_+ \otimes \eta_+^\dagger \ , \qquad \Phi_- = \eta_+ \otimes \eta_-^\dagger \ . \tag{6}$$

(and its complex conjugates). Using Fierz identities, this tensor product can be written in terms of the bilinears by

$$\eta_+ \otimes \eta_\pm^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_\pm^\dagger \gamma_{i_1 \cdots i_k} \eta_+ \gamma^{i_k \cdots i_1} \tag{7}$$

Using the Clifford map (5) backwards, the tensor products in (6) are identified with regular forms. From now on, we will use $\Phi_\pm$ to denote just the forms.

The subindices plus and minus in $\Phi_\pm$ denote the $Spin(6,6)$ chirality: positive corresponds to an even form, and negative to an odd form. Irreducible $Spin(6,6)$ representations are actually "Majorana-Weyl", namely they are of definite parity –"Weyl"– and real –"Majorana"–.

The explicit expression for the $\mathit{Clifford}(6,6)$ spinors in (6) in terms of the defining forms for the $SU(3)$ structure is

$$\Phi_+ = \eta_+ \otimes \eta_+^\dagger = \frac{1}{8} \mathrm{e}^{-iJ} \ , \qquad \Phi_- = \eta_+ \otimes \eta_-^\dagger = -\frac{i}{8} \Omega \ . \tag{8}$$

The forms in (6), (8) are pure. This can be seen from writing the usual gamma matrices acting on the left of Φ (denoted as $\overrightarrow{\gamma}^m$) and on the right (denoted as $\overleftarrow{\gamma}^m$) in terms of the $Clifford(6,6)$ gamma matrices (4)

$$\overrightarrow{\gamma}^m = \frac{1}{2}(\mathrm{d}x^m \wedge + g^{mn}\iota_n) \ , \qquad \overleftarrow{\gamma}^m = \frac{1}{2}(\mathrm{d}x^m \wedge \pm g^{mn}\iota_n) \ , \tag{9}$$

where the $\pm$ sign depends on the parity of the spinor on which $\overleftarrow{\gamma}^m$ acts. We can check now that the forms (6) are indeed pure: the six gamma matrices that annihilate them are

$$(\delta + iI)_m{}^n\gamma_n\,\eta_+ \otimes \eta_\pm^\dagger = 0 \ , \qquad \eta_+ \otimes \eta_\pm^\dagger\,\gamma_n(\delta \mp iI)_m{}^n = 0 \ . \tag{10}$$

where I is the almost complex structure on the tangent bundle.

On a space of $SU(3)$ structure on $\mathcal{TM}$, there are therefore two $SU(3,3)$ invariant pure forms (and their complex conjugates), e^{-iJ} and Ω. This implies that the structure group on $\mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$, which is generically $SO(d,d)$, is reduced in this case to $SU(3) \times SU(3)$.

There is a one to one correspondence between a a pure spinor Φ and a generalized almost complex structure $\mathcal{J}$. It maps the $+i$ eigenspace of $\mathcal{J}$ to the annihilator space of the spinor Φ. Under this correspondence

$$\Phi_- = -\frac{i}{8}\,\Omega \ \leftrightarrow \mathcal{J}_1 \ , \qquad \Phi_+ = \frac{1}{8}\,\mathrm{e}^{-iJ} \leftrightarrow \mathcal{J}_2 \tag{11}$$

where $\mathcal{J}_1$ and $\mathcal{J}_2$ are defined in (2) and (3).

Integrability condition for the generalized complex structure corresponds on the pure spinor side to the condition

$$\mathcal{J} \text{ is integrable} \Leftrightarrow \exists \text{ vector } v \text{ and 1-form } \zeta \text{ such that } d\Phi = (v\llcorner + \zeta\wedge)\Phi$$

A generalized Calabi-Yau is a manifold on which a closed pure spinor exists:

$$\text{Generalized Calabi-Yau} \Leftrightarrow \exists\ \Phi \text{ pure such that } \mathrm{d}\Phi = 0$$

There is also the possibility of twisting by a closed three–form H. Using a three–form, the Courant bracket can be modified[3], and with it the integrability condition. In terms of "integrability" of the pure spinors Φ, adding H amounts to twisting the differential condititions for integrability and for Generalized Calabi-Yau. More precisely,

$$\text{"Twisted" Generalized Calabi-Yau} \Leftrightarrow \exists\ \Phi \text{ pure, and } H \text{ closed such that } (\mathrm{d} - H\wedge)\Phi = 0$$

[3] $[X + \zeta, Y + \eta]_H = [X + \zeta, Y + \eta]_C + \iota_X \iota_Y H.$

計算則

Roughtly speaking, a "primitive form β" means a orthogornal form satisfying $\beta \wedge J = 0$, where J is the fundamental 2-form. For example, a $(2,1)$-form $\beta^{(2,1)}$ includes a primitive $(2,1)$-form $\beta_0^{(2,1)}$ and a (primitive) $(1,0)$-form $\beta_0^{(1,0)}$ such as

$$\beta^{(2,1)} \ = \ \beta_0^{(2,1)} \oplus \beta_0^{(1,0)} \wedge J \ .$$

Now let us express W_a explicitly:

$$\begin{aligned}
\mathrm{d}\Omega_+ \wedge J \ &= \ \Omega_+ \wedge \mathrm{d}J \ \equiv \ W_1^+ J \wedge J \wedge J \ , \qquad \mathrm{d}\Omega_- \wedge J \ = \ \Omega_- \wedge \mathrm{d}J \ \equiv \ W_1^- J \wedge J \wedge J \ , \\
(\mathrm{d}\Omega_+)^{(2,2)} \ &\equiv \ W_1^+ J \wedge J + W_2^+ \wedge J \ , \qquad (\mathrm{d}\Omega_-)^{(2,2)} \ \equiv \ W_1^- J \wedge J + W_2^- \wedge J \ , \\
W_4 \ &= \ \frac{1}{2} J \lrcorner \, \mathrm{d}J \ , \qquad W_5 \ = \ \frac{1}{2} \Omega_+ \lrcorner \, \mathrm{d}\Omega_+ \ , \\
(\mathrm{d}J)^{(2,1)} \ &= \ (J \wedge W_4)^{(2,1)} + W_3 \ .
\end{aligned}$$

where

$$W_a \ = \ W_a^+ + i W_a^- \ ,$$

$$\lrcorner : \ \bigwedge^k T^* \otimes \bigwedge^n T^* \ \longrightarrow \ \bigwedge^{n-k} T^* \ , \qquad (L_{(k)}, M_{(n)}) \ \longmapsto \ \frac{1}{n!} \binom{n}{k} L^{a_1 \cdots a_k} M_{a_1 \cdots a_n} \, e^{a_{k+1}} \cdots e^{a_n} \ .$$

Note that e^i is a vielbein. For instance, $(e^1 \wedge e^2) \lrcorner (e^1 \wedge e^2 \wedge e^3 \wedge e^4) = e^3 \wedge e^4$.

付録

$\mathcal{N} = (2,2),(1,1)$ SUSY

Appendix

$\mathcal{N}=(2,2)$ supersymmetry

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm(\partial_0 \pm \partial_1) , \qquad \overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm(\partial_0 \pm \partial_1) ,$$

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm(\partial_0 \pm \partial_1) , \qquad \overline{Q}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm(\partial_0 \pm \partial_1) .$$

$$Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0 ,$$

$$\{Q_\pm, \overline{Q}_\pm\} = -2i(\partial_0 \pm \partial_1) = 2(H \mp P) ,$$

$$\{\overline{Q}_+, \overline{Q}_-\} = 0 , \quad \{Q_+, Q_-\} = 0 , \quad \{Q_-, \overline{Q}_+\} = 0 , \quad \{Q_+, \overline{Q}_-\} = 0 ,$$

$$\{D_\pm, \overline{D}_\pm\} = 2i(\partial_0 \pm \partial_1) ,$$

$$\{\overline{D}_\alpha, \overline{D}_\beta\} = \{D_\alpha, D_\beta\} = \{D_\pm, \overline{D}_\mp\} = 0 ,$$

$$\{D_\alpha, Q_\beta\} = \{\overline{D}_\alpha, Q_\beta\} = \{D_\alpha, \overline{Q}_\beta\} = \{\overline{D}_\alpha, \overline{Q}_\beta\} = 0 ,$$

$$[M, Q_\pm] = \mp Q_\pm , \quad [M, \overline{Q}_\pm] = \mp\overline{Q}_\pm ,$$

$$[F_V, Q_\pm] = -Q_\pm , \quad [F_V, \overline{Q}_\pm] = \overline{Q}_\pm ,$$

$$[F_A, Q_\pm] = \mp Q_\pm , \quad [F_A, \overline{Q}_\pm] = \pm\overline{Q}_\pm .$$

$\mathcal{N} = (2,2)$ superfields

Here we summarize $\mathcal{N} = (2,2)$ superfields:

chiral superfield Φ	$\overline{D}_\pm \Phi = 0$	$\Phi = \overline{D}_+ \overline{D}_- \Theta$
twisted chiral superfield Y	$\overline{D}_+ Y = D_- Y = 0$	$Y = \overline{D}_+ D_- \Theta$
real linear superfield G	$\overline{D}_+ \overline{D}_- G = D_+ D_- G = 0$	$G = Y + \overline{Y}$
real twisted linear superfield H	$\overline{D}_+ D_- H = D_+ \overline{D}_- H = 0$	$H = \Phi + \overline{\Phi}$
left semi-chiral superfield $\mathbb{X}$	$\overline{D}_+ \mathbb{X} = 0$	$\mathbb{X} = \overline{D}_+ \Theta$
right semi-chiral superfield $\mathbb{Y}$	$\overline{D}_- \mathbb{Y} = 0$	$\mathbb{Y} = \overline{D}_- \Theta$
complex linear superfield Σ	$\overline{D}_+ \overline{D}_- \Sigma = 0$	$\Sigma = a\mathbb{X} + b\mathbb{Y}$
complex twisted linear superfield $\widetilde{\Sigma}$	$\overline{D}_+ D_- \widetilde{\Sigma} = 0$	$\widetilde{\Sigma} = a\mathbb{X} + b\overline{\mathbb{Y}}$

where Θ is an unconstrained superfield; a and b are complex constants.

$$
\begin{aligned}
\Phi &= \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}\theta^-\psi_- + 2i\theta^+\theta^- F \\
&\quad - i\theta^+\overline{\theta}^+(\partial_0+\partial_1)\phi - i\theta^-\overline{\theta}^-(\partial_0-\partial_1)\phi + \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0^2-\partial_1^2)\phi \\
&\quad + \sqrt{2}\theta^+\overline{\theta}^+\theta^-(\partial_0+\partial_1)\psi_- + \sqrt{2}\theta^-\overline{\theta}^-\theta^+(\partial_0-\partial_1)\psi_+ \ ,
\end{aligned}
$$

$$
\begin{aligned}
Y &= y + i\sqrt{2}\theta^+\overline{\chi}_+ + i\sqrt{2}\overline{\theta}^-\chi_- + 2i\theta^+\overline{\theta}^- G \\
&\quad - i\theta^+\overline{\theta}^+(\partial_0+\partial_1)y + i\theta^-\overline{\theta}^-(\partial_0-\partial_1)y - \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0^2-\partial_1^2)y \\
&\quad - \sqrt{2}\theta^-\overline{\theta}^-\theta^+(\partial_0-\partial_1)\overline{\chi}_+ + \sqrt{2}\theta^+\overline{\theta}^+\overline{\theta}^-(\partial_0+\partial_1)\chi_- \ ,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{X} &= \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}(\theta^-\psi_- + \overline{\theta}^-\chi_-) + 2i\theta^+(\theta^- F + \overline{\theta}^- G) \\
&\quad + \theta^-\overline{\theta}^- A_= + 2\theta^+\theta^-\overline{\theta}^-\zeta_- - i\theta^+\overline{\theta}^+(\partial_0+\partial_1)\phi \\
&\quad + \sqrt{2}\theta^+\overline{\theta}^+(\partial_0+\partial_1)(\theta^-\psi_- + \overline{\theta}^-\chi_-) + i\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0+\partial_1)A_= \ ,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{Y} &= \phi + i\sqrt{2}(\theta^+\psi_+ + \overline{\theta}^+\chi_+) + i\sqrt{2}\theta^-\psi_- + 2i\theta^+\theta^- F + 2i\overline{\theta}^+\theta^- N \\
&\quad - i\theta^-\overline{\theta}^-(\partial_0-\partial_1)\phi + \theta^+\overline{\theta}^+ B_{\#} - 2\theta^-\theta^+\overline{\theta}^+\zeta_+ \\
&\quad + \sqrt{2}\theta^-\overline{\theta}^-(\partial_0-\partial_1)(\theta^+\psi_+ + \overline{\theta}^+\chi_+) + i\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0-\partial_1)B_{\#} \ .
\end{aligned}
$$

$\mathcal{N} = (1,1)$ supersymmetry

Here we consider $\mathcal{N} = (1,1)$ supersymmetry which has two real supercharges, one with positive chirality and the other with negative chirality:

$$\theta_1^{\pm} \;\equiv\; -i\mathrm{e}^{-i\nu_{\pm}}\theta^{\pm} \;=\; i\mathrm{e}^{+i\nu_{\pm}}\overline{\theta}^{\pm} \qquad \text{where } \theta_1^{\pm} \text{ is real} \tag{12}$$

We introduce the following differential operators

$$\begin{aligned} Q_{\pm}^1 \;&\equiv\; \frac{1}{\sqrt{2}}\Big\{\mathrm{e}^{i\nu_{\pm}}Q_{\pm} + \mathrm{e}^{-i\nu_{\pm}}\overline{Q}_{\pm}\Big\}\Big|_{\text{eq.(12)}} \;=\; -\frac{i}{\sqrt{2}}\frac{\partial}{\partial\theta_1^{\pm}} + \sqrt{2}\theta_1^{\pm}(\partial_0 \pm \partial_1) \;, \\ D_{\pm}^1 \;&\equiv\; \frac{1}{\sqrt{2}}\Big\{\mathrm{e}^{i\nu_{\pm}}D_{\pm} + \mathrm{e}^{-i\nu_{\pm}}\overline{D}_{\pm}\Big\}\Big|_{\text{eq.(12)}} \;=\; -\frac{i}{\sqrt{2}}\frac{\partial}{\partial\theta_1^{\pm}} - \sqrt{2}\theta_1^{\pm}(\partial_0 \pm \partial_1) \;. \end{aligned}$$

These operators obey the anti-commutation relations

$$\begin{aligned} \{Q_{\pm}^1, Q_{\pm}^1\} \;&=\; -2i(\partial_0 \pm \partial_1) \;=\; 2(H \mp P) \;, \qquad \{Q_+^1, Q_-^1\} \;=\; 0 \;, \\ \{D_{\pm}^1, D_{\pm}^1\} \;&=\; +2i(\partial_0 \pm \partial_1) \;, \qquad \{D_+^1, D_-^1\} \;=\; 0 \;, \qquad \{Q_{\alpha}^1, D_{\beta}^1\} \;=\; 0 \;. \end{aligned}$$

We also define the following "differential operators":

$$\begin{aligned} \widetilde{Q}_{\pm}^1 \;&\equiv\; \frac{i}{\sqrt{2}}\Big\{\mathrm{e}^{i\nu_{\pm}}Q_{\pm} - \mathrm{e}^{-i\nu_{\pm}}\overline{Q}_{\pm}\Big\} \;, \qquad \text{under the constraint (12):} \quad \widetilde{Q}_{\pm}^1\Big| \;=\; 0 \;, \\ \widetilde{D}_{\pm}^1 \;&\equiv\; \frac{i}{\sqrt{2}}\Big\{\mathrm{e}^{i\nu_{\pm}}D_{\pm} - \mathrm{e}^{-i\nu_{\pm}}\overline{D}_{\pm}\Big\} \;, \qquad \text{under the constraint (12):} \quad \widetilde{D}_{\pm}^1\Big| \;=\; 0 \;. \end{aligned}$$

Under the constraint (12) these operators are trivially zero. However, they are another two supercharges and two covariant derivatives in the original $\mathcal{N} = (2,2)$ supersymmetry. We can easily find this "second" $(1,1)$ supersymmetry operators satisfy the following anti-commutation relations in the $(2,2)$ supersymmetry level:

$$\begin{aligned}
\{\widetilde{Q}^1_{\pm}, \widetilde{Q}^1_{\pm}\} &= -2i\big(\partial_0 \pm \partial_1\big) \;=\; 2(H \mp P) \, , \qquad \{\widetilde{Q}^1_{+}, \widetilde{Q}^1_{-}\} \;=\; 0 \, , \\
\{\widetilde{D}^1_{\pm}, \widetilde{D}^1_{\pm}\} &= +2i\big(\partial_0 \pm \partial_1\big) \, , \qquad \{\widetilde{D}^1_{+}, \widetilde{D}^1_{-}\} \;=\; 0 \, , \\
\{\widetilde{Q}^1_{\alpha}, \widetilde{D}^1_{\beta}\} &= 0 \, .
\end{aligned}$$

Furthermore we can check that the first $(1,1)$ supersymmetry and the second $(1,1)$ supersymmetry commute with each other:

$$\{Q^1_{\alpha}, \widetilde{Q}^1_{\beta}\} \;=\; \{D^1_{\alpha}, \widetilde{D}^1_{\beta}\} \;=\; \{Q^1_{\alpha}, \widetilde{D}^1_{\beta}\} \;=\; \{D^1_{\alpha}, \widetilde{Q}^1_{\beta}\} \;=\; 0 \, .$$

This result is consistent with the original $\mathcal{N} = (2,2)$ supersymmetry.

$\mathcal{N}=(1,1)$ scalar/spinor superfields from $\mathcal{N}=(2,2)$ semi-chiral superfields

$$
\begin{aligned}
\mathbb{X}^{(1,1)} &= \mathbb{X}^{(2,2)}\Big| \\
&= \phi + i\sqrt{2}\theta_1^+\widehat{\psi}_+ + i\sqrt{2}\theta_1^-(\widehat{\psi}_- + \widehat{\chi}_-) + 2i\theta_1^+\theta_1^-(\widehat{F}+\widehat{G})\ , \\
\Psi_-^{(1,1)} &= \widetilde{Q}^1_-\mathbb{X}^{(2,2)}\Big| \\
&= i(\widehat{\psi}_- - \widehat{\chi}_-) - i\sqrt{2}\theta_1^+(\widehat{F}-\widehat{G}) + \sqrt{2}\theta_1^- A_{=} + 2\sqrt{2}\theta_1^+\theta_1^-\widehat{\zeta}_-\ ,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{Y}^{(1,1)} &= \mathbb{Y}^{(2,2)}\Big| \\
&= \phi + i\sqrt{2}\theta_1^+(\widehat{\psi}_+ + \widehat{\chi}_+) + i\sqrt{2}\theta_1^-\widehat{\psi}_- + 2i\theta_1^+\theta_1^-(\widehat{F}+\widehat{N})\ , \\
\Upsilon_+^{(1,1)} &= \widetilde{Q}^1_+\mathbb{Y}^{(2,2)}\Big| \\
&= i(\widehat{\psi}_+ - \widehat{\chi}_+) + \sqrt{2}\theta_1^+ B_{+\!\!+} + i\sqrt{2}\theta_1^-(\widehat{F}-\widehat{N}) - 2\sqrt{2}\theta_1^+\theta_1^-\widehat{\zeta}_+\ .
\end{aligned}
$$