Comments on Heterotic Flux Compactifications

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Our "Purpose" is...

Construct a realistic model of 4-dim. particle physics

- matter contents and their interactions
- gauge symmetry and its breaking
- gravity, cosmology
- etc., etc.

We take an approach from STRING/SUPERGRAVITY THEORIES.

- NS H-flux appears in the SUSY variation of the gravitino:

$$\delta\psi_M = \left(\partial_M + (\omega_M{}^{AB} - H_M{}^{AB})\Gamma_{AB}\right)\xi$$

$\delta\psi_M = 0$ gives us information of (compactified) geometry.

If $H_M{}^{AB} \neq 0$, the geometry is no longer a Kähler manifold.

↓

G-structure manifold

Let us consider an effective theory from HETEROTIC STRING.

Why heterotic string theory?

- contains only NS fields
- includes Yang-Mills gauge symmetry
- realizes anomaly cancellation in a miraculous way

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$$\begin{cases} \text{contains only NS fields} \\ \text{includes Yang-Mills gauge symmetry} \\ \text{realizes anomaly cancellation in a miraculous way} \end{cases}$$

A good issue to understand string dualities at deeper level

duality: $\mathrm{II}/\mathrm{CY}_3 \leftrightarrow \mathrm{HE}/K3 \times T^2$

- Why heterotic string theory?

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- A good issue to understand string dualities at deeper level

duality: $\mathrm{II}/\mathrm{CY}_3 \leftrightarrow \mathrm{HE}/K3 \times T^2$

- A new insight on inflation scenario (?)

KKLT model in type II string with fluxes

Contents

G-structure Manfolds

G-structure group on an n-dim. manifold $\mathcal{M}$

$^{\exists}$nowhere vanishing tensors X on $\mathcal{M}$

with satisfying $D_m(\omega)X = 0$:

tensors	G-structure	
η_{ab}	$O(n)$	
η_{ab} $\varepsilon_{a_1 \cdots a_n}$	$SO(n)$	
η_{ab} $J_a{}^b$	$U(m)$	$J^2 = -1$
η_{ab} $J_a{}^b$ $\Omega^{(m)}$	$SU(m)$	$(2m = n)$

6-dim. $SU(3)$-structure on manifold

Consider a geometry $\mathcal{K}_6$ with a Killing spinor equation including torsion

$$^{\exists}\text{complex Weyl } \xi \qquad \text{s.t. } \nabla^{(T)}\xi = 0 \,, \qquad i\gamma^{456789}\xi = \xi$$

This is a definition of the geometry with $SU(3)$-structure.

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Consider a geometry $\mathcal{K}_6$ with a Killing spinor equation including torsion

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This is a definition of the geometry with $SU(3)$-structure.

Invariant p-forms on the $SU(3)$-structure manifold:

2-form in $SO(6)$ ⇨ a real 2-form in $SU(3)$

$${}_6C_2 = 15 \quad = \quad 1 + 3 + \overline{3} + 8 \quad : \quad J_{ab} = i\xi^{\dagger}\Gamma_{ab}\xi$$

3-form in $SO(6)$ ⇨ an (almost) complex 3-form in $SU(3)$

$${}_6C_3 = 20 \quad = \quad 1 + 1 + 3 + \overline{3} + 6 + \overline{6} \quad : \quad \Omega_{abc} = \xi^{\mathrm{T}}\Gamma_{abc}\xi$$

Furthermore

$$J \wedge \Omega = 0 \,, \qquad J \wedge J \wedge J = -\frac{3i}{4}\Omega \wedge \overline{\Omega} = -3! \times (\text{vol.})_{\mathcal{K}_6}$$

Heterotic Theory

The story started from

A. Strominger, "*Superstrings with torsion,*" [Nucl. Phys. B274 (1986) 253]

Heterotic theory on $SU(3)$-structure manifold

Supergravity with anti-hermitian Yang-Mills field

- **Bosonic part of the Lagrangian** (without fermion condensations)

$$\mathscr{L} = \frac{1}{4}\sqrt{-G}\,\mathrm{e}^{-2\Phi}\Big[R(\omega) - \frac{1}{3}H_{MNP}H^{MNP} + 4(\nabla_M\Phi)^2 + \alpha'\Big\{\mathrm{tr}\big(F_{MN}F^{MN}\big)\Big\}\Big]$$

- **Bianchi identity** $[\omega]$

$$\mathrm{d}H = +\alpha'\Big[\mathrm{tr}\{F \wedge F\}\Big]$$

Chapline and Manton [Phys. Lett. B120 (1983) 105]

(supergravity)

Heterotic theory on $SU(3)$-structure manifold

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$$\mathrm{d}H = +\alpha'\Big[\mathrm{tr}\{F \wedge F\} - \mathrm{tr}\{R(\omega) \wedge R(\omega)\}\Big]$$

Green and Schwarz [Phys. Lett. B149 (1984) 117]

(anomaly cancellation)

(worldsheet 1-loop β-function)

Heterotic theory on $SU(3)$-structure manifold

Supergravity with anti-hermitian Yang-Mills field QL

- **Bosonic part of the Lagrangian** (without fermion condensations)

$$\mathscr{L} = \frac{1}{4}\sqrt{-G}\, e^{-2\Phi}\Big[R(\omega) - \frac{1}{3} H_{MNP} H^{MNP} + 4(\nabla_M \Phi)^2 + \alpha' \Big\{ \mathrm{tr}\big(F_{MN} F^{MN}\big) - \mathrm{tr}\big(R_{MN}(\omega_+) R^{MN}(\omega_+)\big) \Big\} \Big]$$

- **Bianchi identity** $[\omega_+ = \omega + H]$

$$dH = +\alpha' \Big[\mathrm{tr}\{F \wedge F\} - \mathrm{tr}\{R(\omega_+) \wedge R(\omega_+)\} \Big]$$

Hull [Phys. Lett. B178 (1986) 357]

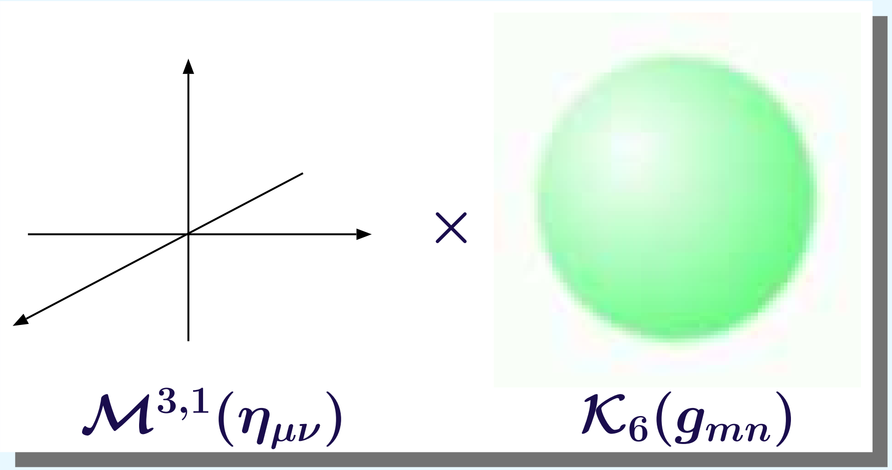
Bergshoeff and de Roo [Nucl. Phys. B328 (1989) 439]

(worldsheet 2-loop β-function)

Towards 4-dim. Physics...

- **Study vacuum configuration**
  - SUSY variations $\longrightarrow$ geometry with $SU(3)$-structure
- **Investigate low energy effective theory**
  - Gauge symmetry
  - Zero mode equations
  - Evaluations

Vacuum Configuration

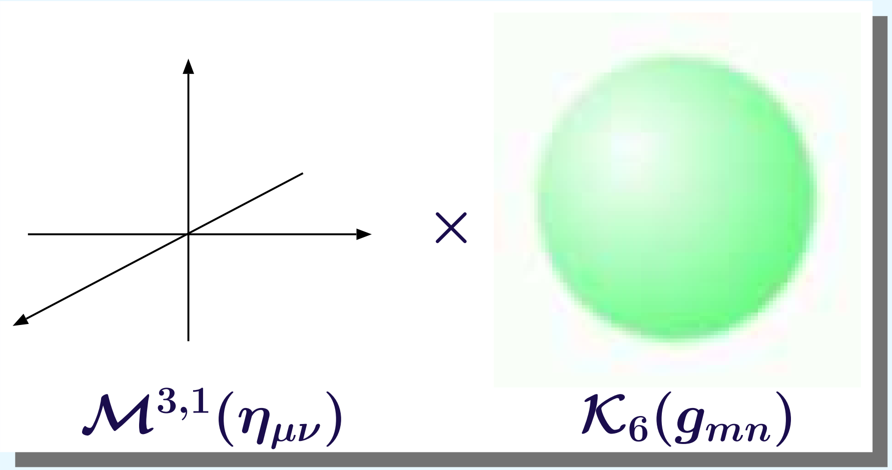


Ansatz:

$$
\begin{aligned}
G_{MN}\,\mathrm{d}x^M \mathrm{d}x^N \\
&= \mathrm{e}^{(\Phi - \widehat{D})/2}\Big(\eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn}\,\mathrm{d}y^m \mathrm{d}y^n\Big)
\end{aligned}
$$

$\mathcal{M}^{3,1}$: maximally symmetric

Vacuum Configuration



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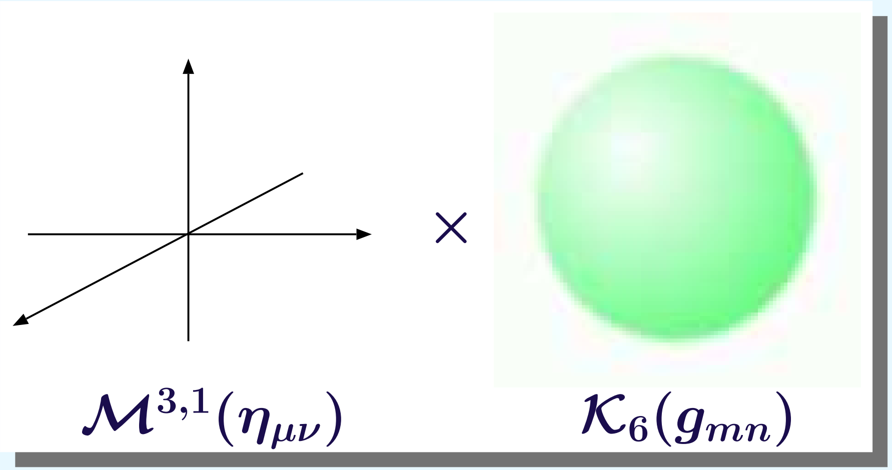
$\mathcal{M}^{3,1}$: maximally symmetric

$$\boxed{Spin(9,1) \;\rightarrow\; SL(2,\mathbb{C}) \times SU(4)}$$

$$16 \;=\; (2,4) + (\overline{2},\overline{4}) \;:\qquad \epsilon_+ \;=\; \eta_+ \otimes \xi_+ + \eta_- \otimes \xi_-$$

$\mathcal{N} = 1$ SUSY on $\mathcal{M}^{3,1}$

Vacuum Configuration



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$$Spin(9,1) \;\rightarrow\; SL(2,\mathbb{C})\times SU(4)$$

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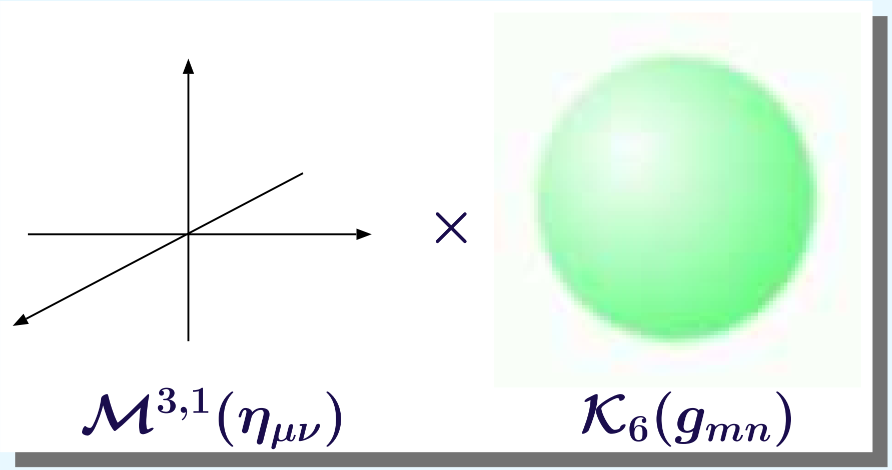
$\mathcal{N}=1$ SUSY on $\mathcal{M}^{3,1}$ ⟷ 1 Killing spinor ξ_+ on $\mathcal{K}_6$

Vacuum Configuration



Ansatz:

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$\mathcal{M}^{3,1}$: maximally symmetric

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$\mathcal{N}=1$ SUSY on $\mathcal{M}^{3,1}$ ⟷ **1 Killing spinor ξ_+ on $\mathcal{K}_6$** ⟷ **$SU(3)$-structure on $\mathcal{K}_6$**

SUSY variations

$\mathbf{0 \equiv \delta\psi_m = D_m(\omega_-)\xi_+}$ ← **Killing spinor eq.** $[\omega_- = \omega - H]$ $\xi_+^\dagger \xi_+ = 1$

$$\begin{aligned} J_{ab} &= i\xi_+^\dagger \Gamma_{ab} \xi_+ &:& \quad D_m(\omega_-) J_{ab} &= 0 \\ \Omega_{abc} &= \xi_+^T \Gamma_{abc} \xi_+ &:& \quad D_m(\omega_-) \Omega_{abc} &= 0 \end{aligned}$$

SUSY variations

$0 \equiv \delta\psi_m = D_m(\omega_-)\xi_+ \leftarrow$ **Killing spinor eq.** $[\omega_- = \omega - H]$ $\xi_+^\dagger \xi_+ = 1$

$$J_{ab} = i\xi_+^\dagger \Gamma_{ab}\xi_+ \quad : \quad D_m(\omega_-)J_{ab} = 0$$
$$\Omega_{abc} = \xi_+^{\mathrm{T}} \Gamma_{abc}\xi_+ \quad : \quad D_m(\omega_-)\Omega_{abc} = 0$$

Furthermore "$0 \equiv \delta(\text{fermions})$" **indicates**

$$R^{ab}{}_{mn}(\omega_-)J_{ab} = 0 \ : \quad c_1(R_-) \text{ vanishes}$$
$$N_{mn}{}^p = 0 \ : \quad \mathcal{K}_6 \text{ is complex}$$

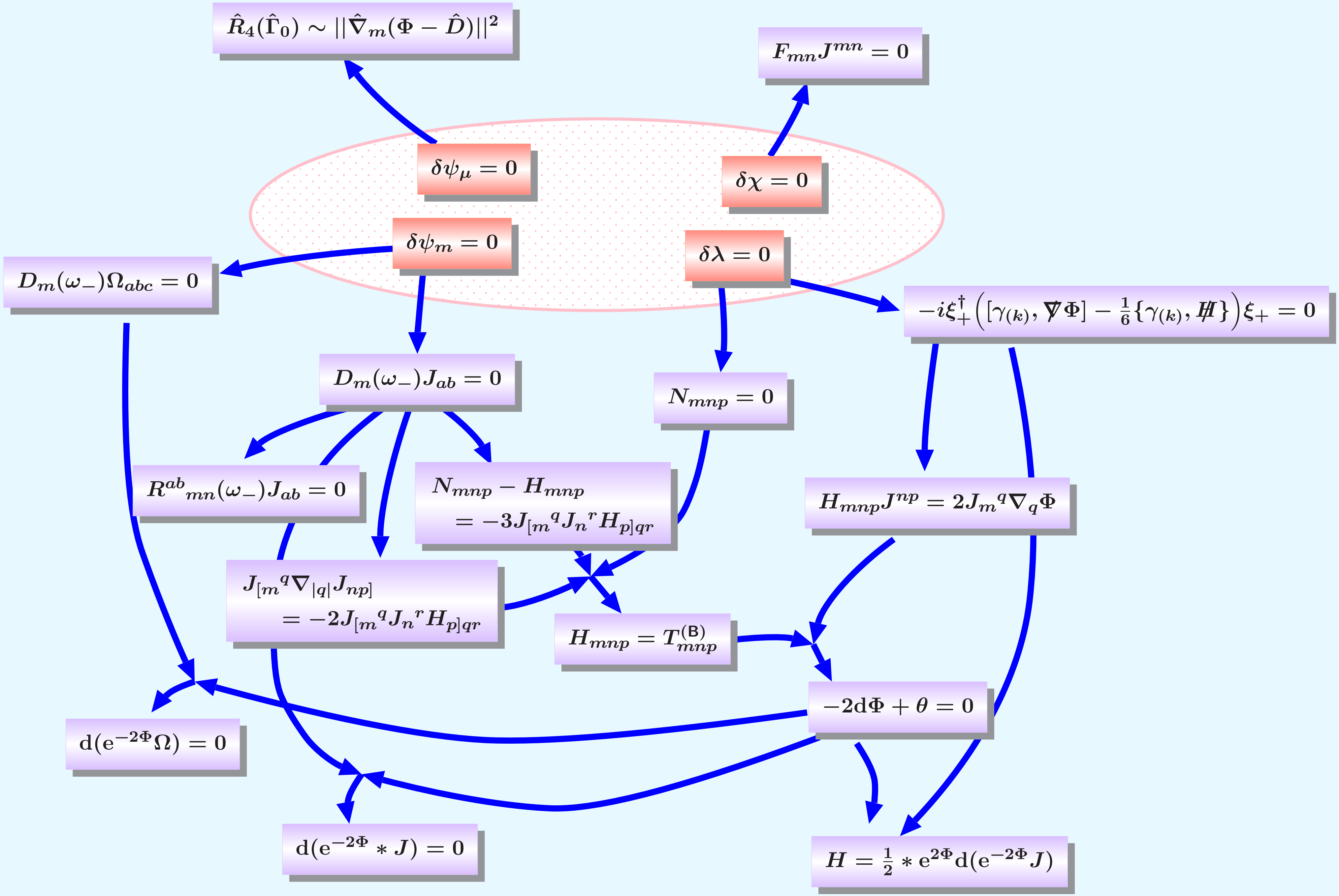
T

$$H = \frac{i}{2}(\partial - \overline{\partial})J \,, \qquad \mathrm{d}H = -i\partial\overline{\partial}J \,, \qquad \mathrm{d}\Phi = \frac{1}{2} J \lrcorner \mathrm{d}J$$
$$H = H^0 + \widehat{H} \,, \qquad J \lrcorner H^0 = 0 \,, \qquad \widehat{H}_{mnp} = \frac{3}{2} J_{[mn} J_{p]}{}^q \nabla_q \Phi$$

$$*(J \wedge \mathrm{d}H) = \nabla_m^2 \Phi - (\nabla_m \Phi)^2 + \frac{1}{3}(H^0_{mnp})^2$$

$\mathrm{d}H = 0$

$\mathrm{d}H \neq 0$



$\hat{R}_4(\hat{\Gamma}_0) \sim ||\hat{\nabla}_m(\Phi - \hat{D})||^2$
$F_{mn}J^{mn} = 0$
$\delta\psi_\mu = 0$
$\delta\chi = 0$
$\delta\psi_m = 0$
$\delta\lambda = 0$
$D_m(\omega_-)\Omega_{abc} = 0$
$-i\xi_+^\dagger\Big([\gamma_{(k)}, \nabla\!\!\!/ \Phi] - \frac{1}{6}\{\gamma_{(k)}, H\!\!\!/\}\Big)\xi_+ = 0$
$D_m(\omega_-)J_{ab} = 0$
$N_{mnp} = 0$
$R^{ab}{}_{mn}(\omega_-)J_{ab} = 0$
$N_{mnp} - H_{mnp}$
$= -3J_{[m}{}^q J_n{}^r H_{p]qr}$
$H_{mnp}J^{np} = 2J_m{}^q\nabla_q\Phi$
$J_{[m}{}^q\nabla_{|q|}J_{np]}$
$= -2J_{[m}{}^q J_n{}^r H_{p]qr}$
$H_{mnp} = T^{(\mathrm{B})}_{mnp}$
$-2\mathrm{d}\Phi + \theta = 0$
$\mathrm{d}(\mathrm{e}^{-2\Phi}\Omega) = 0$
$\mathrm{d}(\mathrm{e}^{-2\Phi} * J) = 0$
$H = \frac{1}{2} * \mathrm{e}^{2\Phi}\mathrm{d}(\mathrm{e}^{-2\Phi}J)$

Gauge Symmetry Breaking

- $E_8 \longrightarrow E_6 \times SU(3)$

$$\mathbf{248} = (\mathbf{78},\mathbf{1}) + (\mathbf{1},\mathbf{8}) + (\mathbf{27},\mathbf{3}) + (\overline{\mathbf{27}},\overline{\mathbf{3}})$$

- $E_8 \longrightarrow SO(16)$

$\longrightarrow SO(10) \times SO(6)$

$$\begin{aligned} \mathbf{248} &= \mathbf{120}_{SO(16)\ \text{adj.}} + \mathbf{128}_{SO(16)\ \text{spinor}} \\ &= (\mathbf{45},\mathbf{1}) + (\mathbf{1},\mathbf{15}) + (\mathbf{10},\mathbf{6}) + (\mathbf{16},\mathbf{4}) + (\overline{\mathbf{16}},\overline{\mathbf{4}}) \end{aligned}$$

Each breaking scenario deeply depends on the way of embedding the holonomy group into the gauge groups.

$$\textbf{Embedding}\quad A \;\leftrightarrow\; \begin{cases} \omega_- \;:\; SU(3)\ \textbf{holonomy} \\ \omega_+ \;:\; SO(6)\ \textbf{holonomy} \\ \quad \textbf{etc.} \end{cases} \qquad [\omega_\pm = \omega \pm \boldsymbol{H}]$$

$$\text{Embedding } A \leftrightarrow \begin{cases} \omega_- : & SU(3) \text{ holonomy} \\ \omega_+ : & SO(6) \text{ holonomy} \\ \text{etc.} & \end{cases} \qquad [\omega_\pm = \omega \pm H]$$

with following constraints

$$R_{mnpq}(\omega_+) = R_{pqmn}(\omega_-) + (\mathrm{d}H)_{pqmn}$$

$$\mathrm{d}H = +\alpha' \Big[\mathrm{tr}\big(F \wedge F\big) - \mathrm{tr}\big\{ R(\omega_+) \wedge R(\omega_+) \big\} \Big] \qquad \boxed{\mathrm{d}H = 0} \;\; \boxed{\mathrm{d}H \neq 0}$$

$R_{pqmn}(\omega_-)$:	type $(1,1)$ w/ indices p,q
F :	$(1,1)$-form
$\mathrm{d}H$:	$(2,2)$-form, higher order in α'
$R_{mnpq}(\omega_+)$:	$(1,1)$-form w/ indices p,q + higher order in α'

$\mathrm{d}H = 0$ case: $A \equiv \omega_+ \quad \Rightarrow \quad E_8 \longrightarrow SO(10) \times \underline{SO(6)}$

$\mathrm{d}H \neq 0$ case: $A =$ others.. $\quad \Rightarrow \quad E_8 \longrightarrow \mathcal{G} \times \underline{\mathcal{H}}$

Minimal embedding: $\mathrm{d}H = 0$

SUSY A

The condition $*(J \wedge \mathrm{d}H) = 0$ denotes

$$0 \;=\; \nabla_m^2 \Phi - (\nabla_m \Phi)^2 + \frac{1}{3}(H^0_{mnp})^2$$

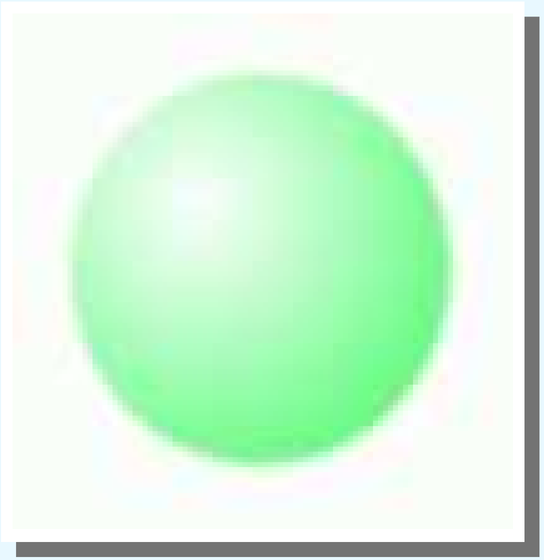
If there are no boundaries/singularities on $\mathcal{K}_6$, then

$$\int_{\mathcal{K}_6} \nabla_m^2 \mathrm{e}^{-\Phi} \;=\; \frac{1}{3}\int_{\mathcal{K}_6} \mathrm{e}^{-\Phi} \left\| H^0_{mnp} \right\|^2$$

$\mathrm{d}H = 0$ without any boundaries ➡ $H^0 = 0$ ➡ $\Phi = \text{const.}$

➡ smooth $\mathcal{K}_6 = \mathrm{CY}_3$

No-go theorem on smooth compactification with fluxes

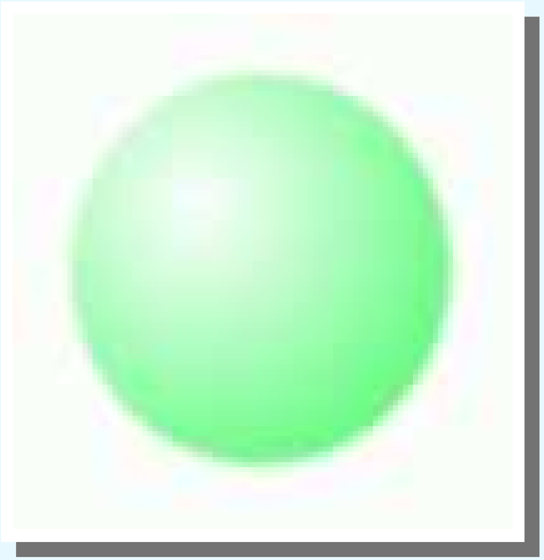


Without boundaries/singularities on $\mathcal{K}_6$:

- all fluxes are trivial $H = \mathrm{d}\Phi = 0$
- $\mathcal{K}_6 = \mathrm{CY}_3$
- $\omega_+ = \omega_- = \omega$, $E_8 \longrightarrow E_6 \times SU(3)$
- $\#$ of zero modes — AS index theorem

Candelas, Horowitz, Strominger and Witten

[Nucl. Phys. B258 (1985) 46]



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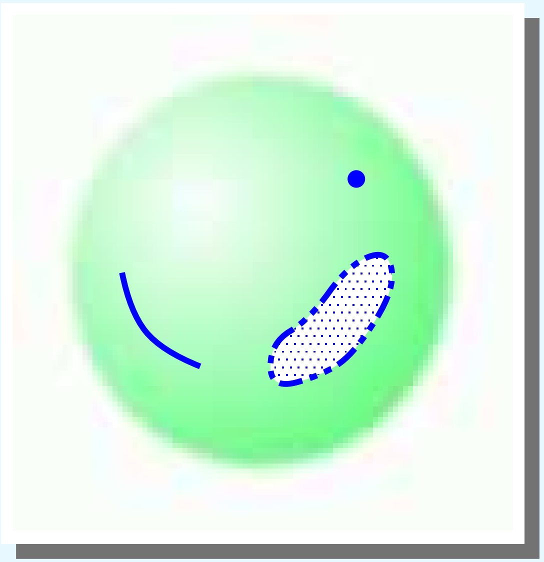

With boundaries/singularities on $\mathcal{K}_6$:

- non-trivial fluxes can exist

  $\partial_m \Phi \neq 0$, $H^0_{mnp} \neq 0$ T
- $E_8 \to SO(10) \times SO(6)$
- $\chi^0_{\mathcal{H}}$ lives in the boundaries
- $\#$ of zero modes — APS index theorem

Non-minimal embedding: $\mathrm{d}H \neq 0$

SUSY A

Combining equations of motion and SUSY conditions, we obtain

$$0 = \frac{1}{2}\nabla_m^2 \mathrm{e}^{-2\Phi} - \frac{1}{3}\mathrm{e}^{-2\Phi}(H_{mnp})^2 + \mathrm{e}^{-2\Phi} * (J \wedge \mathrm{d}H)$$

$$\mathrm{tr}\big(R_{mn}R^{mn}\big) - \mathrm{tr}(F_{mn}F^{mn}) = -2 * \Big[J \wedge \Big(\mathrm{tr}(R \wedge R) - \mathrm{tr}(F \wedge F)\Big)\Big] + \mathcal{O}(\alpha')$$

Non-minimal embedding: $\mathrm{d}H \neq 0$

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Combining equations of motion and SUSY conditions, we obtain

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Then, within the linear order in α', we find

$$\nabla_m^2 e^{-2\Phi} = e^{-2\Phi}\left[\frac{2}{3}\big\|H_{mnp}\big\|^2 + \alpha'\big(\mathrm{tr}\big\|F_{mn}\big\|^2 - \mathrm{tr}\big\|R_{mn}\big\|^2\big)\right]$$

Integral on a smooth manifold $\mathcal{K}_6$:

$$\int_{\mathcal{K}_6} e^{-2\Phi}\left[\frac{2}{3}\big\|H_{mnp}\big\|^2 + \alpha'\mathrm{tr}\big\|F_{mn}\big\|^2\right] = \int_{\mathcal{K}_6} e^{-2\Phi}\left[\alpha'\mathrm{tr}\big\|R_{mn}\big\|^2\right]$$

with $\quad \mathrm{tr}\big\|F_{mn}\big\|^2 \neq \mathrm{tr}\big\|R_{mn}\big\|^2$

Smooth compactification scenario is possible!

Summary and Discussions

Summary and Discussions

- **Vacuum configuration of the flux compactifications in heterotic theory**
- **No-go theorem on smooth manifolds with $H \neq 0$ and $\mathrm{d}H = 0$**
- **Possibility of smooth compactifications with $H \neq 0$ and $\mathrm{d}H \neq 0$**

Summary and Discussions

- Vacuum configuration of the flux compactifications in heterotic theory
- No-go theorem on smooth manifolds with $H \neq 0$ and $\mathrm{d}H = 0$
- Possibility of smooth compactifications with $H \neq 0$ and $\mathrm{d}H \neq 0$
- $\#$ of zero modes under the condition $\mathrm{d}H \neq 0$

  modification of the Atiyah-(Patodi)-Singer index theorem
- other possibilities of gauge symmetry breaking
- compactifications on non-complex geometries SUSY

Frey and Lippert [hep-th/0507202]

Manousselis, Prezas and Zoupanos [hep-th/0511122]

Appendix

Appendix: Quartic effective Lagrangian

$$\mathscr{L}_{\text{total}} = \mathscr{L}_0(R) + \mathscr{L}_\beta(F^2) + \mathscr{L}_\alpha(R^2)$$ L

$$\begin{aligned}
\mathscr{L}_0(R) \;=\; & \frac{1}{2\kappa_{10}^2}\sqrt{-G}\,\mathrm{e}^{-2\Phi}\Big[R(\omega) - \frac{1}{3}H_{MNP}H^{MNP} + 4(\nabla_M\Phi)^2 - \overline{\psi}_M\Gamma^{MNP}D_N(\omega)\psi_P + 16\,\overline{\lambda}\not{D}(\omega)\lambda \\
& + 8\,\overline{\lambda}\Gamma^{MN}D_M(\omega)\psi_N + 8\,\overline{\psi}_M\Gamma^N\Gamma^M\lambda\,(\nabla_N\Phi) - 2\,\overline{\psi}_M\Gamma^M\psi_N\,(\nabla^N\Phi) \\
& + \frac{1}{12}\,H^{PQR}\Big\{\overline{\psi}_M\Gamma^{[M}\Gamma_{PQR}\Gamma^{N]}\psi_N + 8\,\overline{\psi}_M\Gamma^M{}_{PQR}\lambda - 16\,\overline{\lambda}\Gamma_{PQR}\lambda\Big\} \\
& + \frac{1}{48}\,\overline{\psi}^M\Gamma^{ABC}\psi_M\Big\{2\overline{\lambda}\Gamma_{ABC}\lambda + \overline{\lambda}\Gamma_{ABC}\Gamma^N\psi_N - \frac{1}{4}\,\overline{\psi}^N\Gamma_{ABC}\psi_N - \frac{1}{8}\,\overline{\psi}^N\Gamma_N\Gamma_{ABC}\Gamma^P\psi_P\Big\}\Big]
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_\beta(F^2) \;=\; & -\frac{1}{2\kappa_{10}^2}\frac{\kappa_{10}^2}{2g_{10}^2}\sqrt{-G}\,\mathrm{e}^{-2\Phi}\Big[-\operatorname{tr}(F_{MN}F^{MN}) - 2\operatorname{tr}\{\overline{\chi}\not{D}(\omega,A)\chi\} + \frac{1}{6}\operatorname{tr}(\overline{\chi}\Gamma^{ABC}\chi)\hat{H}_{ABC} \\
& - \frac{1}{2}\operatorname{tr}\{\overline{\chi}\Gamma^M\Gamma^{AB}(F_{AB} + \hat{F}_{AB})\}\Big(\psi_M + \frac{2}{3}\Gamma_M\lambda\Big) - \frac{1}{48}\operatorname{tr}(\overline{\chi}\Gamma^{ABC}\chi)\,\overline{\psi}_M\big(4\Gamma_{ABC}\Gamma^M + 3\Gamma^M\Gamma_{ABC}\big)\lambda \\
& + \frac{1}{12}\operatorname{tr}(\overline{\chi}\Gamma^{ABC}\chi)\overline{\lambda}\Gamma_{ABC}\lambda - \frac{\beta}{96}\operatorname{tr}(\overline{\chi}\Gamma^{ABC}\chi)\operatorname{tr}(\overline{\chi}\Gamma_{ABC}\chi)\Big]
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_\alpha(R^2) \;=\; & -\frac{1}{2\kappa_{10}^2}\frac{\kappa_{10}^2}{2g_{10}^2}\sqrt{-G}\,\mathrm{e}^{-2\Phi}\Big[-R_{ABMN}(\omega_+)R^{ABMN}(\omega_+) - 2\overline{\psi}^{AB}\not{D}(\omega(e,\psi),\omega_+)\psi_{AB} + \frac{1}{6}\overline{\psi}^{AB}\Gamma^{MNP}\psi_{AB}\hat{H}_{MNP} \\
& + \frac{1}{2}\overline{\psi}_{AB}\Gamma^M\Gamma^{NP}\big\{R^{AB}{}_{NP}(\omega_+) + \hat{R}^{AB}{}_{NP}(\omega_+)\big\}\Big(\psi_M + \frac{2}{3}\Gamma_M\lambda\Big) \\
& - \frac{1}{48}\,\overline{\psi}_{AB}\Gamma^{CDE}\psi_{AB}\cdot\overline{\psi}_M\big(4\Gamma_{CDE}\Gamma^M + 3\Gamma^M\Gamma_{CDE}\big)\lambda \\
& + \frac{1}{12}\overline{\psi}^{AB}\Gamma^{CDE}\psi_{AB}\big(\overline{\lambda}\Gamma_{CDE}\lambda\big) - \frac{\alpha}{96}\overline{\psi}^{AB}\Gamma^{FGH}\psi_{AB}\big(\overline{\psi}^{CD}\Gamma_{FGH}\psi_{CD}\big)\Big]
\end{aligned}$$

$$
\begin{aligned}
\delta_0 e_M{}^A &= \frac{1}{2}\bar{\epsilon}\Gamma^A\psi_M \\
\delta_0\psi_M &= \Big(\partial_M + \frac{1}{4}\omega_{-M}{}^{AB}\Gamma_{AB}\Big)\epsilon + \Big\{\epsilon\big(\overline{\psi}_M\lambda\big) - \psi_M\big(\bar{\epsilon}\lambda\big) + \Gamma^A\lambda\big(\overline{\psi}_M\Gamma_A\epsilon\big)\Big\} \\
\delta_0 B_{MN} &= \bar{\epsilon}\Gamma_{[M}\psi_{N]} \\
\delta_0\lambda &= -\frac{1}{4}\not{D}\Phi\,\epsilon + \frac{1}{24}\Gamma^{ABC}\epsilon\Big(\hat{H}_{ABC} - \frac{1}{4}\bar{\lambda}\Gamma_{ABC}\lambda\Big) \\
\delta_0\Phi &= -\bar{\epsilon}\lambda \\
\delta_0 A_M &= \frac{1}{2}\bar{\epsilon}\Gamma_M\chi \\
\delta_0\chi &= -\frac{1}{4}\Gamma^{AB}\epsilon\,\hat{F}_{AB} + \Big\{\epsilon\big(\overline{\chi}\lambda\big) - \chi\big(\bar{\epsilon}\lambda\big) + \Gamma^A\lambda\big(\overline{\chi}\Gamma_A\epsilon\big)\Big\} \\
\delta_\beta\psi_M &= \frac{\beta}{192}\,\Gamma^{ABC}\Gamma_M\epsilon\,\mathrm{tr}\big(\overline{\chi}\Gamma_{ABC}\chi\big) \\
\delta_\beta B_{MN} &= -\beta\,\mathrm{tr}\big\{A_{[M}\delta_0 A_{N]}\big\} \\
\delta_\beta\lambda &= \frac{\beta}{384}\,\Gamma^{ABC}\epsilon\,\mathrm{tr}\big(\overline{\chi}\Gamma_{ABC}\chi\big) \\
\delta_\alpha\psi_M &= \frac{\alpha}{192}\Gamma^{CDE}\Gamma_M\epsilon\,\overline{\psi}^{AB}\Gamma_{CDE}\psi_{AB} \\
\delta_\alpha B_{MN} &= -\alpha\,\omega_{+[M}{}^{AB}\,\delta_0\omega_{+N]}{}^{AB} \\
\delta_\alpha\lambda &= \frac{\alpha}{384}\,\Gamma^{CDE}\Gamma_M\epsilon\,\overline{\psi}^{AB}\Gamma_{CDE}\psi_{AB}
\end{aligned}
$$

Bergshoeff and de Roo [Nucl. Phys. B328 (1989) 439]

Equations of Motion

$$\Phi \;:\quad 0 = -R(\omega) + \frac{1}{3}H_{MNP}H^{MNP} + 4(\nabla_M\Phi)^2 - 4\nabla_M^2\Phi - \alpha'\Big[\mathrm{tr}(F_{MN}F^{MN}) - \mathrm{tr}\{R_{MN}(\omega_+)R^{MN}(\omega_+)\}\Big]$$

$$G_{MN} \;:\quad 0 = R_{MN}(\omega) - H_{MPQ}H_N{}^{PQ} + 2\nabla_M\nabla_N\Phi - \frac{1}{2}G_{MN}\Big[R(\omega) - \frac{1}{3}H_{PQR}H^{PQR} - 4(\nabla_P\Phi)^2 + 4\nabla_P^2\Phi\Big] - \frac{\alpha'}{2}G_{MN}\Big[\mathrm{tr}(F_{MN}F^{MN}) - \mathrm{tr}\{R_{MN}(\omega_+)R^{MN}(\omega_+)\}\Big] + 2\alpha'\Big[\mathrm{tr}(F_{MP}F_N{}^P) - \mathrm{tr}\{R_{MP}(\omega_+)R_N{}^P(\omega_+)\}\Big] + 2\alpha' e^{2\Phi}\Big[2\nabla^P\nabla^Q_{(+)}\{e^{-2\Phi}R_{MPNQ}(\omega_+)\} - \nabla^Q_{(+)}\{e^{-2\Phi}R_{MPQR}(\omega_+)\}H_N{}^{PR} - 2\nabla^P\{e^{-2\Phi}R_{MPQR}(\omega_+)H_N{}^{QR}\} - 2e^{-2\Phi}R_{MPQR}(\omega_+)H_N{}^{PS}H_S{}^{QR} - \nabla^P\nabla^Q_{(+)}\{e^{-2\Phi}R_{MNPQ}(\omega_+)\} + \nabla^P\{e^{-2\Phi}R_{MNQR}(\omega_+)H_P{}^{QR}\}\Big]$$

$$B_{MN} \;:\quad 0 = \nabla^M(e^{-2\Phi}H_{MNP})$$

$$\chi \;:\quad 0 = \not{D}(\omega - \tfrac{1}{3}H, A)\chi - \Gamma^M\chi\nabla_M\Phi + \frac{3}{2}\Gamma^M\Gamma^{AB}(F_{AB} + \hat{F}_{AB})\Big(\psi_M + \frac{2}{3}\Gamma_M\lambda\Big)$$

Appendix: Intrinsic torsion

Chiossi and Salamon [math.DG/0202282]

Let κ be a contorsion in $\nabla^{(T)}$ with acting on the $SU(3)$ Killing spinor ξ:

$$0 = \nabla^{(T)}\xi = (\nabla + \kappa^0 + \kappa^{\mathfrak{g}})\xi$$

where we decomposed $\kappa \equiv \kappa^0 + \kappa^{\mathfrak{g}}$ in such a way as $\kappa^{\mathfrak{g}}\,\xi = 0$ (where $\mathfrak{g} = \mathfrak{su}(3)$):

$$\xi = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline * \end{array}\right) \qquad \kappa^{\mathfrak{g}} \equiv \left(\begin{array}{ccc|c} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \qquad \kappa^0 \equiv \left(\begin{array}{ccc|c} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ \hline * & * & * & * \end{array}\right)$$

Then, under the same structure group G we find

$$(\nabla^{(T_1)} - \nabla^{(T_2)})\xi \;\propto\; \kappa^{\mathfrak{g}}\xi \;=\; 0$$

So, from the group-theoretical viewpoint, κ^0 carries an intrinsic part of the contorsion when we consider the classification of the $SU(3)$-structure manifolds!

Torsion $T_{mn} \equiv T^p{}_{mn} \mathrm{d}x^p = \kappa^p{}_{[mn]} \mathrm{d}x^p$ is given in the various representations:

$$T^{\mathfrak{g}}_{mn} = \kappa^{\mathfrak{g}}{}_{[mn]} \sim \mathfrak{su}(3) , \qquad T^0_{mn} = \kappa^0_{[mn]} \sim \mathfrak{so}(6)/\mathfrak{su}(3) \equiv \mathfrak{su}(3)^\perp$$

$$\therefore \qquad (T^0)^p{}_{mn} \in \Lambda^1 \otimes \mathfrak{su}(3)^\perp \quad \text{on } \mathcal{K}_6$$

$$\Lambda^1 \sim 3 \oplus \overline{3} , \qquad \mathfrak{su}(3) \sim 8 , \qquad \mathfrak{su}(3)^\perp = \mathfrak{so}(6)/\mathfrak{su}(3) \sim 1 \oplus 3 \oplus \overline{3}$$

Thus the intrinsic torsion T^0 can be decomposed

$$\begin{aligned}(T^0)^p{}_{mn} \in \Lambda^1 \otimes \mathfrak{su}(3)^\perp &= (3 \oplus \overline{3}) \otimes (1 \oplus 3 \oplus \overline{3}) \\ &= (1 \oplus 1) \oplus (8 \oplus 8) \oplus (6 \oplus \overline{6}) \oplus (3 \oplus \overline{3}) \oplus (3 \oplus \overline{3})' \\ & \quad\; W_1 \qquad\;\; W_2 \qquad\;\; W_3 \qquad\;\; W_4 \qquad\;\; W_5\end{aligned}$$

where

W_1 : complex scalar in $(1 \oplus 1)$

W_2 : complex primitive 2-form in $(8 \oplus 8)$

W_3 : real primitive $(2,1) \oplus (1,2)$-form in $(6 \oplus \overline{6})$

W_4 : real 1-form in $(3 \oplus \overline{3})$

W_5 : complex $(1,0)$-form in $(3 \oplus \overline{3})'$

complex manifolds

SUSY | $\mathrm{d}H = 0$ | $\mathrm{d}H \neq 0$

$W_1 = W_2 = 0$	$T^0 \in W_3 \oplus W_4 \oplus W_5$	hermitian
$W_1 = W_2 = W_4 = 0$	$T^0 \in W_3 \oplus W_5$	balanced
$W_1 = W_2 = W_4 = W_5 = 0$	$T^0 \in W_3$	special-hermitian
$W_1 = W_2 = W_3 = W_4 = 0$	$T^0 \in W_5$	Kähler
$W_1 = W_2 = W_3 = W_4 = W_5 = 0$	$T^0 = 0$	Calabi-Yau
$W_1 = W_2 = W_3 = 3W_4 + 2W_5 = 0$	$T^0 \in W_4 \oplus W_5$	conformally Calabi-Yau

non-complex manifolds

Summary

$W_1 = W_3 = W_4 = 0$	$T^0 \in W_2 \oplus W_5$	symplectic
$W_2 = W_3 = W_4 = W_5 = 0$	$T^0 \in W_1$	nearly-Kähler
$W_1 = W_3 = W_4 = W_5 = 0$	$T^0 \in W_2$	almost-Kähler
$W_3 = W_4 = W_5 = 0$	$T^0 \in W_1 \oplus W_2$	quasi-Kähler
$W_4 = W_5 = 0$	$T^0 \in W_1 \oplus W_2 \oplus W_3$	semi-Kähler
$W_1^- = W_2^- = W_4 = W_5 = 0$	$T^0 \in W_1^+ \oplus W_2^+ \oplus W_3$	half-flat