



Comments on Heterotic Flux Compactifications

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Purpose

Construct a realistic model of 4-dim. particle physics

- **matter contents**
- **gauge symmetry and its breaking**
- **gravity, cosmology**
- **etc., etc.**

Supergravity with fluxes has a long, and interesting story

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▼ Freund-Rubin Ansatz in 11- and 10-dimensional supergravities

$F_p = |F| \times (\text{vol.})$ generates a cosmological constant in AdS_q -space

Supergravity with fluxes has a long, and interesting story

- ▼ Freund-Rubin Ansatz in 11- and 10-dimensional supergravities

$F_p = |F| \times (\text{vol.})$ generates a cosmological constant in AdS_q -space

- ▼ Gauge/Gravity Dualities in type II theories

F_p generates a superpotential in 4-dim. $\mathcal{N} = 1$ SYM (e.g., $W = \int_{CY_3} \Omega \wedge F_3$)

Both solutions have given us new insights in higher-dimensional theories

▼ Flux can be a **torsion** on a (compactified) geometry

$$\delta\psi_M = \left(\partial_M + (\omega_M^{AB} - H_M^{AB})\Gamma_{AB} \right) \xi$$

If $H_M^{AB} \neq 0$, the geometry looks no longer a Kähler manifold.



G -structure manifold

- ▼ Flux can be a **torsion** on a (compactified) geometry

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***G*-structure manifold**

- ▼ How does the flux work in heterotic theory?

{ anomaly cancellation in a miraculous way
{ gauge symmetry $E_8 \times E_8$ or $SO(32)$

Contents

- ▼ $SU(3)$ -structure manifold
- ▼ Heterotic theory on $SU(3)$ -structure manifold
 - Vacuum configuration
 - Towards low energy effective theory: (zero mode eqs., gauge groups)
- ▼ Summary and Discussion

useful Refs. [Becker, Becker, Dasgupta and Green \[hep-th/0301161\]](#)
[Cardoso, Curio, Dall'Agata and Lüst \[hep-th/0306088\]](#)
[Becker and Tseng \[hep-th/0509131\]](#)
etc., etc..

SU(3)-structure Manifold

– mathematics –

G -structure group

on an n -dim. manifold \mathcal{M}

$$\left\{ \begin{array}{l} F(\mathcal{M}) \text{ frame bundle : principal } GL(n) \text{ bundle} \\ G\text{-structure} \quad \quad \quad : \text{ principal } G \text{ sub-bundle of } F(\mathcal{M}) \end{array} \right.$$

\Leftrightarrow nowhere vanishing tensors on \mathcal{M}

tensors	G -structure	
η_{ab}	$O(n)$	
$\eta_{ab} \quad \varepsilon_{a_1 \dots a_n}$	$SO(n)$	
$\eta_{ab} \quad J_a^b$	$U(m)$	$J^2 = -1$
$\eta_{ab} \quad J_a^b \quad \Omega^{(m,0)}$	$SU(m)$	$(2m = n)$

6-dim. $SU(3)$ -structure on manifold

Consider a geometry \mathcal{K}_6 with a Killing spinor equation including torsion

$$\exists \xi \quad \text{s.t.} \quad \nabla^{(T)} \xi = 0$$

This is a definition of the geometry with $SU(3)$ -structure.

6-dim. $SU(3)$ -structure on manifold

Consider a geometry \mathcal{K}_6 with a Killing spinor equation including torsion

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This is a definition of the geometry with $SU(3)$ -structure.

Invariant p -forms on the $SU(3)$ -structure manifold:

2-form in $SO(6)$ \Rightarrow a real 2-form in $SU(3)$

$${}_6C_2 = 15 = \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{8} \quad : \quad J_{ab} = -i\xi^\dagger \Gamma_{ab} \xi$$

3-form in $SO(6)$ \Rightarrow an (almost) complex 3-form in $SU(3)$

$${}_6C_3 = 20 = \mathbf{1} + \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{6} + \bar{\mathbf{6}} \quad : \quad \Omega_{abc} = \xi^T \Gamma_{abc} \xi$$

Furthermore

$$J \wedge \Omega = 0, \quad J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \bar{\Omega} = 3! \times (\text{vol.})_{\mathcal{K}_6}$$

Heterotic Theory

– vacuum, gauge group and zero modes –

Heterotic theory on $SU(3)$ -structure manifold

Supergravity

▼ Bosonic part of the Lagrangian (without fermion condensations)

$$\mathcal{L} = \frac{1}{4} \sqrt{-G} e^{-2\Phi} \left[R(\omega) - \frac{1}{3} H_{MNP} H^{MNP} + 4(\nabla_M \Phi)^2 - \alpha' \left\{ \text{tr}(F_{MN} F^{MN}) \right\} \right]$$

▼ Bianchi identity $[\omega]$

$$dH = -\alpha' \left[\text{tr}\{F \wedge F\} \right]$$

Chapline and Manton [Phys. Lett. B120 (1983) 105]

(supergravity)

Heterotic theory on $SU(3)$ -structure manifold

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Green and Schwarz [Phys. Lett. B149 (1984) 117]

(anomaly cancellation)

(worldsheet 1-loop β -function)

Heterotic theory on $SU(3)$ -structure manifold

Supergravity QL

▼ Bosonic part of the Lagrangian (without fermion condensations)

$$\mathcal{L} = \frac{1}{4} \sqrt{-G} e^{-2\Phi} \left[R(\omega) - \frac{1}{3} H_{MNP} H^{MNP} + 4(\nabla_M \Phi)^2 - \alpha' \left\{ \text{tr}(F_{MN} F^{MN}) - \text{tr}(R_{MN}(\omega_+) R^{MN}(\omega_+)) \right\} \right]$$

▼ Bianchi identity $[\omega_+ = \omega + H]$

$$dH = -\alpha' \left[\text{tr}\{F \wedge F\} - \text{tr}\{R(\omega_+) \wedge R(\omega_+)\} \right]$$

Hull [Phys. Lett. B178 (1986) 357]

Bergshoeff and de Roo [Nucl. Phys. B328 (1989) 439]

(worldsheet 2-loop β -function)

Towards 4-dim. Physics...

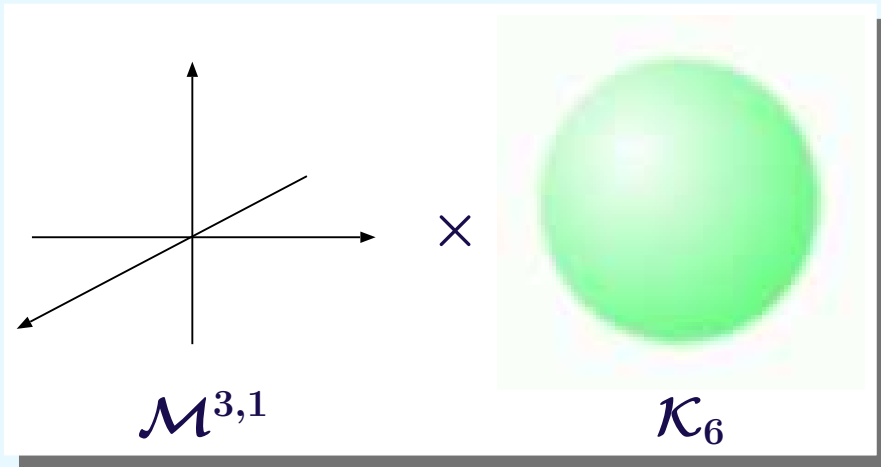
▼ Study vacuum configuration

- SUSY variations → geometry with $SU(3)$ -structure

▼ Investigate low energy effective theory

- Gauge symmetry
- Zero mode equations
- Norm of fields

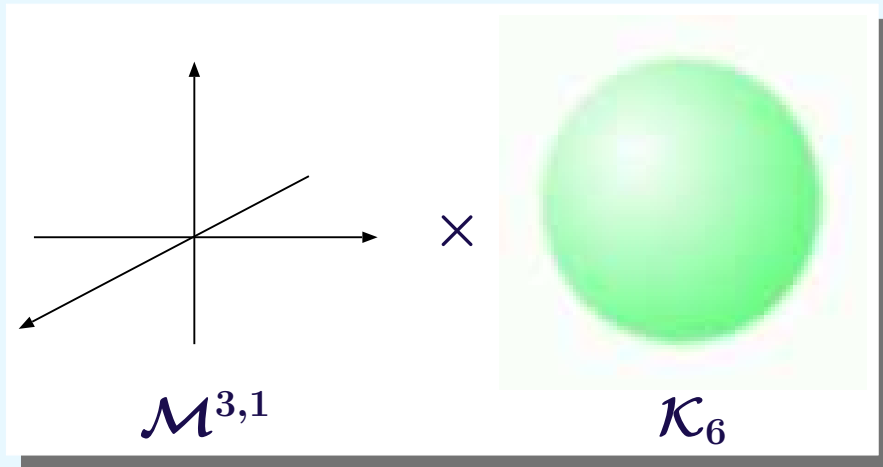
Vacuum Configuration



$$G_{MN} dx^M dx^N = \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n$$

$$G_{MN}^E dx^M dx^N = e^{-\Phi/2} \left(\eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n \right)$$

Vacuum Configuration

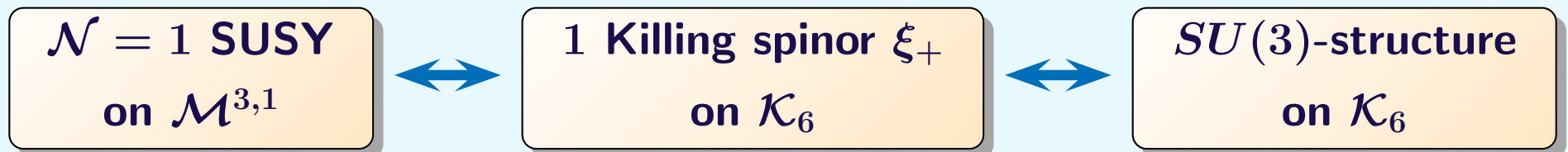


$$G_{MN} dx^M dx^N = \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n$$

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$$Spin(9, 1) \rightarrow SL(2, \mathbb{C}) \times SU(4)$$

$$16 = (2, 4) + (\bar{2}, \bar{4}) : \quad \epsilon_+ = \eta_+ \otimes \xi_+ + \eta_- \otimes \xi_-$$



▼ SUSY variations

$$0 \equiv \delta\psi_m = D_m(\omega_-)\xi_+ \leftarrow \text{Killing spinor eq.} \quad [\omega_- = \omega - H]$$

$$J_{ab} = -i\xi_+^\dagger \Gamma_{ab} \xi_+ \quad : \quad D_m(\omega_-)J_{ab} = 0$$

$$\Omega_{abc} = \xi_+^T \Gamma_{abc} \xi_+ \quad : \quad D_m(\omega_-)\Omega_{abc} = 0$$

▼ SUSY variations

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$$\Omega_{abc} = \xi_+^T \Gamma_{abc} \xi_+ \quad : \quad D_m(\omega_-)\Omega_{abc} = 0$$

Furthermore “ $0 \equiv \delta(\text{fermions})$ ” indicates

$$0 = R^{ab}{}_{mn}(\omega_-)J_{ab} \quad : \quad c_1(R_-) \text{ vanishes}$$

$$0 = N_{mn}{}^p \quad : \quad \mathcal{K}_6 \text{ is complex } \boxed{\tau}$$

$$H_{mnp} = H_{mnp}^0 + \widehat{H}_{mnp}$$

$$0 = H_{mnp}^0 J^{np} \quad \widehat{H}_{mnp} = \frac{3}{2} J_{[mn} J_p]^q \nabla_q \Phi$$

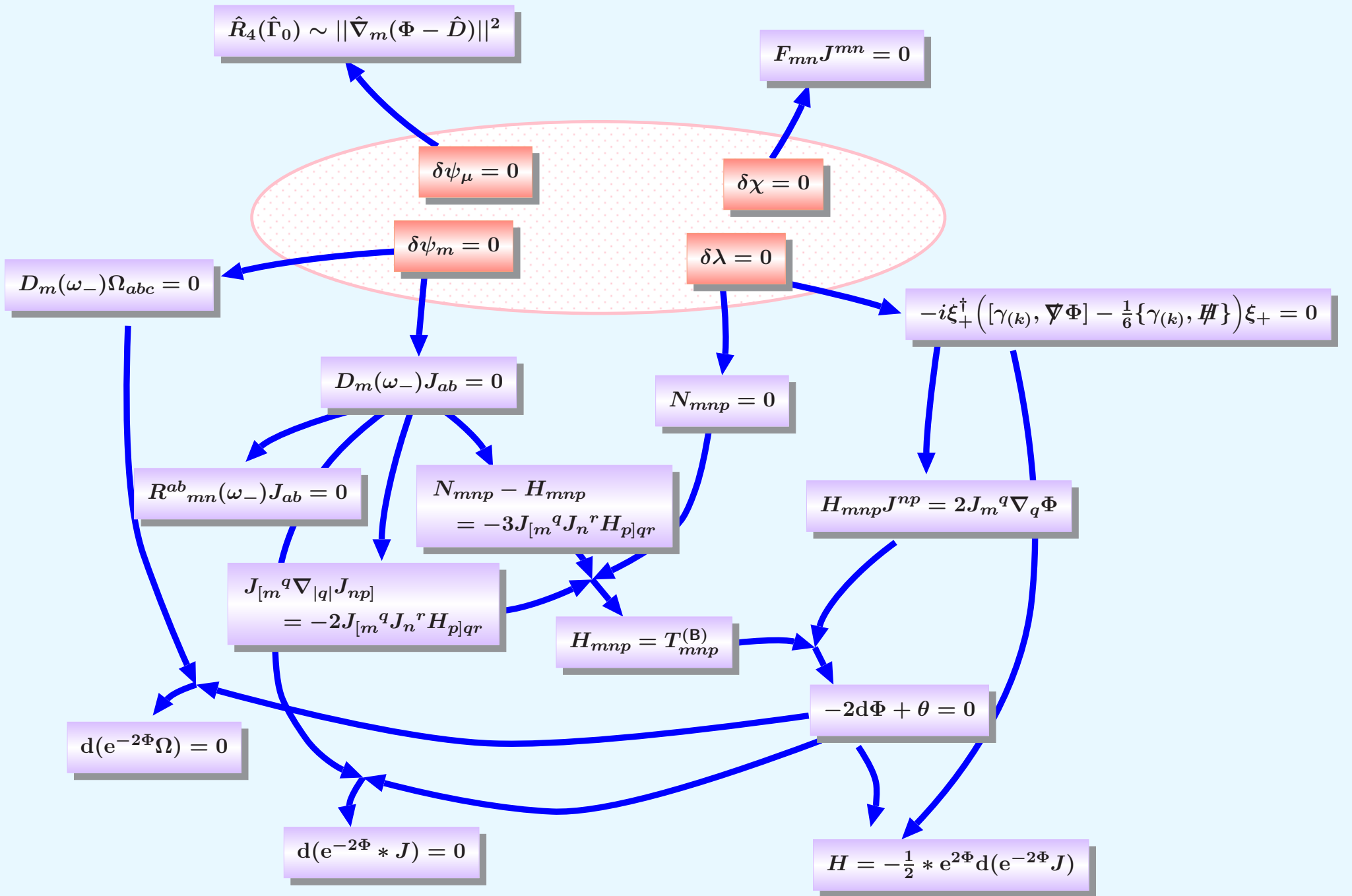
$$0 = F_{mn} J^{mn}$$

$$*(J \wedge dH) = -\nabla_m^2 \Phi + (\nabla_m \Phi)^2 - \frac{1}{3} (H_{mnp}^0)^2$$

$$R(\omega) = -\frac{1}{3} (H_{mnp}^0)^2 - 6\nabla_m^2 \Phi + 7(\nabla_m \Phi)^2$$

$$\boxed{dH = 0}$$

$$\boxed{dH \neq 0}$$



Gauge Symmetry Breaking

gauge algebra $\mathcal{G} = \mathcal{F} \oplus \mathcal{F}_\perp$, $\mathcal{F}_\perp = \mathcal{H} \oplus \mathcal{Q}$, $[\mathcal{H}, \mathcal{F}] = 0$
with F_{mn} taking a value in \mathcal{F}

Gauge Symmetry Breaking

$$\text{gauge algebra } \mathcal{G} = \mathcal{F} \oplus \mathcal{F}_\perp, \quad \mathcal{F}_\perp = \mathcal{H} \oplus \mathcal{Q}, \quad [\mathcal{H}, \mathcal{F}] = 0$$

with F_{mn} taking a value in \mathcal{F}

$$\blacktriangledown E_8 \rightarrow E_6 \times \underline{SU(3)} : \quad (\mathcal{G} = E_8, \quad \mathcal{F} = SU(3), \quad \mathcal{H} = E_6)$$

$$248 = (78, 1) + (1, 8) + (27, 3) + (\overline{27}, \overline{3})$$

$$\blacktriangledown E_8 \rightarrow SO(16)$$

$$\rightarrow SO(10) \times \underline{SO(6)} : \quad (\mathcal{G} = E_8, \quad \mathcal{F} = SO(6), \quad \mathcal{H} = SO(10))$$

$$248 = 120_{SO(16) \text{ adj.}} + 128_{SO(16) \text{ spinor}}$$

$$= (45, 1) + (1, 15) + (10, 6) + (16, 4) + (\overline{16}, \overline{4})$$

Each breaking scenario deeply depends on the way

of embedding the holonomy group into the gauge groups.

Candidates:

$$A \leftrightarrow \begin{cases} \omega_- : & SU(3) \text{ holonomy} \\ \omega_+ : & SO(6) \text{ holonomy} \\ & \text{etc.} \end{cases} \quad [\omega_{\pm} = \omega \pm H]$$

Candidates:

$$A \leftrightarrow \begin{cases} \omega_- : & SU(3) \text{ holonomy} \\ \omega_+ : & SO(6) \text{ holonomy} \\ \text{etc.} \end{cases} \quad [\omega_{\pm} = \omega \pm H]$$

with following constraints

$$R_{mnpq}(\omega_+) = R_{pqmn}(\omega_-) + (dH)_{pqmn}$$

$$dH = -\alpha' \left[\text{tr}(F \wedge F) - \text{tr}\{R(\omega_+) \wedge R(\omega_+)\} \right] \quad \begin{array}{|l} dH = 0 \\ dH \neq 0 \end{array}$$

$$R^{ab}{}_{mn}(\omega_-) : \quad \text{type } (1, 1) \text{ w/ indices } a, b$$

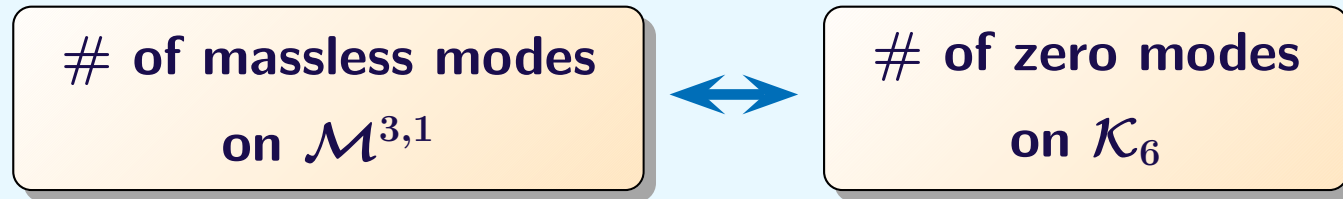
$$F : \quad (1, 1)\text{-form}$$

$$dH : \quad (2, 2)\text{-form, higher order in } \alpha'$$

$$R(\omega_+) : \quad (1, 1)\text{-form} + \text{higher order in } \alpha'$$

Mainly we consider $E_8 \rightarrow SO(10) \times SO(6)$ breaking: $A = \omega_+$

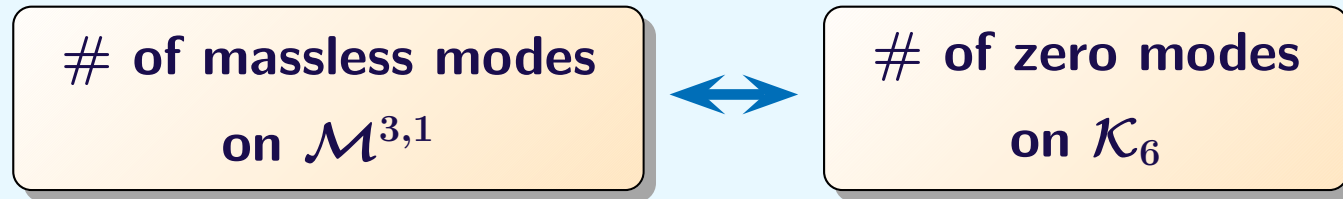
Zero mode equations



Zero mode eq. for gaugino:

$$\begin{aligned} 0 &= \mathcal{D}(\omega, A)\chi^0 - \frac{1}{12}H_{mnp}\Gamma^{mnp}\chi^0 \\ &= \mathcal{D}(\hat{\omega}, A)\chi^0 \quad \left[\hat{\omega} \equiv \omega - \frac{1}{3}H \right] \end{aligned}$$

Zero mode equations



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Decompose χ^0 into

$$0 = \mathcal{D}(\hat{\omega})\chi_{\mathcal{H}}^0 \quad \text{and} \quad 0 = \mathcal{D}(\hat{\omega}, A_{\mathcal{Q}})\chi_{\mathcal{Q}}^0,$$

which correspond to the neutral and charged matter fermions, respectively.

$$[\# \text{ of zero mode } \chi_{\mathcal{H}}^0] = \lim_{\beta \rightarrow 0} \text{Tr } \Gamma_{(6+1)} e^{-\beta \Delta_{\mathcal{H}}} \equiv \text{index } \mathcal{D}(\hat{\omega})$$

Evaluate the square of the Dirac operators with $\hat{\omega} = \omega - \frac{1}{3}H$:

$$\begin{aligned}\Delta_{\mathcal{H}} &\equiv -[\mathcal{D}(\hat{\omega})]^2 \\ &= D_m(\omega_-)^\dagger D^m(\omega_-) + V\end{aligned}$$

$$\begin{aligned}\Delta_{\mathcal{Q}} &\equiv -[\mathcal{D}(\hat{\omega}, A_{\mathcal{Q}})]^2 \\ &= D_m(\omega_-)^\dagger D^m(\omega_-) + V + \frac{i}{2} F_{mn}^{\mathcal{Q}} \Gamma^{mn}\end{aligned}$$

$$V = \frac{1}{4} \left[R(\omega) - \frac{1}{3} H_{mnp} H^{mnp} + \frac{1}{12} (dH)_{mnpq} \Gamma^{mnpq} \right]$$

$dH = 0$

$dH \neq 0$

The “potential” V plays a crucial role in

- the zero mode equations of Klein-Gordon type
- the Atiyah-(Patodi)-Singer index density
- etc.

Minimal embedding: $dH = 0$

SUSY A V

Zero mode equation tells us $[V = \frac{1}{3}(H^0_{mnp})^2]$

$$0 = \left[D_m(\omega_-)^\dagger D^m(\omega_-) + \frac{1}{3}(H^0_{mnp})^2 \right] \chi_{\mathcal{H}}^0$$

$$\therefore 0 = \int_{\mathcal{K}_6} \left[|D_m(\omega_-) \chi_{\mathcal{H}}^0|^2 + \frac{1}{3} |H^0_{mnp}|^2 |\chi_{\mathcal{H}}^0|^2 \right]$$

If no boundaries/singularities $\Rightarrow \begin{cases} H^0 = 0 \text{ or} \\ \chi_{\mathcal{H}}^0 = 0 \Rightarrow \text{no massless modes} \end{cases}$

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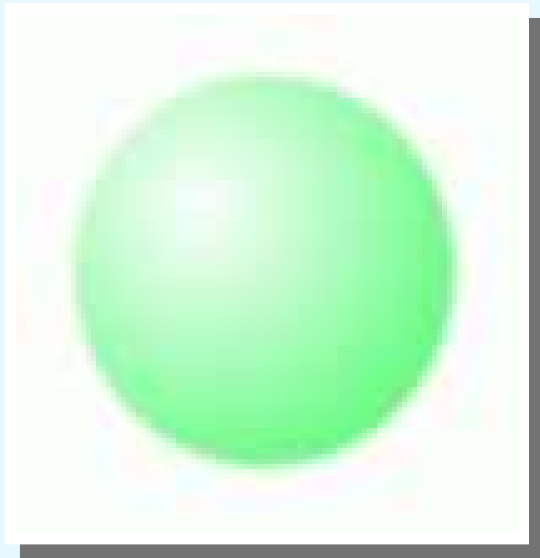
The condition $*(J \wedge dH) = 0$ denotes

$$\frac{1}{2} \nabla_m^2 e^{-\Phi} = \frac{1}{3} e^{-\Phi} (H_{mnp}^0)^2$$

If there are no boundaries/singularities on \mathcal{K}_6 , then

$$\frac{1}{3} \int_{\mathcal{K}_6} e^{-\Phi} |H_{mnp}^0|^2 = \frac{1}{2} \int_{\mathcal{K}_6} \nabla_m^2 e^{-\Phi} = 0$$

This means $H^0 = 0 \Rightarrow \Phi = \text{const.} \Rightarrow \mathcal{K}_6 = \text{CY}_3$



Without boundaries/singularities on \mathcal{K}_6 :

- all fluxes are trivial $H = d\Phi = 0$
- $\mathcal{K}_6 = \text{CY}_3$
- $\omega_+ = \omega_- = \omega$, $E_8 \rightarrow E_6 \times SU(3)$
- # of zero modes — AS index theorem

Candelas, Horowitz, Strominger and Witten

[Nucl. Phys. B258 (1985) 46]

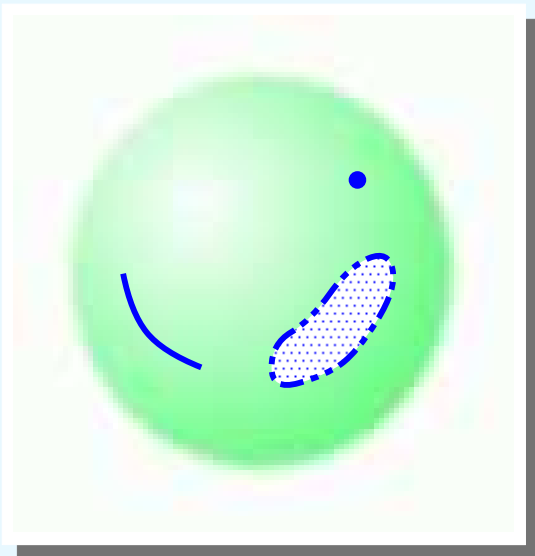


Without boundaries/singularities on \mathcal{K}_6 :

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- # of zero modes — AS index theorem

Candelas, Horowitz, Strominger and Witten

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With boundaries/singularities on \mathcal{K}_6 :

- non-trivial fluxes can exist
 $\partial_m \Phi = 0$, $H_{mnp}^0 \neq 0$: conformally balanced T
- $E_8 \rightarrow SO(10) \times SO(6)$
- $\chi_{\mathcal{H}}^0$ lives in the boundaries
- # of zero modes — APS index theorem

Non-minimal embedding: $dH \neq 0$

SUSY A V

In this case we should notice the α' -ordering in the Lagrangian with keeping

$$\frac{\alpha'}{L^2} \ll 1$$

$L =$ linear size of \mathcal{K}_6

Non-minimal embedding: $dH \neq 0$

SUSY A V

In this case we should notice the α' -ordering in the Lagrangian with keeping

$$\frac{\alpha'}{L^2} \ll 1 \quad L = \text{linear size of } \mathcal{K}_6$$

Notice that the scaling orders of variables:

$$F_{mn} \sim R^p{}_{qmn}(\omega) \sim \frac{1}{L^2}$$

$$(\nabla_m \Phi)^2 \sim dH \sim (H_{mnp})^2 \sim (Ric)_{mn}(\omega) \sim \frac{\alpha'}{L^4}$$

$$R^p{}_{qmn}(\omega_+) = R^p{}_{qmn}(\omega) + 2\nabla_{[m} H^p{}_{|q|n]} + 2H^p{}_{r[m} H^r{}_{|q|n]}$$

Non-minimal embedding: $dH \neq 0$

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$$R^p{}_{qmn}(\omega_+) = R^p{}_{qmn}(\omega) + 2\nabla_{[m} H^p{}_{|q|n]} + 2H^p{}_{r[m} H^r{}_{|q|n]}$$

Then, in the leading order in α' ,

we replace $R_{abmn}(\omega_+)$ to $R_{abmn}(\omega)$ in the Lagrangian and the Bianchi identity

$$\rightarrow dH = -\alpha' \left[\text{tr}(F \wedge F) - \text{tr}\{R(\omega) \wedge R(\omega)\} \right]$$

Combining equations of motion and SUSY conditions, we obtain

$$\begin{aligned} \text{tr}(R_{mn}R^{mn}) - \text{tr}(F_{mn}F^{mn}) &= -2 * \left[J \wedge \left(\text{tr}(R \wedge R) - \text{tr}(F \wedge F) \right) \right] + \mathcal{O}(\alpha') \\ 0 &= \frac{1}{2} \nabla_m^2 e^{-2\Phi} - \frac{1}{3} e^{-2\Phi} (H_{mnp})^2 - e^{-2\Phi} * (J \wedge dH) + \mathcal{O}(\alpha'^2) \end{aligned}$$

Then, within the linear order in α' , we find

$$\nabla_m^2 e^{-2\Phi} = e^{-2\Phi} \left[\frac{2}{3} |H_{mnp}|^2 + \alpha' (\text{tr} |F_{mn}|^2 - \text{tr} |R_{mn}|^2) \right]$$

Integral on a **smooth manifold \mathcal{K}_6** :

$$\int_{\mathcal{K}_6} e^{-2\Phi} \left[\frac{2}{3} |H_{mnp}|^2 + \alpha' \text{tr} |F_{mn}|^2 \right] = \int_{\mathcal{K}_6} e^{-2\Phi} \left[\alpha' \text{tr} |R_{mn}|^2 \right]$$

$$\text{with } \text{tr} |F_{mn}|^2 \neq \text{tr} |R_{mn}|^2$$

Smooth compactification scenario is possible!

Summary and Discussions

Summary and Discussion

- ▼ Vacuum configuration of the flux compactifications in heterotic theory
- ▼ No-go theorem on smooth manifolds with $H \neq 0$ and $dH = 0$
- ▼ Possibility of smooth compactifications with $H \neq 0$ and $dH \neq 0$

Summary and Discussion

- ▼ Vacuum configuration of the flux compactifications in heterotic theory
- ▼ No-go theorem on smooth manifolds with $H \neq 0$ and $dH = 0$
- ▼ Possibility of smooth compactifications with $H \neq 0$ and $dH \neq 0$
- ▼? # of zero modes under the condition $dH \neq 0$
 - modification of the Atiyah-(Patodi)-Singer index theorem
- ▼ other possibilities of gauge symmetry breaking
- ▼ compactifications on non-complex geometries SUSY

Frey and Lippert [hep-th/0507202]

Manousselis, Prezas and Zoupanos [hep-th/0511122]

Appendix

Appendix: Quartic effective Lagrangian

$$\mathcal{L}_{\text{total}} = \mathcal{L}_0(\mathbf{R}) + \mathcal{L}_\beta(\mathbf{F}^2) + \mathcal{L}_\alpha(\mathbf{R}^2) \quad \square$$

$$\begin{aligned} \mathcal{L}_0(\mathbf{R}) = & \frac{1}{2\kappa_{10}^2} \sqrt{-G} e^{-2\Phi} \left[R(\omega) - \frac{1}{3} H_{MNP} H^{MNP} + 4(\nabla_M \Phi)^2 - \bar{\psi}_M \Gamma^{MNP} D_N(\omega) \psi_P + 16 \bar{\lambda} \mathcal{D}(\omega) \lambda \right. \\ & + 8 \bar{\lambda} \Gamma^{MN} D_M(\omega) \psi_N + 8 \bar{\psi}_M \Gamma^N \Gamma^M \lambda (\nabla_N \Phi) - 2 \bar{\psi}_M \Gamma^M \psi_N (\nabla^N \Phi) \\ & + \frac{1}{12} H^{PQR} \left\{ \bar{\psi}_M \Gamma^{[M} \Gamma_{PQR} \Gamma^{N]} \psi_N + 8 \bar{\psi}_M \Gamma^M{}_{PQR} \lambda - 16 \bar{\lambda} \Gamma_{PQR} \lambda \right\} \\ & \left. + \frac{1}{48} \bar{\psi}^M \Gamma^{ABC} \psi_M \left\{ 2 \bar{\lambda} \Gamma_{ABC} \lambda + \bar{\lambda} \Gamma_{ABC} \Gamma^N \psi_N - \frac{1}{4} \bar{\psi}^N \Gamma_{ABC} \psi_N - \frac{1}{8} \bar{\psi}^N \Gamma_N \Gamma_{ABC} \Gamma^P \psi_P \right\} \right] \end{aligned}$$

$$\begin{aligned} \mathcal{L}_\beta(\mathbf{F}^2) = & \frac{1}{2\kappa_{10}^2} \frac{\kappa_{10}^2}{2g_{10}^2} \sqrt{-G} e^{-2\Phi} \left[-\text{tr}(F_{MN} F^{MN}) - 2 \text{tr}\{\bar{\chi} \mathcal{D}(\omega, A) \chi\} + \frac{1}{6} \text{tr}(\bar{\chi} \Gamma^{ABC} \chi) \hat{H}_{ABC} \right. \\ & - \frac{1}{2} \text{tr}\{\bar{\chi} \Gamma^M \Gamma^{AB} (F_{AB} + \hat{F}_{AB})\} \left(\psi_M + \frac{2}{3} \Gamma_M \lambda \right) - \frac{1}{48} \text{tr}(\bar{\chi} \Gamma^{ABC} \chi) \bar{\psi}_M (4 \Gamma_{ABC} \Gamma^M + 3 \Gamma^M \Gamma_{ABC}) \lambda \\ & \left. + \frac{1}{12} \text{tr}(\bar{\chi} \Gamma^{ABC} \chi) \bar{\lambda} \Gamma_{ABC} \lambda - \frac{\beta}{96} \text{tr}(\bar{\chi} \Gamma^{ABC} \chi) \text{tr}(\bar{\chi} \Gamma_{ABC} \chi) \right] \end{aligned}$$

$$\begin{aligned} \mathcal{L}_\alpha(\mathbf{R}^2) = & \frac{1}{2\kappa_{10}^2} \frac{\kappa_{10}^2}{2g_{10}^2} \sqrt{-G} e^{-2\Phi} \left[-R_{ABMN}(\omega_+) R^{ABMN}(\omega_+) - 2 \bar{\psi}^{AB} \mathcal{D}(\omega(e, \psi), \omega_+) \psi_{AB} + \frac{1}{6} \bar{\psi}^{AB} \Gamma^{MNP} \psi_{AB} \hat{H}_{MNP} \right. \\ & + \frac{1}{2} \bar{\psi}_{AB} \Gamma^M \Gamma^{NP} \left\{ R^{AB}{}_{NP}(\omega_+) + \hat{R}^{AB}{}_{NP}(\omega_+) \right\} \left(\psi_M + \frac{2}{3} \Gamma_M \lambda \right) \\ & - \frac{1}{48} \bar{\psi}_{AB} \Gamma^{CDE} \psi_{AB} \cdot \bar{\psi}_M (4 \Gamma_{CDE} \Gamma^M + 3 \Gamma^M \Gamma_{CDE}) \lambda \\ & \left. + \frac{1}{12} \bar{\psi}^{AB} \Gamma^{CDE} \psi_{AB} (\bar{\lambda} \Gamma_{CDE} \lambda) - \frac{\alpha}{96} \bar{\psi}^{AB} \Gamma^{FGH} \psi_{AB} (\bar{\psi}^{CD} \Gamma_{FGH} \psi_{CD}) \right] \end{aligned}$$

$$\begin{aligned}
\delta_0 e_M{}^A &= \frac{1}{2} \bar{\epsilon} \Gamma^A \psi_M \\
\delta_0 \psi_M &= \left(\partial_M + \frac{1}{4} \omega_{-M}{}^{AB} \Gamma_{AB} \right) \epsilon + \left\{ \epsilon (\bar{\psi}_M \lambda) - \psi_M (\bar{\epsilon} \lambda) + \Gamma^A \lambda (\bar{\psi}_M \Gamma_A \epsilon) \right\} \\
\delta_0 B_{MN} &= \bar{\epsilon} \Gamma_{[M} \psi_{N]} \\
\delta_0 \lambda &= -\frac{1}{4} \not{D} \Phi \epsilon + \frac{1}{24} \Gamma^{ABC} \epsilon \left(\hat{H}_{ABC} - \frac{1}{4} \bar{\lambda} \Gamma_{ABC} \lambda \right) \\
\delta_0 \Phi &= -\bar{\epsilon} \lambda \\
\delta_0 A_M &= \frac{1}{2} \bar{\epsilon} \Gamma_M \chi \\
\delta_0 \chi &= -\frac{1}{4} \Gamma^{AB} \epsilon \hat{F}_{AB} + \left\{ \epsilon (\bar{\chi} \lambda) - \chi (\bar{\epsilon} \lambda) + \Gamma^A \lambda (\bar{\chi} \Gamma_A \epsilon) \right\} \\
\delta_\beta \psi_M &= \frac{\beta}{192} \Gamma^{ABC} \Gamma_M \epsilon \operatorname{tr}(\bar{\chi} \Gamma_{ABC} \chi) \\
\delta_\beta B_{MN} &= -\beta \operatorname{tr} \{ A_{[M} \delta_0 A_{N]} \} \\
\delta_\beta \lambda &= \frac{\beta}{384} \Gamma^{ABC} \epsilon \operatorname{tr}(\bar{\chi} \Gamma_{ABC} \chi) \\
\delta_\alpha \psi_M &= \frac{\alpha}{192} \Gamma^{CDE} \Gamma_M \epsilon \bar{\psi}^{AB} \Gamma_{CDE} \psi_{AB} \\
\delta_\alpha B_{MN} &= -\alpha \omega_{+[M}{}^{AB} \delta_0 \omega_{+N]}{}^{AB} \\
\delta_\alpha \lambda &= \frac{\alpha}{384} \Gamma^{CDE} \Gamma_M \epsilon \bar{\psi}^{AB} \Gamma_{CDE} \psi_{AB}
\end{aligned}$$

Bergshoeff and de Roo [Nucl. Phys. B328 (1989) 439]

Heterotic theory on $SU(3)$ -structure manifold

Supergravity QL

▼ Bosonic part of the Lagrangian (without fermion condensations)

$$\mathcal{L} = \frac{1}{4} \sqrt{-G} e^{-2\Phi} \left[R(\omega) - \frac{1}{3} H_{MNP} H^{MNP} + 4(\nabla_M \Phi)^2 - \alpha' \left\{ \text{tr}(F_{MN} F^{MN}) - \text{tr}(R_{MN}(\tilde{\omega}) R^{MN}(\tilde{\omega})) \right\} \right]$$

▼ Bianchi identity [$\omega_+ = \omega + H \rightarrow \tilde{\omega}$]

$$dH = -\alpha' \left[\text{tr}\{F \wedge F\} - \text{tr}\{R(\tilde{\omega}) \wedge R(\tilde{\omega})\} \right]$$

Hull [Phys. Lett. B167 (1986) 51]

(worldsheet 2-loop β -function)

Let κ be a contorsion in $\nabla^{(T)}$ with acting on the $SU(3)$ Killing spinor ξ :

$$0 = \nabla^{(T)}\xi = (\nabla + \kappa^0 + \kappa^{\mathfrak{g}})\xi$$

where we decomposed $\kappa \equiv \kappa^0 + \kappa^{\mathfrak{g}}$ in such a way as $\kappa^{\mathfrak{g}}\xi = 0$ (where $\mathfrak{g} = \mathfrak{su}(3)$):

$$\xi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ * \end{pmatrix} \quad \kappa^{\mathfrak{g}} \equiv \left(\begin{array}{ccc|c} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right) \quad \kappa^0 \equiv \left(\begin{array}{ccc|c} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ \hline * & * & * & * \end{array} \right)$$

Then, under the same structure group G we find

$$(\nabla^{(T_1)} - \nabla^{(T_2)})\xi \propto \kappa^{\mathfrak{g}}\xi = 0$$

So, from the group-theoretical viewpoint, κ^0 carries an **intrinsic** part of the contorsion when we consider the classification of the $SU(3)$ -structure manifolds!

Torsion $T_{mn} \equiv T^p{}_{mn} dx^p = \kappa^p{}_{[mn]} dx^p$ is given in the various representations:

$$T_{mn}^g = \kappa^g{}_{[mn]} \sim \mathfrak{su}(3), \quad T_{mn}^0 = \kappa^0{}_{[mn]} \sim \mathfrak{so}(6)/\mathfrak{su}(3) \equiv \mathfrak{su}(3)^\perp$$

$$\therefore (T^0)^p{}_{mn} \in \Lambda^1 \otimes \mathfrak{su}(3)^\perp \quad \text{on } \mathcal{K}_6$$

$$\Lambda^1 \sim 3 \oplus \bar{3}, \quad \mathfrak{su}(3) \sim 8, \quad \mathfrak{su}(3)^\perp = \mathfrak{so}(6)/\mathfrak{su}(3) \sim 1 \oplus 3 \oplus \bar{3}$$

Thus the **intrinsic torsion** T^0 can be decomposed

$$\begin{aligned} (T^0)^p{}_{mn} \in \Lambda^1 \otimes \mathfrak{su}(3)^\perp &= (3 \oplus \bar{3}) \otimes (1 \oplus 3 \oplus \bar{3}) \\ &= (1 \oplus 1) \oplus (8 \oplus 8) \oplus (6 \oplus \bar{6}) \oplus (3 \oplus \bar{3}) \oplus (3 \oplus \bar{3})' \\ &\quad \underbrace{\hspace{1.5cm}}_{W_1} \quad \underbrace{\hspace{1.5cm}}_{W_2} \quad \underbrace{\hspace{1.5cm}}_{W_3} \quad \underbrace{\hspace{1.5cm}}_{W_4} \quad \underbrace{\hspace{1.5cm}}_{W_5} \end{aligned}$$

where

W_1 : complex scalar in $(1 \oplus 1)$

W_2 : complex primitive 2-form in $(8 \oplus 8)$

W_3 : real primitive $(2, 1) \oplus (1, 2)$ -form in $(6 \oplus \bar{6})$

W_4 : real 1-form in $(3 \oplus \bar{3})$

W_5 : complex $(1, 0)$ -form in $(3 \oplus \bar{3})'$

▼ complex manifolds

SUSY $dH = 0$ $dH \neq 0$

$$W_1 = W_2 = 0$$

$$T^0 \in W_3 \oplus W_4 \oplus W_5$$

hermitian

$$W_1 = W_2 = W_4 = 0$$

$$T^0 \in W_3 \oplus W_5$$

balanced

$$W_1 = W_2 = W_4 = W_5 = 0$$

$$T^0 \in W_3$$

special-hermitian

$$W_1 = W_2 = W_3 = W_4 = 0$$

$$T^0 \in W_5$$

Kähler

$$W_1 = W_2 = W_3 = W_4 = W_5 = 0$$

$$T^0 = 0$$

Calabi-Yau

$$W_1 = W_2 = W_3 = 3W_4 + 2W_5 = 0$$

$$T^0 \in W_4 \oplus W_5$$

conformally Calabi-Yau

▼ non-complex manifolds

Summary

$$W_1 = W_3 = W_4 = 0$$

$$T^0 \in W_2 \oplus W_5$$

symplectic

$$W_2 = W_3 = W_4 = W_5 = 0$$

$$T^0 \in W_1$$

nearly-Kähler

$$W_1 = W_3 = W_4 = W_5 = 0$$

$$T^0 \in W_2$$

almost-Kähler

$$W_3 = W_4 = W_5 = 0$$

$$T^0 \in W_1 \oplus W_2$$

quasi-Kähler

$$W_4 = W_5 = 0$$

$$T^0 \in W_1 \oplus W_2 \oplus W_3$$

semi-Kähler

$$W_1^- = W_2^- = W_4 = W_5 = 0$$

$$T^0 \in W_1^+ \oplus W_2^+ \oplus W_3$$

half-flat