Index Theorems on Torsional Geometries

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Why

Why string

Why string theory?

Why string theory?

✓ Quantum Field Theory

Calculation is systematic!

Symmetry is manifest!

No restriction/prediction of gauge symmetry (Why $SU(3) \times SU(2) \times U(1)$?)

No restriction of spacetime dimensions (Why four dimensions?)

No quantum gravity as a field theory

✓ Quantum Field Theory

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No restriction of spacetime dimensions (Why four dimensions?)

No quantum gravity as a field theory

✓ String Theory

Also quantum field theory! (of 2-dim. string worldsheet)

We can determine the gauge symmetries! ($E_8 \times E_8$ or $SO(32)$!)

We can derive the dimensions of spacetime! (even 26 or 10!)

A candidate of quantum gravity

Maybe four-dimensional spacetime is dynamically favored by stringy quantum effects (but still unknown)

$$Q_{\mathsf{SUSY}} |\, \text{Boson}\, \rangle = |\, \text{Fermion}\, \rangle \qquad Q_{\mathsf{SUSY}} |\, \text{Fermion}\, \rangle = |\, \text{Boson}\, \rangle$$

$$\{Q_{\mathsf{SUSY}}, \overline{Q}_{\mathsf{SUSY}}\} = 2\hbar\, \mathscr{H}$$

Supersymmetry (SUSY) has a central role in theoretical particle physics

✓ boson/fermion symmetry

✓ reduce the divergence of quantum corrections

✓ necessary to Grand Unified Theories

✓ control (non)perturbative dynamics of theories

✓ connect physics and geometry

ψ_M: gravitino = superpartner of the graviton

SUSY condition $\longrightarrow$ $$\mathbf{0} = \langle \delta\psi_M \rangle = \langle \,\text{vac.}\, | \{Q, \psi_M\} | \,\text{vac.}\, \rangle$$
$$= \Big(\partial_M + ({\omega_M}^{AB} - {H_M}^{AB}) \Gamma_{AB} \Big) \xi$$ $\longleftarrow$ Killing spinor eq.

# of SUSY in 4-dim. spacetime $\Leftrightarrow$ # of Killing spinors on a geometry

movie gif

on the web "VISUALIZATION" maintained by Jeff Bryant

http://members.wri.com/jeffb/visualization/index.shtml

Construct a realistic model of particle physics from string theories

An approach: (flux) compactification scenario

movie gif

on the web "VISUALIZATION" maintained by Jeff Bryant

http://members.wri.com/jeffb/visualization/index.shtml

STRATEGY


Effective theory
in 4-dim. spacetime
Inner space
(isospin, gauge groups)

STRATEGY

Superstrings

in 10-dim. spacetime

Effective theory

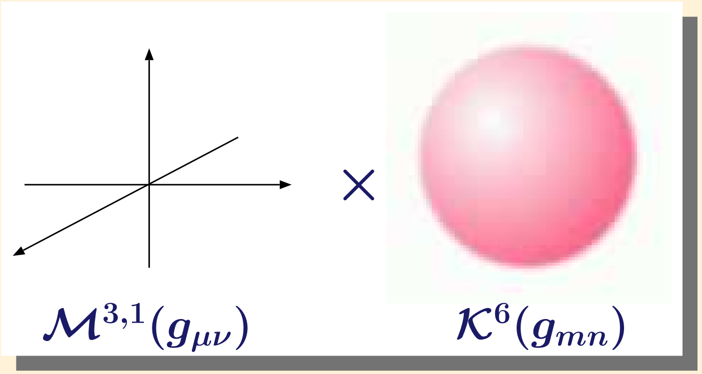
in 4-dim. spacetime

Geometrical aspects of

6-dim. space

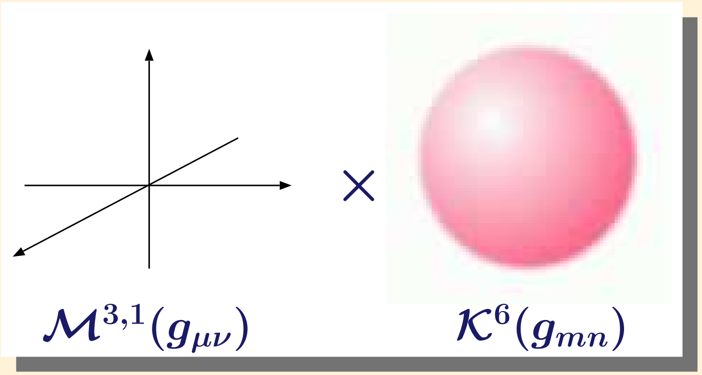
4-dim. $\mathcal{N} = 1$ Physics is given by...

- Vacuum configuration
  - 6-dim. compactified space $\leftarrow$-- SUSY variation of fermionic fields
- Low energy effective theory
  - fields, zero mode equations
  - gauge symmetry (and its breaking)



Ansatz:

$$
\begin{aligned}
g^{\mathrm{E}}_{MN}\,\mathrm{d}x^M \mathrm{d}x^N \\
&= \mathrm{e}^{-\Phi(y)/2}\Big(g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn}\,\mathrm{d}y^m \mathrm{d}y^n\Big) \\
0 &= R(g_{\mu\nu}) \quad \rightsquigarrow \quad \mathcal{M}^{3,1} \;=\; \text{Minkowski}
\end{aligned}
$$



Ansatz:

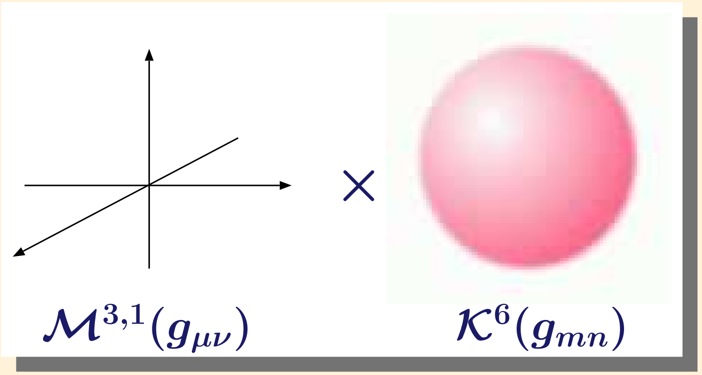
$$g^{\mathrm{E}}_{MN}\,\mathrm{d}x^M\mathrm{d}x^N = \mathrm{e}^{-\Phi(y)/2}\Big(g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu + g_{mn}\,\mathrm{d}y^m\mathrm{d}y^n\Big)$$

$$0 = R(g_{\mu\nu}) \rightsquigarrow \mathcal{M}^{3,1} = \text{Minkowski}$$

Lorentz group: $$Spin(9,1) \to SL(2,\mathbb{C}) \times SU(4)$$

Representations: $\mathbf{16} = (\mathbf{2},\mathbf{4}) + (\overline{\mathbf{2}},\overline{\mathbf{4}})$: $\xi = \eta_+ \otimes \epsilon_+ + \eta_- \otimes \epsilon_-$

$\mathcal{N}=1$ SUSY on $\mathcal{M}^{3,1}$ ⟷ 1 Killing spinor ϵ_+ on $\mathcal{K}_6$ ⟷ $SU(3)$-structure on $\mathcal{K}_6$



Ansatz:

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$$0 = R(g_{\mu\nu}) \rightsquigarrow \mathcal{M}^{3,1} = \text{Minkowski}$$

How many massless modes appear?

It depends on the cohomology classes

via eqs. of motion

Geometrical aspects of conformally balanced manifold

Cohomology classes? topological invariants?

Geometrical aspects of conformally balanced manifold

Cohomology classes? topological invariants?

not completely known, yet

M.Becker, L.-S.Tseng and S.-T.Yau, `hep-th/0612290`

Geometrical aspects of conformally balanced manifold

What should we do?

Investigate the Dirac operator!

Dirac operator in Equations of motion

\# of massless modes
on $\mathcal{M}^{3,1} \times \mathcal{K}^6$
$\Delta_{10}\,\mathcal{F}(x,y) = 0$

\# of massless modes
on $\mathcal{M}^{3,1}$
$\Delta_4\,\Phi(x) = 0$

\# of zero modes
on $\mathcal{K}^6$
$\Delta_6\,\Psi(y) = 0$

ex.) fermion: $$0 \;=\; \not{D}(\omega)\lambda - \frac{1}{12}H_{mnp}\Gamma^{mnp}\lambda \;=\; \not{D}(\omega - \tfrac{1}{3}H)\lambda$$

- Dirac index by $\not{D}(\omega) : C^\infty(S_+) \longrightarrow C^\infty(S_-)$

$$
\begin{aligned}
\mathrm{index}\,\not{D}(\omega) &\equiv (\#\text{ of left chiral}) - (\#\text{ of right chiral}) \\
&= \lim_{\beta\to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D)} \left[\Gamma_{(5)} \mathrm{e}^{-\beta\mathscr{R}} \right]
\end{aligned}
$$

▼ Dirac index by $\not{D}(\omega) : C^\infty(S_+) \longrightarrow C^\infty(S_-)$ result

$$\text{index}\,\not{D}(\omega) \equiv (\#\text{ of left chiral}) - (\#\text{ of right chiral})$$
$$= \lim_{\beta\to 0} \operatorname*{Tr}_{\text{Cliff}(D)} \left[\Gamma_{(5)} e^{-\beta\mathscr{R}} \right]$$

▼ Euler characteristic by $d + d^\dagger : C^\infty(\Lambda^{\text{even}}) \longrightarrow C^\infty(\Lambda^{\text{odd}})$

$$\chi(\mathcal{K}) \equiv (\#\text{ of harm. even-forms}) - (\#\text{ of harm. odd-forms})$$
$$= \lim_{\beta\to 0} \operatorname*{Tr}_{\text{Cliff}(D,D)} \left[\Gamma_{(5)} \widetilde{\Gamma}_{(5)} e^{-\beta\mathscr{R}} \right]$$

▼ Hirzebruch signature by $d + d^\dagger : C^\infty(\Lambda^{\text{SD}}) \longrightarrow C^\infty(\Lambda^{\text{ASD}})$

$$\sigma(\mathcal{K}) \equiv (\#\text{ of self-dual forms}) - (\#\text{ of anti-self dual forms})$$
$$= \lim_{\beta\to 0} \operatorname*{Tr}_{\text{Cliff}(D,D)} \left[\Gamma_{(5)} e^{-\beta\mathscr{R}} \right]$$

topological invariants on geometry

$$\mathrm{index}\,\not{D}(\omega) \;=\; \lim_{\beta\to 0}\ \mathop{\mathrm{Tr}}_{\mathrm{Cliff}(D)}\left[\Gamma_{(5)}\mathrm{e}^{-\beta\mathscr{R}}\right]$$

can be evaluated by Path Integral Formalism!

$\mathrm{index}\,\not{D}(\omega)$: $\mathcal{N} = 1$ quantum mechanics

$\chi(\mathcal{K})$, $\sigma(\mathcal{K})$: $\mathcal{N} = 2$ quantum mechanics

L. Alvarez-Gaumé, Commun. Math. Phys. 90 (1983) 161

Identifications

with $\mathcal{N} = 1$ quantum mechanics with $(\psi^a)^\dagger = \psi^a$:

$$\{\Gamma^a, \Gamma^b\} = 2\delta^{ab} \quad \leftrightarrow \quad \{\psi^a, \psi^b\} = \hbar\,\delta^{ab}$$

$$\not{D}(\omega) \quad \leftrightarrow \quad Q_1 = \psi^m g^{\frac{1}{4}}\Big(p_m - \frac{i}{2}\omega_{mab}\,\psi^{ab}\Big)g^{-\frac{1}{4}}$$

$$\not{D}(\omega)^2 = \Delta \quad \leftrightarrow \quad (Q_1)^2 = \hbar\,\mathscr{H}_1$$

Identifications

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\not{D}(\omega)^2 &= \Delta &\leftrightarrow&\quad (Q_1)^2 = \hbar\,\mathscr{H}_1
\end{aligned}$$

with $\mathcal{N} = 2$ quantum mechanics with $\varphi^a = \frac{1}{\sqrt{2}}(\psi_1^a + i\psi_2^a)$:

$$\begin{aligned}
e^a &&\leftrightarrow&\quad \varphi^a \\
\delta^{ab}\frac{\partial}{\partial e^b} &&\leftrightarrow&\quad \overline{\varphi}^a \\
\mathrm{d} &&\leftrightarrow&\quad Q_2 = \varphi^m g^{\frac{1}{4}}\Big(p_m - i\omega_{mab}\,\varphi^a\overline{\varphi}^b\Big)g^{-\frac{1}{4}} \\
\mathrm{d}^\dagger &&\leftrightarrow&\quad \overline{Q}_2 = \overline{\varphi}^m g^{\frac{1}{4}}\Big(p_m - i\omega_{mab}\,\varphi^a\overline{\varphi}^b\Big)g^{-\frac{1}{4}} \\
\{\mathrm{d}, \mathrm{d}^\dagger\} &= \Delta &\leftrightarrow&\quad \{Q_2, \overline{Q}_2\} = 2\hbar\,\mathscr{H}_2
\end{aligned}$$

$$\mathbf{index}\, \not{D}(\omega) = \lim_{\beta \to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D)} \left\{ \Gamma_{(5)} \mathrm{e}^{-\beta \mathscr{R}} \right\}$$

$$\chi(\mathcal{K}) = \lim_{\beta \to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D,D)} \left\{ \Gamma_{(5)} \widetilde{\Gamma}_{(5)} \mathrm{e}^{-\beta \mathscr{R}} \right\}$$

$$\sigma(\mathcal{K}) = \lim_{\beta \to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D,D)} \left\{ \Gamma_{(5)} \mathrm{e}^{-\beta \mathscr{R}} \right\}$$

$$\operatorname{index} \not{D}(\omega) = \lim_{\beta\to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D)} \left\{ \Gamma_{(5)} \mathrm{e}^{-\beta\mathscr{R}} \right\} = \lim_{\beta\to 0} (-i)^{D/2}\, \mathrm{Tr} \prod_{a=1}^{D} \psi^a \, \exp\left(-\frac{\beta}{\hbar}\mathscr{H}_1\right)$$

$$\chi(\mathcal{K}) = \lim_{\beta\to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D,D)} \left\{ \Gamma_{(5)} \widetilde{\Gamma}_{(5)} \mathrm{e}^{-\beta\mathscr{R}} \right\} = \lim_{\beta\to 0} \mathrm{Tr} \prod_{a=1}^{D} \left(\varphi^a + \overline{\varphi}^a\right) \prod_{b=1}^{D} \left(\varphi^b - \overline{\varphi}^b\right) \exp\left(-\frac{\beta}{\hbar}\mathscr{H}_2\right)$$

$$\sigma(\mathcal{K}) = \lim_{\beta\to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D,D)} \left\{ \Gamma_{(5)} \mathrm{e}^{-\beta\mathscr{R}} \right\} = \lim_{\beta\to 0} (-i)^{D/2} \mathrm{Tr} \prod_{a=1}^{D} \left(\varphi^a + \overline{\varphi}^a\right) \exp\left(-\frac{\beta}{\hbar}\mathscr{H}_2\right)$$

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$$= \lim_{\beta\to 0} \left(\frac{1}{2\pi i}\right)^{D/2} \int \mathrm{d}^D x \sqrt{g(x)} \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}} \left\langle \exp\left(-\frac{1}{\hbar} S_1^{(\mathrm{int})}\right)\right\rangle$$

$$\chi(\mathcal{K}) = \lim_{\beta\to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D,D)} \left\{\Gamma_{(5)} \widetilde{\Gamma}_{(5)} e^{-\beta\mathscr{R}}\right\} = \lim_{\beta\to 0} \mathrm{Tr} \prod_{a=1}^{D} \left(\varphi^a + \overline{\varphi}^a\right) \prod_{b=1}^{D} \left(\varphi^b - \overline{\varphi}^b\right) \exp\left(-\frac{\beta}{\hbar}\mathscr{H}_2\right)$$

$$= \lim_{\beta\to 0} \left(\frac{1}{2\pi}\right)^{D/2} \int \mathrm{d}^D x \sqrt{g(x)} \prod_{a=1}^{D} \mathrm{d}\overline{\varphi}^a_{\mathrm{bg}} \mathrm{d}\varphi^a_{\mathrm{bg}} \left\langle \exp\left(-\frac{1}{\hbar} S_2^{(\mathrm{int})}\right)\right\rangle$$

$$\sigma(\mathcal{K}) = \lim_{\beta\to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D,D)} \left\{\Gamma_{(5)} e^{-\beta\mathscr{R}}\right\} = \lim_{\beta\to 0} (-i)^{D/2} \mathrm{Tr} \prod_{a=1}^{D} \left(\varphi^a + \overline{\varphi}^a\right) \exp\left(-\frac{\beta}{\hbar}\mathscr{H}_2\right)$$

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torsionless case $H = 0$ well-known ($\dim \mathcal{K} = D = 2n$)

Dirac index

$$\text{index}\, \not{D}(\omega) = \int_{\mathcal{K}} \exp\left[\frac{1}{2}\,\text{tr}\log\left(\frac{iR(\omega)/4\pi}{\sinh(iR(\omega)/4\pi)}\right)\right]$$

Euler charcteristic

$$\chi(\mathcal{K}) = \frac{1}{(4\pi)^n n!}\,\mathcal{E}_{A_1\cdots A_{2n}} \int_{\mathcal{K}} R^{A_1 A_2}(\omega) \wedge \cdots \wedge R^{A_{2n-1}A_{2n}}(\omega)$$

Hirzebruch signature

$$\sigma(\mathcal{K}) = \int_{\mathcal{K}} \exp\left[\frac{1}{2}\,\text{tr}\log\left(\frac{iR(\omega)/2\pi}{\tanh(iR(\omega)/2\pi)}\right)\right]$$

How to apply it

to a geometry with torsion?

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to a geometry with torsion?

In case of **index** $\not{D}$:

$$\omega_{mab} \longrightarrow \hat{\omega}_{mab} \equiv \omega_{mab} - \frac{1}{3} H_{mab}$$

$$Q_{1,H} \;=\; \psi^m g^{\frac{1}{4}} \Big(p_m - \frac{i}{2} \hat{\omega}_{mab}\, \psi^{ab} \Big) g^{-\frac{1}{4}} \;\equiv\; \psi^m g^{\frac{1}{4}} \pi_m^{(-1/3)} g^{-\frac{1}{4}}$$

How to apply it

to a geometry with torsion?

In cases of $\chi(\mathcal{K})$, $\sigma(\mathcal{K})$:

a non-trivial extension!

✍ Naive extension $\omega_{mab} \longrightarrow \hat{\omega}_{mab}$ does not yield SUSY algebra

This cannot yield the Witten index corresponding to the topological invariants

How to apply it

to a geometry with torsion?

In cases of $\chi(\mathcal{K})$, $\sigma(\mathcal{K})$:

a non-trivial extension!

Naive extension $\omega_{mab} \longrightarrow \hat{\omega}_{mab}$ does not yield SUSY algebra

$$\downarrow$$

$$\mathrm{d} \;\longrightarrow\; \mathrm{d}_H \;\equiv\; \mathrm{d} + H\wedge \qquad \text{with } \mathrm{d}H = 0$$

$$Q_{2,H} \;=\; \varphi^m g^{\frac{1}{4}}\Big(p_m - i\omega_{mab}\,\varphi^a\overline{\varphi}^b + \frac{i}{3}H_{mab}\,\varphi^{ab}\Big)g^{-\frac{1}{4}} \;=\; \varphi^m g^{\frac{1}{4}}\Big(\pi_m + \frac{i}{3}H_{mab}\,\varphi^{ab}\Big)g^{-\frac{1}{4}}$$

If $\mathrm{d}H \neq 0$, $Q_{2,H}$ does not commute with $\mathscr{H}_{2,H} \sim \{Q_{2,H}, \overline{Q}_{2,H}\}$

Strong Kähler with torsion $H \neq 0$, $\mathrm{d}H = 0$ as a smooth, compact manifold

Index

Dirac index ←-- modified! ($\hat{\omega} = \omega - \frac{1}{3}H$, $\omega_+ = \omega + H$)

$$\mathrm{index}\, \not{D}(\hat{\omega}) \;=\; \int_{\mathcal{K}} \exp\left[\frac{1}{2}\,\mathrm{tr}\log\left(\frac{iR(\omega_+)/4\pi}{\sinh(iR(\omega_+)/4\pi)}\right)\right]$$

Euler charcteristic

$$\chi(\mathcal{K}) \;=\; \frac{1}{(4\pi)^n n!}\,\mathcal{E}_{A_1\cdots A_{2n}} \int_{\mathcal{K}} R^{A_1A_2}(\omega) \wedge \cdots \wedge R^{A_{2n-1}A_{2n}}(\omega)$$

Hirzebruch signature ←-- modified!

$$\sigma(\mathcal{K}) \;=\; \int_{\mathcal{K}} \exp\left[\frac{1}{2}\,\mathrm{tr}\log\left(\frac{iR(\omega_+)/2\pi}{\tanh(iR(\omega_+)/2\pi)}\right)\right]$$

Summary and Discussions

- Modification of index theorems on torsional manifold

  restricted to strong Kähler with torsion; $H \neq 0$, $\mathrm{d}H = 0$

  Dirac index (or Pontrjagin class) and Hirzebruch signature

- Towards a generalization to conformally balanced

  $\mathcal{N} = 1$ QM: no problem (but hard work!!)

  cf.) Dirac index on 4-dim. torsional manifold by Peeters and Waldron

  $\mathcal{N} = 2$ QM: necessary to find a formulation including $\mathrm{d}H \neq 0$

- Dolbeault cohomology class $\dashrightarrow$ $\mathcal{N} = 4$ QM in the case of Kähler

  How to formulate in the case of non-Kähler?

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and more..