Index Theorems on Torsional Geometries

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Construct a realistic model of particle physics from string theories

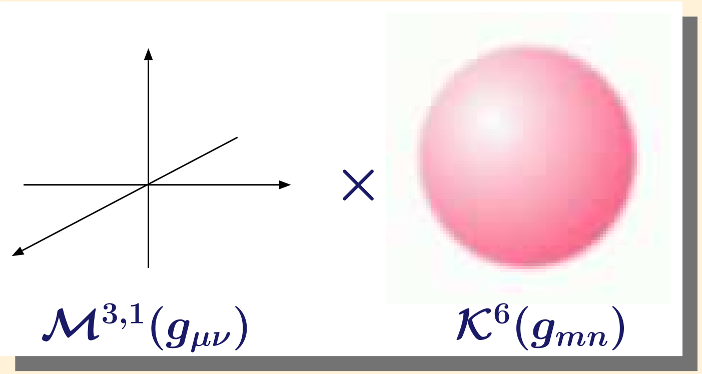
(flux) compactification scenario

(etc.)

4-dim. $\mathcal{N} = 1$ Physics is given by...

- Vacuum configuration
  - 6-dim. compactified space ←– – SUSY variation of fermionic fields
- Low energy effective theory
  - gauge symmetry (and its breaking)
  - moduli, zero mode equations
  - stabilization

In Heterotic String case:



Ansatz:

$$
\begin{aligned}
g^{\mathrm{E}}_{MN}\,\mathrm{d}x^M\mathrm{d}x^N \\
&= \mathrm{e}^{-\Phi(y)/2}\Big(g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu + g_{mn}\,\mathrm{d}y^m\mathrm{d}y^n\Big) \\
0 &= R(g_{\mu\nu}) \quad \rightsquigarrow \quad \mathcal{M}^{3,1} \;=\; \text{Minkowski}
\end{aligned}
$$

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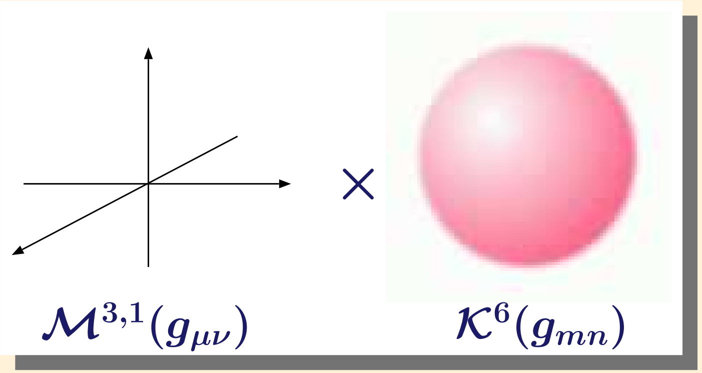

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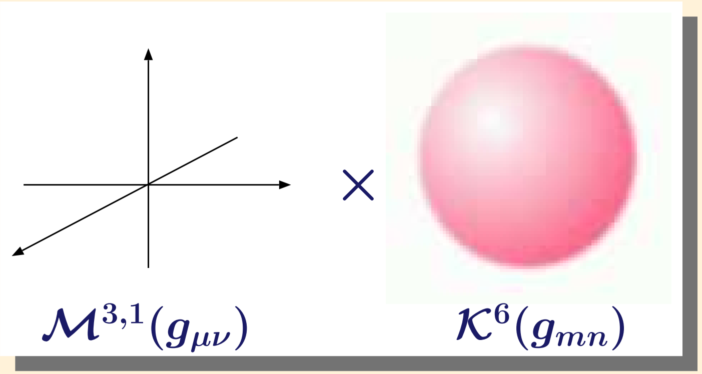
What is $\mathcal{K}^6$?

In Heterotic String case:



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$$g^{\mathrm{E}}_{MN}\, \mathrm{d}x^M \mathrm{d}x^N = \mathrm{e}^{-\Phi(y)/2}\Big(g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn}\, \mathrm{d}y^m \mathrm{d}y^n\Big)$$

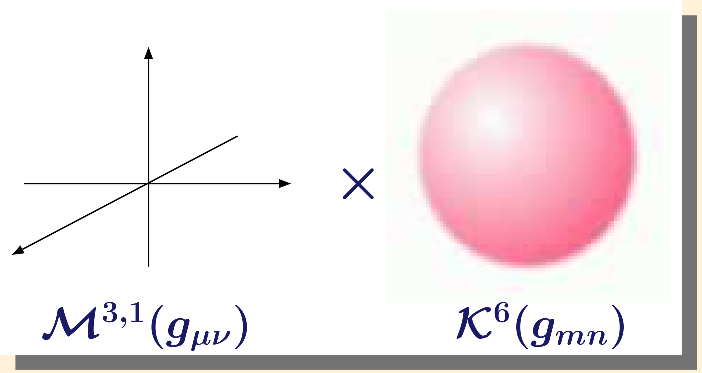
$$0 = R(g_{\mu\nu}) \quad \rightsquigarrow \quad \mathcal{M}^{3,1} = \text{Minkowski}$$

What is $\mathcal{K}^6$?

conformally balanced manifold

$SU(3)$-structure manifold with torsion

In Heterotic String case:



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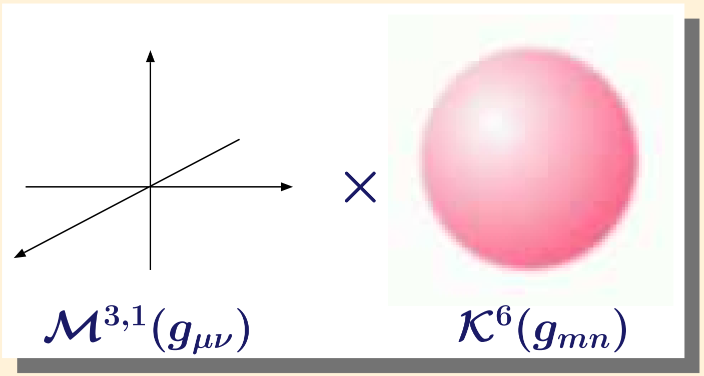
Kähler?

No (in general): $\mathrm{d}J \neq 0$

with $0 = \mathrm{d}\big(\mathrm{e}^{-2\Phi}J\wedge J\big)$, $\quad H = \frac{i}{2}(\partial - \bar{\partial})J$

In Heterotic String case:



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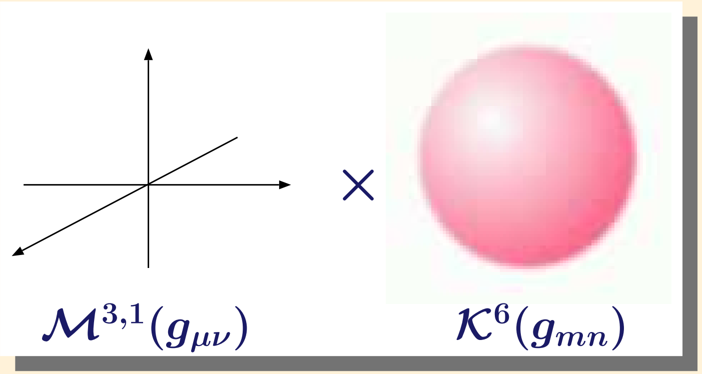
$$g^{\mathrm{E}}_{MN}\,\mathrm{d}x^M\mathrm{d}x^N = \mathrm{e}^{-\Phi(y)/2}\Big(g_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu + g_{mn}\,\mathrm{d}y^m\mathrm{d}y^n\Big)$$

$$0 = R(g_{\mu\nu}) \rightsquigarrow \mathcal{M}^{3,1} = \text{Minkowski}$$

Classification?

if $\mathrm{d}\Phi = 0$	:	balanced
if $\mathrm{d}\big(\mathrm{e}^{-\Phi}J\big) = 0$	:	conformally Kähler
if $\mathrm{d}H = 0$	:	strong Kähler with torsion
if $H = \mathrm{d}\Phi = 0$	:	Calabi–Yau

In Heterotic String case:



Ansatz:

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$$0 = R(g_{\mu\nu}) \quad \rightsquigarrow \quad \mathcal{M}^{3,1} = \text{Minkowski}$$

Gauge symmetry?

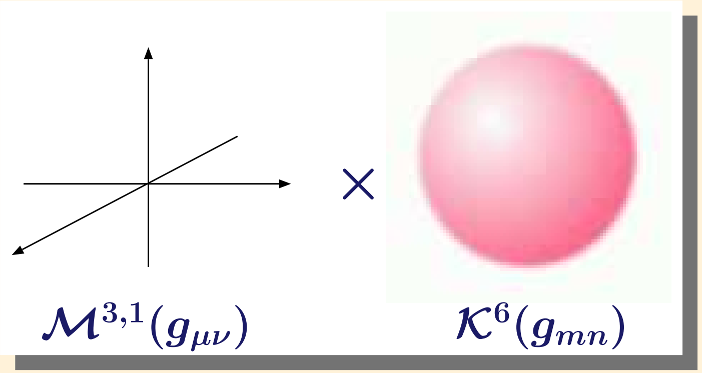
It depends on the cohomology classes

via

$$\mathrm{d}H = \alpha'\Big[\mathrm{tr}\big(F \wedge F\big) - \mathrm{tr}\big\{R(\omega_+) \wedge R(\omega_+)\big\}\Big]$$

$$\omega_+ \equiv \omega + H$$

In Heterotic String case:



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$$0 = R(g_{\mu\nu}) \quad \rightsquigarrow \quad \mathcal{M}^{3,1} = \text{Minkowski}$$

How many massless modes appear?

It depends on the cohomology classes

via eqs. of motion

Geometrical aspects of conformally balanced manifold

☹ Cohomology classes? topological invariants?

Geometrical aspects of conformally balanced manifold

Cohomology classes? topological invariants?

not completely known, yet

M.Becker, L.-S.Tseng and S.-T.Yau, `hep-th/0612290`

Geometrical aspects of conformally balanced manifold

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☹

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☹

Geometrical aspects of conformally balanced manifold

What should we do?

...

Geometrical aspects of conformally balanced manifold

What should we do?

Investigate the Dirac operator!

Dirac operator in Equations of motion

\# of massless modes
on $\mathcal{M}^{3,1} \times \mathcal{K}^6$
$\Delta_{10}\, \mathcal{F}(x,y) = 0$

\# of massless modes
on $\mathcal{M}^{3,1}$
$\Delta_4\, \Phi(x) = 0$

\# of zero modes
on $\mathcal{K}^6$
$\Delta_6\, \Psi(y) = 0$

ex.) dilatino: $$0 \ = \ \not{D}(\omega)\lambda - \frac{1}{12} H_{mnp}\Gamma^{mnp}\lambda \ = \ \not{D}(\omega - \tfrac{1}{3}H)\lambda$$

Dirac index by $\not{D}(\omega) : C^\infty(S_+) \longrightarrow C^\infty(S_-)$

$$\begin{aligned} \text{index}\,\not{D}(\omega) &\equiv (\#\text{ of left chiral}) - (\#\text{ of right chiral}) \\ &= \lim_{\beta\to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D)} \left[\Gamma_{(5)} \mathrm{e}^{-\beta\mathscr{R}} \right] \end{aligned}$$

Dirac index by $\not{D}(\omega) : C^\infty(S_+) \longrightarrow C^\infty(S_-)$ result

$$\begin{aligned} \text{index}\,\not{D}(\omega) &\equiv (\#\text{ of left chiral}) - (\#\text{ of right chiral}) \\ &= \lim_{\beta\to 0} \operatorname*{Tr}_{\text{Cliff}(D)} \left[\Gamma_{(5)} e^{-\beta\mathscr{R}} \right] \end{aligned}$$

Euler characteristic by $\mathbf{d} + \mathbf{d}^\dagger : C^\infty(\Lambda^{\text{even}}) \longrightarrow C^\infty(\Lambda^{\text{odd}})$

$$\begin{aligned} \chi(\mathcal{K}) &\equiv (\#\text{ of harm. even-forms}) - (\#\text{ of harm. odd-forms}) \\ &= \lim_{\beta\to 0} \operatorname*{Tr}_{\text{Cliff}(D,D)} \left[\Gamma_{(5)} \widetilde{\Gamma}_{(5)} e^{-\beta\mathscr{R}} \right] \end{aligned}$$

Hirzebruch signature by $\mathbf{d} + \mathbf{d}^\dagger : C^\infty(\Lambda^{\text{SD}}) \longrightarrow C^\infty(\Lambda^{\text{ASD}})$

$$\begin{aligned} \sigma(\mathcal{K}) &\equiv (\#\text{ of self-dual forms}) - (\#\text{ of anti-self dual forms}) \\ &= \lim_{\beta\to 0} \operatorname*{Tr}_{\text{Cliff}(D,D)} \left[\Gamma_{(5)} e^{-\beta\mathscr{R}} \right] \end{aligned}$$

topological invariants on geometry

$$\updownarrow$$

SUSY (Witten) index in quantum mechanics

$\mathbf{index}\, \not{\mathcal{D}}(\omega)$: $\mathcal{N} = 1$ quantum mechanics

$\chi(\mathcal{K})$, $\sigma(\mathcal{K})$: $\mathcal{N} = 2$ quantum mechanics

L. Alvarez-Gaumé, Commun. Math. Phys. 90 (1983) 161

Identifications

with $\mathcal{N} = 1$ quantum mechanics with $(\psi^a)^\dagger = \psi^a$:

$$
\begin{aligned}
\{\Gamma^a, \Gamma^b\} = 2\delta^{ab} &\quad\leftrightarrow\quad \{\psi^a, \psi^b\} = \hbar\,\delta^{ab} \\
\not{D}(\omega) &\quad\leftrightarrow\quad Q_1 = \psi^m g^{\frac{1}{4}}\Big(p_m - \frac{i}{2}\omega_{mab}\,\psi^{ab}\Big)g^{-\frac{1}{4}} \\
\not{D}(\omega)^2 = \Delta &\quad\leftrightarrow\quad (Q_1)^2 = \hbar\,\mathscr{H}_1
\end{aligned}
$$

with $\mathcal{N} = 2$ quantum mechanics with $\varphi^a = \frac{1}{\sqrt{2}}(\psi_1^a + i\psi_2^a)$:

$$
\begin{aligned}
e^a &\quad\leftrightarrow\quad \varphi^a \\
\delta^{ab}\frac{\partial}{\partial e^b} &\quad\leftrightarrow\quad \overline{\varphi}^a \\
\mathrm{d} &\quad\leftrightarrow\quad Q_2 = \varphi^m g^{\frac{1}{4}}\Big(p_m - i\omega_{mab}\,\varphi^a\overline{\varphi}^b\Big)g^{-\frac{1}{4}} \\
\mathrm{d}^\dagger &\quad\leftrightarrow\quad \overline{Q}_2 = \overline{\varphi}^m g^{\frac{1}{4}}\Big(p_m - i\omega_{mab}\,\varphi^a\overline{\varphi}^b\Big)g^{-\frac{1}{4}} \\
\{\mathrm{d}, \mathrm{d}^\dagger\} = \Delta &\quad\leftrightarrow\quad \{Q_2, \overline{Q}_2\} = 2\hbar\,\mathscr{H}_2
\end{aligned}
$$

$$\mathrm{index}\, \not{D}(\omega) = \lim_{\beta \to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D)} \left\{ \Gamma_{(5)} e^{-\beta \mathscr{R}} \right\}$$

$$\chi(\mathcal{K}) = \lim_{\beta \to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D,D)} \left\{ \Gamma_{(5)} \widetilde{\Gamma}_{(5)} e^{-\beta \mathscr{R}} \right\}$$

$$\sigma(\mathcal{K}) = \lim_{\beta \to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D,D)} \left\{ \Gamma_{(5)} e^{-\beta \mathscr{R}} \right\}$$

$$\mathbf{index}\,\not{D}(\omega) = \lim_{\beta\to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D)} \left\{ \Gamma_{(5)} \mathrm{e}^{-\beta\mathscr{R}} \right\} = \lim_{\beta\to 0} (-i)^{D/2}\, \mathrm{Tr} \prod_{a=1}^{D} \psi^a \, \exp\left(-\frac{\beta}{\hbar} \mathscr{H}_1 \right)$$

$$\chi(\mathcal{K}) = \lim_{\beta\to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D,D)} \left\{ \Gamma_{(5)} \widetilde{\Gamma}_{(5)} \mathrm{e}^{-\beta\mathscr{R}} \right\} = \lim_{\beta\to 0} \mathrm{Tr} \prod_{a=1}^{D} \left(\varphi^a + \overline{\varphi}^a \right) \prod_{b=1}^{D} \left(\varphi^b - \overline{\varphi}^b \right) \exp\left(-\frac{\beta}{\hbar} \mathscr{H}_2 \right)$$

$$\sigma(\mathcal{K}) = \lim_{\beta\to 0} \operatorname*{Tr}_{\mathrm{Cliff}(D,D)} \left\{ \Gamma_{(5)} \mathrm{e}^{-\beta\mathscr{R}} \right\} = \lim_{\beta\to 0} (-i)^{D/2} \mathrm{Tr} \prod_{a=1}^{D} \left(\varphi^a + \overline{\varphi}^a \right) \exp\left(-\frac{\beta}{\hbar} \mathscr{H}_2 \right)$$

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\textbf{index}\, \not{D}(\omega) &= \lim_{\beta\to 0} \operatorname*{Tr}_{\text{Cliff}(D)} \left\{\Gamma_{(5)} \mathrm{e}^{-\beta \mathscr{R}}\right\} = \lim_{\beta\to 0} (-i)^{D/2}\, \mathrm{Tr} \prod_{a=1}^{D} \psi^a \exp\left(-\frac{\beta}{\hbar}\mathscr{H}_1\right) \\
&= \lim_{\beta\to 0} \left(\frac{1}{2\pi i}\right)^{D/2} \int \mathrm{d}^D x \sqrt{g(x)} \prod_{a=1}^{D} \mathrm{d}\psi^a_{1,\mathrm{bg}} \left\langle \exp\left(-\frac{1}{\hbar} S_1^{(\mathrm{int})}\right)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
\chi(\mathcal{K}) &= \lim_{\beta\to 0} \operatorname*{Tr}_{\text{Cliff}(D,D)} \left\{\Gamma_{(5)} \widetilde{\Gamma}_{(5)} \mathrm{e}^{-\beta \mathscr{R}}\right\} = \lim_{\beta\to 0} \mathrm{Tr} \prod_{a=1}^{D} \left(\varphi^a + \overline{\varphi}^a\right) \prod_{b=1}^{D} \left(\varphi^b - \overline{\varphi}^b\right) \exp\left(-\frac{\beta}{\hbar}\mathscr{H}_2\right) \\
&= \lim_{\beta\to 0} \left(\frac{1}{2\pi}\right)^{D/2} \int \mathrm{d}^D x \sqrt{g(x)} \prod_{a=1}^{D} \mathrm{d}\overline{\varphi}^a_{\mathrm{bg}} \mathrm{d}\varphi^a_{\mathrm{bg}} \left\langle \exp\left(-\frac{1}{\hbar} S_2^{(\mathrm{int})}\right)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
\sigma(\mathcal{K}) &= \lim_{\beta\to 0} \operatorname*{Tr}_{\text{Cliff}(D,D)} \left\{\Gamma_{(5)} \mathrm{e}^{-\beta \mathscr{R}}\right\} = \lim_{\beta\to 0} (-i)^{D/2} \mathrm{Tr} \prod_{a=1}^{D} \left(\varphi^a + \overline{\varphi}^a\right) \exp\left(-\frac{\beta}{\hbar}\mathscr{H}_2\right) \\
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\end{aligned}
$$

torsionless case $H = 0$ well-known ($\dim \mathcal{K} = D = 2n$)

Dirac index

$$\text{index}\, D\!\!\!/(\omega) \;=\; \int_{\mathcal{K}} \exp\left[\frac{1}{2}\,\text{tr}\log\left(\frac{iR(\omega)/4\pi}{\sinh(iR(\omega)/4\pi)}\right)\right]$$

Euler charcteristic

$$\chi(\mathcal{K}) \;=\; \frac{1}{(4\pi)^n n!}\,\mathcal{E}_{A_1\cdots A_{2n}} \int_{\mathcal{K}} R^{A_1A_2}(\omega)\wedge\cdots\wedge R^{A_{2n-1}A_{2n}}(\omega)$$

Hirzebruch signature

$$\sigma(\mathcal{K}) \;=\; \int_{\mathcal{K}} \exp\left[\frac{1}{2}\,\text{tr}\log\left(\frac{iR(\omega)/2\pi}{\tanh(iR(\omega)/2\pi)}\right)\right]$$

How to introduce

a (totally anti-symmetric) torsion?

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a (totally anti-symmetric) torsion?

In $\mathcal{N} = 1$ case (index $\not{D}$):

$$\omega_{mab} \longrightarrow \hat{\omega}_{mab} \equiv \omega_{mab} - \frac{1}{3} H_{mab}$$

$$Q_{1,H} \;=\; \psi^m g^{\frac{1}{4}} \Big(p_m - \frac{i}{2} \hat{\omega}_{mab}\, \psi^{ab} \Big) g^{-\frac{1}{4}} \;\equiv\; \psi^m g^{\frac{1}{4}} \pi_m^{(-1/3)} g^{-\frac{1}{4}}$$

How to introduce

a (totally anti-symmetric) torsion?

In $\mathcal{N} = 2$ case $(\chi(\mathcal{K}), \sigma(\mathcal{K}))$:

a non-trivial extension!

✎ Naive extension $\omega_{mab} \longrightarrow \hat{\omega}_{mab}$ does not yield SUSY algebra

This cannot yield the Witten index corresponding to the topological invariants

How to introduce

a (totally anti-symmetric) torsion?

In $\mathcal{N} = 2$ case ($\chi(\mathcal{K})$, $\sigma(\mathcal{K})$):

a non-trivial extension!

- Naive extension $\omega_{mab} \longrightarrow \hat{\omega}_{mab}$ does not yield SUSY algebra

$$\downarrow$$

$$\mathrm{d} \;\longrightarrow\; \mathrm{d}_H \;\equiv\; \mathrm{d} + H\wedge \qquad \text{with } \mathrm{d}H \;=\; 0$$

$$Q_{2,H} \;=\; \varphi^m g^{\frac{1}{4}}\Big(p_m - i\omega_{mab}\,\varphi^a\overline{\varphi}^b + \frac{i}{3}H_{mab}\,\varphi^{ab}\Big)g^{-\frac{1}{4}} \;=\; \varphi^m g^{\frac{1}{4}}\Big(\pi_m + \frac{i}{3}H_{mab}\,\varphi^{ab}\Big)g^{-\frac{1}{4}}$$

- If $\mathrm{d}H \neq 0$, $Q_{2,H}$ does not commute with $\mathscr{H}_{2,H} \sim \{Q_{2,H}, \overline{Q}_{2,H}\}$

Calculate in a straightforward way

Calculate in a straightforward way

(It took more than 6 months...)

Strong Kähler with torsion $H \neq 0$, $\mathrm{d}H = 0$ as a smooth, compact manifold

Index

Dirac index ⇠-- modified! ($\hat{\omega} = \omega - \frac{1}{3}H$, $\omega_+ = \omega + H$)

$$\mathrm{index}\, \not{D}(\hat{\omega}) \;=\; \int_{\mathcal{K}} \exp\left[\frac{1}{2}\,\mathrm{tr}\log\left(\frac{iR(\omega_+)/4\pi}{\sinh(iR(\omega_+)/4\pi)}\right)\right]$$

Euler charcteristic

$$\chi(\mathcal{K}) \;=\; \frac{1}{(4\pi)^n n!}\,\mathcal{E}_{A_1\cdots A_{2n}} \int_{\mathcal{K}} R^{A_1A_2}(\omega) \wedge \cdots \wedge R^{A_{2n-1}A_{2n}}(\omega)$$

Hirzebruch signature ⇠-- modified!

$$\sigma(\mathcal{K}) \;=\; \int_{\mathcal{K}} \exp\left[\frac{1}{2}\,\mathrm{tr}\log\left(\frac{iR(\omega_+)/2\pi}{\tanh(iR(\omega_+)/2\pi)}\right)\right]$$

Summary and Discussions

- Modification of index theorems on torsional manifold

  restricted to strong Kähler with torsion; $H \neq 0$, $\mathrm{d}H = 0$

  Dirac index (or Pontrjagin class) and Hirzebruch signature

- Towards a generalization to conformally balanced

  $\mathcal{N} = 1$ QM: no problem (but hard work!!)

  cf.) Dirac index on 4-dim. torsional manifold by Peeters and Waldron

  $\mathcal{N} = 2$ QM: necessary to find a formulation including $\mathrm{d}H \neq 0$

- Dolbeault cohomology class $\dashrightarrow$ $\mathcal{N} = 4$ QM in the case of Kähler

  How to formulate in the case of non-Kähler?

Appendix

SUSY variations of fermionic fields give rise to the Killing spinor equations

$$0 = \delta\psi_m = \Big(\partial_m + \frac{1}{4}\omega_{-mab}\,\Gamma^{ab}\Big)\eta_+ \equiv D_m(\omega_-)\eta_+$$
$$0 = \delta\lambda = -\frac{1}{4}\Big(\Gamma^m\nabla_m\Phi - \frac{1}{6}H_{mnp}\Gamma^{mnp}\Big)\eta_+$$

η_+: Weyl spinor on 6-dim. manifold; $\omega_{-mab} = \omega_{mab} - H_{mab}$

Analyses of the manifold become much clear by introducing mathematical tools:

almost complex structure : $J_m{}^n \equiv i\eta_+^\dagger\Gamma_m{}^n\,\eta_+$ with $J_m{}^pJ_p{}^n = -\delta_m^n$

Bismut torsion : $T^{(\mathrm{B})}_{mnp} \equiv \frac{3}{2}J_m{}^qJ_n{}^rJ_p{}^s\nabla_{[s}J_{qr]}$

NS-NS 3-form flux behaves as a (con)torsion $H = T^{(\mathrm{B})}$

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and more..


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