Index Theorems on Torsional Geometries

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Construct a realistic model of particle physics from string theories

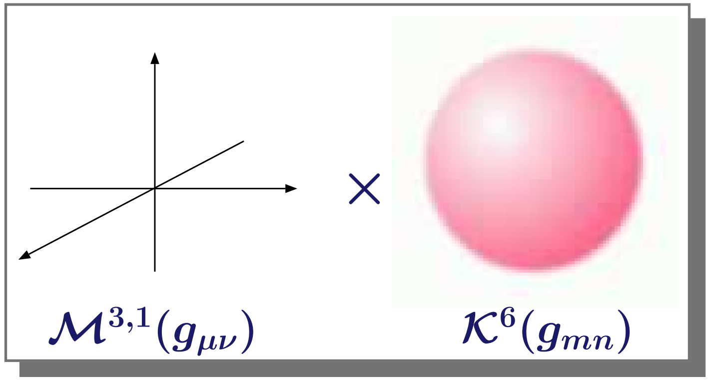
(flux) compactification scenarios

4-dim. $\mathcal{N} = 1$ Physics is given by...

- Vacuum configuration
  - 6-dim. compactified space ←-- SUSY variation of fermionic fields

- Low energy effective theory
  - gauge symmetry (and its breaking)
  - moduli, zero mode equations
  - stabilization

In Heterotic String case:



<u>Ansatz:</u>

$$
\begin{aligned}
g^{\mathrm{E}}_{MN}\,\mathrm{d}x^{M}\mathrm{d}x^{N} &= \mathrm{e}^{-\Phi(y)/2}\Big(g_{\mu\nu}\,\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + g_{mn}\,\mathrm{d}y^{m}\mathrm{d}y^{n}\Big) \\
0 &= R(g_{\mu\nu}) \quad \rightsquigarrow \quad \mathcal{M}^{3,1} = \text{Minkowski}
\end{aligned}
$$

$\mathcal{K}^6$: non-Kähler $\mathrm{d}J \neq 0$

with $0 = \mathrm{d}\big(\mathrm{e}^{-2\Phi} J \wedge J\big)$, $\quad H = \dfrac{i}{2}(\partial - \overline{\partial})J$

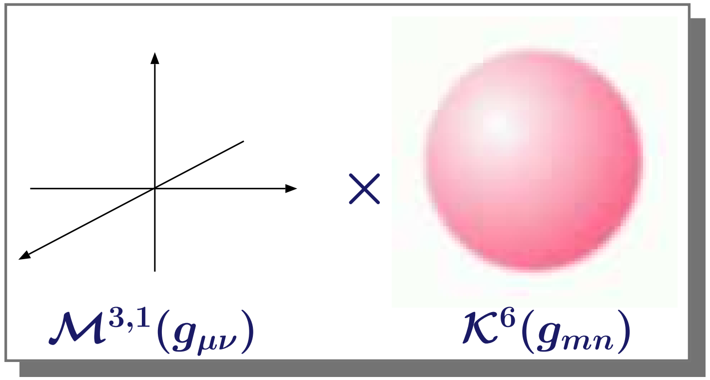
if $\mathrm{d}\Phi = 0$: balanced

if $\mathrm{d}\big(\mathrm{e}^{-\Phi}J\big) = 0$: conformally Kähler

if $\mathrm{d}H = 0$: strong Kähler with torsion

if $H = \mathrm{d}\Phi = 0$: Calabi–Yau

In Heterotic String case:



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\end{aligned}
$$

Gauge symmetry?

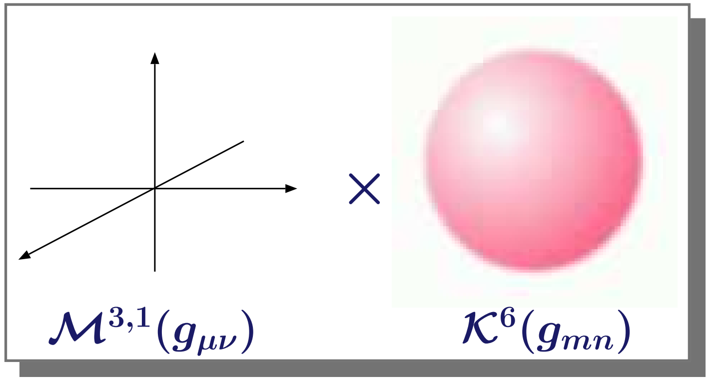
It depends on the cohomology classes

via

$$
\mathrm{d}H = \alpha' \Big[\mathrm{tr}\big(F \wedge F\big) - \mathrm{tr}\big\{R(\omega_+) \wedge R(\omega_+)\big\}\Big]
$$

$$
\omega_+ \equiv \omega + H
$$

In Heterotic String case:



Ansatz:

$$g^{\mathrm{E}}_{MN}\,\mathrm{d}x^M \mathrm{d}x^N = \mathrm{e}^{-\Phi(y)/2}\Big(g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn}\,\mathrm{d}y^m \mathrm{d}y^n\Big)$$

$$0 = R(g_{\mu\nu}) \quad \rightsquigarrow \quad \mathcal{M}^{3,1} = \text{Minkowski}$$

How many massless modes appear?

It depends on the cohomology classes

via eqs. of motion

Dirac operator in Equations of motion

\# of massless modes
on $\mathcal{M}^{3,1} \times \mathcal{K}^6$
$$\Delta_{10}\, \mathcal{F}(x,y) = 0$$

\# of massless modes
on $\mathcal{M}^{3,1}$
$$\Delta_4\, \Phi(x) = 0$$

\# of zero modes
on $\mathcal{K}^6$
$$\Delta_6\, \Psi(y) = 0$$

ex.) dilatino: $$0 \;=\; \not{D}(\omega)\lambda - \frac{1}{12} H_{mnp} \Gamma^{mnp} \lambda \;=\; \not{D}(\omega - \tfrac{1}{3} H)\lambda$$

Dirac index by $\not{D}(\omega) : C^\infty(S_+) \longrightarrow C^\infty(S_-)$ result

$$
\begin{aligned}
\text{index}\,\not{D}(\omega) &\equiv (\#\text{ of left chiral}) - (\#\text{ of right chiral}) \\
&= \lim_{\beta\to 0} \operatorname*{Tr}_{\text{Cliff}(D)} \left[\Gamma_{(5)} e^{-\beta\mathscr{R}} \right]
\end{aligned}
$$

Euler characteristic by $\mathrm{d} + \mathrm{d}^\dagger : C^\infty(\Lambda^{\text{even}}) \longrightarrow C^\infty(\Lambda^{\text{odd}})$

$$
\begin{aligned}
\chi(\mathcal{K}) &\equiv (\#\text{ of harm. even-forms}) - (\#\text{ of harm. odd-forms}) \\
&= \lim_{\beta\to 0} \operatorname*{Tr}_{\text{Cliff}(D,D)} \left[\Gamma_{(5)} \widetilde{\Gamma}_{(5)} e^{-\beta\mathscr{R}} \right]
\end{aligned}
$$

Hirzebruch signature by $\mathrm{d} + \mathrm{d}^\dagger : C^\infty(\Lambda^{\text{SD}}) \longrightarrow C^\infty(\Lambda^{\text{ASD}})$

$$
\begin{aligned}
\sigma(\mathcal{K}) &\equiv (\#\text{ of self-dual forms}) - (\#\text{ of anti-self dual forms}) \\
&= \lim_{\beta\to 0} \operatorname*{Tr}_{\text{Cliff}(D,D)} \left[\Gamma_{(5)} e^{-\beta\mathscr{R}} \right]
\end{aligned}
$$

topological invariants on geometry

$$\updownarrow$$

SUSY (Witten) index in quantum mechanics

$\mathbf{index}\, \not{D}(\omega)$: $\mathcal{N} = 1$ quantum mechanics

$\chi(\mathcal{K})$, $\sigma(\mathcal{K})$: $\mathcal{N} = 2$ quantum mechanics

L. Alvarez-Gaumé, Commun. Math. Phys. 90 (1983) 161

Identifications

with $\mathcal{N}=1$ quantum mechanics with $(\psi^a)^\dagger = \psi^a$:

$$\begin{aligned} \{\Gamma^a,\Gamma^b\} &= 2\delta^{ab} &\leftrightarrow&& \{\psi^a,\psi^b\} &= \hbar\,\delta^{ab} \\ \not{D}(\omega) &&\leftrightarrow&& Q_1 &= \psi^m g^{\frac{1}{4}}\Big(p_m - \frac{i}{2}\omega_{mab}\,\psi^{ab}\Big)g^{-\frac{1}{4}} \\ \not{D}(\omega)^2 &= \Delta &\leftrightarrow&& (Q_1)^2 &= \hbar\,\mathscr{H}_1 \end{aligned}$$

with $\mathcal{N}=2$ quantum mechanics with $\varphi^a = \frac{1}{\sqrt{2}}(\psi_1^a + i\psi_2^a)$:

$$\begin{aligned} e^a &&\leftrightarrow&& \varphi^a \\ \delta^{ab}\frac{\partial}{\partial e^b} &&\leftrightarrow&& \overline{\varphi}^a \\ \mathrm{d} &&\leftrightarrow&& Q_2 &= \varphi^m g^{\frac{1}{4}}\Big(p_m - i\omega_{mab}\,\varphi^a\overline{\varphi}^b\Big)g^{-\frac{1}{4}} \\ \mathrm{d}^\dagger &&\leftrightarrow&& \overline{Q}_2 &= \overline{\varphi}^m g^{\frac{1}{4}}\Big(p_m - i\omega_{mab}\,\varphi^a\overline{\varphi}^b\Big)g^{-\frac{1}{4}} \\ \{\mathrm{d},\mathrm{d}^\dagger\} &= \Delta &\leftrightarrow&& \{Q_2,\overline{Q}_2\} &= 2\hbar\,\mathscr{H}_2 \end{aligned}$$

$$\begin{aligned}\text{index}\,\not{D}(\omega) &= \lim_{\beta\to 0}\operatorname*{Tr}_{\text{Cliff}(D)}\left\{\Gamma_{(5)}\mathrm{e}^{-\beta\mathscr{R}}\right\} = \lim_{\beta\to 0}(-i)^{D/2}\,\mathrm{Tr}\prod_{a=1}^{D}\psi^a\ \exp\left(-\frac{\beta}{\hbar}\mathscr{H}_1\right)\\ &= \lim_{\beta\to 0}\left(\frac{1}{2\pi i}\right)^{D/2}\int \mathrm{d}^D x\sqrt{g(x)}\prod_{a=1}^{D}\mathrm{d}\psi^a_{1,\mathrm{bg}}\left\langle \exp\left(-\frac{1}{\hbar}S_1^{(\mathrm{int})}\right)\right\rangle\end{aligned}$$

$$\begin{aligned}\chi(\mathcal{K}) &= \lim_{\beta\to 0}\operatorname*{Tr}_{\text{Cliff}(D,D)}\left\{\Gamma_{(5)}\widetilde{\Gamma}_{(5)}\mathrm{e}^{-\beta\mathscr{R}}\right\} = \lim_{\beta\to 0}\mathrm{Tr}\prod_{a=1}^{D}\left(\varphi^a+\overline{\varphi}^a\right)\prod_{b=1}^{D}\left(\varphi^b-\overline{\varphi}^b\right)\ \exp\left(-\frac{\beta}{\hbar}\mathscr{H}_2\right)\\ &= \lim_{\beta\to 0}\left(\frac{1}{2\pi}\right)^{D/2}\int \mathrm{d}^D x\sqrt{g(x)}\prod_{a=1}^{D}\mathrm{d}\overline{\varphi}^a_{\mathrm{bg}}\mathrm{d}\varphi^a_{\mathrm{bg}}\left\langle \exp\left(-\frac{1}{\hbar}S_2^{(\mathrm{int})}\right)\right\rangle\end{aligned}$$

$$\begin{aligned}\sigma(\mathcal{K}) &= \lim_{\beta\to 0}\operatorname*{Tr}_{\text{Cliff}(D,D)}\left\{\Gamma_{(5)}\mathrm{e}^{-\beta\mathscr{R}}\right\} = \lim_{\beta\to 0}(-i)^{D/2}\mathrm{Tr}\prod_{a=1}^{D}\left(\varphi^a+\overline{\varphi}^a\right)\ \exp\left(-\frac{\beta}{\hbar}\mathscr{H}_2\right)\\ &= \lim_{\beta\to 0}\left(\frac{1}{2\pi i}\right)^{D/2}\int \mathrm{d}^D x\sqrt{g(x)}\prod_{a=1}^{D}\mathrm{d}\overline{\varphi}^a_{\mathrm{bg}}\mathrm{d}\varphi^a_{\mathrm{bg}}\prod_{b=1}^{D}\left(\varphi^b_{\mathrm{bg}}-\overline{\varphi}^b_{\mathrm{bg}}\right)\left\langle \exp\left(-\frac{1}{\hbar}S_2^{(\mathrm{int})}\right)\right\rangle\end{aligned}$$

torsionless case $H = 0$

well-known ($\dim \mathcal{K} = D = 2n$)

Dirac index

$$\text{index}\, \not{D}(\omega) \;=\; \int_{\mathcal{K}} \exp\left[\frac{1}{2}\,\mathrm{tr}\log\left(\frac{iR(\omega)/4\pi}{\sinh(iR(\omega)/4\pi)}\right)\right]$$

Euler charcteristic

$$\chi(\mathcal{K}) \;=\; \frac{1}{(4\pi)^n n!}\,\mathcal{E}_{A_1\cdots A_{2n}} \int_{\mathcal{K}} R^{A_1 A_2}(\omega) \wedge \cdots \wedge R^{A_{2n-1}A_{2n}}(\omega)$$

Hirzebruch signature

$$\sigma(\mathcal{K}) \;=\; \int_{\mathcal{K}} \exp\left[\frac{1}{2}\,\mathrm{tr}\log\left(\frac{iR(\omega)/2\pi}{\tanh(iR(\omega)/2\pi)}\right)\right]$$

How to introduce

a (totally anti-symmetric) torsion?

In $\mathcal{N} = 1$ case (index $\not{D}$):

$$\omega_{mab} \longrightarrow \hat{\omega}_{mab} \equiv \omega_{mab} - \frac{1}{3} H_{mab}$$

$$Q_{1,H} \;=\; \psi^m g^{\frac{1}{4}} \Big(p_m - \frac{i}{2} \hat{\omega}_{mab} \psi^{ab} \Big) g^{-\frac{1}{4}} \;\equiv\; \psi^m g^{\frac{1}{4}} \pi_m^{(-1/3)} g^{-\frac{1}{4}}$$

☹ How to introduce a (totally anti-symmetric) torsion?

In $\mathcal{N} = 2$ case ($\chi(\mathcal{K})$, $\sigma(\mathcal{K})$): a non-trivial extension! ☺

- Naive extension $\omega_{mab} \longrightarrow \hat{\omega}_{mab}$ does not yield SUSY algebra

$$\downarrow$$

$$\mathrm{d} \;\longrightarrow\; \mathrm{d}_H \;\equiv\; \mathrm{d} + H\wedge \qquad \text{with } \mathrm{d}H = 0$$

$$Q_{2,H} \;=\; \varphi^m g^{\frac{1}{4}}\Big(p_m - i\omega_{mab}\,\varphi^a\overline{\varphi}^b + \frac{i}{3}H_{mab}\,\varphi^{ab}\Big)g^{-\frac{1}{4}} \;=\; \varphi^m g^{\frac{1}{4}}\Big(\pi_m + \frac{i}{3}H_{mab}\,\varphi^{ab}\Big)g^{-\frac{1}{4}}$$

- If $\mathrm{d}H \neq 0$, $Q_{2,H}$ does not commute with $\mathscr{H}_{2,H} \sim \{Q_{2,H}, \overline{Q}_{2,H}\}$

Strong Kähler with torsion $H \neq 0$, $\mathrm{d}H = 0$ as a smooth, compact manifold

Index

Dirac index ←-- modified! $(\hat{\omega} = \omega - \frac{1}{3}H,\ \omega_+ = \omega + H)$

$$\mathrm{index}\, \not{D}(\hat{\omega}) \;=\; \int_{\mathcal{K}} \exp\left[\frac{1}{2}\,\mathrm{tr}\log\left(\frac{iR(\omega_+)/4\pi}{\sinh(iR(\omega_+)/4\pi)}\right)\right]$$

Euler charcteristic

$$\chi(\mathcal{K}) \;=\; \frac{1}{(4\pi)^n n!}\,\mathcal{E}_{A_1\cdots A_{2n}}\int_{\mathcal{K}} R^{A_1A_2}(\omega)\wedge\cdots\wedge R^{A_{2n-1}A_{2n}}(\omega)$$

Hirzebruch signature ←-- modified!

$$\sigma(\mathcal{K}) \;=\; \int_{\mathcal{K}} \exp\left[\frac{1}{2}\,\mathrm{tr}\log\left(\frac{iR(\omega_+)/2\pi}{\tanh(iR(\omega_+)/2\pi)}\right)\right]$$

Summary and Discussions

- Modification of index theorems on torsional manifold

  restricted to strong Kähler with torsion; $H \neq 0$, $\mathrm{d}H = 0$

  Dirac index (or Pontrjagin class) and Hirzebruch signature

- Towards a generalization to conformally balanced

  $\mathcal{N} = 1$ QM: no problem (but hard work!!)

  cf.) Dirac index on 4-dim. torsional manifold by Peeters and Waldron

  $\mathcal{N} = 2$ QM: necessary to find a formulation including $\mathrm{d}H \neq 0$

- Dolbeault cohomology class $\dashrightarrow$ $\mathcal{N} = 4$ QM in the case of Kähler

  How to formulate in the case of non-Kähler?

Appendix

SUSY variations of fermionic fields give rise to the Killing spinor equations

$$0 = \delta\psi_m = \Big(\partial_m + \frac{1}{4}\omega_{-mab}\Gamma^{ab}\Big)\eta_+ \equiv D_m(\omega_-)\eta_+$$
$$0 = \delta\lambda = -\frac{1}{4}\Big(\Gamma^m\nabla_m\Phi - \frac{1}{6}H_{mnp}\Gamma^{mnp}\Big)\eta_+$$

η_+: Weyl spinor on 6-dim. manifold; $\omega_{-mab} = \omega_{mab} - H_{mab}$

Analyses of the manifold become much clear by introducing mathematical tools:

almost complex structure : $J_m{}^n \equiv i\eta_+^\dagger\Gamma_m{}^n\eta_+$ with $J_m{}^p J_p{}^n = -\delta_m^n$

Bismut torsion : $T^{(\mathrm{B})}_{mnp} \equiv \frac{3}{2} J_m{}^q J_n{}^r J_p{}^s \nabla_{[s}J_{qr]}$

NS-NS 3-form flux behaves as a (con)torsion $H = T^{(\mathrm{B})}$

References

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R.Rohm and E.Witten: [AP170 (1986) 454]

N.E.Mavromatos: [JPA21 (1988) 2279]

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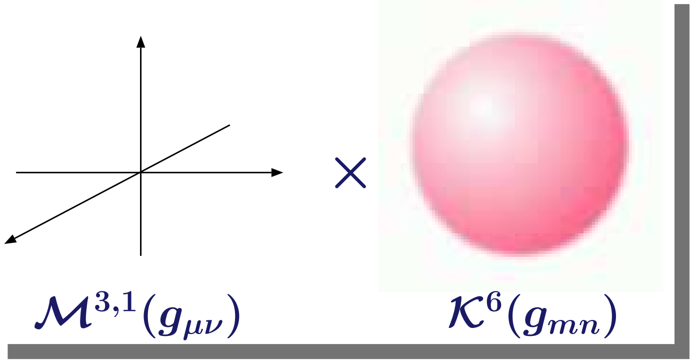
F.Bastianelli and P.van Nieuwenhuizen,
"*Path Integrals and Anomalies in Curved Space*," Cambridge University Press (2006).

and more..

In Heterotic String case:



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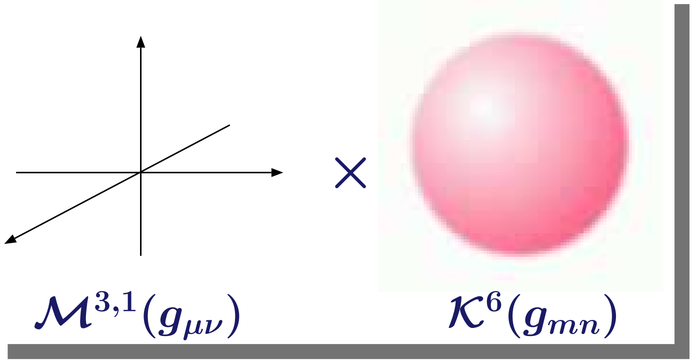
$$0 = R(g_{\mu\nu}) \quad\rightsquigarrow\quad \mathcal{M}^{3,1} = \text{Minkowski}$$

Holonomy group?

$$\mathrm{Hol}(\omega_-) \subseteq SU(3)$$

$$\mathrm{Hol}(\omega_+) \subseteq SO(6)$$

In Heterotic String case:



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How many numbers of generation?

$$\frac{1}{2}|\chi(\mathcal{K}^6)|$$

(if standard embedding is allowed)