Realization of AdS Vacua in Attractor Mechanism on Generalized Geometries

arXiv:0810.0937 [hep-th]

Tetsuji KIMURA

Yukawa Institute for Theoretical Physics, Kyoto University

We are looking for the origin of 4D physics

Physical information

- particle contents and spectra
- (broken) symmetries and interactions
- potential, vacuum and cosmological constant

4D $\mathcal{N} = 1$ supergravity:

$$S = \int \Big(\frac{1}{2} R * \mathbf{1} - \frac{1}{2} F^a \wedge * F^a - K_{\mathcal{M}\overline{\mathcal{N}}} \nabla\phi^{\mathcal{M}} \wedge * \nabla\overline{\phi}^{\overline{\mathcal{N}}} - V * \mathbf{1} \Big)$$

$$V = \mathrm{e}^K \Big(K^{\mathcal{M}\overline{\mathcal{N}}} D_{\mathcal{M}} \mathcal{W} \, \overline{D_{\mathcal{N}} \mathcal{W}} - 3 |\mathcal{W}|^2 \Big) + \frac{1}{2} |D^a|^2$$

K :	Kähler potential		
$\mathcal{W}$:	superpotential	←--	$\delta\psi_\mu = \nabla_\mu \varepsilon - \mathrm{e}^{\frac{K}{2}} \mathcal{W} \gamma_\mu \varepsilon^c$
D^a :	D-term	←--	$\delta\chi^a = \mathrm{Im} F^a_{\mu\nu} \gamma^{\mu\nu} \varepsilon + \mathrm{i} D^a \varepsilon$

Search of vacua $\partial_{\mathcal{P}} V \big|_* = 0$

$V_* > 0$:	de Sitter space
$V_* = 0$:	Minkowski space
$V_* < 0$:	Anti-de Sitter space

10D string theories could provide information

via compactifications

$$10 = 4 + 6$$

A typical success:

$E_8 \times E_8$ heterotic string compactified on Calabi-Yau three-fold

- number of generations $= |\chi(\mathrm{CY}_3)|/2$
- E_6 gauge symmetry
- zero cosmological constant

However, Calabi-Yau manifold is not sufficient --→ Fluxes are highly restricted

common	H_3, $\nabla\phi$, torsion
type IIA	F_0, F_2, F_4, F_6
type IIB	F_1, F_3, F_5

Decompose 10D type IIA SUSY parameters:

$$\epsilon^1 \;=\; \varepsilon_1 \otimes (a\,\eta^1_+) + \varepsilon^c_1 \otimes (\overline{a}\,\eta^1_-)\,, \qquad \epsilon^2 \;=\; \varepsilon_2 \otimes (\overline{b}\,\eta^2_-) + \varepsilon^c_2 \otimes (b\,\eta^2_+)$$

$\delta\psi^{\mathcal{A}}_m = 0$ gives the Killing spinor equation on the 6D compactified space $\mathcal{M}$:

$$\delta\psi^{\mathcal{A}}_m \;=\; \Big(\partial_m + \frac{1}{4}\omega_{mab}\,\gamma^{ab}\Big)\eta^{\mathcal{A}}_+ + (\text{3-form fluxes}\cdot\eta)^{\mathcal{A}} + (\text{other fluxes}\cdot\eta)^{\mathcal{A}} \;=\; 0$$

Information of

6D $SU(3)$ Killing spinors η^1_+, η^2_+:

Calabi-Yau three-fold

↓

$SU(3)$-structure manifold with torsion

↓

generalized geometry

- Calabi-Yau three-folds $\dashrightarrow$ Fluxes are highly restricted

$$\begin{cases} \text{type IIA}: & \text{No fluxes} \\ \text{type IIB}: & F_3 - \tau H \qquad \text{(warped Calabi-Yau)} \\ \text{heterotic}: & \text{No fluxes} \end{cases}$$

- Calabi-Yau three-folds ⇢ Fluxes are highly restricted

$$\begin{cases} \text{type IIA}: & \text{No fluxes} \\ \text{type IIB}: & F_3 - \tau H \qquad \text{(warped Calabi-Yau)} \\ \text{heterotic}: & \text{No fluxes} \end{cases}$$

- $SU(3)$-structure manifolds ⇢ Some components of fluxes can be interpreted as torsion

$$\left.\begin{array}{r} \text{type IIA} \\ \text{type IIB} \\ \text{heterotic}^{1} \end{array}\right\} \qquad \text{restricted fluxes are turned on}^{2}$$

1: Piljin Yi, TK "*Comments on heterotic flux compactifications*," JHEP 0607 (2006) 030, hep-th/0605247

2: TK "*Index theorems on torsional geometries*," JHEP 0708 (2007) 048, arXiv:0704.2111

- Calabi-Yau three-folds --→ Fluxes are highly restricted

$$\begin{cases} \text{type IIA}: & \text{No fluxes} \\ \text{type IIB}: & F_3 - \tau H \qquad \text{(warped Calabi-Yau)} \\ \text{heterotic}: & \text{No fluxes} \end{cases}$$

- $SU(3)$-structure manifolds --→ Some components of fluxes can be interpreted as torsion

$$\left.\begin{array}{r} \text{type IIA} \\ \text{type IIB} \\ \text{heterotic}^{1} \end{array}\right\} \quad \text{restricted fluxes are turned on}^{2}$$

1: Piljin Yi, TK "*Comments on heterotic flux compactifications*," JHEP 0607 (2006) 030, hep-th/0605247

2: TK "*Index theorems on torsional geometries*," JHEP 0708 (2007) 048, arXiv:0704.2111

- Generalized geometries --→ Any types of fluxes can be included

All $\mathcal{N} = 1$ SUSY solutions could be classified

Search 4D SUSY vacua in type IIA theory compactified on generalized geometries

- Moduli stabilization

  We find SUSY AdS (or Minkowski) vacua

- Mathematical feature

  We obtain a powerful rule to evaluate vacua:

  Discriminant of the superpotential governs the cosmological constant

- Stringy effects

  We see that α' corrections are included in certain configurations

$\mathcal{N} = 1$ scalar potential from generalized geometry

Consider a 6D compactified space $\mathcal{M}$

- Ordinary complex structure $J^m{}_n$ in $T\mathcal{M}$ is given by $SU(3)$ invariant Weyl spinor η_+:

$$J^2 \;=\; -\mathbf{1}_6\,, \qquad J^m{}_n \;=\; -2\mathrm{i}\,\eta_+^\dagger\,\gamma^m{}_n\,\eta_+$$

- Generalized complex structures $\mathcal{J}^\Lambda{}_\Sigma$ in $T\mathcal{M} \oplus T^*\mathcal{M}$: basis $\{\mathrm{d}x^m\wedge, \iota_{\partial_n}\}$, $(6,6)$-signature

$$\mathcal{J}_\pm^2 = -\mathbf{1}_{12}\,, \qquad \mathcal{J}_\pm^\Lambda{}_\Sigma = \big\langle \mathrm{Re}\,\Phi_\pm, \Gamma^\Lambda{}_\Sigma\,\mathrm{Re}\,\Phi_\pm \big\rangle$$

$\Phi_\pm$: $SU(3,3)$ invariant Weyl spinors, *isomorphic* to even/odd-forms on $T^*\mathcal{M}$

Γ^Λ : Cliff$(6,6)$ gamma matrix (repr. $= (\mathrm{d}x^m\wedge\,,\,\iota_{\partial_n})$)

Mukai pairing:

even forms: $\big\langle \Psi_+, \Phi_+ \big\rangle = \Psi_6 \wedge \Phi_0 - \Psi_4 \wedge \Phi_2 + \Psi_2 \wedge \Phi_4 - \Psi_0 \wedge \Phi_6$

odd forms: $\big\langle \Psi_-, \Phi_- \big\rangle = \Psi_5 \wedge \Phi_1 - \Psi_3 \wedge \Phi_3 + \Psi_1 \wedge \Phi_5$

Decompose 10D type IIA SUSY parameters:

$$\epsilon^1 \;=\; \varepsilon_1 \otimes (a\,\eta^1_+) + \varepsilon^c_1 \otimes (\overline{a}\,\eta^1_-)\,, \qquad \epsilon^2 \;=\; \varepsilon_2 \otimes (\overline{b}\,\eta^2_-) + \varepsilon^c_2 \otimes (b\,\eta^2_+)$$

⇢ Two $SU(3)$ invariant Weyl spinors on $\mathcal{M}$: η^1_+, η^2_+ with

$$\eta^2_+ \;=\; c_{\|}(y)\eta^1_+ + c_{\perp}(y)(v + \mathrm{i}v')^m\,\gamma_m\,\eta^1_-\,, \qquad (v - \mathrm{i}v')^m \;=\; \eta^{1\dagger}_+\,\gamma^m\,\eta^2_-$$

Decompose 10D type IIA SUSY parameters:

$$\epsilon^1 = \varepsilon_1 \otimes (a\, \eta^1_+) + \varepsilon_1^c \otimes (\overline{a}\, \eta^1_-)\,, \qquad \epsilon^2 = \varepsilon_2 \otimes (\overline{b}\, \eta^2_-) + \varepsilon_2^c \otimes (b\, \eta^2_+)$$

⇢ Two $SU(3)$ invariant Weyl spinors on $\mathcal{M}$: η^1_+, η^2_+ with

$$\eta^2_+ = c_{\parallel}(y)\eta^1_+ + c_{\perp}(y)(v + \mathrm{i}v')^m \gamma_m\, \eta^1_-\,, \qquad (v - \mathrm{i}v')^m = \eta^{1\dagger}_+ \gamma^m\, \eta^2_-$$

▶ On the $SU(3)$ generalized geometry ($\eta^1_+ = \eta^2_+$):

$$\Phi_+ = \mathrm{e}^{-B-\mathrm{i}J}\,, \qquad \Phi_- = \mathrm{e}^{-B}\Omega$$

$$J_{mn} = -2\mathrm{i}\, \eta^\dagger_+ \gamma_{mn}\, \eta_+\,, \qquad \Omega_{mnp} = -2\mathrm{i}\, \eta^\dagger_- \gamma_{mnp}\, \eta_+$$

▶ On the $SU(3) \times SU(3)$ generalized geometry ($\eta^1_+ \neq \eta^2_+$ at some points y):

$$\Phi_+ = \mathrm{e}^{-B}(\overline{c}_{\parallel}\mathrm{e}^{-\mathrm{i}j} - \mathrm{i}\overline{c}_{\perp}w) \wedge \mathrm{e}^{-\mathrm{i}v\wedge v'}\,, \qquad \Phi_- = \mathrm{e}^{-B}(c_{\parallel}\mathrm{e}^{-\mathrm{i}j} + \mathrm{i}c_{\perp}w) \wedge (v + \mathrm{i}v')$$

$$J^{\mathcal{A}} = j \pm v \wedge v'\,, \qquad \Omega^{\mathcal{A}} = w \wedge (v \pm \mathrm{i}v')$$

Moduli spaces of $\mathcal{M}$ are the special Kähler geometries of local type

Kähler potentials, prepotentials, projective coordinates

$$
\begin{aligned}
K_+ &= -\log \mathrm{i} \int_{\mathcal{M}} \langle \Phi_+, \overline{\Phi}_+ \rangle = -\log \mathrm{i} \big(\overline{X}^A \mathcal{F}_A - X^A \overline{\mathcal{F}}_A \big) \\
K_- &= -\log \mathrm{i} \int_{\mathcal{M}} \langle \Phi_-, \overline{\Phi}_- \rangle = -\log \mathrm{i} \big(\overline{Z}^I \mathcal{G}_I - Z^I \overline{\mathcal{G}}_I \big)
\end{aligned}
$$

Expand the even/odd-forms $\Phi_\pm$ by the basis forms:

$$\Phi_+ = X^A \omega_A - \mathcal{F}_A \widetilde{\omega}^A , \qquad \omega_A = (1, \omega_a) , \qquad \widetilde{\omega}^A = (\widetilde{\omega}^a, \mathrm{vol}(\mathcal{M})) \qquad : \quad \text{0,2,4,6-forms}$$

$$\Phi_- = Z^I \alpha_I - \mathcal{G}_I \beta^I , \qquad \alpha_I = (\alpha_0, \alpha_i) , \qquad \beta^I = (\beta^i, \beta^0) \qquad : \quad \text{1,3,5-forms}$$

$\mathcal{N}=1$ Kähler potential and dilaton are given as

$$K = K_+ + 4\varphi$$

$$e^{-2\varphi} = \frac{|\mathcal{C}|^2}{16|a|^2|b|^2} e^{-K_-} = \frac{i}{16|a|^2|b|^2} \int_{\mathcal{M}} \langle \mathcal{C}\Phi_-, \overline{\mathcal{C}\Phi_-} \rangle$$

$\mathcal{N}=1$ SUSY variations yield superpotential and D-term:

$$\delta\psi_\mu = \nabla_\mu \varepsilon - e^{\frac{K}{2}} \mathcal{W} \gamma_\mu \varepsilon^c$$

$$\delta\chi^A = \mathrm{Im} F^A_{\mu\nu} \gamma^{\mu\nu} \varepsilon + i D^A \varepsilon$$

$$\mathcal{W} = \frac{i}{4\overline{a}b} \Big[4i\, e^{\frac{K_-}{2} - \varphi} \int_{\mathcal{M}} \langle \Phi_+, \mathcal{D} \mathrm{Im}(ab\Phi_-) \rangle + \int_{\mathcal{M}} \langle \Phi_+, G \rangle \Big]$$

$$D^A = 2\, e^{K_+} (K_+)^{c\overline{d}} D_c X^A \overline{D_d X^B} \big[\overline{n}^{\mathcal{C}} (\sigma_x)_{\mathcal{C}}{}^{\mathcal{B}} n_{\mathcal{B}} \big] \Big(\mathcal{P}^x_B - \mathcal{N}_{BC} \widetilde{\mathcal{P}}^{xC} \Big)$$

On the $SU(3)$ geometry $(\eta^1_+ = \eta^2_+)$:

$$
\begin{aligned}
\mathrm{d}_H \omega_A &= m_A{}^I \alpha_I - e_{IA} \beta^I & \qquad \mathrm{d}_H \widetilde{\omega}^A &= 0 \\
\mathrm{d}_H \alpha_I &= e_{IA} \widetilde{\omega}^A & \qquad \mathrm{d}_H \beta^I &= m_A{}^I \widetilde{\omega}^A
\end{aligned}
$$

NS three-form H deforms the differential operator:

$$
\mathrm{d}H = 0 \,, \qquad H = H^{\mathsf{fl}} + \mathrm{d}B \,, \qquad H^{\mathsf{fl}} = m_0{}^I \alpha_I - e_{I0} \beta^I
$$

$$
\mathrm{d}_H \equiv \mathrm{d} - H^{\mathsf{fl}} \wedge
$$

background	charges	
NS-flux charges	e_{I0}	$m_0{}^I$
torsion	e_{Ia}	$m_a{}^I$

On the $SU(3) \times SU(3)$ geometry ($\eta_+^1 \neq \eta_+^2$ at some points):

Extend to the generalized differential operator $\mathcal{D}$

$$\mathrm{d}_H = \mathrm{d} - H^{\mathsf{fl}} \wedge \quad \to \quad \mathcal{D} \equiv \mathrm{d} - H^{\mathsf{fl}} \wedge - Q \cdot - R \llcorner$$

$$\mathcal{D}\omega_A \sim m_A{}^I \alpha_I - e_{IA} \beta^I \qquad \mathcal{D}\widetilde{\omega}^A \sim -q^{IA} \alpha_I + p_I{}^A \beta^I$$

$$\mathcal{D}\alpha_I \sim p_I{}^A \omega_A + e_{IA} \widetilde{\omega}^A \qquad \mathcal{D}\beta^I \sim q^{IA} \omega_A + m_A{}^I \widetilde{\omega}^A$$

Necessary to introduce new fluxes Q and R to make a consistent algebra...

On the $SU(3) \times SU(3)$ geometry ($\eta_+^1 \neq \eta_+^2$ at some points):

Extend to the generalized differential operator $\mathcal{D}$

$$\mathrm{d}_H = \mathrm{d} - H^{\mathrm{fl}} \wedge \quad \to \quad \mathcal{D} \equiv \mathrm{d} - H^{\mathrm{fl}} \wedge - Q \cdot - R \llcorner$$

$$\mathcal{D}\omega_A \sim m_A{}^I \alpha_I - e_{IA} \beta^I \qquad \mathcal{D}\widetilde{\omega}^A \sim -q^{IA} \alpha_I + p_I{}^A \beta^I$$

$$\mathcal{D}\alpha_I \sim p_I{}^A \omega_A + e_{IA} \widetilde{\omega}^A \qquad \mathcal{D}\beta^I \sim q^{IA} \omega_A + m_A{}^I \widetilde{\omega}^A$$

Necessary to introduce new fluxes Q and R to make a consistent algebra...

But the compactified geometry becomes nongeometric:

$(Q \cdot C)_{m_1 \cdots m_{k-1}}$	$\equiv$	$Q^{ab}{}_{[m_1} C_{\lvert ab \rvert m_2 \cdots m_{k-1}]}$	feature of T-fold
$(R \llcorner C)_{m_1 \cdots m_{k-3}}$	$\equiv$	$R^{abc} C_{abc m_1 \cdots m_{k-3}}$	locally nongeometric background

Structure group contains Diffeo. + Duality trsf. $\dashrightarrow$ *Doubled formalism*[3]

3: C. Albertsson, R.A. Reid-Edwards, TK "*D-branes and doubled geometry*," arXiv:0806.1783

RR-fluxes on the $SU(3) \times SU(3)$ geometry:

$$G = G^{\text{fl}} + \mathcal{D}A \,, \qquad \mathcal{D}G = 0$$

$$G^{\text{fl}} = m_{\text{RR}}^{A}\,\omega_A - e_{\text{RR}A}\,\widetilde{\omega}^A \,, \qquad A = \xi^I\,\alpha_I - \widetilde{\xi}_I\,\beta^I$$

$$\Downarrow$$

$$G \sim G^A\,\omega_A - \widetilde{G}_A\,\widetilde{\omega}^A$$

$$G^A \sim m_{\text{RR}}^{A} + \xi^I\,p_I{}^A - \widetilde{\xi}_I\,q^{IA} \,, \qquad \widetilde{G}_A \sim e_{\text{RR}A} - \xi^I\,e_{IA} + \widetilde{\xi}_I\,m_A{}^I$$

fluxes	charges	
NS three-form H	e_{I0}	${m_0}^I$
torsion	e_{Ia}	${m_a}^I$
nongeometric fluxes	${p_I}^A$	q^{IA}
RR-fluxes	$e_{\mathrm{RR}A}$	m_{RR}^A

backgrounds	flux charges			
Calabi-Yau	—			
Calabi-Yau with H	e_{I0}	${m_0}^I$		
$SU(3)$ geometry	e_{IA}	${m_A}^I$		
$SU(3) \times SU(3)$ geometry	e_{IA}	${m_A}^I$	${p_I}^A$	q^{IA}

Note: $SU(3)$ generalized geometry without RR-fluxes $\sim SU(3)$-structure manifold

We are ready to search SUSY vacua in 4D $\mathcal{N} = 1$ supergravity.

Consider two typical situations given by

- generalized geometry with RR-flux charges

  $e_{IA},\ m_A{}^I,\ p_I{}^A,\ q^{IA},\ e_{\text{RR}A},\ m_{\text{RR}}^A$

- $SU(3)$-structure manifold

  $e_{IA},\ m_A{}^I$

Search of SUSY vacua: flux vacua attractors

$$V = e^K \Big(K^{\mathcal{M}\overline{\mathcal{N}}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\mathcal{N}} \mathcal{W}} - 3|\mathcal{W}|^2 \Big) + \frac{1}{2} |D^a|^2$$

$$\equiv V_{\mathcal{W}} + V_D$$

Search of vacua $\partial_{\mathcal{P}} V \Big|_* = 0$

$V_* > 0$:	de Sitter space
$V_* = 0$:	Minkowski space
$V_* < 0$:	Anti-de Sitter space

$$0 = \partial_{\mathcal{P}} V_{\mathcal{W}} = e^K \Big\{ K^{\mathcal{M}\overline{\mathcal{N}}} D_{\mathcal{P}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\mathcal{N}} \mathcal{W}} + \partial_{\mathcal{P}} K^{\mathcal{M}\overline{\mathcal{N}}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\mathcal{N}} \mathcal{W}} - 2 \overline{\mathcal{W}} D_{\mathcal{P}} \mathcal{W} \Big\}$$

$$0 = \partial_{\mathcal{P}} V_D \quad \dashrightarrow \quad D^a = 0$$

Consider the SUSY condition $D_{\mathcal{P}} \mathcal{W} \equiv \big(\partial_{\mathcal{P}} + \partial_{\mathcal{P}} K \big) \mathcal{W} = 0$ in various cases.

1. *Set a simple prepotential:* $\mathcal{F} = D_{abc} \dfrac{X^a X^b X^c}{X^0}$

2. *Consider the simplest model: single modulus t of Φ_+ (and U of Φ_-)*

The superpotential is reduced to

$$\begin{aligned} \mathcal{W} &= \mathcal{W}^{\text{RR}} + U\, \mathcal{W}^{\mathbb{Q}} \\ \mathcal{W}^{\text{RR}} &= m_{\text{RR}}^0\, t^3 - 3\, m_{\text{RR}}\, t^2 + e_{\text{RR}}\, t + e_{\text{RR}0} \\ \mathcal{W}^{\mathbb{Q}} &= {p_0}^0\, t^3 - 3\, p_0\, t^2 - e_0\, t - e_{00} \end{aligned}$$

Consider the SUSY condition:

$$\begin{aligned} D_t \mathcal{W} = 0 \;\dashrightarrow\; 0 &= D_t \mathcal{W}^{\text{RR}} + U\, D_t \mathcal{W}^{\mathbb{Q}} \\ D_U \mathcal{W} = 0 \;\dashrightarrow\; 0 &= \frac{\mathrm{i}}{\mathrm{Im} U} \Big(\mathcal{W}^{\text{RR}} + \mathrm{Re} U\, \mathcal{W}^{\mathbb{Q}} \Big) \end{aligned}$$

The discriminant of the superpotential $\mathcal{W}^{\text{RR}}$ (and $\mathcal{W}^{\mathbb{Q}}$) governs the SUSY solutions.

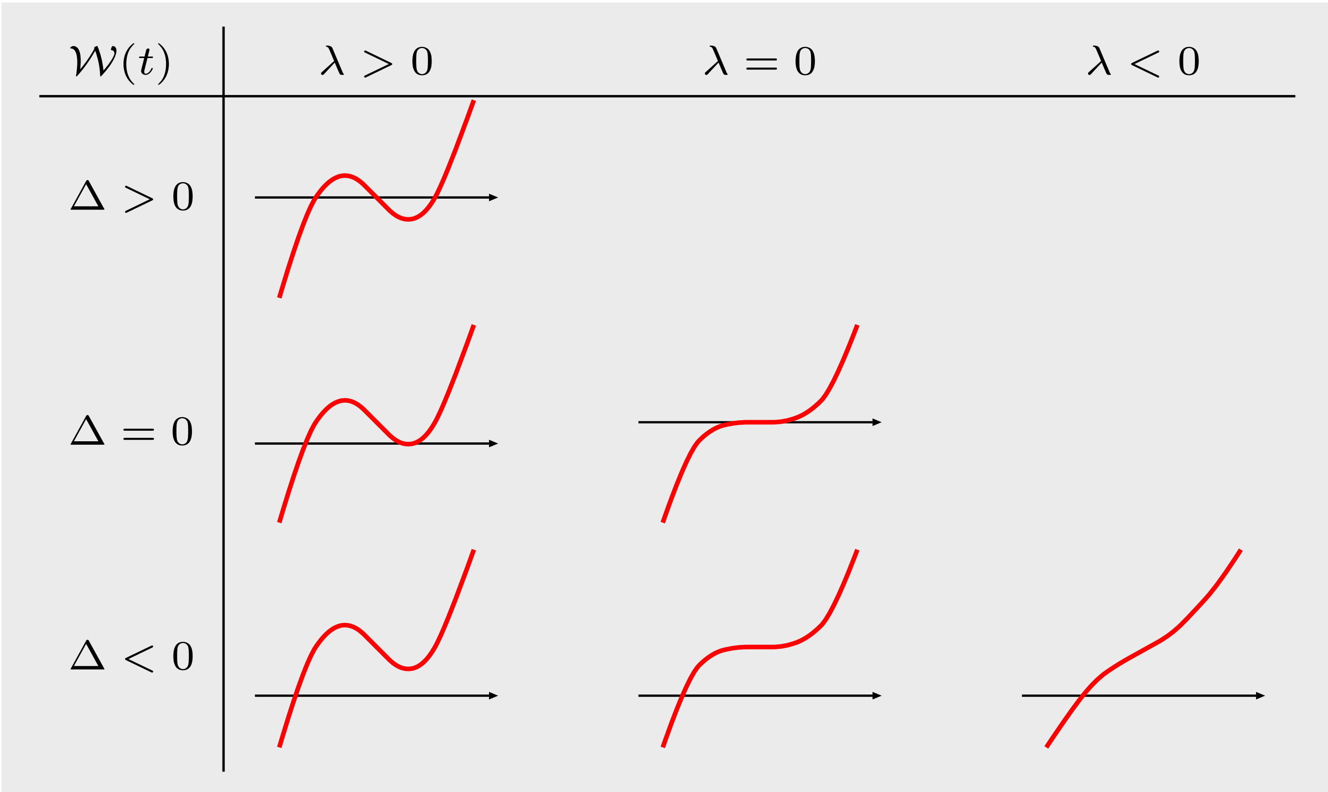
▶ *Discriminant of cubic equation*

Consider a cubic function and its derivative:
$$\begin{cases} \mathcal{W}(t) = a\,t^3 + b\,t^2 + c\,t + d \\ \partial_t \mathcal{W}(t) = 3a\,t^2 + 2b\,t + c \end{cases}$$

Discriminants $\Delta(\mathcal{W})$ *and* $\Delta(\partial_t \mathcal{W})$ *are*

$$\Delta(\mathcal{W}) \equiv \Delta = -4b^3 d + b^2 c^2 - 4ac^3 + 18abcd - 27a^2 d^2$$

$$\Delta(\partial_t \mathcal{W}) \equiv \lambda = 4(b^2 - 3ac)$$



$\Delta^{\mathsf{RR}} > 0$ case: always $\lambda^{\mathsf{RR}} > 0$, and exists a zero point: $D_t \mathcal{W}^{\mathsf{RR}} = 0$

$$\begin{aligned} D_t \mathcal{W}^{\mathsf{RR}}|_* &= 0 \\ t_*^{\mathsf{RR}} &= \frac{6\,(3\,m_{\mathsf{RR}}^0\,e_{\mathsf{RR}0} + m_{\mathsf{RR}}\,e_{\mathsf{RR}})}{\lambda^{\mathsf{RR}}} - 2\mathrm{i}\,\frac{\sqrt{3\,\Delta^{\mathsf{RR}}}}{\lambda^{\mathsf{RR}}} \\ \mathcal{W}_*^{\mathsf{RR}} &= -\frac{24\,\Delta^{\mathsf{RR}}}{(\lambda^{\mathsf{RR}})^3}\Big(36\,(m_{\mathsf{RR}})^3 + 36\,(m_{\mathsf{RR}}^0)^2 e_{\mathsf{RR}0} - 3\,m_{\mathsf{RR}}\lambda^{\mathsf{RR}} - 4\mathrm{i}\,m_{\mathsf{RR}}^0\sqrt{3\,\Delta^{\mathsf{RR}}}\Big) \end{aligned}$$

$\Delta^{\mathsf{RR}} > 0$ case: always $\lambda^{\mathsf{RR}} > 0$, and exists a zero point: $D_t\mathcal{W}^{\mathsf{RR}} = 0$

$$\begin{aligned} D_t\mathcal{W}^{\mathsf{RR}}|_* &= 0 \\ t_*^{\mathsf{RR}} &= \frac{6\,(3\,m^0_{\mathsf{RR}}\,e_{\mathsf{RR}0} + m_{\mathsf{RR}}\,e_{\mathsf{RR}})}{\lambda^{\mathsf{RR}}} - 2\mathrm{i}\,\frac{\sqrt{3\,\Delta^{\mathsf{RR}}}}{\lambda^{\mathsf{RR}}} \\ \mathcal{W}_*^{\mathsf{RR}} &= -\frac{24\,\Delta^{\mathsf{RR}}}{(\lambda^{\mathsf{RR}})^3}\Big(36\,(m_{\mathsf{RR}})^3 + 36\,(m^0_{\mathsf{RR}})^2 e_{\mathsf{RR}0} - 3\,m_{\mathsf{RR}}\lambda^{\mathsf{RR}} - 4\mathrm{i}\,m^0_{\mathsf{RR}}\sqrt{3\,\Delta^{\mathsf{RR}}}\Big) \end{aligned}$$

$\Delta^{\mathsf{RR}} < 0$ case: only $\lambda^{\mathsf{RR}} < 0$ is physically allowed, and exists a zero point: $\mathcal{W}^{\mathsf{RR}} = 0$

$$\begin{aligned} \mathcal{W}_*^{\mathsf{RR}} &= m^0_{\mathsf{RR}}(t_* - e)(t_* - \alpha)(t_* - \overline{\alpha}) \;=\; 0\,, \qquad t_* \;=\; \alpha^{\mathsf{RR}} \;=\; \alpha_1 + \mathrm{i}\,\alpha_2 \\ \alpha_1 &= \frac{\lambda^{\mathsf{RR}} + F^{2/3} + 12\,m_{\mathsf{RR}}\,F^{1/3}}{12\,m^0_{\mathsf{RR}}\,F^{1/3}} \\ (\alpha_2)^2 &= \frac{1}{m^0_{\mathsf{RR}}}\Big(e_{\mathsf{RR}} - 6\,m_{\mathsf{RR}}\,\alpha_1 + 3\,m^0_{\mathsf{RR}}\,(\alpha_1)^2\Big) \\ e &= -\frac{1}{m^0_{\mathsf{RR}}}\Big(-3\,m_{\mathsf{RR}} + 2\,m^0_{\mathsf{RR}}\,\alpha_1\Big) \\ F &= 108\,(m^0_{\mathsf{RR}})^2 e_{\mathsf{RR}0} + 12\,m^0_{\mathsf{RR}}\sqrt{-3\Delta^{\mathsf{RR}}} + 108\,(m_{\mathsf{RR}})^3 - 9\,m_{\mathsf{RR}}\,\lambda^{\mathsf{RR}} \\ D_t\mathcal{W}^{\mathsf{RR}}|_* &= 2\mathrm{i}\,m^0_{\mathsf{RR}}(e - \alpha^{\mathsf{RR}})\alpha_2 \end{aligned}$$

... Analysis of $\mathcal{W}^{\mathbb{Q}}$ is also discussed.

Three types of solutions to satisfy $0 = D_t\mathcal{W}^{\mathsf{RR}} + U\,D_t\mathcal{W}^{\mathbb{Q}}$ and $0 = \mathcal{W}^{\mathsf{RR}} + \mathrm{Re}U\,\mathcal{W}^{\mathbb{Q}}$:

- SUSY AdS vacuum: attractor point

$$\Delta^{\mathsf{RR}} > 0\,, \qquad \Delta^{\mathbb{Q}} > 0\,; \qquad D_t\mathcal{W}^{\mathsf{RR}}|_* = 0 = D_t\mathcal{W}^{\mathbb{Q}}|_*$$

$$t_*^{\mathsf{RR}} = t_*^{\mathbb{Q}}\,, \qquad \mathrm{Re}\,U_* = -\frac{\mathcal{W}_*^{\mathsf{RR}}}{\mathcal{W}_*^{\mathbb{Q}}}$$

$$V_* = -3\,\mathrm{e}^K|\mathcal{W}_*|^2 = -\frac{4}{[\mathrm{Re}(\mathcal{C}\mathcal{G}_0)]^2}\sqrt{\frac{\Delta^{\mathbb{Q}}}{3}} \ll 1$$

- SUSY Minkowski vacuum: attractor point

$$\Delta^{\mathsf{RR}} < 0\,, \qquad \Delta^{\mathbb{Q}} < 0\,; \qquad \mathcal{W}_*^{\mathsf{RR}} = 0 = \mathcal{W}_*^{\mathbb{Q}}$$

$$\alpha^{\mathsf{RR}} = \alpha^{\mathbb{Q}}\,, \qquad U_* = -\frac{D_t\mathcal{W}^{\mathsf{RR}}|_*}{D_t\mathcal{W}^{\mathbb{Q}}|_*} \neq 0$$

$$V_* = 0$$

- SUSY AdS vacua, but moduli t and U are not fixed: non attractor point

$$U = -\frac{D_t\mathcal{W}^{\mathsf{RR}}(t)}{D_t\mathcal{W}^{\mathbb{Q}}(t)}\,, \qquad \mathrm{Re}\,U = -\frac{\mathcal{W}^{\mathsf{RR}}(t)}{\mathcal{W}^{\mathbb{Q}}(t)}$$

1. *Set* $e_{RRA} = 0 = m_{RR}^{A}$ *and* ${p_I}^A = 0 = q^{IA}$
2. *Set a simple prepotential:* $\mathcal{F} = D_{abc} \dfrac{X^a X^b X^c}{X^0}$
3. *Consider the simplest model: single modulus* t *of* Φ_+ *(and* U *of* Φ_-*)*

Functions are reduced to

$$D_t \mathcal{W} \;=\; \frac{U}{t - \bar{t}} \Big(e_0 (2t + \bar{t}) + 3\, e_{00} \Big) \,, \qquad D_U \mathcal{W} \;=\; \mathrm{i} \, \frac{\mathrm{Re} U}{\mathrm{Im} U} \, \mathcal{W}^{\mathbb{Q}}$$

1. *Set* $e_{RRA} = 0 = m_{RR}^{A}$ *and* $p_I{}^{A} = 0 = q^{IA}$
2. *Set a simple prepotential:* $\mathcal{F} = D_{abc}\dfrac{X^a X^b X^c}{X^0}$
3. *Consider the simplest model: single modulus* t *of* Φ_+ *(and* U *of* Φ_-*)*

Functions are reduced to

$$D_t \mathcal{W} \;=\; \frac{U}{t-\bar{t}}\Big(e_0(2t+\bar{t}) + 3\, e_{00}\Big)\,, \qquad D_U \mathcal{W} \;=\; \mathrm{i}\,\frac{\mathrm{Re}U}{\mathrm{Im}U}\,\mathcal{W}^{\mathbb{Q}}$$

There are neither SUSY solutions under the conditions $D_t\mathcal{W} = 0 = D_U\mathcal{W}$

nor non-SUSY solutions satisfying $\partial_{\mathcal{P}} V = 0\,!$

Ansatz 2. "Neglecting all α' corrections on the compactified space" is too strong

2'. *Set a deformed prepotential:* $\mathcal{F} = \frac{(X^t)^3}{X^0} + \sum_n N_n \frac{(X^t)^{n+3}}{(X^0)^{n+1}}$

Consider a simple case as $N_1 \neq 0$, otherwise $N_n = 0$:

$$\begin{aligned} D_t \mathcal{W}^{\mathbb{Q}} &= -e_{00} + \frac{3(t-\bar{t})^2 - \partial_t P}{(t-\bar{t})^3 - P} \Big(e_{00} + e_0 \, t \Big) \\ P &\equiv -2 \Big(N_1 \, t^4 - \overline{N}_1 \, \bar{t}^4 - 2 N_1 \, t^3 \bar{t} + 2 \overline{N}_1 \, t \bar{t}^3 \Big) \end{aligned}$$

SUSY AdS solution appears under the conditions $D_t \mathcal{W} = 0$ and $D_U \mathcal{W} = 0$:

$$t_*^{\mathbb{Q}} = -\frac{2 \, e_{00}}{e_0} \, , \qquad \mathrm{Re} \, U_* = 0$$

$$\mathcal{W}_*^{\mathbb{Q}} = e_{00} \, , \qquad \mathrm{Im} N_1 < 0$$

$$V_* = -3 \, \mathrm{e}^K |\mathcal{W}_*|^2 = \frac{1}{[\mathrm{Re}(\mathcal{C} \mathcal{G}_0)]^2} \frac{3 \, (e_0)^4}{16 \, (e_{00})^2 \, \mathrm{Im} N_1}$$

This is also given by the heterotic string compactifications on $SU(3)$-structure manifolds with torsion, which carries α' corrections.

Summary and Discussions

Summary

- We found SUSY AdS (or Minkowski) vacuum on an attractor point
- We obtained a powerful rule to evaluate the attractor points: Discriminants
- We confirmed that α' corrections are included in certain configurations

Discussions

- Complete stabilization via nonperturbative corrections
- Duality transformations
- Understanding the physical interpretation of nongeometric fluxes
- Connection to doubled formalism

Appendix: compactifications in type II strings

Moduli spaces in $\mathcal{N} = 2$ supergravity are

vector multiplets:	Hodge-Kähler geometry
hypermultiplets:	quaternionic geometry

We look for the origin of the moduli spaces in 10D string theories

Decomposition of vector bundle on 10D spacetime:

$$TM_{9,1} = T_{3,1} \oplus F$$

$$\begin{cases} T_{3,1}: & \text{a real } SO(3,1) \text{ vector bundle} \\ F: & \text{an } SO(6) \text{ vector bundle which admits a pair of } SU(3) \text{ structures} \end{cases}$$

10D spacetime itself is not decomposed yet, i.e., do not yet consider truncation of modes.

Decomposition of vector bundle on 10D spacetime:

$$TM_{9,1} = T_{3,1} \oplus F$$

$$\begin{cases} T_{3,1}: & \text{a real } SO(3,1) \text{ vector bundle} \\ F: & \text{an } SO(6) \text{ vector bundle which admits a pair of } SU(3) \text{ structures} \end{cases}$$

10D spacetime itself is not decomposed yet, i.e., do not yet consider truncation of modes.

Decomposition of Lorentz symmetry:

$$Spin(9,1) \to Spin(3,1) \times Spin(6) = SL(2,\mathbb{C}) \times SU(4)$$

$$\mathbf{16} = (\mathbf{2},\mathbf{4}) \oplus (\overline{\mathbf{2}},\overline{\mathbf{4}}) \qquad \mathbf{16} = (\mathbf{2},\overline{\mathbf{4}}) \oplus (\overline{\mathbf{2}},\mathbf{4})$$

Decomposition of supersymmetry parameters (with $a, b \in \mathbb{C}$):

$$\begin{cases} \epsilon^1_{\rm IIA} = \varepsilon_1 \otimes (a\eta^1_+) + \varepsilon^c_1 \otimes (\overline{a}\eta^1_-) \\ \epsilon^2_{\rm IIA} = \varepsilon_2 \otimes (\overline{b}\eta^2_-) + \varepsilon^c_2 \otimes (b\eta^2_+) \end{cases} \qquad \begin{cases} \epsilon^1_{\rm IIB} = \varepsilon_1 \otimes (a\eta^1_+) + \varepsilon^c_1 \otimes (\overline{a}\eta^1_-) \\ \epsilon^2_{\rm IIB} = \varepsilon_2 \otimes (b\eta^2_+) + \varepsilon^c_2 \otimes (\overline{b}\eta^2_-) \end{cases}$$

Decomposition of vector bundle on 10D spacetime:

$$TM_{9,1} = T_{3,1} \oplus F$$

$$\begin{cases} T_{3,1}: & \text{a real } SO(3,1) \text{ vector bundle} \\ F: & \text{an } SO(6) \text{ vector bundle which admits a pair of } SU(3) \text{ structures} \end{cases}$$

10D spacetime itself is not decomposed yet, i.e., do not yet consider truncation of modes.

Decomposition of Lorentz symmetry:

$$Spin(9,1) \to Spin(3,1) \times Spin(6) = SL(2,\mathbb{C}) \times SU(4)$$

$$\mathbf{16} = (\mathbf{2},\mathbf{4}) \oplus (\overline{\mathbf{2}},\overline{\mathbf{4}}) \qquad \mathbf{16} = (\mathbf{2},\overline{\mathbf{4}}) \oplus (\overline{\mathbf{2}},\mathbf{4})$$

Decomposition of supersymmetry parameters (with $a, b \in \mathbb{C}$):

$$\begin{cases} \epsilon^1_{\rm IIA} = \varepsilon_1 \otimes (a\eta^1_+) + \varepsilon^c_1 \otimes (\overline{a}\eta^1_-) \\ \epsilon^2_{\rm IIA} = \varepsilon_2 \otimes (\overline{b}\eta^2_-) + \varepsilon^c_2 \otimes (b\eta^2_+) \end{cases} \qquad \begin{cases} \epsilon^1_{\rm IIB} = \varepsilon_1 \otimes (a\eta^1_+) + \varepsilon^c_1 \otimes (\overline{a}\eta^1_-) \\ \epsilon^2_{\rm IIB} = \varepsilon_2 \otimes (b\eta^2_+) + \varepsilon^c_2 \otimes (\overline{b}\eta^2_-) \end{cases}$$

Set $SU(3)$ invariant spinor $\eta^{\mathcal{A}}_+$ s.t. $\nabla^{(T)}\eta^{\mathcal{A}}_+ = 0$ $(\mathcal{A} = 1,2)$

a pair of $SU(3)$ on F (η^1_+, η^2_+) $\longleftrightarrow$ a single $SU(3)$ on F $(\eta^1_+ = \eta^2_+ = \eta_+)$

Requirement that we have a pair of $SU(3)$ structures means there is a sub-supermanifold

$$\mathcal{N}^{9,1|4+4} \subset \mathcal{M}^{9,1|16+16}$$

$$\left(\begin{array}{ll} (9,1): & \text{bosonic degrees} \\ 4+4: & \text{eight Grassmann variables as spinors of } Spin(3,1) \text{ and singlet of } SU(3)\text{s} \end{array} \right)$$

Equivalence such as

eight SUSY theory reformulation of type II supergravity

$\Updownarrow$

a pair of $SU(3)$ structures on vector bundle F

$\Updownarrow$

an $SU(3) \times SU(3)$ structures on extended $F \oplus F^*$

10D spinors in type IIA in Einstein frame

$$\delta\Psi_M^{\mathcal{A}} = \nabla_M \epsilon^{\mathcal{A}} - \frac{1}{96} e^{-\phi} \Big(\Gamma_M{}^{PQR} H_{PQR} - 9 \Gamma^{PQ} H_{MPQ} \Big) \Gamma_{(11)} \epsilon^{\mathcal{A}}$$
$$- \sum_{n=0,2,4,6,8} \frac{1}{64 n!} e^{\frac{5-n}{4}\phi} \Big[(n-1) \Gamma_M{}^{N_1 \cdots N_n} - n(9-n) \delta_M{}^{N_1} \Gamma^{N_2 \cdots N_n} \Big] F_{N_1 \cdots N_n} (\Gamma_{(11)})^{n/2} (\sigma^1 \epsilon)^{\mathcal{A}}$$

Split spacetime $10 = 4 + 6$

$$\epsilon^1 = \varepsilon_1 \otimes (a \eta_+^1) + \varepsilon_1^c \otimes (\overline{a} \eta_-^1) \, , \qquad \epsilon^2 = \varepsilon_2 \otimes (\overline{b} \eta_-^2) + \varepsilon_2^c \otimes (b \eta_+^2)$$

$$0 \equiv \delta\psi_m^{\mathcal{A}} = \nabla_m \eta_+^{\mathcal{A}} + (\text{NS-fluxes} \cdot \eta)^{\mathcal{A}} + (\text{RR-fluxes} \cdot \eta)^{\mathcal{A}}$$

Information of

6D $SU(3)$ Killing spinors $\eta_+^{\mathcal{A}}$

Calabi-Yau three-fold

↓

$SU(3)$-structure manifold with torsion

↓

generalized geometry

- on a single $SU(3)$:

a real two-form	$J_{mn} = \mp 2\mathrm{i}\, \eta_{\pm}^{\dagger}\, \gamma_{mn}\, \eta_{\pm}$
a complex three-form	$\Omega_{mnp} = -2\mathrm{i}\, \eta_{-}^{\dagger}\, \gamma_{mnp}\, \eta_{+}$

- on a pair of $SU(3)$:

two real vectors	$(v - \mathrm{i}v')^m = \eta_{+}^{1\dagger} \gamma^m\, \eta_{-}^{2}$
(J^A, Ω^A)	$J^1 = j + v \wedge v' \quad \Omega^1 = w \wedge (v + \mathrm{i}v')$ $J^2 = j - v \wedge v' \quad \Omega^2 = w \wedge (v - \mathrm{i}v')$ (j, w): local $SU(2)$-invariant forms

If $\eta_+^1 \neq \eta_+^2$ globally, a pair of $SU(3)$ is reduced to global single $SU(2)$ w/ (j, w, v, v')

If $\eta_+^1 = \eta_+^2$ globally, a pair of $SU(3)$ is reduced to a single global $SU(3)$ w/ (J, Ω)

$$\eta_+^2 \; = \; c_{\parallel} \eta_+^1 + c_{\perp} (v + \mathrm{i}v')^m\, \gamma_m\, \eta_-^1 \,, \qquad |c_{\parallel}|^2 + |c_{\perp}|^2 = 1$$

a pair of $SU(3)$ on $T\mathcal{M}$ $\sim$ an $SU(3) \times SU(3)$ on $T\mathcal{M} \oplus T^*\mathcal{M}$

Appendix: Calabi-Yau compactifications

One can embed 4D $\mathcal{N} = 2$ theory into 10D type II theory

compactified on Calabi-Yau three-fold

	vector multiplets	hypermultiplets
generic	coord. of Hodge-Kähler	coord. of quaternionic
IIA on Calabi-Yau	Kähler moduli	complex moduli + RR
IIB on Calabi-Yau	complex moduli	Kähler moduli + RR

NS-NS fields in ten-dimensional spacetime are expanded as

$$\begin{aligned} \phi(x,y) &= \varphi(x) \\ G_{m\overline{n}}(x,y) &= \mathrm{i}\, v^a(x)(\omega_a)_{m\overline{n}}(y), \quad G_{mn}(x,y) \;=\; \mathrm{i}\,\overline{z}^k(x)\left(\frac{(\overline{\chi}_k)_{m\overline{pq}}\Omega^{\overline{pq}}{}_n}{||\Omega||^2}\right)(y) \\ B_2(x,y) &= B_2(x) + b^a(x)\omega_a(y) \end{aligned}$$

RR fields in type IIA are

$$\begin{aligned} C_1(x,y) &= C_1^0(x) \\ C_3(x,y) &= C_1^a(x)\omega_a(y) + \xi^K(x)\alpha_K(y) - \widetilde{\xi}_K(x)\beta^K(y) \end{aligned}$$

RR fields in type IIB are

$$\begin{aligned} C_0(x,y) &= C_0(x) \\ C_2(x,y) &= C_2(x) + c^a(x)\omega_a(y) \\ C_4(x,y) &= V_1^K(x)\alpha_K(y) + \rho_a(x)\widetilde{\omega}^a(y) \end{aligned}$$

cohomology class	basis	
$H^{(1,1)}$	ω_a	$a = 1,\ldots,h^{(1,1)}$
$H^{(0)} \oplus H^{(1,1)}$	$\omega_A = (1,\omega_a)$	$A = 0,1,\ldots,h^{(1,1)}$
$H^{(2,2)}$	$\widetilde{\omega}^a$	$a = 1,\ldots,h^{(1,1)}$
$H^{(2,1)}$	χ_k	$k = 1,\ldots,h^{(2,1)}$
$H^{(3)}$	(α_K,β^K)	$K = 0,1,\ldots,h^{(2,1)}$

4D type IIA $\mathcal{N} = 2$ ungauged supergravity action of bosonic fields is

$$S^{(4)}_{\rm IIA} = \int_{\mathcal{M}_{3,1}} \Big(-\frac{1}{2} R * \mathbf{1} + \frac{1}{2} {\rm Re}\mathcal{N}_{AB} F^A \wedge F^B + \frac{1}{2} {\rm Im}\mathcal{N}_{AB} F^A \wedge * F^B - G_{a\bar{b}}\, {\rm d}t^a \wedge * {\rm d}\bar{t}^{\bar{b}} - h_{uv}\, {\rm d}q^u \wedge * {\rm d}q^v \Big)$$

gravity multiplet	$g_{\mu\nu}\,,\ C_1^0$	1
vector multiplet	$C_1^a\,,\ v^a\,,\ b^a$	$a = 1, \ldots, h^{(1,1)}$
hypermultiplet	$z^k\,,\ \xi^k\,,\ \widetilde{\xi}_k$	$k = 1, \ldots, h^{(2,1)}$
tensor multiplet	$B_2\,,\ \varphi\,,\ \xi^0\,,\ \widetilde{\xi}_0$	1

Various functions in the actions:

$$B + {\rm i}J = (b^a + {\rm i}v^a)\,\omega_a = t^a \omega_a \qquad K^{\sf KS} = -\log\Big(\tfrac{4}{3}\int_{\mathcal{M}_6} J \wedge J \wedge J\Big)$$

$$\mathcal{K}_{abc} = \int_{\mathcal{M}_6} \omega_a \wedge \omega_b \wedge \omega_c \qquad \mathcal{K}_{ab} = \int_{\mathcal{M}_6} \omega_a \wedge \omega_b \wedge J = \mathcal{K}_{abc} v^c$$

$$\mathcal{K}_a = \int_{\mathcal{M}_6} \omega_a \wedge J \wedge J = \mathcal{K}_{abc} v^b v^c \qquad \mathcal{K} = \int_{\mathcal{M}_6} J \wedge J \wedge J = \mathcal{K}_{abc} v^a v^b v^c$$

$${\rm Re}\mathcal{N}_{AB} = \begin{pmatrix} -\frac{1}{3}\mathcal{K}_{abc} b^a b^b b^c & \frac{1}{2}\mathcal{K}_{abc} b^b b^c \\ \frac{1}{2}\mathcal{K}_{abc} b^b b^c & -\mathcal{K}_{abc} b^c \end{pmatrix} \qquad {\rm Im}\mathcal{N}_{AB} = -\frac{\mathcal{K}}{6} \begin{pmatrix} 1 + 4G_{ab} b^a b^b & -4G_{ab} b^b \\ -4G_{ab} b^b & 4G_{ab} \end{pmatrix}$$

$$G_{a\bar{b}} = \frac{3}{2} \frac{\int \omega_a \wedge * \omega_b}{\int J \wedge J \wedge J} = \partial_{t^a} \overline{\partial}_{\bar{t}^{\bar{b}}} K^{\sf KS}$$

4D type IIB $\mathcal{N}=2$ ungauged supergravity action of bosonic fields is

$$S^{(4)}_{\rm IIB} = \int_{\mathcal{M}_{3,1}} \Big(-\frac{1}{2} R * \mathbf{1} + \frac{1}{2} {\rm Re}\mathcal{M}_{KL} F^K \wedge F^L + \frac{1}{2} {\rm Im}\mathcal{M}_{KL} F^K \wedge * F^L - G_{k\bar{l}}\, {\rm d}z^k \wedge * {\rm d}\bar{z}^{\bar{l}} - h_{pq}\, {\rm d}\widetilde{q}^p \wedge * {\rm d}\widetilde{q}^q \Big)$$

gravity multiplet	$g_{\mu\nu}\,,\ V_1^0$	1
vector multiplet	$V_1^k\,,\ z^k$	$k = 1, \ldots, h^{(2,1)}$
hypermultiplet	$v^a\,,\ b^a\,,\ c^a\,,\ \rho_a$	$a = 1, \ldots, h^{(1,1)}$
tensor multiplet	$B_2\,,\ C_2\,,\ \varphi\,,\ C_0$	1

Various functions in the actions:

$$\Omega = Z^K \alpha_K - \mathcal{G}_K \beta^K \qquad z^k = Z^K / Z^0 \qquad \mathcal{G}_{KL} = \partial_L \mathcal{G}_K$$

$$K^{\rm CS} = -\log\Big({\rm i} \int_{\mathcal{M}_6} \Omega \wedge \overline{\Omega} \Big) \qquad G_{k\bar{l}} = - \frac{\int \chi_k \wedge \overline{\chi}_{\bar{l}}}{\int \Omega \wedge \overline{\Omega}} = \partial_{z^k} \overline{\partial}_{\bar{z}^{\bar{l}}} K^{\rm CS}$$

$$\mathcal{M}_{KL} = \overline{\mathcal{G}}_{KL} + 2{\rm i}\, \frac{({\rm Im}\mathcal{G})_{KM} Z^M ({\rm Im}\mathcal{G})_{LN} Z^N}{Z^N ({\rm Im}\mathcal{G})_{NM} Z^M}$$

Appendix: $SU(3)$-structure manifold with torsion

Information from Killing spinor eqs. with torsion $D^{(T)}\eta_\pm = 0$ ($^\exists$complex Weyl $\eta_\pm$)

- Invariant p-forms on $SU(3)$-structure manifold:

a real two-form $J_{mn} = \mp 2\mathrm{i}\,\eta_\pm^\dagger\,\gamma_{mn}\,\eta_\pm$

a holomorphic three-form $\Omega_{mnp} = -2\mathrm{i}\,\eta_-^\dagger\,\gamma_{mnp}\,\eta_+$

$$\mathrm{d}J = \frac{3}{2}\mathrm{Im}(\overline{\mathcal{W}}_1\Omega) + \mathcal{W}_4 \wedge J + \mathcal{W}_3 \qquad \mathrm{d}\Omega = \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \overline{\mathcal{W}}_5 \wedge \Omega$$

- Five classes of (intrinsic) torsion are given as

components	interpretations	$SU(3)$-representations
$\mathcal{W}_1$	$J \wedge \mathrm{d}\Omega$ or $\Omega \wedge \mathrm{d}J$	$\mathbf{1} \oplus \mathbf{1}$
$\mathcal{W}_2$	$(\mathrm{d}\Omega)_0^{2,2}$	$\mathbf{8} \oplus \mathbf{8}$
$\mathcal{W}_3$	$(\mathrm{d}J)_0^{2,1} + (\mathrm{d}J)_0^{1,2}$	$\mathbf{6} \oplus \overline{\mathbf{6}}$
$\mathcal{W}_4$	$J \wedge \mathrm{d}J$	$\mathbf{3} \oplus \overline{\mathbf{3}}$
$\mathcal{W}_5$	$(\mathrm{d}\Omega)^{3,1}$	$\mathbf{3} \oplus \overline{\mathbf{3}}$

- Vanishing torsion classes in $SU(3)$-structure manifolds:

complex	hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = 0$
	balanced	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = 0$
	special hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	Kähler	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	conformally Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = 3\mathcal{W}_4 + 2\mathcal{W}_5 = 0$
almost complex	symplectic	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	nearly Kähler	$\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	almost Kähler	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	quasi Kähler	$\mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	semi Kähler	$\mathcal{W}_4 = \mathcal{W}_5 = 0$
	half-flat	$\mathrm{Im}\mathcal{W}_1 = \mathrm{Im}\mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$

Appendix: generalized geometry

Introduce a generalized almost complex structure $\mathcal{J}$ on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$ s.t.

$$\mathcal{J} : \; T\mathcal{M}_d \oplus T^*\mathcal{M}_d \;\longrightarrow\; T\mathcal{M}_d \oplus T^*\mathcal{M}_d$$

$$\mathcal{J}^2 \;=\; -\mathbf{1}_{2d}$$

$^\exists$ $O(d,d)$ invariant metric L, s.t. $\mathcal{J}^T L \mathcal{J} \;=\; L$

Structure group on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$:

$^\exists L$	$GL(2d)$	$\dashrightarrow$	$O(d,d)$
$\mathcal{J}^2 = -\mathbf{1}_{2d}$	$O(d,d)$	$\dashrightarrow$	$U(d/2,d/2)$
$\mathcal{J}_1, \mathcal{J}_2$	$U_1(d/2,d/2) \cap U_2(d/2,d/2)$	$\dashrightarrow$	$U(d/2) \times U(d/2)$
integrable $\mathcal{J}_{1,2}$	$U(d/2) \times U(d/2)$	$\dashrightarrow$	$SU(d/2) \times SU(d/2)$

- Integrability is discussed by "$(0,1)$" part of the complexified $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$:

$$\Pi \equiv \frac{1}{2}(\mathbf{1}_{2d} - \mathrm{i}\mathcal{J})$$

$$\Pi A = A \quad \text{where } A = v + \zeta \text{ is a section of } T\mathcal{M}_d \oplus T^*\mathcal{M}_d$$

We call this A i-eigenbundle $L_{\mathcal{J}}$, whose dimension is $\dim L_{\mathcal{J}} = d$.

Integrability condition of $\mathcal{J}$ is

$$\overline{\Pi}\big[\Pi(v+\zeta), \Pi(w+\eta)\big]_{\mathrm{C}} = 0 \qquad v, w \in T\mathcal{M}_d \qquad \zeta, \eta \in T^*\mathcal{M}_d$$

$$[v+\zeta, w+\eta]_{\mathrm{C}} = [v,w] + \mathcal{L}_v \eta - \mathcal{L}_w \zeta - \frac{1}{2}\mathrm{d}(\iota_v \eta - \iota_w \zeta) : \text{ Courant bracket}$$

- Two typical examples of generalized almost complex structures:

$$\mathcal{J}_- = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I^T \end{pmatrix} \qquad \text{w/ } I^2 = -\mathbf{1}_d\text{: almost complex structure}$$

$$\mathcal{J}_+ = \begin{pmatrix} \mathbf{0} & -J^{-1} \\ J & \mathbf{0} \end{pmatrix} \qquad \text{w/ } J\text{: almost symplectic form}$$

integrable $\mathcal{J}_-$ $\leftrightarrow$ integrable I

integrable $\mathcal{J}_+$ $\leftrightarrow$ integrable J

On a usual geometry, $J_{mn} = I_m{}^p g_{pn}$ is given by an $SU(3)$ invariant (pure) spinor η_+ as

$$J_{mn} = -2\mathrm{i}\, \eta_+^\dagger \gamma_{mn} \eta_+ \qquad \gamma^i \eta_+ = 0 \qquad \gamma^{\bar{\imath}} \eta_+ \neq 0$$

In a similar analogy, we want to find Cliff$(6,6)$ pure spinor(s) Φ.

$\because$) Compared to almost complex structures, (pure) spinors can be easily utilized in supergravity framework.

On $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$, we can define Cliff$(6,6)$ algebra and $Spin(6,6)$ spinor Φ:

$$\{\Gamma^m, \Gamma^n\} = 0 \qquad \{\Gamma^m, \widetilde{\Gamma}_n\} = \delta^m_n \qquad \{\widetilde{\Gamma}_m, \widetilde{\Gamma}_n\} = 0$$

Irreducible repr. of $Spin(6,6)$ spinor is a Majorana-Weyl

$\rightarrow$ a generic $Spin(6,6)$ spinor bundle S splits to $S^{\pm}$ (Weyl)

On $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$, we can define Cliff$(6,6)$ algebra and $Spin(6,6)$ spinor Φ:

$$\{\Gamma^m, \Gamma^n\} = 0 \qquad \{\Gamma^m, \widetilde{\Gamma}_n\} = \delta^m_n \qquad \{\widetilde{\Gamma}_m, \widetilde{\Gamma}_n\} = 0$$

Irreducible repr. of $Spin(6,6)$ spinor is a Majorana-Weyl

$\rightarrow$ a generic $Spin(6,6)$ spinor bundle S splits to $S^{\pm}$ (Weyl)

Weyl spinor bundles $S^{\pm}$ are isomorphic to bundles of forms on $T^*\mathcal{M}_6$:

$$S^+ \text{ on } T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \quad \sim \quad \wedge^{\text{even}} T^*\mathcal{M}_6$$
$$S^- \text{ on } T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \quad \sim \quad \wedge^{\text{odd}} T^*\mathcal{M}_6$$

Thus we often regard a Cliff$(6,6)$ spinor as a form on $\wedge^{\text{even/odd}} T^*\mathcal{M}_6$

A form-valued representation of the algebra

$$\Gamma^m = \mathrm{d}x^m \wedge \,, \qquad \widetilde{\Gamma}_n = \iota_{\partial_n}$$

On $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$, we can define Cliff$(6,6)$ algebra and $Spin(6,6)$ spinor Φ:

$$\{\Gamma^m, \Gamma^n\} = 0 \qquad \{\Gamma^m, \widetilde{\Gamma}_n\} = \delta^m_n \qquad \{\widetilde{\Gamma}_m, \widetilde{\Gamma}_n\} = 0$$

Irreducible repr. of $Spin(6,6)$ spinor is a Majorana-Weyl

$\rightarrow$ a generic $Spin(6,6)$ spinor bundle S splits to $S^{\pm}$ (Weyl)

Weyl spinor bundles $S^{\pm}$ are isomorphic to bundles of forms on $T^*\mathcal{M}_6$:

$$S^+ \text{ on } T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \quad \sim \quad \wedge^{\text{even}} T^*\mathcal{M}_6$$
$$S^- \text{ on } T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \quad \sim \quad \wedge^{\text{odd}} T^*\mathcal{M}_6$$

Thus we often regard a Cliff$(6,6)$ spinor as a form on $\wedge^{\text{even/odd}} T^*\mathcal{M}_6$

A form-valued representation of the algebra

$$\Gamma^m = \mathrm{d}x^m \wedge \, , \qquad \widetilde{\Gamma}_n = \iota_{\partial_n}$$

IF Φ is annihilated by half numbers of the Cliff$(6,6)$ generators:

$\rightarrow$ Φ is called a pure spinor

cf.) $SU(3)$ invariant spinor η_+ is a Cliff(6) pure spinor: $\gamma^i \eta_+ = 0$

An equivalent definition of a Cliff$(6,6)$ pure spinor is given by "Clifford action":

$$(v+\zeta)\cdot\Phi \;=\; v^m \iota_{\partial_m}\Phi + \zeta_n \, \mathrm{d}x^n \wedge \Phi \qquad \text{w/ } v\text{: vector} \qquad \zeta\text{: one-form}$$

Define the annihilator of a spinor as

$$L_\Phi \;\equiv\; \Big\{ v+\zeta \in T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \,\Big|\, (v+\zeta)\cdot\Phi = 0 \Big\}$$

$$\dim L_\Phi \;\leq\; d$$

If $\dim L_\Phi = 6$ (maximally isotropic) $\rightarrow$ Φ is a pure spinor

Correspondence between pure spinors and generalized almost complex structures:

$$\mathcal{J} \leftrightarrow \Phi \qquad \text{if} \;\; L_{\mathcal{J}} = L_{\Phi} \quad \text{with} \;\; \dim L_{\Phi} = 6$$

More precisely: $\mathcal{J} \leftrightarrow$ a line bundle of pure spinor Φ

$\because$) rescaling Φ does not change its annihilator L_{Φ}

Correspondence between pure spinors and generalized almost complex structures:

$$\mathcal{J} \leftrightarrow \Phi \qquad \text{if} \;\; L_{\mathcal{J}} = L_{\Phi} \quad \text{with} \;\; \dim L_{\Phi} = 6$$

More precisely: $\mathcal{J} \leftrightarrow$ a line bundle of pure spinor Φ

$\because$) rescaling Φ does not change its annihilator L_{Φ}

Then, we can rewrite the generalized almost complex structure as

$$\boxed{\mathcal{J}_{\pm \Pi\Sigma} \;=\; \big\langle \mathrm{Re}\Phi_{\pm}, \Gamma_{\Pi\Sigma}\, \mathrm{Re}\Phi_{\pm} \big\rangle}$$

w/ Mukai pairing:

even forms: $\big\langle \Psi_{+}, \Phi_{+} \big\rangle = \Psi_6 \wedge \Phi_0 - \Psi_4 \wedge \Phi_2 + \Psi_2 \wedge \Phi_4 - \Psi_0 \wedge \Phi_6$

odd forms: $\big\langle \Psi_{-}, \Phi_{-} \big\rangle = \Psi_5 \wedge \Phi_1 - \Psi_3 \wedge \Phi_3 + \Psi_1 \wedge \Phi_5$

Correspondence between pure spinors and generalized almost complex structures:

$$\mathcal{J} \leftrightarrow \Phi \qquad \text{if} \;\; L_{\mathcal{J}} = L_{\Phi} \quad \text{with} \;\; \dim L_{\Phi} = 6$$

More precisely: $\mathcal{J} \leftrightarrow$ a line bundle of pure spinor Φ

$\because$) rescaling Φ does not change its annihilator L_{Φ}

Then, we can rewrite the generalized almost complex structure as

$$\boxed{\mathcal{J}_{\pm\Pi\Sigma} \;=\; \big\langle \mathrm{Re}\Phi_{\pm}, \Gamma_{\Pi\Sigma}\, \mathrm{Re}\Phi_{\pm} \big\rangle}$$

w/ Mukai pairing:

even forms: $\big\langle \Psi_+, \Phi_+ \big\rangle = \Psi_6 \wedge \Phi_0 - \Psi_4 \wedge \Phi_2 + \Psi_2 \wedge \Phi_4 - \Psi_0 \wedge \Phi_6$

odd forms: $\big\langle \Psi_-, \Phi_- \big\rangle = \Psi_5 \wedge \Phi_1 - \Psi_3 \wedge \Phi_3 + \Psi_1 \wedge \Phi_5$

$\mathcal{J}$ is integrable $\longleftrightarrow$ $^\exists$ vector v and one-form ζ s.t. $\mathrm{d}\Phi = (v \llcorner + \zeta \wedge)\Phi$

generalized CY $\longleftrightarrow$ $^\exists\Phi$ is pure s.t. $\mathrm{d}\Phi = 0$

"twisted" GCY $\longleftrightarrow$ $^\exists\Phi$ is pure, and H is closed s.t. $(\mathrm{d} - H\wedge)\Phi = 0$

A Cliff$(6,6)$ spinor can also be mapped to a bispinor:

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{m_1 \cdots m_k} \, \mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_k} \qquad \longleftrightarrow \qquad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{m_1 \cdots m_k} \, \gamma_{\alpha\beta}^{m_1 \cdots m_k}$$

A Cliff$(6,6)$ spinor can also be mapped to a bispinor:

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{m_1 \cdots m_k} \mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_k} \qquad \longleftrightarrow \qquad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{m_1 \cdots m_k} \gamma^{m_1 \cdots m_k}_{\alpha\beta}$$

On a geometry of a single $SU(3)$-structure, the following two $SU(3,3)$ spinors:

$$\Phi_{0+} = \eta_+ \otimes \eta_+^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_+^\dagger \gamma_{m_k \cdots m_1} \eta_+ \, \gamma^{m_1 \cdots m_k} = \frac{1}{8} \mathrm{e}^{-\mathrm{i}J}$$

$$\Phi_{0-} = \eta_+ \otimes \eta_-^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_-^\dagger \gamma_{m_k \cdots m_1} \eta_+ \, \gamma^{m_1 \cdots m_k} = -\frac{\mathrm{i}}{8} \Omega$$

Check purity: $(\delta + \mathrm{i}J)_m{}^n \gamma_n \eta_+ \otimes \eta_\pm^\dagger = 0 = \eta_+ \otimes \eta_\pm^\dagger \gamma_n (\delta \mp \mathrm{i}J)^n{}_m$

One-to-one correspondence: $\Phi_{0-} \leftrightarrow \mathcal{J}_1, \quad \Phi_{0+} \leftrightarrow \mathcal{J}_2$

A Cliff$(6,6)$ spinor can also be mapped to a bispinor:

$$C \equiv \sum_k \frac{1}{k!} C^{(k)}_{m_1 \cdots m_k} \mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_k} \qquad \longleftrightarrow \qquad \not{C} \equiv \sum_k \frac{1}{k!} C^{(k)}_{m_1 \cdots m_k} \gamma^{m_1 \cdots m_k}_{\alpha\beta}$$

On a geometry of a single $SU(3)$-structure, the following two $SU(3,3)$ spinors:

$$\Phi_{0+} = \eta_+ \otimes \eta_+^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_+^\dagger \gamma_{m_k \cdots m_1} \eta_+ \, \gamma^{m_1 \cdots m_k} = \frac{1}{8} \mathrm{e}^{-\mathrm{i}J}$$

$$\Phi_{0-} = \eta_+ \otimes \eta_-^\dagger = \frac{1}{4} \sum_{k=0}^{6} \frac{1}{k!} \eta_-^\dagger \gamma_{m_k \cdots m_1} \eta_+ \, \gamma^{m_1 \cdots m_k} = -\frac{\mathrm{i}}{8} \Omega$$

Check purity: $(\delta + \mathrm{i}J)_m{}^n \gamma_n \eta_+ \otimes \eta_\pm^\dagger = 0 = \eta_+ \otimes \eta_\pm^\dagger \gamma_n (\delta \mp \mathrm{i}J)^n{}_m$

One-to-one correspondence: $\Phi_{0-} \leftrightarrow \mathcal{J}_1, \quad \Phi_{0+} \leftrightarrow \mathcal{J}_2$

On a generic geometry of a pair of $SU(3)$-structures defined by (η_+^1, η_+^2)

$$\Phi_{0+} = \eta_+^1 \otimes \eta_+^{2\dagger} = \frac{1}{8} (\bar{c}_\parallel \mathrm{e}^{-\mathrm{i}j} - \mathrm{i}\bar{c}_\perp w) \wedge \mathrm{e}^{-\mathrm{i}v \wedge v'}$$

$$\Phi_{0-} = \eta_+^1 \otimes \eta_-^{2\dagger} = -\frac{1}{8} (c_\perp \mathrm{e}^{-\mathrm{i}j} + \mathrm{i}c_\parallel w) \wedge (v + \mathrm{i}v')$$

$$|c_\parallel|^2 + |c_\perp|^2 = 1$$

$$\Phi_\pm = \mathrm{e}^{-B} \Phi_{0\pm}$$

Each $\Phi_\pm$ defines an $SU(3,3)$ structure on E. Common structure is $SU(3) \times SU(3)$.

(F is extended to E by including e^{-B})

Compatibility requires

$$\left\langle \Phi_+, V \cdot \Phi_- \right\rangle \;=\; \left\langle \overline{\Phi}_+, V \cdot \Phi_- \right\rangle \;=\; 0 \qquad \text{for } \forall V = x + \xi$$

$$\left\langle \Phi_+, \overline{\Phi}_+ \right\rangle \;=\; \left\langle \Phi_-, \overline{\Phi}_- \right\rangle$$

Start with a real form $\chi_f \in \wedge^{\text{even/odd}} F^*$ (associated with a real $Spin(6,6)$ spinor χ_s)

Regard χ_f as a stable form satisfying

$$
\begin{aligned}
q(\chi_f) &= -\frac{1}{4} \langle \chi_f, \Gamma_{\Pi\Sigma} \chi_f \rangle \langle \chi_f, \Gamma^{\Pi\Sigma} \chi_f \rangle \ \in \ \wedge^6 F^* \otimes \wedge^6 F^* \\
U &= \big\{ \chi_f \in \wedge^{\text{even/odd}} F^* \ : \ q(\chi_f) < 0 \big\}
\end{aligned}
$$

Start with a real form $\chi_f \in \wedge^{\text{even/odd}} F^*$ (associated with a real $Spin(6,6)$ spinor χ_s)

Regard χ_f as a stable form satisfying

$$q(\chi_f) \;=\; -\frac{1}{4}\langle \chi_f, \Gamma_{\Pi\Sigma}\chi_f\rangle\langle \chi_f, \Gamma^{\Pi\Sigma}\chi_f\rangle \;\in\; \wedge^6 F^* \otimes \wedge^6 F^*$$
$$U \;=\; \{\chi_f \in \wedge^{\text{even/odd}} F^* \;:\; q(\chi_f) < 0\}$$

Define a Hitchin function

$$H(\chi_f) \;\equiv\; \sqrt{-\frac{1}{3}q(\chi_f)} \;\in\; \wedge^6 F^*$$

which gives an integrable complex structure on U

Start with a real form $\chi_f \in \wedge^{\text{even/odd}} F^*$ (associated with a real $Spin(6,6)$ spinor χ_s)

Regard χ_f as a stable form satisfying

$$q(\chi_f) = -\frac{1}{4}\langle \chi_f, \Gamma_{\Pi\Sigma}\chi_f\rangle\langle \chi_f, \Gamma^{\Pi\Sigma}\chi_f\rangle \in \wedge^6 F^* \otimes \wedge^6 F^*$$

$$U = \{\chi_f \in \wedge^{\text{even/odd}} F^* \;:\; q(\chi_f) < 0\}$$

Define a Hitchin function

$$H(\chi_f) \equiv \sqrt{-\frac{1}{3}q(\chi_f)} \in \wedge^6 F^*$$

which gives an integrable complex structure on U

Then we can get another real form $\hat{\chi}_f$ and a complex form Φ_f by Mukai pairing

$$\langle \hat{\chi}_f, \chi_f \rangle = -\mathrm{d}H(\chi_f) \qquad \text{i.e.,} \quad \hat{\chi}_f = -\frac{\partial H(\chi_f)}{\partial \chi_f}$$

$$\dashrightarrow \qquad \Phi_f \equiv \frac{1}{2}(\chi_f + \mathrm{i}\hat{\chi}_f) \qquad H(\Phi_f) = \mathrm{i}\langle \Phi_f, \overline{\Phi}_f \rangle$$

Hitchin showed: Φ_f is a (form corresponding to) pure spinor!

N.J. Hitchin math/0010054, math/0107101, math/0209099

Consider the space of pure spinors Φ ...

Mukai pairing $\langle *, * \rangle$	$\longrightarrow$	symplectic structure
Hitchin function $H(*)$	$\longrightarrow$	complex structure

$$\Downarrow$$

The space of pure spinor is Kähler

Consider the space of pure spinors Φ ...

Mukai pairing $\langle *, * \rangle$	$\longrightarrow$	symplectic structure
Hitchin function $H(*)$	$\longrightarrow$	complex structure

$\Downarrow$

The space of pure spinor is Kähler

Quotienting this space by the $\mathbb{C}^*$ action $\Phi \to \lambda\Phi$ for $\lambda\mathbb{C}^*$

--→ The space becomes a local special Kähler geometry with Kähler potential K:

$$\mathrm{e}^{-K} \;=\; H(\Phi) \;=\; \mathrm{i}\langle \Phi, \overline{\Phi} \rangle \;=\; \mathrm{i}\big(\overline{X}^A \mathcal{F}_A - X^A \overline{\mathcal{F}}_A \big) \;\in\; \wedge^6 F^*$$

X^A : holomorphic projective coordinates

$\mathcal{F}_A$: derivative of prepotential $\mathcal{F}$, i.e., $\mathcal{F}_A = \partial \mathcal{F} / \partial X^A$

These are nothing but objects which we want to introduce in $\mathcal{N} = 2$ supergravity!

Spaces of pure spinors $\Phi_\pm$ on $F \oplus F^*$ with $SU(3) \times SU(3)$ structures

$\|$

special Kähler geometries of local type $=$ Hodge-Kähler geometries

For the single $SU(3)$-structure case:

$$\Phi_+ = \frac{1}{8} \mathrm{e}^{-B-\mathrm{i}J} \qquad K_+ = -\log \left(\frac{1}{48} J \wedge J \wedge J \right)$$

$$\Phi_- = -\frac{\mathrm{i}}{8} \mathrm{e}^{-B} \Omega \qquad K_- = -\log \left(\frac{\mathrm{i}}{64} \Omega \wedge \overline{\Omega} \right)$$

Structure of forms is exactly same as the one in the case of Calabi-Yau compactification!

We should truncate Kaluza-Klein massive modes from these forms to obtain 4D supergravity.

Appendix: setup in $\mathcal{N} = 1$ theory

Functionals are given by two Kähler potentials on two Hodge-Kähler geometries of $\Phi_\pm$:

$$
\begin{aligned}
K &= K_+ + 4\varphi \\
K_+ &= -\log \mathrm{i} \int_{\mathcal{M}} \langle \Phi_+, \overline{\Phi}_+ \rangle = -\log \mathrm{i} \big(\overline{X}^A \mathcal{F}_A - X^A \overline{\mathcal{F}}_A \big) \\
K_- &= -\log \mathrm{i} \int_{\mathcal{M}} \langle \Phi_-, \overline{\Phi}_- \rangle = -\log \mathrm{i} \big(\overline{Z}^I \mathcal{G}_I - Z^I \overline{\mathcal{G}}_I \big) \\
\int_{\mathcal{M}} \mathrm{vol}_6 &= \frac{1}{8} \mathrm{e}^{-K_\pm} = \mathrm{e}^{-2\varphi + 2\phi^{(10)}}
\end{aligned}
$$

Introduce $\mathcal{C} = \sqrt{2} ab \, \mathrm{e}^{-\phi^{(10)}} = 4ab \, \mathrm{e}^{\frac{K_-}{2} - \varphi}$

$$
\begin{aligned}
\therefore \quad \mathrm{e}^{-2\varphi} &= \frac{|\mathcal{C}|^2}{16|a|^2|b|^2} \mathrm{e}^{-K_-} = \frac{\mathrm{i}}{16|a|^2|b|^2} \int_{\mathcal{M}} \langle \mathcal{C}\Phi_-, \overline{\mathcal{C}\Phi_-} \rangle \\
&= \frac{1}{8|a|^2|b|^2} \Big[\mathrm{Im}(\mathcal{C} Z^I) \mathrm{Re}(\mathcal{C} \mathcal{G}_I) - \mathrm{Re}(\mathcal{C} Z^I) \mathrm{Im}(\mathcal{C} \mathcal{G}_I) \Big]
\end{aligned}
$$

See the SUSY variation of 4D $\mathcal{N} = 2$ gravitinos:

$$\delta\psi_{\mathcal{A}\mu} = \nabla_\mu \varepsilon_{\mathcal{A}} - S_{\mathcal{A}\mathcal{B}}\, \gamma_\mu\, \varepsilon^{\mathcal{B}} + \ldots$$

$$S_{\mathcal{A}\mathcal{B}} = \frac{\mathrm{i}}{2} \mathrm{e}^{\frac{K_+}{2}} \begin{pmatrix} \mathcal{P}^1 - \mathrm{i}\mathcal{P}^2 & -\mathcal{P}^3 \\ -\mathcal{P}^3 & -\mathcal{P}^1 - \mathrm{i}\mathcal{P}^2 \end{pmatrix}_{\mathcal{A}\mathcal{B}}$$

The components are also written by $\Phi_\pm$:

$$\mathcal{P}^1 - \mathrm{i}\mathcal{P}^2 = 2\,\mathrm{e}^{\frac{K_-}{2}+\varphi} \int_{\mathcal{M}} \langle \Phi_+, \mathcal{D}\Phi_- \rangle\,, \qquad \mathcal{P}^1 + \mathrm{i}\mathcal{P}^2 = 2\,\mathrm{e}^{\frac{K_-}{2}+\varphi} \int_{\mathcal{M}} \langle \Phi_+, \mathcal{D}\overline{\Phi}_- \rangle$$

$$\mathcal{P}^3 = -\frac{1}{\sqrt{2}} \mathrm{e}^{2\varphi} \int_{\mathcal{M}} \langle \Phi_+, G \rangle$$

Note: $\hat{\Psi}_{\mathcal{A}\mu} = \Psi_{\mathcal{A}\mu} + \frac{1}{2}\Gamma_\mu{}^m \Psi_m^{\mathcal{A}} = \psi_{\mathcal{A}\mu\pm} \otimes \eta_+ + \psi_{\mathcal{A}\mu\mp} \otimes \eta_- + \ldots$

4D $\mathcal{N} = 1$ fermions given by the SUSY truncation from 4D $\mathcal{N} = 2$ system:

SUSY parameter : $\varepsilon \equiv \overline{n}^{\mathcal{A}} \varepsilon_{\mathcal{A}} = a\, \varepsilon_1 + \bar{b}\, \varepsilon_2$

gravitino : $\psi_\mu \equiv \overline{n}^{\mathcal{A}} \psi_{\mathcal{A}\mu} = a\, \psi_{1\mu} + \bar{b}\, \psi_{2\mu}\,,$ $\qquad \widetilde{\psi}_\mu \equiv \big(b\, \psi_{1\mu} - \bar{a}\, \psi_{2\mu}\big)$

gauginos : $\chi^A \equiv -2\, \mathrm{e}^{\frac{K_+}{2}} D_b X^A \big(\overline{n}^{\mathcal{C}}\, \epsilon_{\mathcal{C}\mathcal{E}}\, \chi^{\mathcal{E}b}\big)$

where $\overline{n}^{\mathcal{A}} = \big(a\,,\, \bar{b}\big)\,, \qquad \epsilon_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

SUSY variations yield the superpotential and the D-term:

$$\delta\psi_\mu = \nabla_\mu\varepsilon - \overline{n}^{\mathcal{A}} S_{\mathcal{AB}} n^{*\mathcal{B}} \gamma_\mu \varepsilon^c \equiv \nabla_\mu\varepsilon - \mathrm{e}^{\frac{K}{2}} \mathcal{W} \gamma_\mu \varepsilon^c$$

$$\delta\widetilde{\psi}_\mu = 0$$

$$\delta\chi^A = \mathrm{Im}F^A_{\mu\nu} \gamma^{\mu\nu} \varepsilon + \mathrm{i}\, D^A \varepsilon$$

$$\mathcal{W} = \frac{\mathrm{i}}{4\overline{a}b}\Big[4\mathrm{i}\,\mathrm{e}^{\frac{K_-}{2}-\varphi} \int_{\mathcal{M}} \langle \Phi_+, \mathcal{D}\mathrm{Im}(ab\Phi_-)\rangle + \frac{1}{\sqrt{2}} \int_{\mathcal{M}} \langle \Phi_+, G\rangle\Big]$$

$$\equiv \mathcal{W}^{\mathsf{RR}} + U^I \mathcal{W}_I^{\mathbb{Q}} + \widetilde{U}_I \widetilde{\mathcal{W}}^I_{\mathbb{Q}}$$

$$\mathcal{W}^{\mathsf{RR}} = -\frac{\mathrm{i}}{4\overline{a}b}\Big[X^A e_{\mathsf{RR}A} - \mathcal{F}_A m^A_{\mathsf{RR}}\Big]$$

$$\mathcal{W}_I^{\mathbb{Q}} = \frac{\mathrm{i}}{4\overline{a}b}\Big[X^A e_{IA} + \mathcal{F}_A p_I{}^A\Big], \qquad \widetilde{\mathcal{W}}^I_{\mathbb{Q}} = -\frac{\mathrm{i}}{4\overline{a}b}\Big[X^A m_A{}^I + \mathcal{F}_A q^{IA}\Big]$$

$$D^A = 2\,\mathrm{e}^{K_+}(K_+)^{c\overline{d}} D_c X^A \overline{D_d X^B} \big[\overline{n}^{\mathcal{C}} (\sigma_x)_{\mathcal{C}}{}^{\mathcal{B}} n_{\mathcal{B}}\big] \Big(\mathcal{P}^x_B - \mathcal{N}_{BC}\widetilde{\mathcal{P}}^{xC}\Big)$$

$\mathcal{N} = 2$ multiplets:

$(t^a = X^a/X^0,\ z^i = Z^i/Z^0)$

gravity multiplet	$g_{\mu\nu},\ A^0_\mu$	
vector multiplets	$A^a_\mu,\ t^a = b^a + \mathrm{i}v^a$	$a = 1, \ldots, b^+$
hypermultiplets	$z^i,\ \xi^i,\ \widetilde{\xi}_i$	$i = 1, \ldots, b^-$
tensor multiplet	$B_{\mu\nu},\ \varphi,\ \xi^0,\ \widetilde{\xi}_0$	

orientifold projection: $\mathcal{O} \equiv \Omega_{\mathsf{WS}}\,(-1)^{F_L}\,\sigma$

$\mathcal{N} = 1$ multiplets:

gravity multiplet	$g_{\mu\nu}$	
vector multiplets	$A^{\hat{a}}_\mu$	$\hat{a} = 1, \ldots, \hat{n}_v = b^+ - n_{ch}$
chiral multiplets	$t^{\check{a}} = b^{\check{a}} + \mathrm{i}v^{\check{a}}$	$\check{a} = 1, \ldots, n_{ch}$
chiral/linear multiplets	$U^{\check{I}} = \xi^{\check{I}} + \mathrm{i}\,\mathrm{Im}(\mathcal{C}Z^{\check{I}})$ $\widetilde{U}_{\hat{I}} = \widetilde{\xi}_{\hat{I}} + \mathrm{i}\,\mathrm{Im}(\mathcal{C}\mathcal{G}_{\hat{I}})$	$I = (\check{I}, \hat{I}) = 0, 1, \ldots, b^-$
(projected out)	$B_{\mu\nu},\ A^0_\mu,\ A^{\check{a}}_\mu,\ t^{\hat{a}},\ U^{\hat{I}},\ \widetilde{U}_{\check{I}}$	

Parameters are restricted as $a = \bar{b}\,\mathrm{e}^{i\theta}$ and $|a|^2 = |b|^2 = \frac{1}{2}$

T.W. Grimm hep-th/0507153

Classification of SUSY solutions on the $SU(3)$ generalized geometries ($\eta^1_+ = \eta^2_+$):

M. Graña, R. Minasian, M. Petrini, A. Tomasiello hep-th/0407249 M. Graña hep-th/0509003

IIA	$a=0$ or $b=0$ (type A)	$a = b\,\mathrm{e}^{\mathrm{i}\beta}$ (type BC)	
1	$\mathcal{W}_1 = H_3^{(1)} = 0$		
	$F_0^{(1)} = \mp F_2^{(1)} = F_4^{(1)} = \mp F_6^{(1)}$	$F_{2n}^{(1)} = 0$	
8		generic β	$\beta = 0$
	$\mathcal{W}_2 = F_2^{(8)} = F_4^{(8)} = 0$	$\mathrm{Re}\mathcal{W}_2 = \mathrm{e}^{\phi} F_2^{(8)}$ $\mathrm{Im}\mathcal{W}_2 = 0$	$\mathrm{Re}\mathcal{W}_2 = \mathrm{e}^{\phi} F_2^{(8)} + \mathrm{e}^{\phi} F_4^{(8)}$ $\mathrm{Im}\mathcal{W}_2 = 0$
6	$\mathcal{W}_3 = \mp *_6 H_3^{(6)}$	$\mathcal{W}_3 = H_3^{(6)} = 0$	
3	$\overline{\mathcal{W}}_5 = 2\mathcal{W}_4 = \mp 2\mathrm{i} H_3^{(3)} = \bar{\partial}\phi$ $\bar{\partial}A = \bar{\partial}a = 0$	$F_2^{(3)} = 2\mathrm{i}\overline{\mathcal{W}}_5 = -2\mathrm{i}\bar{\partial}A = \frac{2\mathrm{i}}{3}\bar{\partial}\phi$ $\mathcal{W}_4 = 0$	

type A	NS-flux only (common to IIA, IIB, heterotic) $\mathcal{W}_1 = \mathcal{W}_2 = 0$, $\mathcal{W}_3 \neq 0$: complex
type BC	RR-flux only $\mathcal{W}_1 = \mathrm{Im}\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$, $\mathrm{Re}\mathcal{W}_2 \neq 0$, $\mathcal{W}_5 \neq 0$: symplectic

For $\mathcal{N}=1$ AdS_4 vacua: hep-th/0403049, hep-th/0407263, hep-th/0412250, hep-th/0502154, hep-th/0609124, etc.

$SU(3) \times SU(3)$ generalized geometries ($\eta^1_+ \neq \eta^2_+$ at some points)

would complete the classification. (But, it's quite hard to find all solutions.)

Appendix: generalization of the differential operator

M. Graña, J. Louis, D. Waldram hep-th/0612237

Recall that $\Phi_\pm$ *are expanded in terms of truncation bases* Σ_+ *and* Σ_-.

Whenever $c_{\|} \neq 0$, *the structure* Φ_+ *contains a scalar. This implies that at least one of the forms in the basis* Σ_+ *contains a scalar. Let us call this element* Σ_+^1, *and take the simple case where the only non-zero elements of* $\mathbb{Q}$ *are those of the form* $\mathbb{Q}_{\hat{I}}{}^1$ *(where* $\hat{I} = 1, \ldots, 2b^- + 2$*).*

Thus $\mathrm{d}(\Sigma_-)_{\hat{I}} = \mathbb{Q}_{\hat{I}}{}^1 \Sigma_+^1$ *and so if* $\mathbb{Q}_{\hat{I}}{}^1 \neq 0$ *then* $(\mathrm{d}\Sigma_-)_{\hat{I}}$ *contains a scalar.*

But this is not possible if d *is an honest exterior derivative, acting as* $\mathrm{d} : \Lambda^p \longrightarrow \Lambda^{p+1}$.

The same is true if $c_{\|}$ *is zero. In this case, there may be no scalars in any of the even forms* Σ_-, *and for an "honest"* d *operator, there should be then no one-forms in* $\mathrm{d}\Sigma_-$. *But we again see from that* Φ_- *contains a one-form, and as a consequence so do some of the elements in* Σ_-.

One way to generate a completely general charge matrix $\mathbb{Q}$ in this picture is to consider a modified operator d which is now a generic map $\mathrm{d} : U^+ \longrightarrow U^-$ which satisfies $\mathrm{d}^2 = 0$ but does not transform the degree of a form properly.

In particular, the operator d can map a p-form to a $(p-1)$-form.
Of course, this d does not act this way in conventional geometrical compactifications.

One is thus led to conjecture that to obtain a generic $\mathbb{Q}$ we must consider non-geometrical compactifications. One can still use the structures

$$\mathrm{d}\Sigma_- \ \sim \ \mathbb{Q}\Sigma_+ \, , \qquad \mathrm{d}\Sigma_+ \ \sim \ \mathcal{S}_+ \mathbb{Q}^T (\mathcal{S}_-)^{-1} \Sigma_-$$

to derive sensible effective actions, expanding in bases Σ_+ and Σ_- with a generalised d operator, but there is of course now no interpretation in terms of differential forms and the exterior derivative.

$\dashrightarrow$ introduce generalized fluxes
(not only geometrical H- and f-fluxes, but also Q- and R-fluxes)

For a geometrical background it is natural to consider forms of the type

$$\omega \;=\; \mathrm{e}^{-B}\omega_{m_1\cdots m_p}\, e^{m_1}\wedge\cdots\wedge e^{m_p} \qquad \text{w/ } \omega_{m_1\cdots m_p} \text{ constant}$$

Action of d on ω is

$$\mathrm{d}\omega \;=\; -H^{\mathsf{fl}}\wedge\omega + f\cdot\omega\,, \qquad (f\cdot\omega)_{m_1\cdots m_{p+1}} \;=\; f^a{}_{[m_1 m_2|}\omega_{a|m_3\cdots m_{p+1}]}$$

The natural nongeometric extension is then to an operator $\mathcal{D}$ such that

$$\mathcal{D} \;:=\; \mathrm{d} - H^{\mathsf{fl}}\wedge - f\cdot - Q\cdot - R\llcorner$$

$$(Q\cdot\omega)_{m_1\cdots m_{p-1}} \;=\; Q^{ab}{}_{[m_1}\omega_{|ab|m_2\cdots m_{p-1}]}\,, \qquad (R\llcorner\omega)_{m_1\cdots m_{p-3}} \;=\; R^{abc}\omega_{abcm_1\cdots m_{p-3}}$$

Requiring $\mathcal{D}^2 = 0$ implies that same conditions on fluxes as arose from the Jacobi identities for the extended Lie algebra

$$\begin{aligned}
[Z_a, Z_b] &= f_{ab}{}^c Z_c + H_{abc}X^c \\
[X^a, X^b] &= Q^{ab}{}_c X^c + R^{abc} Z_c \\
[X^a, Z_b] &= f^a{}_{bc}X^c - Q^{ac}{}_b Z_c
\end{aligned}$$

We can see nongeometric information in $\mathbb{Q}$ as contribution from Q and R.

References

(Lower dimensional) supergravity related to this topic

J. Maharana, J.H. Schwarz hep-th/9207016

L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré, T. Magri hep-th/9605032 P. Fré hep-th/9512043

N. Kaloper, R.C. Myers hep-th/9901045

E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, A. Van Proeyen hep-th/0103233

M.B. Schulz hep-th/0406001 S. Gurrieri hep-th/0408044 T.W. Grimm hep-th/0507153

B. de Wit, H. Samtleben, M. Trigiante hep-th/0507289

篁 羇篁 葋茳若

EOM, SUSY, and Bianchi identities on generalized geometry

M. Graña, R. Minasian, M. Petrini, A. Tomasiello hep-th/0407249 hep-th/0505212

M. Graña, J. Louis, D. Waldram hep-th/0505264 hep-th/0612237

D. Cassani, A. Bilal arXiv:0707.3125 D. Cassani arXiv:0804.0595

P. Koerber, D. Tsimpis arXiv:0706.1244

A.K. Kashani-Poor, R. Minasian hep-th/0611106 A. Tomasiello arXiv:0704.2613 B.y. Hou, S. Hu, Y.h. Yang arXiv:0806.3393

M. Graña, R. Minasian, M. Petrini, D. Waldram arXiv:0807.4527

SUSY AdS$_4$ vacua

D. Lüst, D. Tsimpis hep-th/0412250

C. Kounnas, D. Lüst, P.M. Petropoulos, D. Tsimpis arXiv:0707.4270 P. Koerber, D. Lüst, D. Tsimpis arXiv:0804.0614

C. Caviezel, P. Koerber, S. Kors, D. Lüst, D. Tsimpis, M. Zagermann arXiv:0806.3458

D-branes, orientifold projection, calibration, and smeared sources

B.S. Acharya, F. Benini, R. Valandro hep-th/0607223

M. Graña, R. Minasian, M. Petrini, A. Tomasiello hep-th/0609124

L. Martucci, P. Smyth hep-th/0507099 P. Koerber, D. Tsimpis arXiv:0706.1244 P. Koerber, L. Martucci arXiv:0707.1038

M. Cederwall, A. von Gussich, B.E.W. Nilsson, P. Sundell, A. Westerberg hep-th/9611159

E. Bergshoeff, P.K. Townsend hep-th/9611173

Mathematics

N.J. Hitchin math/0209099

M. Gualtieri math/0401221

Doubled formalism

C.M. Hull hep-th/0406102 hep-th/0605149 hep-th/0701203 C.M. Hull, R.A. Reid-Edwards hep-th/0503114 arXiv:0711.4818

J. Shelton, W. Taylor, B. Wecht hep-th/0508133 A. Dabholkar, C.M. Hull hep-th/0512005

A. Lawrence, M.B. Schulz, B. Wecht hep-th/0602025

G. Dall'Agata, S. Ferrara hep-th/0502066

G. Dall'Agata, M. Prezas, H. Samtleben, M. Trigiante arXiv:0712.1026 G. Dall'Agata, N. Prezas arXiv:0806.2003

C. Albertsson, R.A. Reid-Edwards, TK arXiv:0806.1783

and more...