YITP informal seminar
March 19th, 2007

Generalized Complex Geometries and Supersymmetric Theories

Tetsuji Kimura

References:

hep-th/0401100, 0405085 and 0411186

Motivation

Flux Compactifications and Moduli Stabilization

Strominger [Nucl. Phys. B274 (1986) 253]

Giddings, Kachru and Polchinski [hep-th/0105097]

Kachru, Kallosh, Linde and Trivedi [hep-th/0301240]

Flux Compactifications and Moduli Stabilization

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Giddings, Kachru and Polchinski [hep-th/0105097]

Kachru, Kallosh, Linde and Trivedi [hep-th/0301240]

Modification of Mirror Symmetry

Gurrieri, Louis, Micu and Waldram [hep-th/0211102]

Cardoso, Curio, Dall'Agata, Lüst, Manousselis and Zoupanos [hep-th/0211118]

Fidanza, Minasian and Tomasiello [hep-th/0311122]

Review on superstring, CY moduli spaces, mirror symmetry, moduli stabilization:

Graña [hep-th/0509003]

Hitchin extended the almost complex structure to

generalized complex structure (GCS) and **generalized complex geometry (GCG)**

Hitchin [math.DG/0209099]

Gualtieri [math.DG/0401221]

- **almost complex structure:**

  a mapping: tangent bundle $\mathcal{TM} \longrightarrow$ tangent bundle $\mathcal{TM}$

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generalized complex structure (GCS) and generalized complex geometry (GCG)

Hitchin [math.DG/0209099]

Gualtieri [math.DG/0401221]

- **almost complex structure:**

  a mapping: tangent bundle $\mathcal{TM}$ $\rightarrow$ tangent bundle $\mathcal{TM}$

- **generalized complex structure:**

  a mapping: $\mathcal{TM} \oplus \mathcal{T}^*\mathcal{M} \rightarrow \mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$

Ordinary complex geometry

$$\left.\begin{array}{l}\text{complex structure } J\text{: } \mathcal{TM} \to \mathcal{TM} \text{ with } J^2 = -1 \\ \text{projection operator } \pi_\pm \equiv \frac{1}{2}(1 \pm iJ) \\ X, Y \in \mathcal{TM}\end{array}\right\} \Longrightarrow \begin{array}{c}\pi_\mp[\pi_\pm X, \pi_\pm Y] \ = \ 0 \\ \updownarrow \\ \mathcal{N}(J) \ = \ 0\end{array}$$

Generalized complex geometry

$$X, Y \in \mathcal{TM} \qquad \xi, \eta \in \mathcal{T}^*\mathcal{M}$$

natural pairing $(X+\xi, X+\xi) = \iota_X \xi$ gives a metric $\mathcal{I}$ on $\mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$

generalize the Lie bracket to the Courant bracket defined by

$$[X+\xi, Y+\eta]_c \equiv [X,Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2}\mathrm{d}(\iota_X \eta - \iota_Y \xi)$$

generalized complex structure $\mathcal{J}$: $\mathcal{J}^2 = -1$, $\mathcal{J}^{\mathrm{T}} \mathcal{I} \mathcal{J} = \mathcal{I}$

generalized projection operator $\Pi_\pm \equiv \frac{1}{2}(1 \pm i\mathcal{J})$

Then the integrability condition is generalized to

$$\Pi_\mp [\Pi_\pm (X+\xi), \Pi_\pm (Y+\eta)]_c = 0$$

This gives constraints on the generalized complex structure:

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix} \quad \rightarrow \quad \begin{cases} J^2 + PL = -1_d \\ JP + PK = 0 \\ KL + LJ = 0 \\ LP + K^2 = -1_d \end{cases} \quad \text{with} \quad \begin{cases} K = -J^{\mathrm{T}} \\ P^{\mathrm{T}} = -P \\ L^{\mathrm{T}} = -L \end{cases}$$

[Generalized complex structure]

Structures on a Kähler manifold (J, g, ω) correspond to the GCS:

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^{\mathrm{T}} \end{pmatrix} , \quad \mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} , \quad \mathcal{G} = -\mathcal{J}_J \mathcal{J}_\omega = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} .$$

[generalized Kähler geometry]

$$(\mathcal{J}^{(\pm)})^2 = -1 , \quad [\mathcal{J}^{(+)}, \mathcal{J}^{(-)}] = 0 , \quad \mathcal{G} = -\mathcal{J}^{(-)}\mathcal{J}^{(+)} .$$

Mirror Symmetry is an exchanging rule bewteen 2 GCS: $\mathcal{J}_1 \leftrightarrow \mathcal{J}_2$

(ex.) $$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ * & -J^{\mathrm{T}} \end{pmatrix} , \quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & * \end{pmatrix}$$

complex strucuture symplectic structure

We want to realize this duality in terms of sigma model formulation.

Sigma Models

Worldsheet Theories

The way how to generalize a string sigma model

to a sigma model on generalized complex geometries

- **introduce not only** $\mathrm{d}X^A \in \mathcal{TM}$ **but also** $\eta_A \in \mathcal{T}^*\mathcal{M}$ **(1st order action)**
-
-

Worldsheet Theories

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- **supersymmetrize**
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Worldsheet Theories

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to a sigma model on generalized complex geometries

- **introduce not only** $\mathrm{d}X^A \in \mathcal{TM}$ **but also** $\eta_A \in \mathcal{T}^*\mathcal{M}$ **(1st order action)**
- **supersymmetrize**
- **extend the SUSY transformation rule**

Extension of nonlinear sigma models

ordinary string sigma model given by $\mathrm{d}X^\mu \in \mathcal{TM}$ and $g_{\mu\nu}$, $B_{\mu\nu}$ (2nd order action):

$$S = \frac{1}{2} \int \Big\{ g_{\mu\nu}(X)\, \mathrm{d}X^\mu \wedge *\mathrm{d}X^\nu + B_{\mu\nu}(X)\, \mathrm{d}X^\mu \wedge \mathrm{d}X^\nu \Big\}$$

Extension of nonlinear sigma models

ordinary string sigma model given by $\mathrm{d}X^\mu \in \mathcal{T}\mathcal{M}$ and $g_{\mu\nu}$, $B_{\mu\nu}$ (2nd order action):

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introduce $\eta_\mu \in \mathcal{T}^*\mathcal{M}$ as a Lagrange multiplier (1st order action):

$$S = \frac{1}{2}\int \Big\{ \eta_\mu \wedge \mathrm{d}X^\mu + \frac{1}{2}\theta^{\mu\nu}\,\eta_\mu \wedge \eta_\nu + \frac{1}{2}G^{\mu\nu}\,\eta_\mu \wedge *\eta_\nu + \frac{1}{2}(B-b)_{\mu\nu}\,\mathrm{d}X^\mu \wedge \mathrm{d}X^\nu \Big\}$$

$$E_{\mu\nu} = g_{\mu\nu} + b_{\mu\nu}\,, \qquad E^{\mu\lambda}E_{\lambda\nu} = \delta^\mu{}_\nu$$

$$G^{\mu\nu} = \frac{1}{2}(E^{\mu\nu} + E^{\nu\mu})\,, \qquad \theta^{\mu\nu} = \frac{1}{2}(E^{\mu\nu} - E^{\nu\mu})$$

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supersymmetrize this $\longrightarrow$ a new sigma model on GCG

Lindström, Minasian, Tomasiello and Zabzine [hep-th/0405085]

Lindström [hep-th/0401100]

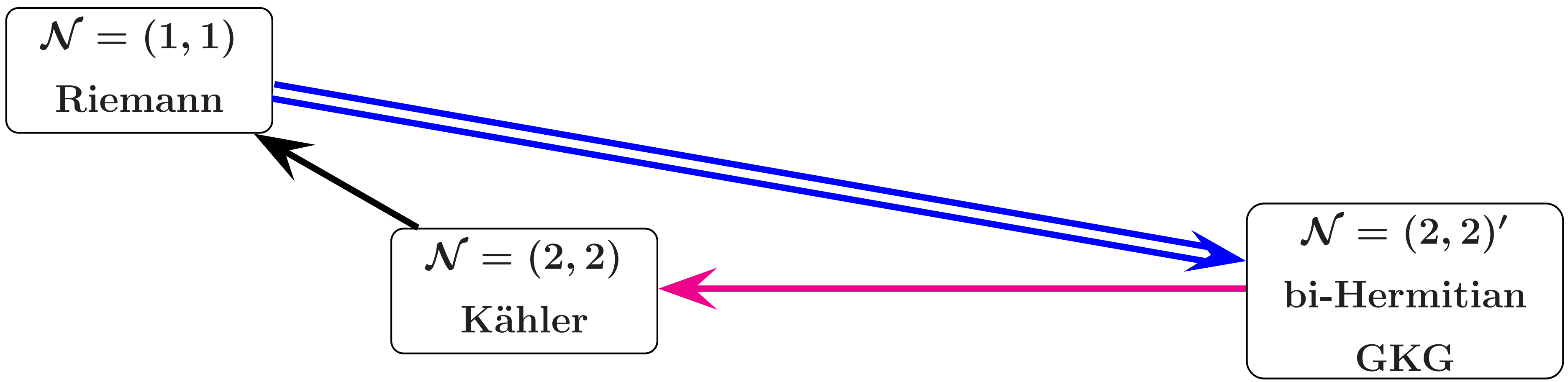
Lindström, Roček, Unge and Zabzine [hep-th/0411186]

Supersymmetric sigma models

$\mathcal{N} = (2, 2)$

Kähler

Supersymmetric sigma models

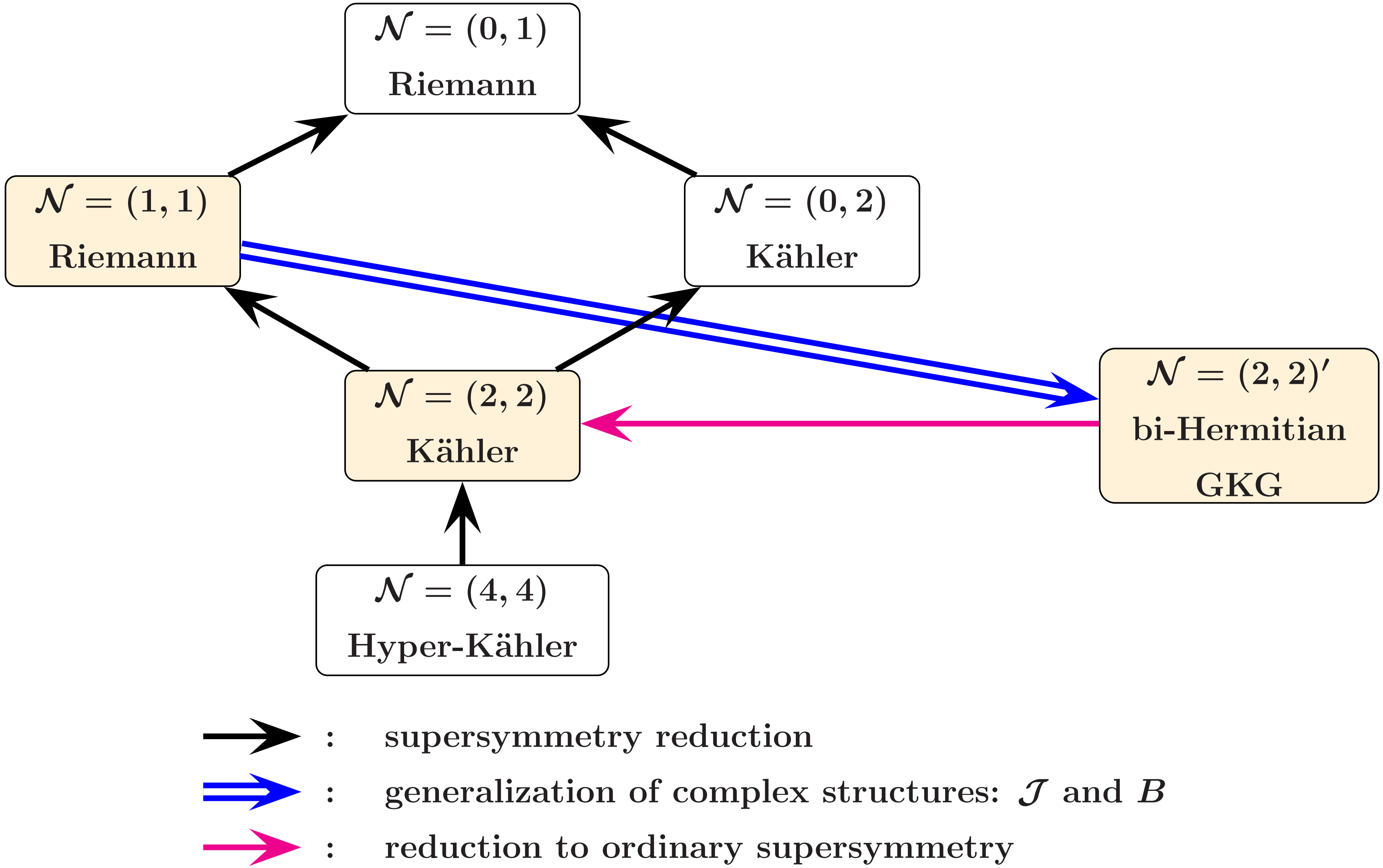


: supersymmetry reduction

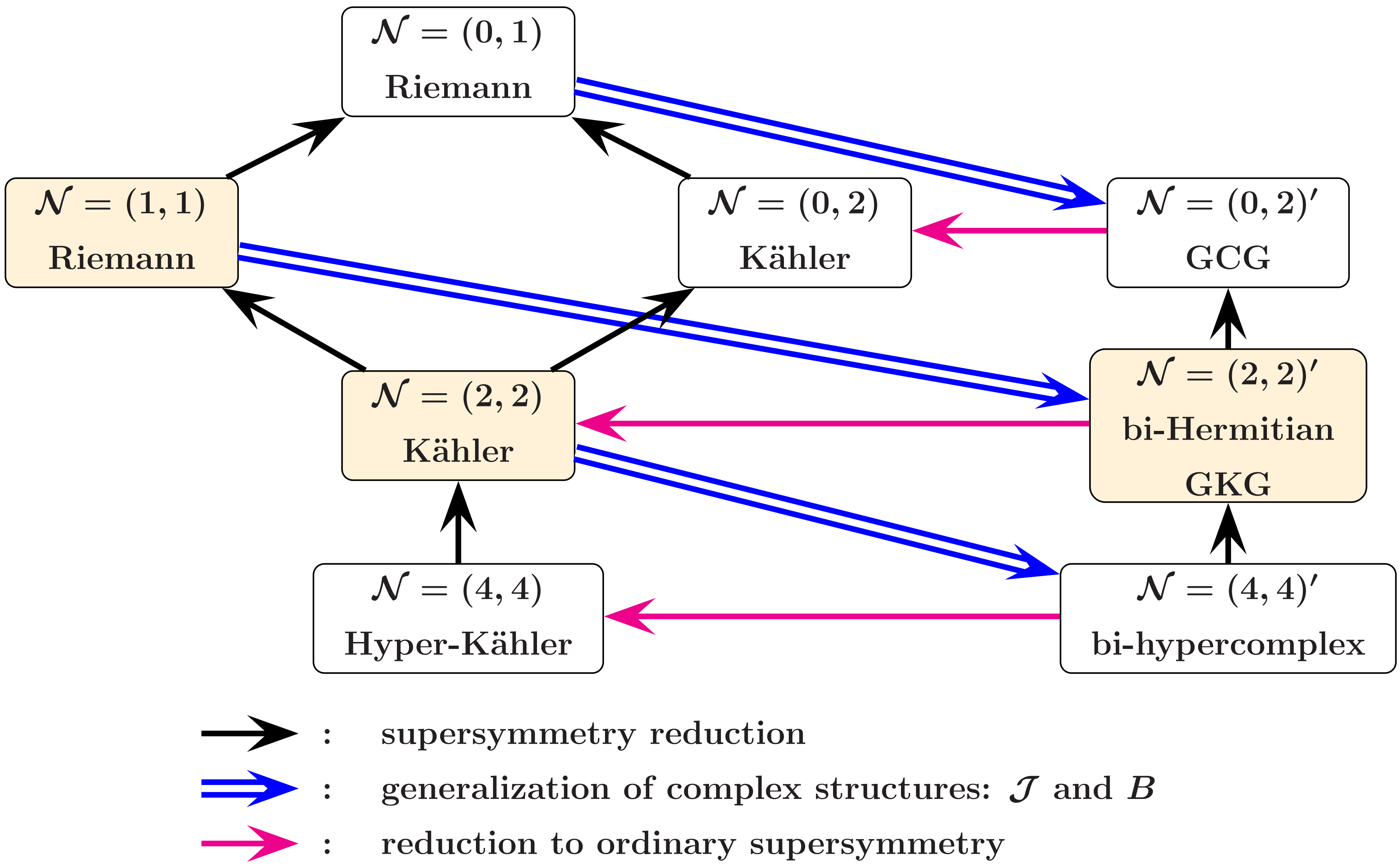
: generalization of complex structures: $\mathcal{J}$ and B

: reduction to ordinary supersymmetry

Supersymmetric sigma models



Supersymmetric sigma models



Superfields

$\mathcal{N} = (2,2)$ superfields

chiral superfield $\Phi^{(2,2)}$ $\qquad \overline{D}_{\pm}\Phi^{(2,2)} = 0 \qquad \Phi^{(2,2)} = \overline{D}_{+}\overline{D}_{-}\Theta$

left semi-chiral superfield $\mathbb{X}^{(2,2)}$ $\qquad \overline{D}_{+}\mathbb{X}^{(2,2)} = 0 \qquad \mathbb{X}^{(2,2)} = \overline{D}_{+}\Theta$

$$\begin{aligned} \Phi^{(2,2)} &= \phi + i\sqrt{2}\theta^{+}\psi_{+} + i\sqrt{2}\theta^{-}\psi_{-} + 2i\theta^{+}\theta^{-}F \\ &\quad +\{(\partial_0 \pm \partial_1)\text{-derivative terms}\} \end{aligned}$$

$$\begin{aligned} \mathbb{X}^{(2,2)} &= \phi + i\sqrt{2}\theta^{+}\psi_{+} + i\sqrt{2}(\theta^{-}\psi_{-} + \overline{\theta}^{-}\chi_{-}) + 2i\theta^{+}(\theta^{-}F + \overline{\theta}^{-}G) \\ &\quad + \theta^{-}\overline{\theta}^{-}A_{=} + 2\theta^{+}\theta^{-}\overline{\theta}^{-}\zeta_{-} \\ &\quad +\{(\partial_0 + \partial_1)\text{-derivative terms}\} \end{aligned}$$

$\mathcal{N} = (1,1)$ scalar/spinor superfields from $\mathcal{N} = (2,2)$ semi-chiral superfields

$$\begin{aligned} \mathbb{X}^{(1,1)} &= \phi + i\sqrt{2}\theta_1^{+}\widehat{\psi}_{+} + i\sqrt{2}\theta_1^{-}(\widehat{\psi}_{-} + \widehat{\chi}_{-}) + 2i\theta_1^{+}\theta_1^{-}(\widehat{F} + \widehat{G}) \\ \Psi_{-}^{(1,1)} &= i(\widehat{\psi}_{-} - \widehat{\chi}_{-}) - i\sqrt{2}\theta_1^{+}(\widehat{F} - \widehat{G}) + \sqrt{2}\theta_1^{-}A_{=} + 2\sqrt{2}\theta_1^{+}\theta_1^{-}\widehat{\zeta}_{-} \end{aligned}$$

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$$\Phi^{(2,2)} = \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}\theta^-\psi_- + 2i\theta^+\theta^- F + \{(\partial_0 \pm \partial_1)\text{-derivative terms}\}$$

$$\mathbb{X}^{(2,2)} = \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}(\theta^-\psi_- + \overline{\theta}^-\chi_-) + 2i\theta^+(\theta^- F + \overline{\theta}^- G) + \theta^-\overline{\theta}^- A_= + 2\theta^+\theta^-\overline{\theta}^-\zeta_- + \{(\partial_0 + \partial_1)\text{-derivative terms}\}$$

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$$\mathbb{X}^{(1,1)} = \phi + i\sqrt{2}\theta_1^+\psi_+ + i\sqrt{2}\theta_1^-(\widehat{\psi}_- + \widehat{\chi}_-) + 2i\theta_1^+\theta_1^-(\widehat{F} + \widehat{G})$$

$$\Psi_-^{(1,1)} = i(\widehat{\psi}_- - \widehat{\chi}_-) - i\sqrt{2}\theta_1^+(\widehat{F} - \widehat{G}) + \sqrt{2}\theta_1^- A_= + 2\sqrt{2}\theta_1^+\theta_1^-\widehat{\zeta}_-$$

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Sigma Models on Generalized Complex Geometries

$\mathcal{N} = (2,2)$ supersymmetric sigma models of semi-chiral superfields

We use semi-chiral superfields $\mathbb{X}$, $\mathbb{Y}$ defined by

$$\overline{D}_+ \mathbb{X} = 0 \, , \qquad \overline{D}_- \mathbb{Y} = 0$$

reduce $\mathcal{N} = (2,2)$ Lagrangian to $\mathcal{N} = (1,1)$ Lagrangian:

$$\mathscr{L} = \int d^4\theta \, K(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}) = -\frac{1}{8} \int d\theta_1^+ d\theta_1^- \, \widetilde{Q}_+^1 \widetilde{Q}_-^1 \, K(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}})$$

reduce $\mathcal{N} = (2,2)$ to $\mathcal{N} = (1,1)$ (i.e., reduce complex $\theta^\pm$ to real $\theta_1^\pm$):

$$\theta_1^\pm \equiv -i e^{-i\nu_\pm} \theta^\pm = i e^{+i\nu_\pm} \overline{\theta}^\pm$$

Semi-chiral superfields are decomposed into 2 independent $\mathcal{N} = (1,1)$ superfields:

$$\mathbb{X}^{(2,2)} \longrightarrow \{ \mathbb{X}^{(1,1)}, \, \Psi_-^{(1,1)} \} \qquad \Psi_-^{(1,1)} \equiv \widetilde{Q}_-^1 \mathbb{X}^{(2,2)} |$$

$$\mathbb{Y}^{(2,2)} \longrightarrow \{ \mathbb{Y}^{(1,1)}, \, \Upsilon_+^{(1,1)} \} \qquad \Upsilon_+^{(1,1)} \equiv \widetilde{Q}_+^1 \mathbb{Y}^{(2,2)} |$$

Buscher, Lindström and Roček [Phys. Lett. B202 (1988) 94]

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“Topological” sigma models (useful to look at the generalized complex structures)

Consider an $\mathcal{N}=(1,1)$ Lagrangian including only $\mathbb{X}^A = \{\mathbb{X}^a, \overline{\mathbb{X}^a}\}$:

$$\mathscr{L}_{\mathbb{X}} \;=\; \int \mathrm{d}^4\theta\, K(\mathbb{X}, \overline{\mathbb{X}}) \;=\; -\frac{1}{4}\int \mathrm{d}\theta_1^+ \mathrm{d}\theta_1^- \Big\{ S_{A-} D^1_+ \mathbb{X}^A \Big\}$$

re-definition of superfields (from $\mathcal{T}\mathcal{M}$ to $\mathcal{T}^*\mathcal{M}$):

$$S_{A-} \;=\; \Psi^B_- \,\omega_{BA}\,, \qquad 2\omega_{AB} \;\equiv\; J_A{}^C K_{CB} - K_{AC} J^C{}_B\,, \qquad K_{AB} \;\equiv\; \frac{\partial^2 K}{\partial \mathbb{X}^A \partial \mathbb{X}^B}$$

$$D^1_+ \mathbb{X}^A \;\in\; \mathcal{T}\mathcal{M}\,, \qquad S_{A-} \;\in\; \mathcal{T}^*\mathcal{M}$$

introduce another $\mathcal{N}=(1,1)$ SUSY to extend the model to a new $\mathcal{N}=(2,2)'$:

$$\widetilde{\delta}^{(+)}\mathbb{X}^A \;=\; \widetilde{\varepsilon}^+ J^A{}_B D^1_+ \mathbb{X}^B\,, \qquad \widetilde{\delta}^{(-)}\mathbb{X}^A \;=\; -\widetilde{\varepsilon}^- \omega^{AB} (S_{B-})^{\mathrm{T}}$$

$$\widetilde{\delta}^{(+)} S_{A-} \;=\; -\widetilde{\varepsilon}^+ D^1_+ S_{B-} J^B{}_A + (\omega_{AB})^{\mathrm{T}} \Big\{ (\widetilde{\delta}^{(+)}\mathbb{X}^E)^{\mathrm{T}} \delta^G_E - \widetilde{\varepsilon}^+ (D^1_+ \mathbb{X}^E)^{\mathrm{T}} J^G{}_E \Big\} \partial_G (\omega^{BC})^{\mathrm{T}} S_{C-}$$

$$\widetilde{\delta}^{(-)} S_{A-} \;=\; -i\widetilde{\varepsilon}^- \big\{ \omega_{AC} (\partial_0 - \partial_1) \mathbb{X}^C \big\}^{\mathrm{T}} + S_{C-} \omega^{CB} \partial_E(\omega_{BA}) (\widetilde{\delta}^{(-)}\mathbb{X}^E)$$

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$$\widetilde{\delta}^{(+)} S_{A-} \;=\; -\widetilde{\varepsilon}^+\, D_+^1 S_{B-}\, J^B{}_A$$

$$+(\omega_{AB})^{\mathrm{T}}\left\{ (\widetilde{\delta}^{(+)}\mathbb{X}^E)^{\mathrm{T}}\, \delta_E^G - \widetilde{\varepsilon}^+\, (D_+^1\mathbb{X}^E)^{\mathrm{T}}\, J^G{}_E \right\} \partial_G(\omega^{BC})^{\mathrm{T}}\, S_{C-}$$

$$\widetilde{\delta}^{(-)} S_{A-} \;=\; -i\widetilde{\varepsilon}^-\, \left\{ \omega_{AC}\, (\partial_0 - \partial_1)\mathbb{X}^C \right\}^{\mathrm{T}}$$

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Consider an $\mathcal{N} = (1,1)$ Lagrangian including only $\mathbb{X}^A = \{\mathbb{X}^a, \overline{\mathbb{X}}^{\overline{a}}\}$:

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re-definition of superfields (from $\mathcal{TM}$ to $\mathcal{T}^*\mathcal{M}$):

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$$D_+^1 \mathbb{X}^A \in \mathcal{TM} \, , \qquad S_{A-} \in \mathcal{T}^*\mathcal{M}$$

introduce another $\mathcal{N} = (1,1)$ SUSY to extend the model to a new $\mathcal{N} = (2,2)'$:

$$\widetilde{\delta}^{(+)} \mathbb{X}^A = \widetilde{\varepsilon}^+ J^A{}_B \, D_+^1 \mathbb{X}^B \, , \qquad \widetilde{\delta}^{(-)} \mathbb{X}^A = -\widetilde{\varepsilon}^- \, \omega^{AB} \, (S_{B-})^{\mathrm{T}}$$

$$\begin{aligned} \widetilde{\delta}^{(+)} S_{A-} &= -\widetilde{\varepsilon}^+ \, D_+^1 S_{B-} \, J^B{}_A \\ &\quad + (\omega_{AB})^{\mathrm{T}} \left\{ (\widetilde{\delta}^{(+)} \mathbb{X}^E)^{\mathrm{T}} \, \delta_E^G - \widetilde{\varepsilon}^+ \, (D_+^1 \mathbb{X}^E)^{\mathrm{T}} \, J^G{}_E \right\} \partial_G (\omega^{BC})^{\mathrm{T}} \, S_{C-} \\ \widetilde{\delta}^{(-)} S_{A-} &= -i \widetilde{\varepsilon}^- \left\{ \omega_{AC} \, (\partial_0 - \partial_1) \mathbb{X}^C \right\}^{\mathrm{T}} \\ &\quad + S_{C-} \, \omega^{CB} \, \partial_E (\omega_{BA}) \, (\widetilde{\delta}^{(-)} \mathbb{X}^E) \end{aligned}$$

Embedded generalized complex structures on $\mathcal{N} = (2,2)$

standard complex structure $\mathcal{J}_1$ and symplectic form $\mathcal{J}_2$ are extended:

$$\begin{cases} \widetilde{\delta}^{(+)}\mathbb{X}^A = \widetilde{\varepsilon}^{+} J^A{}_B D^1_+ \mathbb{X}^B \\ \widetilde{\delta}^{(+)}(S_{A-})^{\mathrm{T}} = -\widetilde{\varepsilon}^{+} J_A{}^B (D^1_+ S_{B-})^{\mathrm{T}} + \cdots \end{cases} \longleftrightarrow \quad \mathcal{J}_1 = \begin{pmatrix} J & 0 \\ 0 & -J^{\mathrm{T}} \end{pmatrix}$$

$$\begin{cases} \widetilde{\delta}^{(-)}\mathbb{X}^A = -\widetilde{\varepsilon}^{-} \omega^{AB} (S_{B-})^{\mathrm{T}} \\ \widetilde{\delta}^{(-)}(S_{A-})^{\mathrm{T}} = -i\widetilde{\varepsilon}^{-} \omega_{AB} (\partial_0 - \partial_1)\mathbb{X}^B + \cdots \end{cases} \longleftrightarrow \quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

$$\begin{cases} \widetilde{\delta}^{(+)}\mathbb{Y}^{A'} = -\widetilde{\varepsilon}^{+} \omega^{A'B'} (S_{B'+})^{\mathrm{T}} \\ \widetilde{\delta}^{(+)}(S_{A'+})^{\mathrm{T}} = -i\widetilde{\varepsilon}^{+} \omega_{A'B'} (\partial_0 + \partial_1)\mathbb{Y}^{B'} + \cdots \end{cases} \longleftrightarrow \quad \mathcal{J}'_2 = \begin{pmatrix} 0 & -\omega'^{-1} \\ \omega' & 0 \end{pmatrix}$$

$$\begin{cases} \widetilde{\delta}^{(-)}\mathbb{Y}^{A'} = \widetilde{\varepsilon}^{-} J^{A'}{}_{B'} D^1_- \mathbb{Y}^{B'} \\ \widetilde{\delta}^{(-)}(S_{A'+})^{\mathrm{T}} = -\widetilde{\varepsilon}^{-} J_{A'}{}^{B'} (D^1_- S_{B'+})^{\mathrm{T}} + \cdots \end{cases} \longleftrightarrow \quad \mathcal{J}'_1 = \begin{pmatrix} J' & 0 \\ 0 & -J'^{\mathrm{T}} \end{pmatrix}$$

Lindström, Roček, Unge and Zabzine [hep-th/0411186]

Embedded generalized complex structures on $\mathcal{N} = (2,2)'$

standard complex structure $\mathcal{J}_1$ and symplectic form $\mathcal{J}_2$ are extended:

$$\left\{\begin{aligned} \widetilde{\delta}^{(+)}\mathbb{X}^A &= \widetilde{\varepsilon}^+ J^A{}_B\, D^1_+ \mathbb{X}^B \\ \widetilde{\delta}^{(+)}(S_{A-})^{\mathrm{T}} &= -\widetilde{\varepsilon}^+ J_A{}^B\, (D^1_+ S_{B-})^{\mathrm{T}} + \cdots \end{aligned}\right. \quad\longleftrightarrow\quad \mathcal{J}_1 = \begin{pmatrix} J & 0 \\ * & -J^{\mathrm{T}} \end{pmatrix}$$

$$\left\{\begin{aligned} \widetilde{\delta}^{(-)}\mathbb{X}^A &= -\widetilde{\varepsilon}^- \omega^{AB} (S_{B-})^{\mathrm{T}} \\ \widetilde{\delta}^{(-)}(S_{A-})^{\mathrm{T}} &= -i\widetilde{\varepsilon}^- \omega_{AB}(\partial_0 - \partial_1)\mathbb{X}^B + \cdots \end{aligned}\right. \quad\longleftrightarrow\quad \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & * \end{pmatrix}$$

$$\left\{\begin{aligned} \widetilde{\delta}^{(+)}\mathbb{Y}^{A'} &= -\widetilde{\varepsilon}^+ \omega^{A'B'} (S_{B'+})^{\mathrm{T}} \\ \widetilde{\delta}^{(+)}(S_{A'+})^{\mathrm{T}} &= -i\widetilde{\varepsilon}^+ \omega_{A'B'}(\partial_0 + \partial_1)\mathbb{Y}^{B'} + \cdots \end{aligned}\right. \quad\longleftrightarrow\quad \mathcal{J}'_2 = \begin{pmatrix} 0 & -\omega'^{-1} \\ \omega' & * \end{pmatrix}$$

$$\left\{\begin{aligned} \widetilde{\delta}^{(-)}\mathbb{Y}^{A'} &= \widetilde{\varepsilon}^- J^{A'}{}_{B'}\, D^1_- \mathbb{Y}^{B'} \\ \widetilde{\delta}^{(-)}(S_{A'+})^{\mathrm{T}} &= -\widetilde{\varepsilon}^- J_{A'}{}^{B'}\, (D^1_- S_{B'+})^{\mathrm{T}} + \cdots \end{aligned}\right. \quad\longleftrightarrow\quad \mathcal{J}'_1 = \begin{pmatrix} J' & 0 \\ * & -J'^{\mathrm{T}} \end{pmatrix}$$

Lindström, Roček, Unge and Zabzine [hep-th/0411186]

A conjecture on the mirror symmetry

As mentioned, the mirror symmetry should be interpreted as

$$\mathcal{J}_1 = \begin{pmatrix} J & 0 \\ * & -J^{\mathrm{T}} \end{pmatrix} \longleftrightarrow \mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & * \end{pmatrix}$$

This can be regarded as exchange between $\mathbb{X}^A$ and $\mathbb{Y}^{A'}$(?)

If the mirror dual transformation means the mapping from $(\mathbb{X}^A, \mathbb{Y}^{A'})$ to $(\widehat{\mathbb{Y}}^{B'}, \widehat{\mathbb{X}}^B)$, we can insist that

> A theory $\mathscr{L}(\mathbb{X}^A, \mathbb{Y}^{A'})$ should be mapped
> to another theory $\mathscr{L}(\widehat{\mathbb{Y}}^{B'}, \widehat{\mathbb{X}}^B)$, and vice versa.

We wish to consider the duality transformation procedure.

[Note] The mirror dual in two-dimensional worldsheet:

$$\begin{matrix} (\Phi, \overline{\Phi}) & & (Y, \overline{Y}) \\ & \longleftrightarrow & \\ (c, c)\text{-ring} & & (a, c)\text{-ring} \end{matrix}$$

On dualities among various $\mathcal{N} = (2, 2)$ SUSY theories $\cdots$

- **We have known the duality transformation rule**

  between $(\boldsymbol{\Phi}, \overline{\boldsymbol{\Phi}})$-model and $(\boldsymbol{Y}, \overline{\boldsymbol{Y}})$-model

- **We have also known the duality transformation rule**

  between $(\mathbb{X}, \mathbb{Y}, \overline{\mathbb{X}}, \overline{\mathbb{Y}})$-model and $(\boldsymbol{\Phi}, \overline{\boldsymbol{\Phi}}, \boldsymbol{Y}, \overline{\boldsymbol{Y}})$-model

Roček and Verlinde [hep-th/9110053]

Ivanov, Kim and Roček [hep-th/9406063]

Penati and Zanon [hep-th/9712137]

Grisaru, Massar, Sevrin and Troost [hep-th/9801080]

However, we have not yet understood the dualities

between $(\mathbb{X}, \overline{\mathbb{X}})$-model and $(\mathbb{Y}, \overline{\mathbb{Y}})$-model,

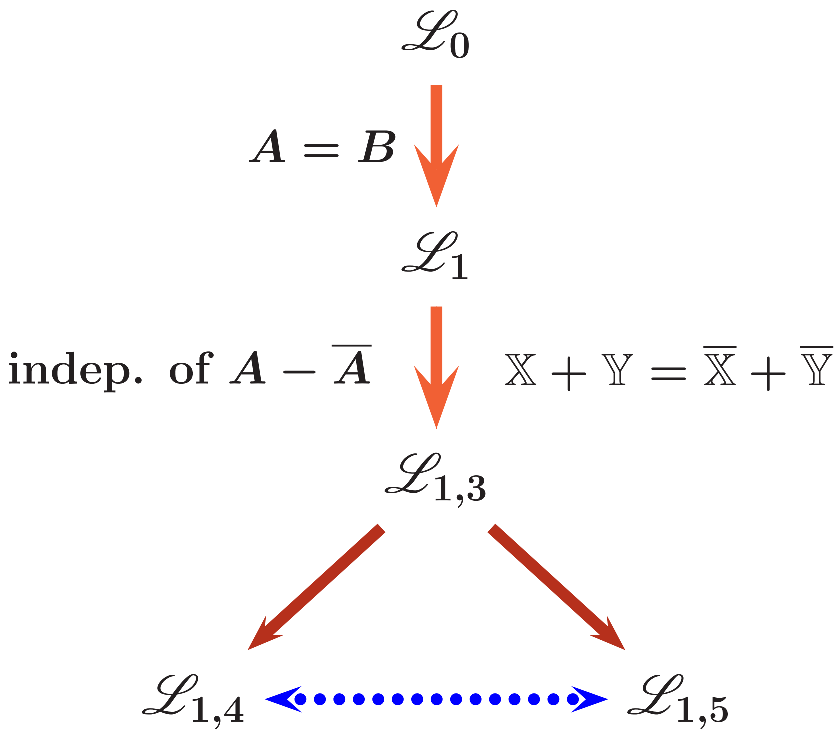
or between $(\mathbb{X}, \mathbb{Y}, \overline{\mathbb{X}}, \overline{\mathbb{Y}})$-model and $(\mathbb{X}, \mathbb{Y}, \overline{\mathbb{X}}, \overline{\mathbb{Y}})$-model

Duality Transformation I

$$\mathscr{L}(\mathbf{\Phi}, \overline{\mathbf{\Phi}}) \leftrightarrow \mathscr{L}(\boldsymbol{Y}, \overline{\boldsymbol{Y}})$$

Duality transformation 1

$$\mathscr{L}_0 \;=\; \int d^4\theta \left\{ K_0(A, \overline{A}, B, \overline{B}; \cdots) - \mathbb{X}A - \overline{\mathbb{X}A} - \mathbb{Y}B - \overline{\mathbb{Y}B} \right\}$$



$$\Phi_1 + \overline{\Phi}_1 \;=\; A + \overline{A} \;=\; \widetilde{A}(Y_1, \overline{Y}_1; \cdots)\,, \qquad \mathbb{X} + \mathbb{Y} \;=\; Y_1 + \overline{Y}_1$$

$$\mathscr{L}_{1,4} \;\equiv\; \int d^4\theta \left\{ K_{1,3}(A + \overline{A}; \cdots) - (Y_1 + \overline{Y}_1)(A + \overline{A}) \right\}\Big|_{\text{EOM of } Y_1 + \overline{Y}_1} \;\equiv\; \int d^4\theta \, K_{1,4}(\Phi_1, \overline{\Phi}_1; \cdots)$$

$$\mathscr{L}_{1,5} \;\equiv\; \int d^4\theta \left\{ K_{1,3}(A + \overline{A}; \cdots) - (Y_1 + \overline{Y}_1)(A + \overline{A}) \right\}\Big|_{\text{EOM of } A + \overline{A}} \;\equiv\; \int d^4\theta \, K_{1,5}(Y_1, \overline{Y}_1; \cdots)$$

Duality Transformation II

$$\mathscr{L}(\boldsymbol{\Phi}, \overline{\boldsymbol{\Phi}}, \boldsymbol{Y}, \overline{\boldsymbol{Y}}) \leftrightarrow \mathscr{L}(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}})$$

Duality transformation 2

$$\mathscr{L}_0 \;=\; \int d^4\theta \left\{ K_0(A, \overline{A}, B, \overline{B}; \cdots) - \mathbb{X}A - \overline{\mathbb{X}A} - \mathbb{Y}B - \overline{\mathbb{Y}B} \right\}$$

$$\mathscr{L}_0$$

$$A + \overline{B} = \overline{A} + B \quad \downarrow \quad A = A' + B',\; B = A' - \overline{B}' + C'$$

$$\mathscr{L}_7$$

$$\swarrow \qquad \searrow$$

$$\mathscr{L}_{7,1} \longleftrightarrow \mathscr{L}_{7,2}$$

$$\Phi_7 \;=\; A' \;=\; A'(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}; \cdots)\,, \qquad Y_7 \;=\; B' \;=\; B'(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}; \cdots)$$

$$0 \;=\; C' \;=\; C'(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}; \cdots)$$

$$\mathscr{L}_{7,1} \;\equiv\; \mathscr{L}_7\Big|_{\text{EOM of } \mathbb{X},\, \mathbb{Y}} \;\equiv\; \int d^4\theta\, K_{7,1}(\Phi_7, \overline{\Phi}_7, Y_7, \overline{Y}_7; \cdots)$$

$$\mathscr{L}_{7,2} \;\equiv\; \mathscr{L}_7\Big|_{\text{EOM of } A',\, B',\, C'} \;\equiv\; \int d^4\theta\, K_{7,2}(\mathbb{X}, \overline{\mathbb{X}}, \mathbb{Y}, \overline{\mathbb{Y}}; \cdots)$$

Duality Transformation III

from $\mathscr{L}_{\mathbb{X}}$-theory to $\mathscr{L}_{\mathbb{Y}}$-theory

Duality transformation: from $\mathscr{L}_{\mathbb{X}}$-theory to $\mathscr{L}_{\mathbb{Y}}$-theory

— idea —

$$\mathscr{L}_{\mathbf{7}}$$

$$\mathscr{L}_{\mathbf{7,2}}(\mathbb{X}, \mathbb{Y}) \longleftrightarrow \mathscr{L}_{\mathbf{7,1}}(\boldsymbol{\Phi}_{\mathbf{7}}, \boldsymbol{Y}_{\mathbf{7}})$$

Duality 2

Duality transformation: from $\mathscr{L}_{\mathbb{X}}$-theory to $\mathscr{L}_{\mathbb{Y}}$-theory

— idea —

$\mathscr{L}_{\mathbf{7}}$

$\mathscr{L}_{\mathbf{7,2}}(\mathbb{X}, \mathbb{Y}) \longleftrightarrow \mathscr{L}_{\mathbf{7,1}}(\boldsymbol{\Phi}_7, \boldsymbol{Y}_7)$

Duality 2

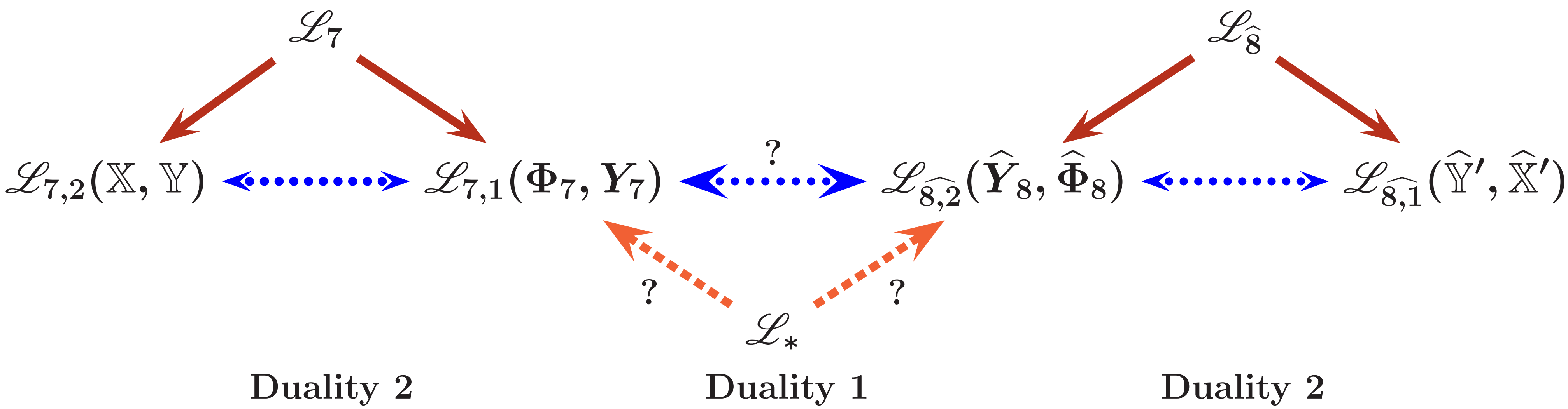
$\mathscr{L}_{\widehat{\mathbf{8}}}$

$\mathscr{L}_{\widehat{\mathbf{8,2}}}(\widehat{\boldsymbol{Y}}_8, \widehat{\boldsymbol{\Phi}}_8) \longleftrightarrow \mathscr{L}_{\widehat{\mathbf{8,1}}}(\widehat{\mathbb{Y}}', \widehat{\mathbb{X}}')$

Duality 2

Duality transformation: from $\mathscr{L}_{\mathbb{X}}$-theory to $\mathscr{L}_{\mathbb{Y}}$-theory

— idea —



Discussions

PROBLEMS

- Doubling problem of degrees of freedom

  Which DOF corresponds to "coordinates" of geometry? (combination of $\mathbb{X}^A$ and $\mathbb{Y}^{A'}$?)

- How to relax the integrability condition? (to realize the half-flat manifold)

CHECK

- Relation between GCG and CY with H_3-flux (or G-structure manifolds)

- Consistency check of T-duality on $\mathcal{TM} \oplus \mathcal{T}^*\mathcal{M}$

  Dabholkar and Hull [hep-th/0512005]

APPLICATIONS

- Sigma models of heterotic strings

  $\mathcal{N} = (2,2)' \longrightarrow \mathcal{N} = (0,2)'$?

- Topological strings on generalized complex manifolds

  Kapustin and Li [hep-th/0407249]

Appendix

$\mathcal{N} = (2, 2), (1, 1)$ SUSY

Appendix

$\mathcal{N}=(2,2)$ supersymmetry

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm(\partial_0 \pm \partial_1)\,, \qquad \overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm(\partial_0 \pm \partial_1)\,,$$

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm(\partial_0 \pm \partial_1)\,, \qquad \overline{Q}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm(\partial_0 \pm \partial_1)\,.$$

$$Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0\,,$$

$$\{Q_\pm, \overline{Q}_\pm\} = -2i(\partial_0 \pm \partial_1) = 2(H \mp P)\,,$$

$$\{\overline{Q}_+, \overline{Q}_-\} = 0\,, \qquad \{Q_+, Q_-\} = 0\,, \qquad \{Q_-, \overline{Q}_+\} = 0\,, \qquad \{Q_+, \overline{Q}_-\} = 0\,,$$

$$\{D_\pm, \overline{D}_\pm\} = 2i(\partial_0 \pm \partial_1)\,,$$

$$\{\overline{D}_\alpha, \overline{D}_\beta\} = \{D_\alpha, D_\beta\} = \{D_\pm, \overline{D}_\mp\} = 0\,,$$

$$\{D_\alpha, Q_\beta\} = \{\overline{D}_\alpha, Q_\beta\} = \{D_\alpha, \overline{Q}_\beta\} = \{\overline{D}_\alpha, \overline{Q}_\beta\} = 0\,,$$

$$[M, Q_\pm] = \mp Q_\pm\,, \qquad [M, \overline{Q}_\pm] = \mp\overline{Q}_\pm\,,$$

$$[F_V, Q_\pm] = -Q_\pm\,, \qquad [F_V, \overline{Q}_\pm] = \overline{Q}_\pm\,,$$

$$[F_A, Q_\pm] = \mp Q_\pm\,, \qquad [F_A, \overline{Q}_\pm] = \pm\overline{Q}_\pm\,.$$

$\mathcal{N} = (2,2)$ superfields

Here we summarize $\mathcal{N} = (2,2)$ superfields:

chiral superfield Φ	$\overline{D}_{\pm}\Phi = 0$	$\Phi = \overline{D}_+\overline{D}_-\Theta$
twisted chiral superfield Y	$\overline{D}_+ Y = D_- Y = 0$	$Y = \overline{D}_+ D_-\Theta$
real linear superfield G	$\overline{D}_+\overline{D}_- G = D_+ D_- G = 0$	$G = Y + \overline{Y}$
real twisted linear superfield H	$\overline{D}_+ D_- H = D_+\overline{D}_- H = 0$	$H = \Phi + \overline{\Phi}$
left semi-chiral superfield $\mathbb{X}$	$\overline{D}_+\mathbb{X} = 0$	$\mathbb{X} = \overline{D}_+\Theta$
right semi-chiral superfield $\mathbb{Y}$	$\overline{D}_-\mathbb{Y} = 0$	$\mathbb{Y} = \overline{D}_-\Theta$
complex linear superfield Σ	$\overline{D}_+\overline{D}_-\Sigma = 0$	$\Sigma = a\mathbb{X} + b\mathbb{Y}$
complex twisted linear superfield $\widetilde{\Sigma}$	$\overline{D}_+ D_-\widetilde{\Sigma} = 0$	$\widetilde{\Sigma} = a\mathbb{X} + b\overline{\mathbb{Y}}$

where Θ is an unconstrained superfield; a and b are complex constants.

$$\begin{aligned}
\Phi &= \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}\theta^-\psi_- + 2i\theta^+\theta^- F \\
&- i\theta^+\overline{\theta}^+(\partial_0+\partial_1)\phi - i\theta^-\overline{\theta}^-(\partial_0-\partial_1)\phi + \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0^2-\partial_1^2)\phi \\
&+ \sqrt{2}\theta^+\overline{\theta}^+\theta^-(\partial_0+\partial_1)\psi_- + \sqrt{2}\theta^-\overline{\theta}^-\theta^+(\partial_0-\partial_1)\psi_+ \ ,
\end{aligned}$$

$$\begin{aligned}
Y &= y + i\sqrt{2}\theta^+\overline{\chi}_+ + i\sqrt{2}\overline{\theta}^-\chi_- + 2i\theta^+\overline{\theta}^- G \\
&- i\theta^+\overline{\theta}^+(\partial_0+\partial_1)y + i\theta^-\overline{\theta}^-(\partial_0-\partial_1)y - \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0^2-\partial_1^2)y \\
&- \sqrt{2}\theta^-\overline{\theta}^-\theta^+(\partial_0-\partial_1)\overline{\chi}_+ + \sqrt{2}\theta^+\overline{\theta}^+\overline{\theta}^-(\partial_0+\partial_1)\chi_- \ ,
\end{aligned}$$

$$\begin{aligned}
\mathbb{X} &= \phi + i\sqrt{2}\theta^+\psi_+ + i\sqrt{2}(\theta^-\psi_- + \overline{\theta}^-\chi_-) + 2i\theta^+(\theta^- F + \overline{\theta}^- G) \\
&+ \theta^-\overline{\theta}^- A_= + 2\theta^+\theta^-\overline{\theta}^-\zeta_- - i\theta^+\overline{\theta}^+(\partial_0+\partial_1)\phi \\
&+ \sqrt{2}\theta^+\overline{\theta}^+(\partial_0+\partial_1)(\theta^-\psi_- + \overline{\theta}^-\chi_-) + i\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0+\partial_1)A_= \ ,
\end{aligned}$$

$$\begin{aligned}
\mathbb{Y} &= \phi + i\sqrt{2}(\theta^+\psi_+ + \overline{\theta}^+\chi_+) + i\sqrt{2}\theta^-\psi_- + 2i\theta^+\theta^- F + 2i\overline{\theta}^+\theta^- N \\
&- i\theta^-\overline{\theta}^-(\partial_0-\partial_1)\phi + \theta^+\overline{\theta}^+ B_{\#} - 2\theta^-\theta^+\overline{\theta}^+\zeta_+ \\
&+ \sqrt{2}\theta^-\overline{\theta}^-(\partial_0-\partial_1)(\theta^+\psi_+ + \overline{\theta}^+\chi_+) + i\theta^+\theta^-\overline{\theta}^+\overline{\theta}^-(\partial_0-\partial_1)B_{\#} \ .
\end{aligned}$$

$\mathcal{N} = (1, 1)$ supersymmetry

Here we consider $\mathcal{N} = (1, 1)$ supersymmetry which has two real supercharges, one with positive chirality and the other with negative chirality:

$$\theta_1^\pm \equiv -i\mathrm{e}^{-i\nu_\pm}\theta^\pm = i\mathrm{e}^{+i\nu_\pm}\overline{\theta}^\pm \qquad \text{where } \theta_1^\pm \text{ is real} \tag{1}$$

We introduce the following differential operators

$$\begin{aligned}
Q_\pm^1 &\equiv \frac{1}{\sqrt{2}}\Big\{\mathrm{e}^{i\nu_\pm}Q_\pm + \mathrm{e}^{-i\nu_\pm}\overline{Q}_\pm\Big\}\Big|_{\text{eq.(1)}} = -\frac{i}{\sqrt{2}}\frac{\partial}{\partial\theta_1^\pm} + \sqrt{2}\theta_1^\pm(\partial_0 \pm \partial_1) \ , \\
D_\pm^1 &\equiv \frac{1}{\sqrt{2}}\Big\{\mathrm{e}^{i\nu_\pm}D_\pm + \mathrm{e}^{-i\nu_\pm}\overline{D}_\pm\Big\}\Big|_{\text{eq.(1)}} = -\frac{i}{\sqrt{2}}\frac{\partial}{\partial\theta_1^\pm} - \sqrt{2}\theta_1^\pm(\partial_0 \pm \partial_1) \ .
\end{aligned}$$

These operators obey the anti-commutation relations

$$\begin{aligned}
&\{Q_\pm^1, Q_\pm^1\} = -2i(\partial_0 \pm \partial_1) = 2(H \mp P) \ , \qquad \{Q_+^1, Q_-^1\} = 0 \ , \\
&\{D_\pm^1, D_\pm^1\} = +2i(\partial_0 \pm \partial_1) \ , \qquad \{D_+^1, D_-^1\} = 0 \ , \qquad \{Q_\alpha^1, D_\beta^1\} = 0 \ .
\end{aligned}$$

We also define the following "differential operators":

$$\begin{aligned}
\widetilde{Q}_\pm^1 &\equiv \frac{i}{\sqrt{2}}\Big\{\mathrm{e}^{i\nu_\pm}Q_\pm - \mathrm{e}^{-i\nu_\pm}\overline{Q}_\pm\Big\} \ , \qquad \text{under the constraint (1):} \quad \widetilde{Q}_\pm^1\Big| = 0 \ , \\
\widetilde{D}_\pm^1 &\equiv \frac{i}{\sqrt{2}}\Big\{\mathrm{e}^{i\nu_\pm}D_\pm - \mathrm{e}^{-i\nu_\pm}\overline{D}_\pm\Big\} \ , \qquad \text{under the constraint (1):} \quad \widetilde{D}_\pm^1\Big| = 0 \ .
\end{aligned}$$

Under the constraint (1) these operators are trivially zero. However, they are another two supercharges and two covariant derivatives in the original $\mathcal{N} = (2,2)$ supersymmetry. We can easily find this "second" $(1,1)$ supersymmetry operators satisfy the following anti-commutation relations in the $(2,2)$ supersymmetry level:

$$\begin{aligned} \{\widetilde{Q}^1_{\pm}, \widetilde{Q}^1_{\pm}\} &= -2i\big(\partial_0 \pm \partial_1\big) = 2(H \mp P) \, , \qquad \{\widetilde{Q}^1_{+}, \widetilde{Q}^1_{-}\} = 0 \, , \\ \{\widetilde{D}^1_{\pm}, \widetilde{D}^1_{\pm}\} &= +2i\big(\partial_0 \pm \partial_1\big) \, , \qquad \{\widetilde{D}^1_{+}, \widetilde{D}^1_{-}\} = 0 \, , \\ \{\widetilde{Q}^1_{\alpha}, \widetilde{D}^1_{\beta}\} &= 0 \, . \end{aligned}$$

Furthermore we can check that the first $(1,1)$ supersymmetry and the second $(1,1)$ supersymmetry commute with each other:

$$\{Q^1_{\alpha}, \widetilde{Q}^1_{\beta}\} = \{D^1_{\alpha}, \widetilde{D}^1_{\beta}\} = \{Q^1_{\alpha}, \widetilde{D}^1_{\beta}\} = \{D^1_{\alpha}, \widetilde{Q}^1_{\beta}\} = 0 \, .$$

This result is consistent with the original $\mathcal{N} = (2,2)$ supersymmetry.

$\mathcal{N} = (1,1)$ scalar/spinor superfields from $\mathcal{N} = (2,2)$ semi-chiral superfields

$$\begin{aligned}
\mathbb{X}^{(1,1)} &= \mathbb{X}^{(2,2)}\Big| \\
&= \phi + i\sqrt{2}\theta_1^+\widehat{\psi}_+ + i\sqrt{2}\theta_1^-(\widehat{\psi}_- + \widehat{\chi}_-) + 2i\theta_1^+\theta_1^-(\widehat{F} + \widehat{G}) \ , \\
\Psi_-^{(1,1)} &= \widetilde{Q}_-^1\mathbb{X}^{(2,2)}\Big| \\
&= i(\widehat{\psi}_- - \widehat{\chi}_-) - i\sqrt{2}\theta_1^+(\widehat{F} - \widehat{G}) + \sqrt{2}\theta_1^- A_= + 2\sqrt{2}\theta_1^+\theta_1^-\widehat{\zeta}_- \ ,
\end{aligned}$$

$$\begin{aligned}
\mathbb{Y}^{(1,1)} &= \mathbb{Y}^{(2,2)}\Big| \\
&= \phi + i\sqrt{2}\theta_1^+(\widehat{\psi}_+ + \widehat{\chi}_+) + i\sqrt{2}\theta_1^-\widehat{\psi}_- + 2i\theta_1^+\theta_1^-(\widehat{F} + \widehat{N}) \ , \\
\Upsilon_+^{(1,1)} &= \widetilde{Q}_+^1\mathbb{Y}^{(2,2)}\Big| \\
&= i(\widehat{\psi}_+ - \widehat{\chi}_+) + \sqrt{2}\theta_1^+ B_{\#} + i\sqrt{2}\theta_1^-(\widehat{F} - \widehat{N}) - 2\sqrt{2}\theta_1^+\theta_1^-\widehat{\zeta}_+ \ .
\end{aligned}$$