Index Theorems on Torsional Geometries

arXiv:0704.2111 [hep-th]　to appear in JHEP

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基研研究会「弦理論と場の理論 — 量子と時空の最前線」於 近畿大学　2007 年 8 月 8 日

Motivation

Towards a Realistic Physical Model from SUPERGRAVITY as effective theory of STRINGS

- 物質場とその相互作用　• 統一対称性とその破れ　• 重力, 宇宙項　• etc.

運動方程式：

$$\text{dilatino}\quad 0 = \not{D}(\omega)\lambda - \frac{1}{12}H_{MNP}\Gamma^{MNP}\lambda \quad = \not{D}(\omega - \tfrac{1}{3}H)\lambda$$

$$\text{gaugino}\quad 0 = \not{D}(\omega, A)\chi - \frac{1}{12}H_{MNP}\Gamma^{MNP}\chi \quad = \not{D}(\omega - \tfrac{1}{3}H, A)\chi$$

\# of massless modes on $\mathcal{M}^{3,1}\times\mathcal{K}^6$, $\Delta_{10}\,\mathcal{F}(x,y)=0$ → \# of massless modes on $\mathcal{M}^{3,1}$, $\Delta_4\,\Phi(x)=0$ ↔ \# of zero modes on $\mathcal{K}^6$, $\Delta_6\,\Psi(y)=0$

3 種のスピン接続：$\widehat{\omega} \equiv \omega - \frac{1}{3}H,\ \omega_{\pm} \equiv \omega \pm H$ の関係は？
ゼロモードの数、指数は変わるのか？
4 次元有効理論に登場する世代数は変わるのか？

SUGRA vs SUSY QM

様々な指数定理　(参照： Green-Schwarz-Witten 14 章)

- **Dirac index** by $\not{D}(\omega) : C^\infty(S_+) \to C^\infty(S_-)$

  index $\not{D}(\omega) \equiv$ (# of left chiral) − (# of right chiral)

  $\longrightarrow \mathcal{N}=1$ QM : $\not{D}(\omega) \sim Q^1$

- **Euler characteristic** by $\mathrm{d}+\mathrm{d}^\dagger : C^\infty(\Lambda^{\text{even}}) \to C^\infty(\Lambda^{\text{odd}})$

  $\chi(\mathcal{M}) \equiv$ (# of harm. even-forms) − (# of harm. odd-forms)

  $\longrightarrow \mathcal{N}=2$ QM : $\mathrm{d}\sim Q,\ \mathrm{d}^\dagger \sim \overline{Q}$

- **Hirzebruch signature** by $\mathrm{d}+\mathrm{d}^\dagger : C^\infty(\Lambda^{\text{SD}}) \to C^\infty(\Lambda^{\text{ASD}})$

  $\sigma(\mathcal{M}) \equiv$ (# of self-dual forms) − (# of anti-self dual forms)

  $\longrightarrow \mathcal{N}=2$ QM : $\mathrm{d}\sim Q,\ \mathrm{d}^\dagger \sim \overline{Q}$

$\mathcal{N}=1$ SUSY Quantum Mechanics

$$\{\psi^A,\psi^B\} = \hbar\,\delta^{AB}$$

$$Q^1 = \psi^M g^{\frac14}\pi_M g^{-\frac14} = \psi^M g^{\frac14}\Big(p_M - \frac{i}{2}\omega_{MAB}\,\psi^{AB}\Big)g^{-\frac14}$$

$$\{Q^1,Q^1\} = 2\hbar\,\mathscr{H}^1\,,\quad \mathscr{H}^1 = \frac12 g^{-\frac14}\pi_M g^{MN}\sqrt{g}\,\pi_N\,g^{-\frac14} + \frac{\hbar^2}{8}R(\omega)$$

$\mathcal{N}=2$ SUSY Quantum Mechanics

$$\{\varphi^A,\overline{\varphi}^B\} = \hbar\,\delta^{AB}\,,\quad \{\varphi^A,\varphi^B\} = 0$$

$$Q = \varphi^M g^{\frac14}\pi_M g^{-\frac14} = \varphi^M g^{\frac14}\Big(p_M - i\omega_{MAB}\,\varphi^A\overline{\varphi}^B\Big)g^{-\frac14}$$

$$\{Q,Q\} = 0\,,\quad \{Q,\overline{Q}\} = 2\hbar\,\mathscr{H}$$

$$\mathscr{H} = \frac12 g^{-\frac14}\pi_M g^{MN}\sqrt{g}\,\pi_N\,g^{-\frac14} - \frac12 R_{ABMN}(\omega)\,\varphi^M\overline{\varphi}^N\varphi^A\overline{\varphi}^B$$

$$\text{index}\,\not{D}(\omega) = \lim_{\beta\to0}\mathrm{Tr}\{\Gamma_{(5)}\mathrm{e}^{-\beta\mathscr{R}}\} = \lim_{\beta\to0}\frac{(-i)^{D/2}}{2^{D/2}}\,\mathrm{Tr}\prod_{A=1}^{D}(\varphi^A+\overline{\varphi}^A)\,\exp\Big(-\frac{\beta}{\hbar}\mathscr{H}^1\Big)$$

$$\chi(\mathcal{M}) = \lim_{\beta\to0}\mathrm{Tr}\{\Gamma_{(5)}\widetilde{\Gamma}_{(5)}\mathrm{e}^{-\beta\mathscr{R}}\} = \lim_{\beta\to0}\mathrm{Tr}\prod_{A=1}^{D}(\varphi^A+\overline{\varphi}^A)\prod_{B=1}^{D}(\varphi^B-\overline{\varphi}^B)\,\exp\Big(-\frac{\beta}{\hbar}\mathscr{H}\Big)$$

$$\sigma(\mathcal{M}) = \lim_{\beta\to0}\mathrm{Tr}\{\Gamma_{(5)}\mathrm{e}^{-\beta\mathscr{R}}\} = \lim_{\beta\to0}(-i)^{D/2}\mathrm{Tr}\prod_{A=1}^{D}(\varphi^A+\overline{\varphi}^A)\,\exp\Big(-\frac{\beta}{\hbar}\mathscr{H}\Big)$$

Extend!　(Ansatz: $\mathrm{d}H = 0$)

$\mathcal{N}=1$ QM :　$\omega \to \widehat{\omega} = \omega - \frac13 H,\ \not{D}(\omega) \to \not{D}(\widehat{\omega}) \sim Q^1_H$　　$\{Q^1_H, Q^1_H\} = 2\hbar\mathscr{H}^1_H$

$\mathcal{N}=2$ QM :　$\mathrm{d} \to \mathrm{d}_H = \mathrm{d} + \frac13 H\wedge \sim Q_H$　　$\{Q_H,Q_H\} \sim (\mathrm{d}H) = 0,\ \{Q_H,\overline{Q}_H\} = 2\hbar\mathscr{H}_H$

Dirac index

$$\text{index}\,\not{D}(\hat{\omega}, A) = \int_{\mathcal{M}}\exp\left[\frac12\,\mathrm{tr}\log\left(\frac{iR(\omega_+)/4\pi}{\sinh(iR(\omega_+)/4\pi)}\right)\right]\mathrm{Tr}_R\exp\Big(\frac{i}{2\pi}F\Big)$$

Euler charcteristic

$$\chi(\mathcal{M}) = \frac{1}{(4\pi)^n n!}\,\mathcal{E}_{A_1\cdots A_{2n}}\int_{\mathcal{M}}R^{A_1A_2}(\omega)\wedge\cdots\wedge R^{A_{2n-1}A_{2n}}(\omega)$$

Hirzebruch signature

$$\sigma(\mathcal{M}) = \int_{\mathcal{M}}\exp\left[\frac12\,\mathrm{tr}\log\left(\frac{iR(\omega_+)/2\pi}{\tanh(iR(\omega_+)/2\pi)}\right)\right]$$

Comments

- $\widehat{\omega}$ in EOM,　ω_- in the Hamiltonian,　ω_+ in the Index
- ゼロモードの数が変わる！　• 世代数は変わらない !!