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Generalized Complex Geometries and Supersymmetric Nonlinear Sigma Models

— *inspired by flux compactification* —

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The title is changed from

“On gauged linear sigma models for generalized complex geometries”

Worldsheet Theories

The main ideas to costruct nonlinear sigma models on generalized complex geometries are

- **extension of the worldsheet action: 1st order action**
- **supersymetrization**
- **closure condition of extended supersymmetry**

Extension of nonlinear sigma models

First we prepare a standard bosonic nonlinear sigma model with a B-field (2nd order action):

$$S \;=\; \frac{1}{2}\int \big\{ g_{\mu\nu}(X)\, \mathrm{d}X^{\mu} \wedge *\mathrm{d}X^{\nu} + B_{\mu\nu}(X)\, \mathrm{d}X^{\mu} \wedge \mathrm{d}X^{\nu} \big\}$$

and generalize this by introducing auxiliary fields η_{μ} (1st order action):

$$S \;=\; \frac{1}{2}\int \Big\{ \eta_{\mu} \wedge \mathrm{d}X^{\mu} + \frac{1}{2}\theta^{\mu\nu}\, \eta_{\mu} \wedge \eta_{\nu} + \frac{1}{2} G^{\mu\nu}\, \eta_{\mu} \wedge *\eta_{\nu} + \frac{1}{2}(B-b)_{\mu\nu}\, \mathrm{d}X^{\mu} \wedge \mathrm{d}X^{\nu} \Big\}$$

$$E_{\mu\nu} \;=\; g_{\mu\nu} + b_{\mu\nu} \, , \qquad E^{\mu\lambda} E_{\lambda\nu} \;=\; \delta^{\mu}{}_{\nu}$$

$$G^{\mu\nu} \;=\; \frac{1}{2}(E^{\mu\nu} + E^{\nu\mu}) \, , \qquad \theta^{\mu\nu} \;=\; \frac{1}{2}(E^{\mu\nu} - E^{\nu\mu})$$

They depend on the Neveu-Schwarz closed 3-form flux $H = \mathrm{d}B$.

We obtain the 2nd order action from the 1st order action when we integrate out η_{μ}.

In this formulation, we can define a (con)torsion tensor of the target space

$$\nabla^{(T)} = \nabla^0 + \kappa = \nabla^0 + H \ ,$$

where ∇^0 is the Levi-Civita connection and H is the NS-flux.

Supersymmetrization and closure condition of extra supersymmetry

We introduce a supersymmetry to obtain a standard almost complex structure I.

We also introduce an extra supersymmetry in order to extend the standard almost complex structure to generalized almost complex structures $\mathcal{J}_i$

The most simple example: $\mathcal{J}_1$ and $\mathcal{J}_2$ consist of I and J, respectively

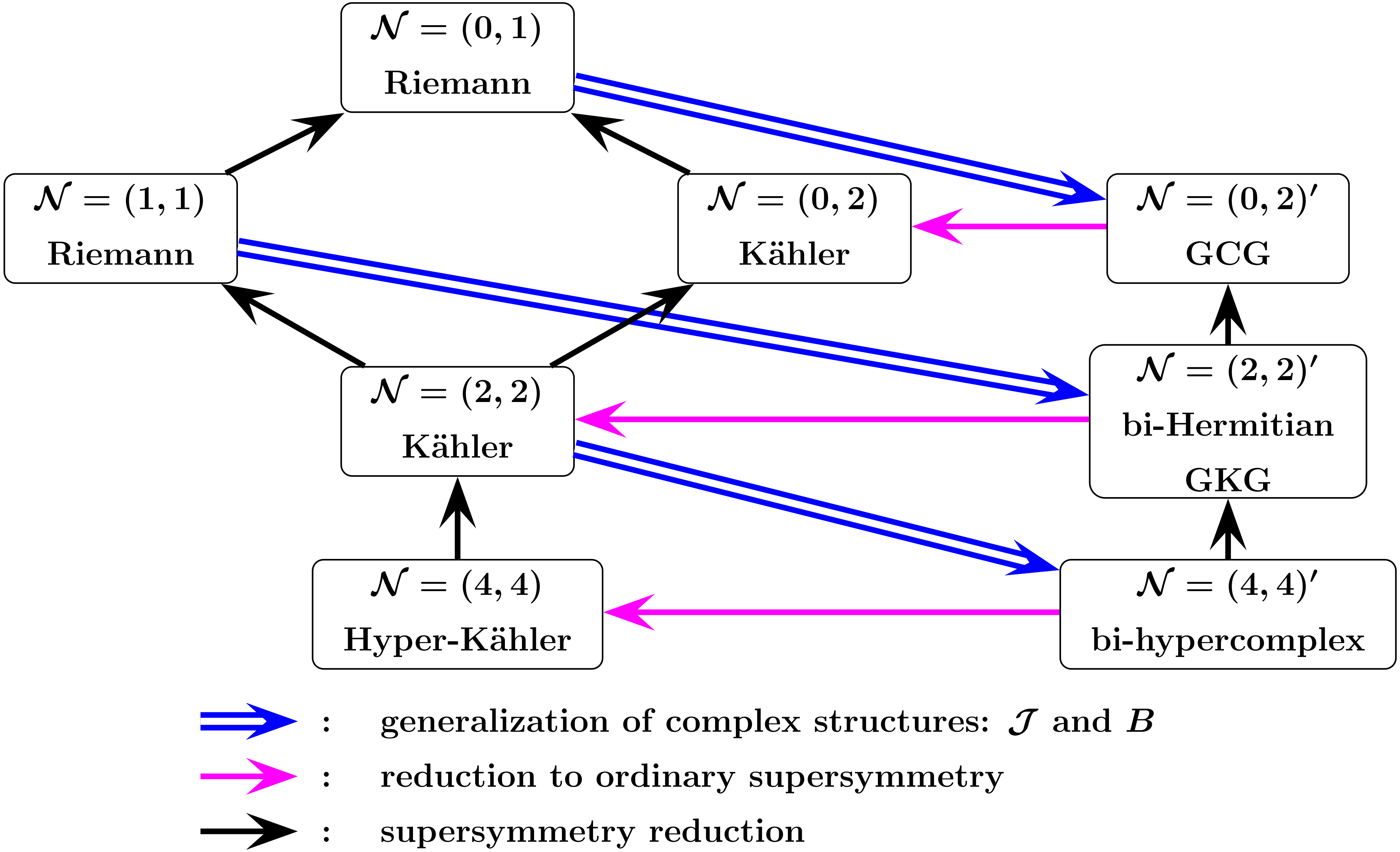
[Lindstrom, Minasian, Tomasiello, Zabzine]:

$(0,2)$ nonlinear sigma models from $(0,1)$ supersymmetric theories

[Lindstrom, Rocek, von Unge, Zabzine]:

$(2,2)$ nonlinear sigma models from $(2,1)$ supersymmetric theories

Relations among various supersymmetric theories



Application to mirror symmetry

Minasian et al suggested a new type of mirror symmetry $e^{B-iJ} \leftrightarrow \Omega$ on generalized complex geometries.

Lindström et al discussed $\mathcal{N} = (2,2)$ sigma models described only one function K, which is a function of semi-chiral superfields.

We wish to consider the duality transformation procedure discussed by Roček-Verlinde (and Hori-Vafa as a gauge theory).

Let us first discuss mirror symmetry (or 3 T-dualities) in terms of some objects of the structures J and Ω:

$$\begin{aligned} \Phi_+ &\longleftrightarrow \Phi_- \\ e^{B-iJ} &\longleftrightarrow \Omega \end{aligned}$$

where in the second line we have specialized to $SU(3)$ structure manifolds.

We expect it will also hold for the general case of $SU(3) \times SU(3)$ structures on $T \oplus T^*$.

Sigma Models on Generalized Complex Geometries

$\mathcal{N} = (2,2)$ supersymmetric sigma models of semi-chiral superfields

Prepare an $\mathcal{N} = (2,2)$ supersymmetric Lagrangian of semi-chiral superfields $\mathbb{X}$ and $\mathbb{Y}$:

$$\overline{D}_+ \mathbb{X} \;=\; 0\,, \qquad \overline{D}_- \mathbb{Y} \;=\; 0$$

Reduce it to the $\mathcal{N} = (1,1)$ theory ($\widetilde{Q}^1_\pm$ are extra $\mathcal{N} = (1,1)$ superchages):

$$\mathscr{L} \;=\; \int \mathrm{d}^4\theta\, K(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}) \;=\; -\frac{1}{8} \int \mathrm{d}\theta^+_1 \mathrm{d}\theta^-_1\, \widetilde{Q}^1_+ \widetilde{Q}^1_- K(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}})$$

Semi-chiral superfields decompose into two independent $\mathcal{N} = (1,1)$ superfields:

$$\begin{aligned}
\mathbb{X}^{(2,2)} \;&\longrightarrow\; \big\{ \mathbb{X}^{(1,1)}\,,\; \Psi^{(1,1)}_- \big\} & \qquad \Psi^{(1,1)}_- \;&\equiv\; \widetilde{Q}^1_- \mathbb{X}^{(2,2)}\big| \\
\mathbb{Y}^{(2,2)} \;&\longrightarrow\; \big\{ \mathbb{Y}^{(1,1)}\,,\; \Upsilon^{(1,1)}_+ \big\} & \qquad \Upsilon^{(1,1)}_+ \;&\equiv\; \widetilde{Q}^1_+ \mathbb{Y}^{(2,2)}\big|
\end{aligned}$$

We obtain a general $\mathcal{N} = (1,1)$ supersymmetric Lagrangian such as

$$\begin{aligned}
\mathscr{L} \;&=\; -\frac{1}{8} \int \mathrm{d}\theta^+_1 \mathrm{d}\theta^-_1 \widetilde{Q}^1_+ \widetilde{Q}^1_- K \\
&=\; -\frac{1}{8} \int \mathrm{d}\theta^+_1 \mathrm{d}\theta^-_1 \Big\{ - D^1_+ \mathbb{X}^A\, m_{AB'}\, D^1_- \mathbb{Y}^{B'} + \Upsilon^{A'}_+\, n_{A'B}\, \Psi^B_- \\
&\qquad\qquad + \Psi^A_- \big(2\omega_{AB}\, D^1_+ \mathbb{X}^B + i p_{AB'}\, D^1_+ \mathbb{Y}^{B'} \big) + \Upsilon^{A'}_+ \big(i q_{A'B}\, D^1_- \mathbb{X}^B - 2\omega_{A'B'}\, D^1_- \mathbb{Y}^{B'} \big) \Big\}
\end{aligned}$$

Now we introduced the following matrix-valued functions:

$$\boldsymbol{J^A{}_B} \;=\; \begin{pmatrix} J^a{}_b & 0 \\ 0 & \overline{J}^{\overline{a}}{}_{\overline{b}} \end{pmatrix} \;, \qquad (J^B{}_A)^{\mathfrak{T}} \;=\; J_A{}^B \;=\; \begin{pmatrix} J_a{}^b & 0 \\ 0 & \overline{J}_{\overline{a}}{}^{\overline{b}} \end{pmatrix}$$

$$m_{AB'} \;\equiv\; \begin{pmatrix} -J_a{}^c\, K_{cd'}\, J^{d'}{}_{b'} & -J_a{}^c\, K_{c\overline{d}'}\, \overline{J}^{\overline{d}'}{}_{\overline{b}'} \\ -\overline{J}_{\overline{a}}{}^{\overline{c}}\, K_{\overline{c}d'}\, J^{d'}{}_{b'} & -\overline{J}_{\overline{a}}{}^{\overline{c}}\, K_{\overline{c}\overline{d}'}\, \overline{J}^{\overline{d}'}{}_{\overline{b}'} \end{pmatrix} \;=\; -J_A{}^C\, K_{CD'}\, J^{D'}{}_{B'}$$

$$n_{A'B} \;\equiv\; \begin{pmatrix} K_{ab'} & K_{a\overline{b}'} \\ K_{\overline{a}b'} & K_{\overline{a}\overline{b}'} \end{pmatrix} \;=\; K_{A'B}$$

$$p_{AB'} \;\equiv\; \begin{pmatrix} -iJ_a{}^c\, K_{cb'} & -iJ_a{}^c\, K_{c\overline{b}'} \\ -i\overline{J}_{\overline{a}}{}^{\overline{c}}\, K_{\overline{c}b'} & -i\overline{J}_{\overline{a}}{}^{\overline{c}}\, K_{\overline{c}\overline{b}'} \end{pmatrix} \;=\; -iJ_A{}^C\, K_{CB'}$$

$$q_{A'B} \;\equiv\; \begin{pmatrix} iJ_{a'}{}^{c'}\, K_{c'b} & iJ_{a'}{}^{c'}\, K_{c'\overline{b}} \\ i\overline{J}_{\overline{a}'}{}^{\overline{c}'}\, K_{\overline{c}'b} & i\overline{J}_{\overline{a}'}{}^{\overline{c}'}\, K_{\overline{c}'\overline{b}} \end{pmatrix} \;=\; iJ_{A'}{}^{C'}\, K_{C'B}$$

$$2\omega_{AB} \;\equiv\; \begin{pmatrix} J_a{}^c\, K_{cb} - K_{ac}\, J^c{}_b & J_a{}^c\, K_{c\overline{b}} - K_{a\overline{c}}\, \overline{J}^{\overline{c}}{}_{\overline{b}} \\ \overline{J}_{\overline{a}}{}^{\overline{c}}\, K_{\overline{c}b} - K_{\overline{a}c}\, J^c{}_b & \overline{J}_{\overline{a}}{}^{\overline{c}}\, K_{\overline{c}\overline{b}} - K_{\overline{a}\overline{c}}\, \overline{J}^{\overline{c}}{}_{\overline{b}} \end{pmatrix} \;=\; J_A{}^C\, K_{CB} - K_{AC}\, J^C{}_B$$

$$2\omega_{A'B'} \;\equiv\; \begin{pmatrix} J_{a'}{}^{c'}\, K_{c'b'} - K_{a'c'}\, J^{c'}{}_{b'} & J_{a'}{}^{c'}\, K_{c'\overline{b}'} - K_{a'\overline{c}'}\, \overline{J}^{\overline{c}'}{}_{\overline{b}'} \\ \overline{J}_{\overline{a}'}{}^{\overline{c}'}\, K_{\overline{c}'b'} - K_{\overline{a}'c'}\, J^{c'}{}_{b'} & \overline{J}_{\overline{a}'}{}^{\overline{c}'}\, K_{\overline{c}'\overline{b}'} - K_{\overline{a}'\overline{c}'}\, \overline{J}^{\overline{c}'}{}_{\overline{b}'} \end{pmatrix} \;=\; J_{A'}{}^{C'}\, K_{C'B'} - K_{A'C'}\, J^{C'}{}_{B'}$$

$$D^1_+\mathbb{X}^A \;=\; \begin{pmatrix} D^1_+\mathbb{X}^a \\ D^1_+\overline{\mathbb{X}}^{\overline{a}} \end{pmatrix} \;, \quad \Psi^A_- \;=\; \left(\Psi^a_- \,,\; \overline{\Psi}^{\overline{a}}_- \right) , \qquad D^1_+\mathbb{Y}^{A'} \;=\; \begin{pmatrix} D^1_+\mathbb{Y}^{a'} \\ D^1_+\overline{\mathbb{Y}}^{\overline{a}'} \end{pmatrix} \;, \quad \Upsilon^{A'}_+ \;=\; \left(\Upsilon^{a'}_+ \,,\; \overline{\Upsilon}^{\overline{a}'}_+ \right)$$

Topological sigma models

We can easily obtain a "topological" theory written only by $\mathbb{X}^a$ or only by $\mathbb{Y}^{b'}$:

$$\mathscr{L}_{\mathbb{X}} = \int d^4\theta \, K(\mathbb{X}, \overline{\mathbb{X}}) = -\frac{1}{4} \int d\theta_1^+ d\theta_1^- \Big\{ \Psi_-^A \, \omega_{AB} \, D_+^1 \mathbb{X}^B \Big\} = -\frac{1}{4} \int d\theta_1^+ d\theta_1^- \Big\{ S_{A-} D_+^1 \mathbb{X}^A \Big\}$$

$$\mathscr{L}_{\mathbb{Y}} = \int d^4\theta \, K(\mathbb{Y}, \overline{\mathbb{Y}}) = \frac{1}{4} \int d\theta_1^+ d\theta_1^- \Big\{ \Upsilon_+^{A'} \, \omega_{A'B'} \, D_-^1 \mathbb{Y}^{B'} \Big\} = \frac{1}{4} \int d\theta_1^+ d\theta_1^- \Big\{ S_{A'+} D_-^1 \mathbb{Y}^{A'} \Big\}$$

where we re-defined $S_{A-} = \Psi_-^B \, \omega_{BA}$, $\Psi_-^A = S_{B-} \, \omega^{BA}$, $S_{A'+} = \Upsilon_+^{B'} \, \omega_{B'A'}$ $\Upsilon_+^{A'} = S_{B'+} \, \omega^{B'A'}$.

We determine the second, non-manifest supersymmetry transformation rules:

$$\widetilde{\delta}^{(+)} \mathbb{X}^A = \widetilde{\varepsilon}^+ \, J^A{}_B \, D_+^1 \mathbb{X}^B \,, \qquad \widetilde{\delta}^{(-)} \mathbb{X}^A = -\widetilde{\varepsilon}^- \, \omega^{AB} \, (S_{B-})^{\mathfrak{T}}$$

$$\begin{aligned}
\widetilde{\delta}^{(+)} S_{A-} &= -\widetilde{\varepsilon}^+ \, D_+^1 S_{B-} \, J^B{}_A \\
&\quad + (1 - \alpha_+) S_{C-} \, \omega^{CB} \Big\{ \partial_E(\omega_{BA}) \, (\widetilde{\delta}^{(+)} \mathbb{X}^E) + \partial_L(\omega_{BE}) \, \widetilde{\varepsilon}^+ \, J^L{}_A \, (D_+^1 \mathbb{X}^E) \Big\} \\
&\quad + \beta \, S_{C-} \, \omega^{CB} \Big\{ \partial_A(\omega_{BE}) \, (\widetilde{\delta}^{(+)} \mathbb{X}^E) + \partial_E(\omega_{BL}) \, \widetilde{\varepsilon}^+ \, J^L{}_A \, D_+^1 \mathbb{X}^E \Big\} \\
\widetilde{\delta}^{(-)} S_{A-} &= -i \widetilde{\varepsilon}^- \, \big\{ \omega_{AC} \, (\partial_0 - \partial_1) \mathbb{X}^C \big\}^{\mathfrak{T}} + S_{C-} \, \omega^{CB} \, \partial_E(\omega_{BA}) \, (\widetilde{\delta}^{(-)} \mathbb{X}^E)
\end{aligned}$$

The transformation rules of $\mathbb{Y}$-theory are

$$\widetilde{\delta}^{(+)} \mathbb{Y}^{A'} = -\widetilde{\varepsilon}^+ \, \omega^{A'C'} \, (S_{C'+})^{\mathfrak{T}} \,, \qquad \widetilde{\delta}^{(-)} \mathbb{Y}^{A'} = \widetilde{\varepsilon}^- \, J^{A'}{}_{C'} \, D_-^1 \mathbb{Y}^{C'}$$

$$\begin{aligned}
\widetilde{\delta}^{(+)} S_{A'+} &= -i \widetilde{\varepsilon}^+ \, \big\{ \omega_{A'C'} \, (\partial_0 + \partial_1) \mathbb{Y}^{C'} \big\}^{\mathfrak{T}} + S_{C'+} \, \omega^{C'B'} \, \partial_{E'}(\omega_{B'A'}) \, (\widetilde{\delta}^{(+)} \mathbb{Y}^{E'}) \\
\widetilde{\delta}^{(-)} S_{A'+} &= -\widetilde{\varepsilon}^- \, D_-^1 S_{B'+} \, J^{B'}{}_{A'} + (\text{still undetermined terms (2005 9/20)})
\end{aligned}$$

Embedded generalized complex structures

In these models, the additional supersymmetry transformations are expressed (we abbrebiated the terms described by the derivatives of complex structures and symplectic forms):

$$\left\{ \begin{array}{rcl} \widetilde{\delta}^{(+)} \mathbb{X}^A &=& J^A{}_B D^1_+ \mathbb{X}^B \\ \widetilde{\delta}^{(+)} (S_{A-})^{\mathfrak{T}} &=& -J_A{}^B \, (D^1_+ S_{B-})^{\mathfrak{T}} \end{array} \right. \quad \longleftrightarrow \quad \mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^{\mathfrak{T}} \end{pmatrix}$$

$$\left\{ \begin{array}{rcl} \widetilde{\delta}^{(-)} \mathbb{X}^A &=& -\omega^{AB} (S_{B-})^{\mathfrak{T}} \\ \widetilde{\delta}^{(-)} (S_{A-})^{\mathfrak{T}} &=& -i \omega_{AB} (\partial_0 - \partial_1) \mathbb{X}^B \end{array} \right. \quad \longleftrightarrow \quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

$$\left\{ \begin{array}{rcl} \widetilde{\delta}^{(+)} \mathbb{Y}^{A'} &=& -\omega^{A'B'} (S_{B'+})^{\mathfrak{T}} \\ \widetilde{\delta}^{(+)} (S_{A'+})^{\mathfrak{T}} &=& -i \omega_{A'B'} (\partial_0 + \partial_1) \mathbb{Y}^{B'} \end{array} \right. \quad \longleftrightarrow \quad \widetilde{\mathcal{J}}_2 = \begin{pmatrix} 0 & -\omega'^{-1} \\ \omega' & 0 \end{pmatrix}$$

$$\left\{ \begin{array}{rcl} \widetilde{\delta}^{(-)} \mathbb{Y}^{A'} &=& J^{A'}{}_{B'} D^1_- \mathbb{Y}^{B'} \\ \widetilde{\delta}^{(-)} (S_{A'+})^{\mathfrak{T}} &=& -J_{A'}{}^{B'} \, (D^1_- S_{B'+})^{\mathfrak{T}} \end{array} \right. \quad \longleftrightarrow \quad \widetilde{\mathcal{J}}_1 = \begin{pmatrix} J' & 0 \\ 0 & -J'^{\mathfrak{T}} \end{pmatrix}$$

A conjecture on the mirror symmetry

The mirror symmetry means the symmetry under the exchange of two Clifford$(6,6)$ (pure) spinors $\Phi_{\pm}$ and the exchange of two generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$:

(pure) spinors	$SU(3)$-invariant tensors	$SU(3)$-invariant bispinors	generalized complex structures
Φ_-	$-\frac{i}{8}\Omega$	$\eta_+ \otimes \eta_+^{\dagger}$	$\mathcal{J}_1 = \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix}$
Φ_+	$\frac{1}{8}\mathrm{e}^{-iJ}$	$\eta_+ \otimes \eta_-^{\dagger}$	$\mathcal{J}_2 = \begin{pmatrix} 0 & -J^{-1} \\ J & 0 \end{pmatrix}$

Applying the above conjecture to the two topological theories, we find a candidate of the mirror transformations:

$$\mathcal{J}_1 \longleftrightarrow \widetilde{\mathcal{J}}_2 \, , \qquad \mathcal{J}_2 \longleftrightarrow \widetilde{\mathcal{J}}_1$$

This means exchange between $\mathbb{X}^A$ and $\mathbb{Y}^{A'}$

Note that we have already understood that, in two-dimensional worldsheet theory, the mirror dual is represented by the exchange of a (c,c)-ring and a (a,c)-ring

$$(\Phi, \overline{\Phi}) \longleftrightarrow (Y, \overline{Y})$$

Φ and Y are a chiral superfield and a twisted chiral superfield defined by

$$\overline{D}_{\pm}\Phi = 0 \, , \qquad \overline{D}_+ Y = D_- Y = 0$$

The semi-chiral superfields $\mathbb{X}$ and $\mathbb{Y}$ can be represented as functions of the chiral and twisted chiral superfields:

$$\mathbb{X} \;=\; \mathbb{X}(\boldsymbol{\Phi}, \boldsymbol{Y}) \;, \qquad \mathbb{Y} \;=\; \mathbb{Y}(\boldsymbol{\Phi}, \overline{\boldsymbol{Y}})$$

We can easily check the above relations from the constraints $\overline{\boldsymbol{D}}_{+}\mathbb{X} = \overline{\boldsymbol{D}}_{-}\mathbb{Y} = 0$. So, if the mirror dual transformation means the mapping from $(\mathbb{X}, \overline{\mathbb{X}})$ to $(\mathbb{Y}, \overline{\mathbb{Y}})$, this mirror dual also means the mapping from $(\boldsymbol{\Phi}, \overline{\boldsymbol{\Phi}})$ to $(\boldsymbol{Y}, \overline{\boldsymbol{Y}})$ and vice versa.

Thus we insist that

> **The topological theory $\mathscr{L}_{\mathbb{X}}$ should be mapped to the topological theory $\mathscr{L}_{\mathbb{Y}}$, and vice versa.**

A conjecture on the reduction of the degrees of freedom

If the theory is given only by semi-chiral superfields $\mathbb{X}^a(\Phi, Y)$ and $\overline{\mathbb{X}}^{\overline{a}}(\overline{\Phi}, \overline{Y})$, a half of the degrees of freedom should be irrelevant and we should fix them. The half of the degrees of freedom in the physical theory should also be fixed in order to describe a target space of suitable dimensions[1].

Now we propose a reduction rule of the semi-chiral superfields as follows:

$$\begin{aligned} \overline{D}_- \mathbb{X}^a &= 0 \, , \qquad & D_- \overline{\mathbb{X}}^{\overline{a}} &= 0 \\ \overline{D}_+ \mathbb{Y}^{a'} &= 0 \, , \qquad & D_+ \overline{\mathbb{Y}}^{\overline{a}'} &= 0 \end{aligned}$$

This means that the Y dependence in $\mathbb{X}$ disappears under this constraint. In other words, the semi-chiral superfields $\mathbb{X}^A$ and $\mathbb{Y}^{A'}$ become chiral superfields, which probably represent the complex coordinates of the target space.

We can find the above constraints from the F-term in the $\mathcal{N} = (2,2)$ theory such as

$$\begin{aligned} \mathcal{L}_{\rm c} &\equiv \int d^2\theta \, \mathbb{X}^a \widetilde{\mathbb{Y}}^{a'} + \int d^2\overline{\theta} \, \overline{\mathbb{X}}^{\overline{a}} \overline{\widetilde{\mathbb{Y}}}^{\overline{a}'} \\ &= \frac{1}{2} e^{-i\nu_+ - i\nu_-} \int d\theta_1^+ d\theta_1^- \, \mathbb{X}^{(1,1)} \widetilde{\mathbb{Y}}^{(1,1)} - \frac{1}{2} e^{+i\nu_+ + i\nu_-} \int d\theta_1^+ d\theta_1^- \, \overline{\mathbb{X}}^{(1,1)} \overline{\widetilde{\mathbb{Y}}}^{(1,1)} \end{aligned}$$

[1] We guess, for example, that a generalized Calabi-Yau 3-fold should be defined as a complex three-dimensional geometry as in the case of a ordinary Calabi-Yau 3-fold.

If we integrate out $\widetilde{\mathbb{Y}}^{(1,1)}$ as a Lagrange multiplier, we obtain $0 = \mathrm{e}^{-i\nu_-}\overline{D}_-\mathbb{X}^{(1,1)}$. From this constraint we find that the following relations:

$$\widehat{\psi}_- + \widehat{\chi}_- \ = \ \widehat{F} + \widehat{G} \ = \ (\partial_0 - \partial_1)\phi \ = \ (\partial_0 - \partial_1)\psi_+ \ = \ 0$$

We also find that spinor superfields Ψ_-^a and $\Upsilon_+^{a'}$ are still independent of the scalar superfields $\mathbb{X}^a$ and $\mathbb{Y}^{a'}$.

Because of this property, the $\mathcal{N} = (2,2)$ sigma models $\mathscr{L}_{\mathbb{X}}$ and $\mathscr{L}_{\mathbb{Y}}$ with the F-terms can be reduced to the meaningful $\mathcal{N} = (1,1)$ sigma models, in which we can introduce the generalized complex structures.

[Notes]: $\mathcal{N} = (2,2)$ and $(1,1)$ superfields $(\theta_1^{\pm} \equiv -i\mathrm{e}^{i\nu_\pm}\theta)$

$$\begin{aligned}
\mathbb{X}^{(2,2)} \ &= \ \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}(\theta^-\psi_- + \overline{\theta}^-\chi_-) + 2i\theta^+(\theta^-F + \overline{\theta}^-G) + \theta^-\overline{\theta}^-(A_0 - A_1) + 2\theta^+\theta^-\overline{\theta}^-\zeta_- \\
&\quad - i\theta^+\overline{\theta}^+(\partial_0 + \partial_1)\phi + \sqrt{2}\theta^+\overline{\theta}^+(\partial_0 + \partial_1)(\theta^-\psi_- + \overline{\theta}^-\chi_-) + i\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0 + \partial_1)(A_0 - A_1) \\
\mathbb{X}^{(1,1)} \ &= \ \phi + i\sqrt{2}\theta_1^+\widehat{\psi}_+ + i\sqrt{2}\theta_1^-(\widehat{\psi}_- + \widehat{\chi}_-) + 2i\theta_1^+\theta_1^-(\widehat{F} + \widehat{G}) \\
\Psi_-^{(1,1)} \ &= \ i(\widehat{\psi}_- - \widehat{\chi}_-) - i\sqrt{2}\theta_1^+(\widehat{F} - \widehat{G}) + \sqrt{2}\theta_1^-(A_0 - A_1) + 2\sqrt{2}\theta_1^+\theta_1^-\widehat{\zeta}_-
\end{aligned}$$

Comment on $\boldsymbol{U(1)}$ gauge symmetries

A chiral superfield $\boldsymbol{\Phi}$ can possess a vector $\boldsymbol{U(1)_{\mathrm{V}}}$ gauge symmetry.

A twisted chiral superfield can possess an axial $\boldsymbol{U(1)_{\mathrm{A}}}$ gauge symmetry.

As we mentioned before, a pair $(\boldsymbol{\Phi}, \overline{\boldsymbol{\Phi}})$ is mapped to a pair $(\boldsymbol{Y}, \overline{\boldsymbol{Y}})$ and vice versa. This means that the $\boldsymbol{U(1)_{\mathrm{V}}}$ gauge symmetry is mapped to another $\boldsymbol{U(1)_{\mathrm{A}}}$ gauge symmetry and vice versa.

Suppose

- **$\mathscr{L}_{\mathbb{X}}(\mathbb{X}, \overline{\mathbb{X}})$ with F-term has an $\boldsymbol{U(1)_{\mathrm{V}}}$ gauge symmetry from the $\boldsymbol{\Phi}$ dependence in $\mathbb{X}$**
- **$\mathscr{L}_{\mathbb{Y}}(\widetilde{\mathbb{Y}}, \overline{\widetilde{\mathbb{Y}}})$ with F-term has an $\widetilde{\boldsymbol{U(1)}}_{\mathrm{V}}$ gauge symmetry from the $\widetilde{\boldsymbol{\Phi}}$ dependence in $\widetilde{\mathbb{Y}}$**

Under the mirror dual mapping from $\mathscr{L}_{\mathbb{X}}$ to $\mathscr{L}_{\mathbb{Y}}$, we guess that the $\boldsymbol{U(1)_{\mathrm{V}}}$ gauge symmetry in $\mathscr{L}_{\mathbb{X}}$ should be fixed and the fixed $\boldsymbol{U(1)_{\mathrm{A}}}$ gauge symmetry by is mapped to the unfixed $\widetilde{\boldsymbol{U(1)}}_{\mathrm{V}}$ gauge symmetry in the $\mathscr{L}_{\mathbb{Y}}$ theory.

theory	variables	$U(1)$ gauge		constraint	reduced variables	reduced $U(1)$ gauge
$\mathscr{L}_{\mathbb{X}}$	$(\mathbb{X}(\boldsymbol{\Phi}, \boldsymbol{Y}), \ \overline{\mathbb{X}}(\overline{\boldsymbol{\Phi}}, \overline{\boldsymbol{Y}}))$	$U(1)_{\mathrm{V}}$	$U(1)_{\mathrm{A}}$	$\overline{\boldsymbol{D}}_{-}\mathbb{X} = 0$	$(\mathbb{X}(\boldsymbol{\Phi}), \ \overline{\mathbb{X}}(\overline{\boldsymbol{\Phi}}))$	$U(1)_{\mathrm{V}}$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathscr{L}_{\mathbb{Y}}$	$(\widetilde{\mathbb{Y}}(\overline{\widetilde{\boldsymbol{Y}}}, \widetilde{\boldsymbol{\Phi}}), \ \overline{\widetilde{\mathbb{Y}}}(\widetilde{\boldsymbol{Y}}, \overline{\widetilde{\boldsymbol{\Phi}}}))$	$\widetilde{U(1)}_{\mathrm{A}}$	$\widetilde{U(1)}_{\mathrm{V}}$	$\overline{\boldsymbol{D}}_{+}\widetilde{\mathbb{Y}} = 0$	$(\widetilde{\mathbb{Y}}(\widetilde{\boldsymbol{\Phi}}), \ \overline{\widetilde{\mathbb{Y}}}(\overline{\widetilde{\boldsymbol{\Phi}}}))$	$\widetilde{U(1)}_{\mathrm{V}}$

A Naive Construction of Gauge Theory

We consider a naive construction of topological theory in terms of the $U(1)_{\rm V}$ vector superfield. For simplicity we only consider a semi-chiral theory:

$$\begin{aligned}
\mathscr{L}_0 \ &= \ \int \mathrm{d}^4\theta \Big\{ -\frac{1}{e^2}\overline{\Sigma}\Sigma + \sum_a \overline{\mathbb{X}^a}\, \mathrm{e}^{2Q_a V} \mathbb{X}^a \Big\} - \Big(\frac{\tau}{\sqrt{2}} \int \mathrm{d}^2\widetilde{\theta}\, \Sigma + h.c. \Big) \\
&= \ \frac{1}{e^2}\Big\{ \frac{1}{2}(D^2 + F_{01}^2) - \overline{\sigma}(\partial_0^2 - \partial_1^2)\sigma \Big\} - \frac{1}{2}\sum_a \Big\{ i(A_0^a - A_1^a)(\partial_0 + \partial_1)(\phi^a - \overline{\phi^a}) - 2\overline{F^a}F^a + 2\overline{G^a}G^a \Big\} \\
&\quad - \frac{1}{2}\sum_a Q_a \Big\{ -(A_0^a - A_1^a)(\phi^a + \overline{\phi^a})(v_0 + v_1) - 2i\phi^a(\partial_0 + \partial_1)\overline{\phi^a}(v_0 - v_1) \\
&\qquad\qquad - 2\sqrt{2}i\overline{\phi^a}\sigma G^a + 2\sqrt{2}i\phi^a\overline{\sigma}\overline{G^a} - 2D\overline{\phi^a}\phi^a \Big\} \\
&\quad - \sum_a Q_a^2 \Big\{ -\overline{\phi^a}\phi^a(v_0^2 - v_1^2) + 2\overline{\phi^a}\phi^a\overline{\sigma}\sigma \Big\} - rD + \theta F_{01} + (\text{fermionic terms})
\end{aligned}$$

Solve the equations of motion of F^a, G^a and D, we obtain the potential density

$$\mathcal{U} \ = \ \frac{1}{2e^2}D^2 \ = \ e^2\Big\{ r - \sum_a Q_a|\phi^a|^2 \Big\}$$

there does not appear the term $2|\sigma|^2\sum_a Q_a^2|\phi^a|^2$ which exist in the usual gauged linear sigma model. This means that there cannot appear the Coulomb branch, which plays a significant role when we consider the effective theory around the singularity point in the moduli space. Because of the existence of the Coulomb branch, we can smoothly connect the effective theories with the positive FI parameters to the effective theories with the negative FI parameter.