Index Theorems on Torsional Geometries

— 高次元時空の超弦理論から現実的な統一理論へ —

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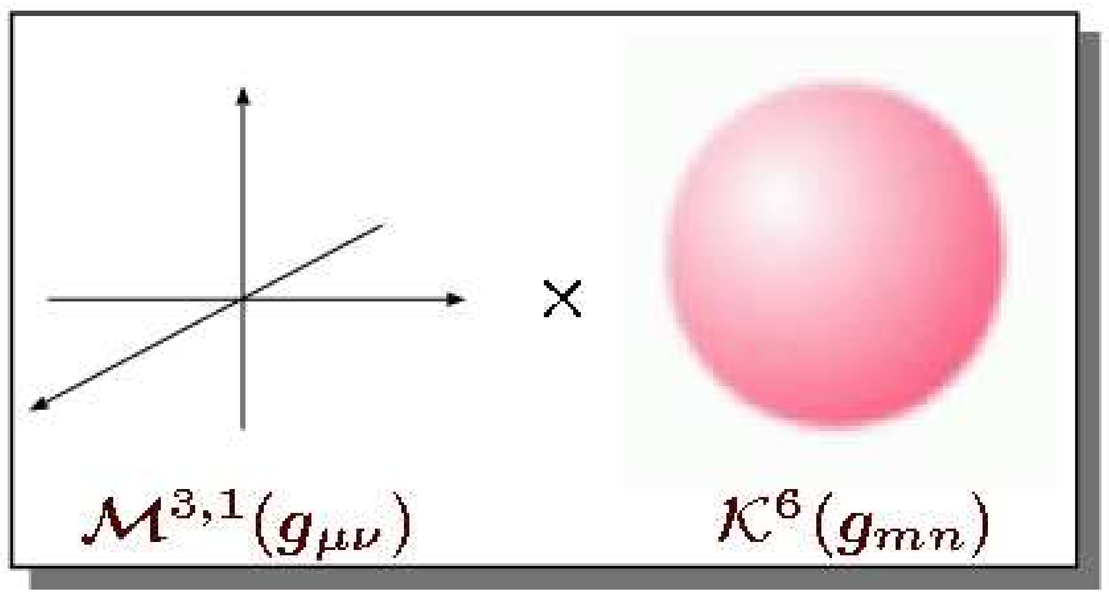
based on the work T. Kimura, "Index Theorems on Torsional Geometries," JHEP08(2007)048

Towards Realistic Physical Model from SUPERGRAVITY as effective theory of STRINGS

- Matters and their interactions
- Unified symmetry and its breaking
- Gravity, cosmological constants
- etc.



$$g^{\rm E}_{MN}\,{\rm d}x^M{\rm d}x^N = {\rm e}^{-\Phi(y)/2}\Big(g_{\mu\nu}\,{\rm d}x^\mu{\rm d}x^\nu + g_{mn}\,{\rm d}y^m{\rm d}y^n\Big)$$

$$0 = R(g_{\mu\nu}) \quad : \quad \mathcal{M}^{3,1} \rightsquigarrow \text{Minkowski}$$

$$0 = {\rm d}\big({\rm e}^{-2\Phi}J\wedge J\big) : \quad \mathcal{K}^6 \rightsquigarrow \text{conformally balanced with } H = T^{(\rm B)}$$

if ${\rm d}\Phi = 0$:	$J \lrcorner {\rm d}J = 0$	balanced manifold
if $H^0 = 0,\ {\rm d}\Phi \neq 0$:	${\rm d}({\rm e}^{-\Phi}J) = 0$	conformally Kähler manifold
if ${\rm d}H = 0$:		strong Kähler with torsion
if $H = {\rm d}\Phi = 0$:		Calabi–Yau manifold

# of massless modes on $\mathcal{M}^{3,1}\times\mathcal{K}^6$: $\Delta_{10}\,\mathcal{F}(x,y) = 0$ → # of massless modes on $\mathcal{M}^{3,1}$: $\Delta_4\,\Phi(x) = 0$ ↔ # of zero modes on $\mathcal{K}^6$: $\Delta_6\,\Psi(y) = 0$

運動方程式：

$$\begin{cases} \text{dilatino} \quad 0 = \not{D}(\omega)\lambda - \dfrac{1}{12}H_{MNP}\Gamma^{MNP}\lambda \\ \text{gaugino} \quad 0 = \not{D}(\omega, A)\chi - \dfrac{1}{12}H_{MNP}\Gamma^{MNP}\chi \end{cases}$$

[ゼロモードの数？
４次元有効理論に登場する世代数及びゲージ対称性？]

SUGRA vs SUSY QM

様々な指数定理　（参照： Green-Schwarz-Witten 14 章）

- **Dirac index** by $\not{D}(\omega) : C^\infty(S_+) \to C^\infty(S_-)$

  index $\not{D}(\omega) \equiv$ (# of left chiral) − (# of right chiral)

  $\longrightarrow \mathcal{N}=1$ QM : $\not{D}(\omega) \sim Q^1$

- **Euler characteristic** by ${\rm d}+{\rm d}^\dagger : C^\infty(\Lambda^{\rm even}) \to C^\infty(\Lambda^{\rm odd})$

  $\chi(\mathcal{M}) \equiv$ (# of harm. even-forms) − (# of harm. odd-forms)

  $\longrightarrow \mathcal{N}=2$ QM : ${\rm d}\sim Q,\ {\rm d}^\dagger \sim \overline{Q}$

- **Hirzebruch signature** by ${\rm d}+{\rm d}^\dagger : C^\infty(\Lambda^{\rm SD}) \to C^\infty(\Lambda^{\rm ASD})$

  $\sigma(\mathcal{M}) \equiv$ (# of self-dual forms) − (# of anti-self dual forms)

  $\longrightarrow \mathcal{N}=2$ QM : ${\rm d}\sim Q,\ {\rm d}^\dagger \sim \overline{Q}$

$\mathcal{N}=1$ SUSY Quantum Mechanics

$$\{\psi^A, \psi^B\} = \hbar\,\delta^{AB}$$

$$Q^1 = \psi^M g^{\frac14}\pi_M g^{-\frac14} = \psi^M g^{\frac14}\Big(p_M - \frac{i}{2}\omega_{MAB}\,\psi^{AB}\Big)g^{-\frac14}$$

$$\{Q^1, Q^1\} = 2\hbar\mathscr{H}^1\,, \quad \mathscr{H}^1 = \frac12 g^{-\frac14}\pi_M g^{MN}\sqrt{g}\,\pi_N\, g^{-\frac14} + \frac{\hbar^2}{8}R(\omega)$$

$\mathcal{N}=2$ SUSY Quantum Mechanics

$$\{\varphi^A, \overline{\varphi}^B\} = \hbar\,\delta^{AB}\,, \quad \{\varphi^A, \varphi^B\} = 0$$

$$Q = \varphi^M g^{\frac14}\pi_M g^{-\frac14} = \varphi^M g^{\frac14}\Big(p_M - i\omega_{MAB}\,\varphi^A\overline{\varphi}^B\Big)g^{-\frac14}$$

$$\{Q, Q\} = 0\,, \quad \{Q, \overline{Q}\} = 2\hbar\mathscr{H}$$

$$\mathscr{H} = \frac12 g^{-\frac14}\pi_M g^{MN}\sqrt{g}\,\pi_N\, g^{-\frac14} - \frac12 R_{ABMN}(\omega)\,\varphi^M\overline{\varphi}^N\varphi^A\overline{\varphi}^B$$

$$\text{index}\,\not{D}(\omega) = \lim_{\beta\to0}{\rm Tr}\big\{\Gamma_{(5)}{\rm e}^{-\beta\mathscr{R}}\big\} = \lim_{\beta\to0}\frac{(-i)^{D/2}}{2^{D/2}}{\rm Tr}\prod_{A=1}^{D}\big(\varphi^A+\overline{\varphi}^A\big)\exp\Big(-\frac{\beta}{\hbar}\mathscr{H}^1\Big)$$

$$\chi(\mathcal{M}) = \lim_{\beta\to0}{\rm Tr}\big\{\Gamma_{(5)}\widetilde{\Gamma}_{(5)}{\rm e}^{-\beta\mathscr{R}}\big\} = \lim_{\beta\to0}{\rm Tr}\prod_{A=1}^{D}\big(\varphi^A+\overline{\varphi}^A\big)\prod_{B=1}^{D}\big(\varphi^B-\overline{\varphi}^B\big)\exp\Big(-\frac{\beta}{\hbar}\mathscr{H}\Big)$$

$$\sigma(\mathcal{M}) = \lim_{\beta\to0}{\rm Tr}\big\{\Gamma_{(5)}{\rm e}^{-\beta\mathscr{R}}\big\} = \lim_{\beta\to0}(-i)^{D/2}{\rm Tr}\prod_{A=1}^{D}\big(\varphi^A+\overline{\varphi}^A\big)\exp\Big(-\frac{\beta}{\hbar}\mathscr{H}\Big)$$

Dirac index

$$\text{index}\,\not{D}(\widehat{\omega}, A) = \int_{\mathcal{M}}\exp\left[\frac12\,{\rm tr}\log\left(\frac{iR(\omega_+)/4\pi}{\sinh(iR(\omega_+)/4\pi)}\right)\right]{\rm Tr}_R\exp\Big(\frac{i}{2\pi}F\Big)$$

Euler charcteristic

$$\chi(\mathcal{M}) = \frac{1}{(4\pi)^n n!}\,\mathcal{E}_{A_1\cdots A_{2n}}\int_{\mathcal{M}} R^{A_1A_2}(\omega)\wedge\cdots\wedge R^{A_{2n-1}A_{2n}}(\omega)$$

Hirzebruch signature

$$\sigma(\mathcal{M}) = \int_{\mathcal{M}}\exp\left[\frac12\,{\rm tr}\log\left(\frac{iR(\omega_+)/2\pi}{\tanh(iR(\omega_+)/2\pi)}\right)\right]$$