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Atiyah-Singer Index Theorem in Heterotic String Flux Compactifications

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- **What is heterotic string theory?**

$$\begin{cases} \text{contains only NS fields : } g_{MN}, B_{MN}, \Phi \\ \text{includes Yang-Mills gauge symmetry : } E_8 \times E_8 \text{ or } SO(32) \\ \text{realizes anomaly cancellation in a miraculous way} \end{cases}$$

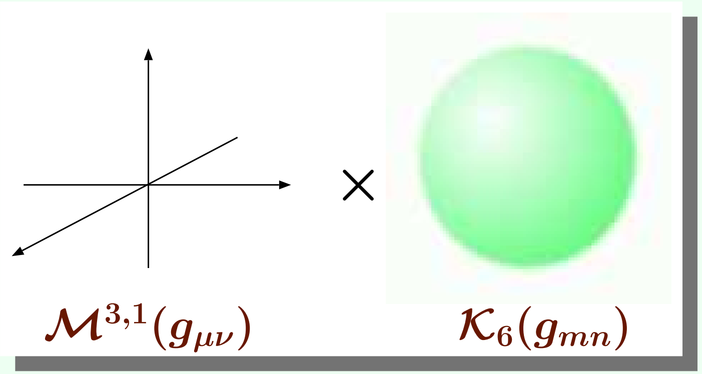
- **A good issue to understand string dualities at deeper levels**

duality: $\mathrm{II}/\mathrm{CY}_3 \leftrightarrow \mathrm{HE}/[\mathrm{K3} \times T^2]$

Towards 4-dim. Physics...

- **Vacuum configuration**
  - compactified geometry with $\boldsymbol{SU(3)}$-structure
- **Investigate low energy effective theory**
  - gauge symmetry breaking
  - evaluation of background fields
  - moduli, zero mode equations

Appearing spacetime geometry in Heterotic flux compactification



$$g_{MN}\,\mathrm{d}x^M \mathrm{d}x^N = \mathrm{e}^{(\Phi-\widehat{D})/2}\Big(g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn}\,\mathrm{d}y^m \mathrm{d}y^n\Big)$$

$$0 = R(g_{\mu\nu}) \qquad \mathcal{M}^{3,1} = \text{Minkowski} \ \& \ \Phi(y) = \widehat{D}(y)$$

$$0 = \mathrm{d}\big(\mathrm{e}^{-2\Phi} J \wedge J\big) \qquad \mathcal{K}_6 = \text{conformally balanced}$$

Furthermore, $\mathcal{K}_6$ can be reduced to...

if $\partial_m \Phi = 0$	:	$J \lrcorner \mathrm{d}J = 0$	balanced
if $H^0_{mnp} = 0,\ \partial_m \Phi \neq 0$	:	$\mathrm{d}\big(\mathrm{e}^{-\Phi} J\big) = 0$	conformally Kähler
if $\mathrm{d}H = 0$	:		strong Kähler with torsion
if $H_{mnp} = \partial_m \Phi = 0$	:		Calabi–Yau

Summary of my previous work

JHEP 0607 (2006) 030 [hep-th/0605247]

- Vacuum configuration in HETEROTIC STRING FLUX COMPACTIFICATIONS
- No-go theorem on a smooth manifold with $\boldsymbol{H} \neq 0$, $\mathrm{d}\boldsymbol{H} = 0$
- Possibility of smooth compactifications with $\boldsymbol{H} \neq 0$, $\underline{\mathrm{d}\boldsymbol{H} \neq 0}$
- Moduli space of conformally balanced manifolds (??)
- # of zero modes under the condition $\mathrm{d}\boldsymbol{H} \neq 0$ (??)

Necessary to investigate the torsionful geometry itself!

Moduli space of conformally balanced manifolds

M.Becker, L.-S.Tseng and S.-T.Yau [hep-th/0612290]

There are a number of interesting open questions.

- In our analysis we have kept for simplicity the complex structure fixed.
  It is well known that for Calabi-Yau compactifications the moduli space is a direct product of complex structure and Kähler structure deformations.
  For non-Kähler manifolds with torsion, this likely is not the case and it would be interesting to allow for a simultaneous variation of the complex structure and the hermitian form.
- It would be interesting to analyze the geometry of the moduli space and to determine if powerful tools such as the well known ‘‘special geometry’’ of Calabi-Yau compactifications can be derived in this case.
- Counting techniques for moduli fields need to be developed and we expect that the number of moduli can be characterized in terms of an index or some topological invariants of the manifold.

Index of Dirac operator

of massless modes on $\mathcal{M}^{3,1}$ ⟷ # of zero modes on $\mathcal{K}_6$

Eq. of motion for gaugino: $\left[\,\hat{\omega} \equiv \omega - \frac{1}{3}H\,\right]$

$$0 \;=\; \not{D}(\omega, A)\chi^0 - \frac{1}{12}H_{mnp}\,\Gamma^{mnp}\,\chi^0 \;\equiv\; \not{D}(\hat{\omega}, A)\chi^0$$

$$\text{index}\,\not{D}(\hat{\omega}, A) \;\equiv\; \dim(\ker D_L) - \dim(\ker D_R) \;=\; \text{Tr}\,\Gamma_{(5)}\text{e}^{-\beta\mathscr{R}}$$

$$\not{D}^2 \;=\; \frac{1}{\sqrt{g}}D_m(\omega_-, A)g^{mn}\sqrt{g}D_n(\omega_-, A) - \frac{1}{4}R(\omega) + \frac{1}{12}H_{mnp}H^{mnp} - \frac{1}{48}(\text{d}H)_{mnpq}\Gamma^{mnpq} + \frac{1}{2}F_{mn}\Gamma^{mn}$$

Witten Index

Dirac index on a geometry can be computed

as Witten index in SUSY quantum mechanics (in path integral formalism).

$$\Gamma^a \;\to\; \sqrt{\frac{2}{\hbar}}\,\psi^a(\tau)\,, \qquad -i\hbar\,\partial_m \;\to\; g^{\frac{1}{4}}\,p_m(\tau)\,g^{-\frac{1}{4}}\,, \qquad \not{D}^2 \;\to\; -\frac{2}{\hbar^2}\mathscr{H}$$

$$\mathrm{Tr}\left[\Gamma_{(5)}\mathrm{e}^{-\beta\mathscr{R}}\right] \;\to\; \mathrm{Tr}\left[(-)^{\mathcal{F}}\mathrm{e}^{-\beta\mathscr{H}}\right]$$

$\begin{cases} \mathcal{N}=1: & \text{Dirac genus } \widehat{\mathcal{A}}(R), \text{ Chern character } \mathrm{ch}(F) \\ \mathcal{N}=2: & \text{Euler characteristic } \chi(\mathcal{M}), \text{ Hirzebruch signature } \sigma(\mathcal{M}) \end{cases}$

Results

- **Riemann geometry** $(H = 0)$ *well-known*

- **Strong Kähler with torsion** $(\mathrm{d}H = 0)$ *improved* from the works by S.Yajima et. al.

$$\mathrm{index}\, \not{D}(\hat{\omega}, A) = \int \exp\left[\frac{1}{2}\mathrm{tr}\log\left(\frac{i\mathcal{R}/4\pi}{\sinh(i\mathcal{R}/4\pi)}\right)\right]\mathrm{Tr}_R \exp\left(\frac{i}{2\pi}F_2\right)$$

$$\mathrm{tr}(\mathcal{R}^k) = \mathcal{R}_{m_1 n_1}\mathcal{R}_{m_2 n_2}\cdots\mathcal{R}_{m_k n_k}\, g^{n_1 m_2} g^{n_2 m_3}\cdots g^{n_k m_1}$$

$$\mathcal{R}_{mn} = \frac{1}{2}\Big\{R_{mnab}(\omega) - 2\widetilde{D}_{[m}H_{n]ab}\Big\}e^a \wedge e^b \,, \qquad F_2 = \frac{1}{2}F_{ab}\, e^a \wedge e^b$$

- **Conformally balanced** $(H \neq 0,\ \mathrm{d}H \neq 0)$ ... *in progress*

  (cf.) 4-dim. manifold: K.Peeters and A.Waldron

Example: Dirac index in 4-dim. strong Kähler with torsion $(\mathrm{d}H = 0)$

$$\text{index}\, \not{D}(\hat{\omega}, A) = \frac{1}{192\pi^2}\int_{\mathcal{M}_4} \mathrm{tr}\Big\{R_2(\omega)\wedge R_2(\omega)\Big\} - \frac{1}{8\pi^2}\int_{\mathcal{M}_4} \mathrm{Tr}_R\Big\{F_2(A)\wedge F_2(A)\Big\}$$
$$-\frac{1}{8\pi^2}\int_{\mathcal{M}_4} \widetilde{F}_2(\widetilde{H})\wedge\widetilde{F}_2(\widetilde{H}) + \int_{\mathcal{M}_4}\Big(\text{total derivative terms}\Big)$$

where

$$H_{abc} \equiv -\sqrt{\frac{3}{2}}\,\mathcal{E}_{abcd}\,\widetilde{H}^d\,, \qquad \widetilde{F}_{ab}(\widetilde{H}) \equiv \partial_a\widetilde{H}_b - \partial_b\widetilde{H}_a$$

Summary

- Now is the time to investigate properties of a torsionful geometry
- Dirac index on the geometry via Witten index in SUSY quantum mechanics
- We find a remarkable result on 1st Pontrjagin class in 4-dim. strong Kähler with torsion
- We'll soon find new results in a generic conformally balanced manifold

    Modified Dirac genus, Euler characteristic, etc..