

Realization of AdS Vacua on Generalized Geometry

based on [arXiv:0810.0937](https://arxiv.org/abs/0810.0937) [hep-th]

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We are looking for the origin of 4D physics

Physical information

- particle contents and spectra
- (broken) symmetries and interactions
- potential, vacuum and cosmological constant

4D $\mathcal{N} = 1$ supergravity:

$$S = \int \left(\frac{1}{2} R * \mathbf{1} - \frac{1}{2} F^a \wedge * F^a - K_{\mathcal{M}\bar{\mathcal{N}}} \nabla \phi^{\mathcal{M}} \wedge * \nabla \bar{\phi}^{\bar{\mathcal{N}}} - V * \mathbf{1} \right)$$

$$V = e^K \left(K^{\mathcal{M}\bar{\mathcal{N}}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\bar{\mathcal{N}}} \mathcal{W}} - 3 |\mathcal{W}|^2 \right) + \frac{1}{2} |D^a|^2$$

K : Kähler potential

\mathcal{W} : superpotential ←-- $\delta\psi_{\mu} = \nabla_{\mu} \varepsilon - e^{\frac{K}{2}} \mathcal{W} \gamma_{\mu} \varepsilon^c$

D^a : D-term ←-- $\delta\chi^a = \text{Im} F_{\mu\nu}^a \gamma^{\mu\nu} \varepsilon + i D^a \varepsilon$

10D string theories provide information of K, \mathcal{W}, D^a

via compactifications: $10 = 4 + 6$

Decompose 10D type IIA SUSY parameters:

$$\epsilon^1 = \epsilon_1 \otimes (\bar{a} \eta_-^1) + \epsilon_1^c \otimes (a \eta_+^1), \quad \epsilon^2 = \epsilon_2 \otimes (b \eta_+^2) + \epsilon_2^c \otimes (\bar{b} \eta_-^2)$$

$\delta(\text{fermions}) = 0$ provide **spinor equations** on the 6D compactified space \mathcal{M} :

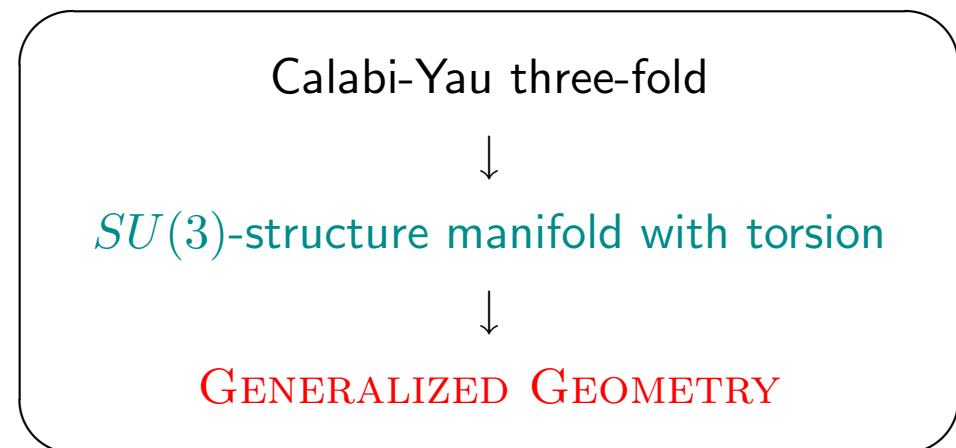
$$\delta\psi_m^{\mathcal{A}} = \left(\partial_m + \frac{1}{4} \omega_{mab} \gamma^{ab} \right) \eta^{\mathcal{A}} + (\text{3-form fluxes} \cdot \eta)^{\mathcal{A}} + (\text{other fluxes} \cdot \eta)^{\mathcal{A}} = 0$$

with a pair of $SU(3)$ invariant Weyl spinors η_+^1, η_+^2 :

$$\eta_+^2 = c_{\parallel}(y) \eta_+^1 + c_{\perp}(y) (v + iv')^m \gamma_m \eta_-^1, \quad (v - iv')^m \equiv \eta_+^{1\dagger} \gamma^m \eta_-^2$$

Information of

6D $SU(3)$ Killing spinors η_+^1, η_+^2 :



► Calabi-Yau three-folds \dashrightarrow Fluxes are highly restricted

$$\left\{ \begin{array}{ll} \text{type IIA :} & \text{No fluxes} \\ \text{type IIB :} & F_3 - \tau H \quad (\text{warped Calabi-Yau}) \\ \text{heterotic :} & \text{No fluxes} \end{array} \right.$$

► $SU(3)$ -structure manifolds \dashrightarrow Some components of fluxes can be interpreted as torsion

1: Piljin Yi, TK “*Comments on heterotic flux compactifications,*” JHEP 0607 (2006) 030, [hep-th/0605247](https://arxiv.org/abs/hep-th/0605247)

2: TK “*Index theorems on torsional geometries,*” JHEP 0708 (2007) 048, [arXiv:0704.2111](https://arxiv.org/abs/0704.2111)

► Generalized geometries \dashrightarrow Any types of fluxes can be included

All $\mathcal{N} = 1$ SUSY solutions could be classified

Search 4D SUSY vacua in type IIA theory compactified on generalized geometries

- **Moduli stabilization**

We find SUSY AdS (or Minkowski) vacua

- **Mathematical feature**

We obtain a powerful rule to evaluate vacua:

Discriminant of the superpotential governs the cosmological constant

- **Stringy effects**

We see that α' **corrections** are included in certain configurations

$\mathcal{N} = 1$ functions from generalized geometry

$$10\text{D} = 4\text{D} (\Lambda_{\text{cosmo.}} = -|\mu|^2) + 6\text{D}: \quad ds_{10}^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + ds_6^2$$

Consider polyforms Φ_{\pm}^0 on the compactified space \mathcal{M}_6 which satisfy

$$\begin{aligned} e^{-2A+\phi}(d - H\wedge)(e^{2A-\phi}\Phi_+^0) &= -2\bar{\mu} \text{Re}\Phi_-^0 \\ e^{-2A+\phi}(d - H\wedge)(e^{2A-\phi}\Phi_-^0) &= -3i \text{Im}(\mu\Phi_+^0) - dA \wedge \overline{\Phi_-^0} \\ &\quad - \frac{1}{16}e^\phi \left[(|a|^2 - |b|^2)F + i(|a|^2 + |b|^2) * \lambda(F) \right] \end{aligned}$$

► On Calabi-Yau ($d\Phi_{\pm}^0 = 0$) or $SU(3)$ -structure manifolds ($d\Phi_{\pm}^0 \neq 0$) ($\eta_+^1 = \eta_+^2$):

$$\begin{aligned} \Phi_+^0 &= e^{-iJ}, & \Phi_-^0 &= -\Omega \\ J_{mn} &= -2i \eta_+^\dagger \gamma_{mn} \eta_+, & \Omega_{mnp} &= -2i \eta_-^\dagger \gamma_{mnp} \eta_+ \end{aligned}$$

► On $SU(3) \times SU(3)$ generalized geometries ($\eta_+^1 \neq \eta_+^2$ at some points y):

$$\begin{aligned} \Phi_+^0 &= (\bar{c}_{\parallel} e^{-ij} - i\bar{c}_{\perp} w) \wedge e^{-iv\wedge v'}, & \Phi_-^0 &= (c_{\parallel} e^{-ij} + ic_{\perp} w) \wedge (v + iv') \\ J^{\mathcal{A}} &= j \pm v \wedge v', & \Omega^{\mathcal{A}} &= w \wedge (v \pm iv') \end{aligned}$$

$\Phi_{\pm} (= e^{-B} \Phi_{\pm}^0)$ are even/odd-forms *isomorphic* to $SU(3,3)$ invariant Weyl spinors



The compactified space \mathcal{M} admits Hitchin's generalized complex structures

$\mathcal{J}^{\Lambda}_{\Sigma} : T\mathcal{M} \oplus T^*\mathcal{M} \rightarrow T\mathcal{M} \oplus T^*\mathcal{M}$ with basis $\{dx^m \wedge, \iota_{\partial_n}\}$ and $(6,6)$ -signature

$$\mathcal{J}^2 = -\mathbf{1}_{12}, \quad \mathcal{J}^{\Lambda}_{\Sigma} = \langle \text{Re } \Phi_{\pm}, \Gamma^{\Lambda}_{\Sigma} \text{Re } \Phi_{\pm} \rangle$$

Φ_{\pm} : $SU(3,3)$ invariant (pure) spinors

Γ^{Λ} : $\text{Cliff}(6,6)$ gamma matrices



For details of generalized complex structures, see [APPENDIX](#)

cf.) ordinary complex structure $J^m_n : T\mathcal{M} \rightarrow T\mathcal{M}$

$$J^2 = -\mathbf{1}_6, \quad J^m_n = -2i \eta_+^{\dagger} \gamma^m_n \eta_+$$

η_+ : $SU(3)$ invariant (pure) spinor

Moduli spaces of \mathcal{M} are the special Kähler geometries of local type
 Kähler potentials, prepotentials, projective coordinates

$$K_+ = -\log i \int_{\mathcal{M}} \langle \Phi_+, \bar{\Phi}_+ \rangle = -\log i (\bar{X}^A \mathcal{F}_A - X^A \bar{\mathcal{F}}_A)$$

$$K_- = -\log i \int_{\mathcal{M}} \langle \Phi_-, \bar{\Phi}_- \rangle = -\log i (\bar{Z}^I \mathcal{G}_I - Z^I \bar{\mathcal{G}}_I)$$

Expand the even/odd-forms Φ_{\pm} by the basis forms:

$$\Phi_+ = X^A \omega_A - \mathcal{F}_A \tilde{\omega}^A, \quad \omega_A = (1, \omega_a), \quad \tilde{\omega}^A = (\tilde{\omega}^a, \text{vol}(\mathcal{M})) \quad : \quad 0, 2, 4, 6\text{-forms}$$

$$\Phi_- = Z^I \alpha_I - \mathcal{G}_I \beta^I, \quad \alpha_I = (\alpha_0, \alpha_i), \quad \beta^I = (\beta^i, \beta^0) \quad : \quad 1, 3, 5\text{-forms}$$

(if $\eta_+^1 \neq \eta_+^2$ at some points)

In the presence of fluxes, the differential operator should be **Extended**

$$d \rightarrow d_H = d - H^{\text{fl}} \wedge \rightarrow \mathcal{D} \equiv d - H^{\text{fl}} \wedge - Q \cdot - R_{\perp}$$

$$dH = 0, \quad H = H^{\text{fl}} + dB, \quad H^{\text{fl}} = m_0^I \alpha_I - e_{I0} \beta^I$$

$$\mathcal{D}\omega_A \sim m_A^I \alpha_I - e_{IA} \beta^I \quad \mathcal{D}\tilde{\omega}^A \sim -q^{IA} \alpha_I + p_I^A \beta^I$$

$$\mathcal{D}\alpha_I \sim p_I^A \omega_A + e_{IA} \tilde{\omega}^A \quad \mathcal{D}\beta^I \sim q^{IA} \omega_A + m_A^I \tilde{\omega}^A$$

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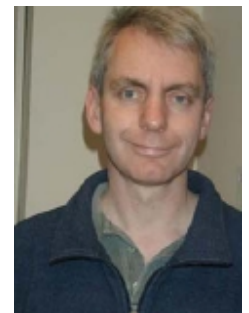
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6D compactified space \mathcal{M} is regarded as **nongeometric** (if $\eta_+^1 \neq \eta_+^2$ at some points):

$$(Q \cdot C)_{m_1 \dots m_{k-1}} \equiv Q^{ab} [{}_{m_1} C_{|ab| m_2 \dots m_{k-1}}] \quad \text{feature of T-fold}$$

$$(R_{\perp} C)_{m_1 \dots m_{k-3}} \equiv R^{abc} C_{abcm_1 \dots m_{k-3}} \quad \text{locally nongeometric background}$$



Structure group = Diffeo. + **Duality trsf.** \rightarrow *Hull's Doubled formalism*

RR-fluxes on $SU(3) \times SU(3)$ generalized geometries:

$$G = G^{\text{fl}} + \mathcal{D}A, \quad \mathcal{D}G = 0$$

$$G^{\text{fl}} = m_{\text{RR}}^A \omega_A - e_{\text{RR}A} \tilde{\omega}^A, \quad A = \xi^I \alpha_I - \tilde{\xi}_I \beta^I$$



$$G \sim G^A \omega_A - \tilde{G}_A \tilde{\omega}^A$$

$$G^A \sim m_{\text{RR}}^A + \xi^I p_I^A - \tilde{\xi}_I q^{IA}, \quad \tilde{G}_A \sim e_{\text{RR}A} - \xi^I e_{IA} + \tilde{\xi}_I m_A^I$$

Flux charges
on generalized geometries:

fluxes	charges	
NS three-form H	e_{I0}	m_0^I
torsion	e_{Ia}	m_a^I
nongeometric fluxes	p_I^A	q^{IA}
RR-fluxes	$e_{\text{RR}A}$	m_{RR}^A

$\mathcal{N} = 1$ Kähler potential and dilaton are given as

$$K = K_+ + 4\varphi$$

$$e^{-2\varphi} = \frac{|\mathcal{C}|^2}{16|a|^2|b|^2} e^{-K_-} = \frac{i}{16|a|^2|b|^2} \int_{\mathcal{M}} \langle \mathcal{C}\Phi_-, \overline{\mathcal{C}\Phi_-} \rangle$$

$\mathcal{N} = 1$ SUSY variations yield superpotential and D-term:

$$\mathcal{W} = \frac{i}{4\bar{a}b} \left[4i e^{\frac{K_-}{2} - \varphi} \int_{\mathcal{M}} \langle \Phi_+, \mathbf{D} \text{Im}(ab\Phi_-) \rangle + \frac{1}{\sqrt{2}} \int_{\mathcal{M}} \langle \Phi_+, \mathbf{G} \rangle \right]$$

$$\equiv \mathcal{W}^{\text{RR}} + U^I \mathcal{W}_I^{\text{Q}} + \tilde{U}_I \tilde{\mathcal{W}}_Q^I$$

$$\mathcal{W}^{\text{RR}} = -\frac{i}{4\bar{a}b} \left[X^A e_{\text{RRA}} - \mathcal{F}_A m_{\text{RR}}^A \right]$$

$$\mathcal{W}_I^{\text{Q}} = \frac{i}{4\bar{a}b} \left[X^A e_{IA} + \mathcal{F}_A p_I^A \right], \quad \tilde{\mathcal{W}}_Q^I = -\frac{i}{4\bar{a}b} \left[X^A m_A^I + \mathcal{F}_A q^{IA} \right]$$

$$D^A = 2e^{K_+} (K_+)^{c\bar{d}} D_c X^A \overline{D_d X^B} \left[\bar{n}^c (\sigma_x)_c^{\mathcal{B}} n_{\mathcal{B}} \right] \left(\mathcal{P}_B^x - \mathcal{N}_{BC} \tilde{\mathcal{P}}^{xC} \right)$$

$\mathcal{N} = 2$ multiplets:

$(t^a = X^a/X^0, z^i = Z^i/Z^0)$

gravity multiplet	$g_{\mu\nu}, A_\mu^0$	
vector multiplets	$A_\mu^a, t^a = b^a + iv^a$	$a = 1, \dots, b^+$
hypermultiplets	$z^i, \xi^i, \tilde{\xi}_i$	$i = 1, \dots, b^-$
tensor multiplet	$B_{\mu\nu}, \varphi, \xi^0, \tilde{\xi}_0$	



O6 orientifold projection: $\mathcal{O} \equiv \Omega_{\text{WS}} (-1)^{F_L} \sigma$

$\mathcal{N} = 1$ multiplets:

gravity multiplet	$g_{\mu\nu}$	
vector multiplets	$A_\mu^{\hat{a}}$	$\hat{a} = 1, \dots, \hat{n}_v = b^+ - n_{ch}$
chiral multiplets	$t^{\check{a}} = b^{\check{a}} + iv^{\check{a}}$	$\check{a} = 1, \dots, n_{ch}$
chiral/linear multiplets	$U^{\check{I}} = \xi^{\check{I}} + i \text{Im}(\mathcal{C}Z^{\check{I}})$ $\tilde{U}_{\hat{I}} = \tilde{\xi}_{\hat{I}} + i \text{Im}(\mathcal{C}\mathcal{G}_{\hat{I}})$	$I = (\check{I}, \hat{I}) = 0, 1, \dots, b^-$
(projected out)	$B_{\mu\nu}, A_\mu^0, A_\mu^{\check{a}}, t^{\hat{a}}, U^{\hat{I}}, \tilde{U}_{\hat{I}}$	

Parameters are restricted as $a = \bar{b}e^{i\theta}$ and $|a|^2 = |b|^2 = \frac{1}{2}$

T.W. Grimm [hep-th/0507153](https://arxiv.org/abs/hep-th/0507153)

We are ready to search SUSY vacua in 4D $\mathcal{N} = 1$ theory given by K, \mathcal{W}, D^a

NEXT: Consider **two** typical situations

- generalized geometry with RR-flux charges

$$e_{IA}, m_A^I, p_I^A, q^{IA}, e_{\text{RR}A}, m_{\text{RR}}^A$$

- $SU(3)$ -structure manifold

$$e_{IA}, m_A^I$$

Search of SUSY AdS vacua

$$\begin{aligned}
V &= e^K \left(K^{\mathcal{M}\mathcal{N}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\mathcal{N}} \mathcal{W}} - 3|\mathcal{W}|^2 \right) + \frac{1}{2} |D^a|^2 \\
&\equiv V_{\mathcal{W}} + V_D
\end{aligned}$$

Search of vacua $\partial_{\mathcal{P}} V|_* = 0$

$V_* > 0$: de Sitter space

$V_* = 0$: Minkowski space

$V_* < 0$: Anti-de Sitter space

$$0 = \partial_{\mathcal{P}} V_{\mathcal{W}} = e^K \left\{ K^{\mathcal{M}\mathcal{N}} D_{\mathcal{P}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\mathcal{N}} \mathcal{W}} + \partial_{\mathcal{P}} K^{\mathcal{M}\mathcal{N}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\mathcal{N}} \mathcal{W}} - 2\overline{\mathcal{W}} D_{\mathcal{P}} \mathcal{W} \right\}$$

$$0 = \partial_{\mathcal{P}} V_D \quad \rightarrow \quad D^a = 0$$

Consider the SUSY condition $D_{\mathcal{P}} \mathcal{W} \equiv (\partial_{\mathcal{P}} + \partial_{\mathcal{P}} K) \mathcal{W} = 0$ in various cases.

1. Set a simple prepotential: $\mathcal{F} = D_{abc} \frac{X^a X^b X^c}{X^0}$
2. Consider the simplest model: single modulus t of Φ_+ (and U of Φ_-)

The superpotential is reduced to

$$\mathcal{W} = \mathcal{W}^{\text{RR}} + U \mathcal{W}^{\text{Q}}$$

$$\mathcal{W}^{\text{RR}} = m_{\text{RR}}^0 t^3 - 3 m_{\text{RR}} t^2 + e_{\text{RR}} t + e_{\text{RR}0}$$

$$\mathcal{W}^{\text{Q}} = p_0^0 t^3 - 3 p_0 t^2 - e_0 t - e_{00}$$

Consider the SUSY condition:

$$D_t \mathcal{W} = 0 \quad \rightarrow \quad 0 = D_t \mathcal{W}^{\text{RR}} + U D_t \mathcal{W}^{\text{Q}}$$

$$D_U \mathcal{W} = 0 \quad \rightarrow \quad 0 = \frac{i}{\text{Im}U} \left(\mathcal{W}^{\text{RR}} + \text{Re}U \mathcal{W}^{\text{Q}} \right)$$

The discriminant of the superpotential \mathcal{W}^{RR} (and \mathcal{W}^{Q}) governs the SUSY solutions.

$\Delta^{\text{RR}} > 0$ case: always $\lambda^{\text{RR}} > 0$, and exists a zero point: $D_t \mathcal{W}^{\text{RR}} = 0$

$$\begin{aligned}
 D_t \mathcal{W}^{\text{RR}}|_* &= 0 \\
 t_*^{\text{RR}} &= \frac{6(3m_{\text{RR}}^0 e_{\text{RR}0} + m_{\text{RR}} e_{\text{RR}})}{\lambda^{\text{RR}}} - 2i \frac{\sqrt{3\Delta^{\text{RR}}}}{\lambda^{\text{RR}}} \\
 \mathcal{W}_*^{\text{RR}} &= -\frac{24\Delta^{\text{RR}}}{(\lambda^{\text{RR}})^3} \left(36(m_{\text{RR}})^3 + 36(m_{\text{RR}}^0)^2 e_{\text{RR}0} - 3m_{\text{RR}} \lambda^{\text{RR}} - 4i m_{\text{RR}}^0 \sqrt{3\Delta^{\text{RR}}} \right)
 \end{aligned}$$

$\Delta^{\text{RR}} < 0$ case: only $\lambda^{\text{RR}} < 0$ is physically allowed, and exists a zero point: $\mathcal{W}^{\text{RR}} = 0$

$$\begin{aligned}
 \mathcal{W}_*^{\text{RR}} &= m_{\text{RR}}^0 (t_* - e)(t_* - \alpha)(t_* - \bar{\alpha}) = 0, \quad t_* = \alpha^{\text{RR}} = \alpha_1 + i\alpha_2 \\
 \alpha_1 &= \frac{\lambda^{\text{RR}} + F^{2/3} + 12m_{\text{RR}} F^{1/3}}{12m_{\text{RR}}^0 F^{1/3}} \\
 (\alpha_2)^2 &= \frac{1}{m_{\text{RR}}^0} \left(e_{\text{RR}} - 6m_{\text{RR}} \alpha_1 + 3m_{\text{RR}}^0 (\alpha_1)^2 \right) \\
 e &= -\frac{1}{m_{\text{RR}}^0} \left(-3m_{\text{RR}} + 2m_{\text{RR}}^0 \alpha_1 \right) \\
 F &= 108(m_{\text{RR}}^0)^2 e_{\text{RR}0} + 12m_{\text{RR}}^0 \sqrt{-3\Delta^{\text{RR}}} + 108(m_{\text{RR}})^3 - 9m_{\text{RR}} \lambda^{\text{RR}} \\
 D_t \mathcal{W}^{\text{RR}}|_* &= 2i m_{\text{RR}}^0 (e - \alpha^{\text{RR}}) \alpha_2
 \end{aligned}$$

... Analysis of \mathcal{W}^{Q} is also discussed.

Three types of solutions to satisfy $0 = D_t \mathcal{W}^{\text{RR}} + U D_t \mathcal{W}^{\text{Q}}$ and $0 = \mathcal{W}^{\text{RR}} + \text{Re}U \mathcal{W}^{\text{Q}}$:

- SUSY AdS vacuum: moduli are (almost) stabilized

$$\Delta^{\text{RR}} > 0, \quad \Delta^{\text{Q}} > 0; \quad D_t \mathcal{W}^{\text{RR}}|_* = 0 = D_t \mathcal{W}^{\text{Q}}|_*$$

$$t_*^{\text{RR}} = t_*^{\text{Q}}, \quad \text{Re}U_* = -\frac{\mathcal{W}_*^{\text{RR}}}{\mathcal{W}_*^{\text{Q}}}$$

$$V_* = -3e^K |\mathcal{W}_*|^2 = -\frac{4}{[\text{Re}(\mathcal{C}\mathcal{G}_0)]^2} \sqrt{\frac{\Delta^{\text{Q}}}{3}} \ll \mathcal{O}(1)$$

- SUSY Minkowski vacuum: moduli are stabilized

$$\Delta^{\text{RR}} < 0, \quad \Delta^{\text{Q}} < 0; \quad \mathcal{W}_*^{\text{RR}} = 0 = \mathcal{W}_*^{\text{Q}}$$

$$\alpha^{\text{RR}} = \alpha^{\text{Q}}, \quad U_* = -\frac{D_t \mathcal{W}^{\text{RR}}|_*}{D_t \mathcal{W}^{\text{Q}}|_*} \neq 0$$

$$V_* = 0$$

- SUSY AdS vacua, but moduli t and U are not fixed: non-stabilized point

$$U = -\frac{D_t \mathcal{W}^{\text{RR}}(t)}{D_t \mathcal{W}^{\text{Q}}(t)}, \quad \text{Re}U = -\frac{\mathcal{W}^{\text{RR}}(t)}{\mathcal{W}^{\text{Q}}(t)}$$

1. Set $e_{\text{RRA}} = 0 = m_{\text{RR}}^A$, $p_I^A = 0 = q^{IA}$, and single modulus t of Φ_+ (and U of Φ_-)

2. Set a deformed prepotential: $\mathcal{F} = \frac{(X^t)^3}{X^0} + \sum_n N_n \frac{(X^t)^{n+3}}{(X^0)^{n+1}}$

Superpotential $\mathcal{W} = U\mathcal{W}^{\mathbb{Q}}$ with a simple setting $N_1 \neq 0$, $N_n = 0$:

$$D_t \mathcal{W}^{\mathbb{Q}} = -e_{00} + \frac{3(t - \bar{t})^2 - \partial_t P}{(t - \bar{t})^3 - P} (e_{00} + e_0 t)$$

$$P \equiv -2(N_1 t^4 - \bar{N}_1 \bar{t}^4 - 2N_1 t^3 \bar{t} + 2\bar{N}_1 t \bar{t}^3)$$

SUSY condition

$$D_t \mathcal{W} = D_U \mathcal{W} = 0$$

has a solution

$$t_*^{\mathbb{Q}} = -\frac{2e_{00}}{e_0}, \quad \text{Re} U_* = 0$$

$$\mathcal{W}_*^{\mathbb{Q}} = e_{00}, \quad \text{Im} N_1 < 0$$

$$V_* = -3e^K |\mathcal{W}_*|^2 = \frac{1}{[\text{Re}(\mathcal{CG}_0)]^2} \frac{3(e_0)^4}{16(e_{00})^2 \text{Im} N_1}$$

This is also given by the **heterotic** string compactifications on $SU(3)$ -structure manifolds **with torsion**, which carries α' corrections.

Summary and Discussions

Summary

- We found SUSY AdS (or Minkowski) vacuum on an **moduli stabilized** point
- We obtained a powerful rule to evaluate the moduli stabilized points: **Discriminants**
- We confirmed that α' **corrections** are included in certain configurations

Discussions

- More generic configurations
- Understanding the physical interpretation of **nongeometric fluxes**
- Connection to doubled formalism

Appendix: compactifications in type II strings

Moduli spaces in $\mathcal{N} = 2$ supergravity are

vector multiplets: Hodge-Kähler geometry
hypermultiplets: quaternionic geometry

We look for the origin of the moduli spaces in 10D string theories

Decomposition of vector bundle on 10D spacetime:

$$T\mathcal{M}_{9,1} = T_{3,1} \oplus F$$
$$\left\{ \begin{array}{l} T_{3,1} : \text{ a real } SO(3,1) \text{ vector bundle} \\ F : \text{ an } SO(6) \text{ vector bundle which admits a pair of } SU(3) \text{ structures} \end{array} \right.$$

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Decomposition of Lorentz symmetry:

$$Spin(9,1) \rightarrow Spin(3,1) \times Spin(6) = SL(2, \mathbb{C}) \times SU(4)$$

$$\mathbf{16} = (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}}) \quad \mathbf{16} = (\mathbf{2}, \bar{\mathbf{4}}) \oplus (\bar{\mathbf{2}}, \mathbf{4})$$

Decomposition of supersymmetry parameters (with $a, b \in \mathbb{C}$):

$$\begin{cases} \epsilon_{\text{IIA}}^1 = \varepsilon_1 \otimes (\bar{a} \eta_-^1) + \varepsilon_1^c \otimes (a \eta_+^1) \\ \epsilon_{\text{IIA}}^2 = \varepsilon_2 \otimes (b \eta_+^2) + \varepsilon_2^c \otimes (\bar{b} \eta_-^2) \end{cases} \quad \begin{cases} \epsilon_{\text{IIB}}^1 = \varepsilon_1 \otimes (\bar{a} \eta_-^1) + \varepsilon_1^c \otimes (a \eta_+^1) \\ \epsilon_{\text{IIB}}^2 = \varepsilon_2 \otimes (\bar{b} \eta_-^2) + \varepsilon_2^c \otimes (b \eta_+^2) \end{cases}$$

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$$\mathbf{16} = (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}}) \quad \mathbf{16} = (\mathbf{2}, \bar{\mathbf{4}}) \oplus (\bar{\mathbf{2}}, \mathbf{4})$$

Decomposition of supersymmetry parameters (with $a, b \in \mathbb{C}$):

$$\begin{cases} \epsilon_{\text{IIA}}^1 = \varepsilon_1 \otimes (\bar{a} \eta_-^1) + \varepsilon_1^c \otimes (a \eta_+^1) \\ \epsilon_{\text{IIA}}^2 = \varepsilon_2 \otimes (b \eta_+^2) + \varepsilon_2^c \otimes (\bar{b} \eta_-^2) \end{cases} \quad \begin{cases} \epsilon_{\text{IIB}}^1 = \varepsilon_1 \otimes (\bar{a} \eta_-^1) + \varepsilon_1^c \otimes (a \eta_+^1) \\ \epsilon_{\text{IIB}}^2 = \varepsilon_2 \otimes (\bar{b} \eta_-^2) + \varepsilon_2^c \otimes (b \eta_+^2) \end{cases}$$

Set $SU(3)$ invariant spinor $\eta_+^{\mathcal{A}}$ s.t. $\nabla^{(T)} \eta_+^{\mathcal{A}} = 0$ ($\mathcal{A} = 1, 2$)

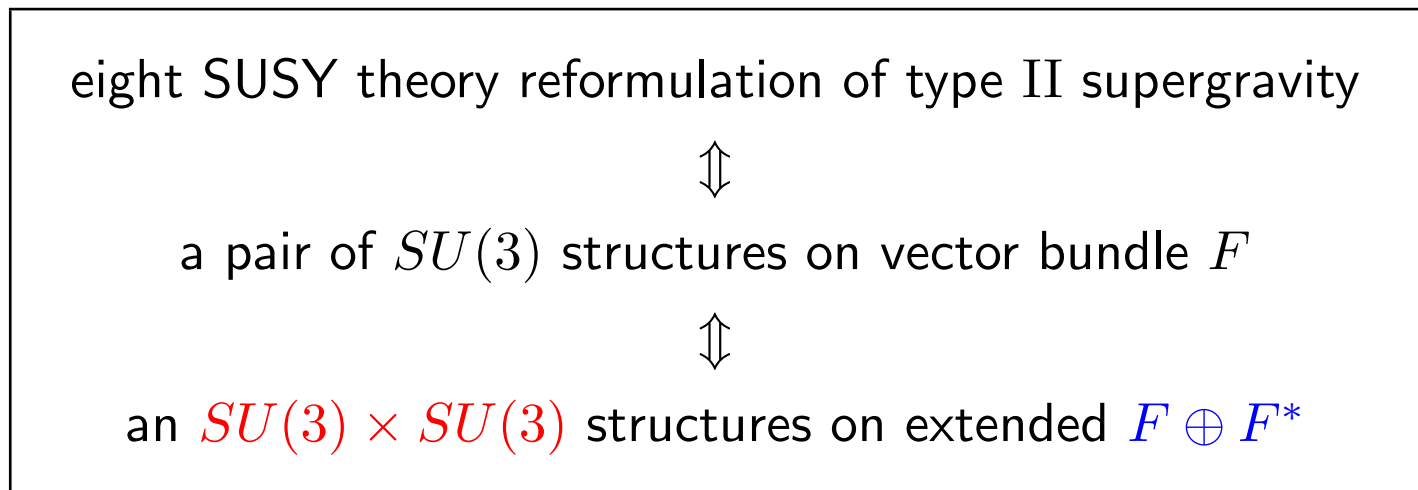
a pair of $SU(3)$ on F (η_+^1, η_+^2) \longleftrightarrow a single $SU(3)$ on F $(\eta_+^1 = \eta_+^2 = \eta_+)$

Requirement that we have a pair of $SU(3)$ structures means there is a sub-supermanifold

$$\mathcal{N}^{9,1|4+4} \subset \mathcal{M}^{9,1|16+16}$$

$$\left(\begin{array}{l} (9, 1) : \text{ bosonic degrees} \\ 4 + 4 : \text{ eight Grassmann variables as spinors of } Spin(3, 1) \text{ and singlet of } SU(3)\text{s} \end{array} \right)$$

Equivalence such as



10D spinors in type IIA in Einstein frame

$$\begin{aligned} \delta\Psi_M^{\mathcal{A}} &= \nabla_M \epsilon^{\mathcal{A}} - \frac{1}{96} e^{-\phi} \left(\Gamma_M^{PQR} H_{PQR} - 9\Gamma^{PQ} H_{MPQ} \right) \Gamma_{(11)} \epsilon^{\mathcal{A}} \\ &\quad - \sum_{n=0,2,4,6,8} \frac{1}{64n!} e^{\frac{5-n}{4}\phi} \left[(n-1)\Gamma_M^{N_1 \dots N_n} - n(9-n)\delta_M^{N_1} \Gamma^{N_2 \dots N_n} \right] F_{N_1 \dots N_n} (\Gamma_{(11)})^{n/2} (\sigma^1 \epsilon)^{\mathcal{A}} \end{aligned}$$

Split spacetime $10 = 4 + 6$

$$\epsilon^1 = \varepsilon_1 \otimes (\bar{a} \eta_-^1) + \varepsilon_1^c \otimes (a \eta_+^1), \quad \epsilon^2 = \varepsilon_2 \otimes (b \eta_+^2) + \varepsilon_2^c \otimes (\bar{b} \eta_-^2)$$

$$0 \equiv \delta\psi_m^{\mathcal{A}} = \nabla_m \eta^{\mathcal{A}} + (\text{NS-fluxes} \cdot \eta)^{\mathcal{A}} + (\text{RR-fluxes} \cdot \eta)^{\mathcal{A}}$$

Information of

6D $SU(3)$ Killing spinors η_+^1, η_+^2

Calabi-Yau three-fold



$SU(3)$ -structure manifold with torsion



generalized geometry

▶ on a single $SU(3)$:	a real two-form	$J_{mn} = \mp 2i \eta_{\pm}^{\dagger} \gamma_{mn} \eta_{\pm}$
	a complex three-form	$\Omega_{mnp} = -2i \eta_{-}^{\dagger} \gamma_{mnp} \eta_{+}$
▶ on a pair of $SU(3)$:	two real vectors	$(v - iv')^m = \eta_{+}^{1\dagger} \gamma^m \eta_{-}^2$
	(J^A, Ω^A)	$J^1 = j + v \wedge v' \quad \Omega^1 = w \wedge (v + iv')$
		$J^2 = j - v \wedge v' \quad \Omega^2 = w \wedge (v - iv')$
		(j, w) : local $SU(2)$ -invariant forms

If $\eta_{+}^1 \neq \eta_{+}^2$ globally, a pair of $SU(3)$ is reduced to global single $SU(2)$ w/ (j, w, v, v')

If $\eta_{+}^1 = \eta_{+}^2$ globally, a pair of $SU(3)$ is reduced to a single global $SU(3)$ w/ (J, Ω)

$$\eta_{+}^2 = c_{\parallel} \eta_{+}^1 + c_{\perp} (v + iv')^m \gamma_m \eta_{-}^1, \quad |c_{\parallel}|^2 + |c_{\perp}|^2 = 1$$

a pair of $SU(3)$ on $T\mathcal{M} \sim$ an $SU(3) \times SU(3)$ on $T\mathcal{M} \oplus T^*\mathcal{M}$

Appendix: Calabi-Yau compactifications

One can embed 4D $\mathcal{N} = 2$ theory into 10D type II theory
compactified on Calabi-Yau three-fold

	vector multiplets	hypermultiplets
generic	coord. of Hodge-Kähler	coord. of quaternionic
IIA on Calabi-Yau	Kähler moduli	complex moduli + RR
IIB on Calabi-Yau	complex moduli	Kähler moduli + RR

NS-NS fields in ten-dimensional spacetime are expanded as

$$\begin{aligned}\phi(x, y) &= \varphi(x) \\ G_{m\bar{n}}(x, y) &= i v^a(x) (\omega_a)_{m\bar{n}}(y), \quad G_{mn}(x, y) = i \bar{z}^k(x) \left(\frac{(\bar{\chi}_k)_{m\bar{p}\bar{q}} \Omega^{\bar{p}\bar{q}}}{\|\Omega\|^2} \right)_n(y) \\ B_2(x, y) &= B_2(x) + b^a(x) \omega_a(y)\end{aligned}$$

RR fields in type IIA are

$$\begin{aligned}C_1(x, y) &= C_1^0(x) \\ C_3(x, y) &= C_1^a(x) \omega_a(y) + \xi^K(x) \alpha_K(y) - \tilde{\xi}_K(x) \beta^K(y)\end{aligned}$$

RR fields in type IIB are

$$\begin{aligned}C_0(x, y) &= C_0(x) \\ C_2(x, y) &= C_2(x) + c^a(x) \omega_a(y) \\ C_4(x, y) &= V_1^K(x) \alpha_K(y) + \rho_a(x) \tilde{\omega}^a(y)\end{aligned}$$

cohomology class	basis	
$H^{(1,1)}$	ω_a	$a = 1, \dots, h^{(1,1)}$
$H^{(0)} \oplus H^{(1,1)}$	$\omega_A = (1, \omega_a)$	$A = 0, 1, \dots, h^{(1,1)}$
$H^{(2,2)}$	$\tilde{\omega}^a$	$a = 1, \dots, h^{(1,1)}$
$H^{(2,1)}$	χ_k	$k = 1, \dots, h^{(2,1)}$
$H^{(3)}$	(α_K, β^K)	$K = 0, 1, \dots, h^{(2,1)}$

4D type IIA $\mathcal{N} = 2$ ungauged supergravity action of bosonic fields is

$$S_{\text{IIA}}^{(4)} = \int_{\mathcal{M}_{3,1}} \left(-\frac{1}{2} R * \mathbf{1} + \frac{1}{2} \text{Re} \mathcal{N}_{AB} F^A \wedge F^B + \frac{1}{2} \text{Im} \mathcal{N}_{AB} F^A \wedge * F^B - G_{a\bar{b}} dt^a \wedge * d\bar{t}^{\bar{b}} - h_{uv} dq^u \wedge * dq^v \right)$$

gravity multiplet	$g_{\mu\nu}, C_1^0$	1
vector multiplet	C_1^a, v^a, b^a	$a = 1, \dots, h^{(1,1)}$
hypermultiplet	$z^k, \xi^k, \tilde{\xi}_k$	$k = 1, \dots, h^{(2,1)}$
tensor multiplet	$B_2, \varphi, \xi^0, \tilde{\xi}_0$	1

Various functions in the actions:

$$\begin{aligned}
 B + iJ &= (b^a + iv^a) \omega_a = t^a \omega_a & K^{\text{KS}} &= -\log \left(\frac{4}{3} \int_{\mathcal{M}_6} J \wedge J \wedge J \right) \\
 \mathcal{K}_{abc} &= \int_{\mathcal{M}_6} \omega_a \wedge \omega_b \wedge \omega_c & \mathcal{K}_{ab} &= \int_{\mathcal{M}_6} \omega_a \wedge \omega_b \wedge J = \mathcal{K}_{abc} v^c \\
 \mathcal{K}_a &= \int_{\mathcal{M}_6} \omega_a \wedge J \wedge J = \mathcal{K}_{abc} v^b v^c & \mathcal{K} &= \int_{\mathcal{M}_6} J \wedge J \wedge J = \mathcal{K}_{abc} v^a v^b v^c \\
 \text{Re} \mathcal{N}_{AB} &= \begin{pmatrix} -\frac{1}{3} \mathcal{K}_{abc} b^a b^b b^c & \frac{1}{2} \mathcal{K}_{abc} b^b b^c \\ \frac{1}{2} \mathcal{K}_{abc} b^b b^c & -\mathcal{K}_{abc} b^c \end{pmatrix} & \text{Im} \mathcal{N}_{AB} &= -\frac{\mathcal{K}}{6} \begin{pmatrix} 1 + 4G_{ab} b^a b^b & -4G_{ab} b^b \\ -4G_{ab} b^b & 4G_{ab} \end{pmatrix} \\
 G_{a\bar{b}} &= \frac{3}{2} \frac{\int_{\mathcal{M}_6} \omega_a \wedge * \omega_b}{\int_{\mathcal{M}_6} J \wedge J \wedge J} = \partial_{t^a} \bar{\partial}_{\bar{t}^{\bar{b}}} K^{\text{KS}}
 \end{aligned}$$

4D type IIB $\mathcal{N} = 2$ ungauged supergravity action of bosonic fields is

$$S_{\text{IIB}}^{(4)} = \int_{\mathcal{M}_{3,1}} \left(-\frac{1}{2} R * \mathbf{1} + \frac{1}{2} \text{Re} \mathcal{M}_{KL} F^K \wedge F^L + \frac{1}{2} \text{Im} \mathcal{M}_{KL} F^K \wedge *F^L - G_{k\bar{l}} dz^k \wedge *d\bar{z}^{\bar{l}} - h_{pq} d\tilde{q}^p \wedge *d\tilde{q}^q \right)$$

gravity multiplet	$g_{\mu\nu}, V_1^0$	1
vector multiplet	V_1^k, z^k	$k = 1, \dots, h^{(2,1)}$
hypermultiplet	v^a, b^a, c^a, ρ_a	$a = 1, \dots, h^{(1,1)}$
tensor multiplet	B_2, C_2, φ, C_0	1

Various functions in the actions:

$$\Omega = Z^K \alpha_K - \mathcal{G}_K \beta^K \quad z^k = Z^K / Z^0 \quad \mathcal{G}_{KL} = \partial_L \mathcal{G}_K$$

$$K^{\text{CS}} = -\log \left(i \int_{\mathcal{M}_6} \Omega \wedge \bar{\Omega} \right) \quad G_{k\bar{l}} = -\frac{\int \chi_k \wedge \bar{\chi}_{\bar{l}}}{\int \Omega \wedge \bar{\Omega}} = \partial_{z^k} \bar{\partial}_{\bar{z}^{\bar{l}}} K^{\text{CS}}$$

$$\mathcal{M}_{KL} = \bar{\mathcal{G}}_{KL} + 2i \frac{(\text{Im} \mathcal{G})_{KM} Z^M (\text{Im} \mathcal{G})_{LN} Z^N}{Z^N (\text{Im} \mathcal{G})_{NM} Z^M}$$

Appendix: $SU(3)$ -structure manifold with torsion

i Information from Killing spinor eqs. with torsion $D^{(T)}\eta_{\pm} = 0$ (\exists complex Weyl η_{\pm})

► Invariant p -forms on $SU(3)$ -structure manifold:

a real two-form $J_{mn} = \mp 2i \eta_{\pm}^{\dagger} \gamma_{mn} \eta_{\pm}$

a holomorphic three-form $\Omega_{mnp} = -2i \eta_{-}^{\dagger} \gamma_{mnp} \eta_{+}$

$$dJ = \frac{3}{2} \text{Im}(\overline{\mathcal{W}}_1 \Omega) + \mathcal{W}_4 \wedge J + \mathcal{W}_3 \quad d\Omega = \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \overline{\mathcal{W}}_5 \wedge \Omega$$

► Five classes of (intrinsic) torsion are given as

components	interpretations	$SU(3)$ -representations
\mathcal{W}_1	$J \wedge d\Omega$ or $\Omega \wedge dJ$	$\mathbf{1} \oplus \mathbf{1}$
\mathcal{W}_2	$(d\Omega)_0^{2,2}$	$\mathbf{8} \oplus \mathbf{8}$
\mathcal{W}_3	$(dJ)_0^{2,1} + (dJ)_0^{1,2}$	$\mathbf{6} \oplus \overline{\mathbf{6}}$
\mathcal{W}_4	$J \wedge dJ$	$\mathbf{3} \oplus \overline{\mathbf{3}}$
\mathcal{W}_5	$(d\Omega)^{3,1}$	$\mathbf{3} \oplus \overline{\mathbf{3}}$

► Vanishing torsion classes in $SU(3)$ -structure manifolds:

complex	hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = 0$
	balanced	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = 0$
	special hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	Kähler	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	conformally Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = 3\mathcal{W}_4 + 2\mathcal{W}_5 = 0$
almost complex	symplectic	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	nearly Kähler	$\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	almost Kähler	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	quasi Kähler	$\mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	semi Kähler	$\mathcal{W}_4 = \mathcal{W}_5 = 0$
	half-flat	$\text{Im}\mathcal{W}_1 = \text{Im}\mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$

Appendix: generalized geometry

Introduce a generalized almost complex structure \mathcal{J} on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$ s.t.

$$\mathcal{J} : T\mathcal{M}_d \oplus T^*\mathcal{M}_d \longrightarrow T\mathcal{M}_d \oplus T^*\mathcal{M}_d$$

$$\mathcal{J}^2 = -\mathbf{1}_{2d}$$

$$\exists O(d, d) \text{ invariant metric } L, \text{ s.t. } \mathcal{J}^T L \mathcal{J} = L$$

Structure group on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$:

$\exists L$	$GL(2d)$	\dashrightarrow	$O(d, d)$
$\mathcal{J}^2 = -\mathbf{1}_{2d}$	$O(d, d)$	\dashrightarrow	$U(d/2, d/2)$
$\mathcal{J}_1, \mathcal{J}_2$	$U_1(d/2, d/2) \cap U_2(d/2, d/2)$	\dashrightarrow	$U(d/2) \times U(d/2)$
integrable $\mathcal{J}_{1,2}$	$U(d/2) \times U(d/2)$	\dashrightarrow	$SU(d/2) \times SU(d/2)$

► Integrability is discussed by “(0, 1)” part of the complexified $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$:

$$\Pi \equiv \frac{1}{2}(\mathbf{1}_{2d} - i\mathcal{J})$$

$$\Pi A = A \quad \text{where } A = v + \zeta \text{ is a section of } T\mathcal{M}_d \oplus T^*\mathcal{M}_d$$

We call this A **i-eigenbundle** $L_{\mathcal{J}}$, whose dimension is $\dim L_{\mathcal{J}} = d$.

Integrability condition of \mathcal{J} is

$$\bar{\Pi}[\Pi(v + \zeta), \Pi(w + \eta)]_{\mathbb{C}} = 0 \quad v, w \in T\mathcal{M}_d \quad \zeta, \eta \in T^*\mathcal{M}_d$$

$$[v + \zeta, w + \eta]_{\mathbb{C}} = [v, w] + \mathcal{L}_v\eta - \mathcal{L}_w\zeta - \frac{1}{2}d(\iota_v\eta - \iota_w\zeta) : \text{ Courant bracket}$$

- ▶ Two typical examples of generalized almost complex structures:

$$\mathcal{J}_- = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I^T \end{pmatrix} \quad \text{w/ } I^2 = -\mathbf{1}_d: \text{ almost complex structure}$$

$$\mathcal{J}_+ = \begin{pmatrix} \mathbf{0} & -J^{-1} \\ J & \mathbf{0} \end{pmatrix} \quad \text{w/ } J: \text{ almost symplectic form}$$

$$\text{integrable } \mathcal{J}_- \quad \leftrightarrow \quad \text{integrable } I$$

$$\text{integrable } \mathcal{J}_+ \quad \leftrightarrow \quad \text{integrable } J$$

On a usual geometry, $J_{mn} = g_{mp} I^p_n$ is given by an $SU(3)$ invariant (pure) spinor η_+ as

$$J_{mn} = -2i \eta_+^\dagger \gamma_{mn} \eta_+ \quad \gamma^i \eta_+ = 0 \quad \gamma^{\bar{i}} \eta_+ \neq 0$$

In a similar analogy, we want to find $\text{Cliff}(6, 6)$ pure spinor(s) Φ .

∴ Compared to almost complex structures, (pure) spinors can be easily utilized in supergravity framework.

On $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$, we can define Cliff(6, 6) algebra and $Spin(6, 6)$ spinor Φ :

$$\{\Gamma^m, \Gamma^n\} = 0 \quad \{\Gamma^m, \tilde{\Gamma}_n\} = \delta_n^m \quad \{\tilde{\Gamma}_m, \tilde{\Gamma}_n\} = 0$$

Irreducible repr. of $Spin(6, 6)$ spinor is a Majorana-**Weyl**

→ a generic $Spin(6, 6)$ spinor bundle S splits to S^\pm (Weyl)

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→ a generic $Spin(6, 6)$ spinor bundle S splits to S^\pm (Weyl)

Weyl spinor bundles S^\pm are isomorphic to bundles of forms $T^*\mathcal{M}_6$:

$$\Phi_+ \in S^+ \quad \sim \quad \text{section of } \wedge^{\text{even}} T^*\mathcal{M}_6$$

$$\Phi_- \in S^- \quad \sim \quad \text{section of } \wedge^{\text{odd}} T^*\mathcal{M}_6$$

Thus we often regard a Cliff(6, 6) spinor as a section of $\wedge^{\text{even/odd}} T^*\mathcal{M}_6$

A form-valued representation of the algebra

$$\Gamma^m = dx^m \wedge, \quad \tilde{\Gamma}_n = \iota_{\partial_n}$$

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$$\Gamma^m = dx^m \wedge, \quad \tilde{\Gamma}_n = \iota_{\partial_n}$$

IF Φ is annihilated by half numbers of the Cliff(6, 6) generators:

→ Φ is called a **pure spinor**

cf.) $SU(3)$ invariant spinor η_+ is a Cliff(6) pure spinor: $\gamma^i \eta_+ = 0$

An equivalent definition of a $\text{Cliff}(6, 6)$ pure spinor is given by “Clifford action”:

$$(v + \zeta) \cdot \Phi = v^m \iota_{\partial_m} \Phi + \zeta_n dx^n \wedge \Phi \quad \text{w/ } v: \text{ vector} \quad \zeta: \text{ one-form}$$

Define the annihilator of a spinor as

$$L_\Phi \equiv \{v + \zeta \in T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \mid (v + \zeta) \cdot \Phi = 0\}$$

$$\dim L_\Phi \leq 6$$

If $\dim L_\Phi = 6$ (maximally isotropic) $\rightarrow \Phi$ is a **pure spinor**

Correspondence between pure spinors and generalized almost complex structures:

$$\mathcal{J} \leftrightarrow \Phi \quad \text{if } L_{\mathcal{J}} = L_{\Phi} \quad \text{with } \dim L_{\Phi} = 6$$

More precisely: $\mathcal{J} \leftrightarrow$ a line bundle of pure spinor Φ

\therefore) rescaling Φ does not change its annihilator L_{Φ}

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Then, we can rewrite the generalized almost complex structure as

$$\mathcal{J}_{\pm\Pi\Sigma} = \langle \text{Re}\Phi_{\pm}, \Gamma_{\Pi\Sigma} \text{Re}\Phi_{\pm} \rangle$$

w/ Mukai pairing:

$$\text{even forms: } \langle \Psi_+, \Phi_+ \rangle = \Psi_6 \wedge \Phi_0 - \Psi_4 \wedge \Phi_2 + \Psi_2 \wedge \Phi_4 - \Psi_0 \wedge \Phi_6$$

$$\text{odd forms: } \langle \Psi_-, \Phi_- \rangle = \Psi_5 \wedge \Phi_1 - \Psi_3 \wedge \Phi_3 + \Psi_1 \wedge \Phi_5$$

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$$\text{odd forms: } \langle \Psi_-, \Phi_- \rangle = \Psi_5 \wedge \Phi_1 - \Psi_3 \wedge \Phi_3 + \Psi_1 \wedge \Phi_5$$

$$\mathcal{J} \text{ is integrable} \iff \exists \text{ vector } v \text{ and one-form } \zeta \text{ s.t. } d\Phi = (v_{\perp} + \zeta \wedge)\Phi$$

$$\text{generalized CY} \iff \exists \Phi \text{ is pure s.t. } d\Phi = 0$$

$$\text{"twisted" GCY} \iff \exists \Phi \text{ is pure, and } H \text{ is closed s.t. } (d - H \wedge)\Phi = 0$$

A $\text{Cliff}(6, 6)$ spinor can also be mapped to a bispinor:

$$C \equiv \sum_k \frac{1}{k!} C_{m_1 \dots m_k}^{(k)} dx^{m_1} \wedge \dots \wedge dx^{m_k} \quad \longleftrightarrow \quad \mathcal{C} \equiv \sum_k \frac{1}{k!} C_{m_1 \dots m_k}^{(k)} \gamma_{\alpha\beta}^{m_1 \dots m_k}$$

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On a geometry of a **single** $SU(3)$ -structure, the following two $SU(3, 3)$ spinors:

$$\begin{aligned} \Phi_{0+} &= \eta_+ \otimes \eta_+^\dagger = \frac{1}{4} \sum_{k=0}^6 \frac{1}{k!} \eta_+^\dagger \gamma_{m_k \dots m_1} \eta_+ \gamma^{m_1 \dots m_k} = \frac{1}{8} e^{-iJ} \\ \Phi_{0-} &= \eta_+ \otimes \eta_-^\dagger = \frac{1}{4} \sum_{k=0}^6 \frac{1}{k!} \eta_-^\dagger \gamma_{m_k \dots m_1} \eta_+ \gamma^{m_1 \dots m_k} = -\frac{i}{8} \Omega \end{aligned}$$

Check purity: $(\delta + iJ)_m{}^n \gamma_n \eta_+ \otimes \eta_\pm^\dagger = 0 = \eta_+ \otimes \eta_\pm^\dagger \gamma_n (\delta \mp iJ)^n{}_m$

One-to-one correspondence: $\Phi_{0-} \leftrightarrow \mathcal{J}_1, \quad \Phi_{0+} \leftrightarrow \mathcal{J}_2$

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One-to-one correspondence: $\Phi_{0-} \leftrightarrow \mathcal{J}_1, \quad \Phi_{0+} \leftrightarrow \mathcal{J}_2$

On a generic geometry of a **pair** of $SU(3)$ -structures defined by (η_+^1, η_+^2)

$$\begin{aligned} \Phi_{0+} &= \eta_+^1 \otimes \eta_+^{2\dagger} = \frac{1}{8} (\bar{c}_\parallel e^{-ij} - i\bar{c}_\perp w) \wedge e^{-iv \wedge v'} & |c_\parallel|^2 + |c_\perp|^2 &= 1 \\ \Phi_{0-} &= \eta_+^1 \otimes \eta_-^{2\dagger} = -\frac{1}{8} (c_\perp e^{-ij} + ic_\parallel w) \wedge (v + iv') \\ && \Phi_\pm &= e^{-B} \Phi_{0\pm} \end{aligned}$$

Each Φ_{\pm} defines an $SU(3, 3)$ structure on E . Common structure is $SU(3) \times SU(3)$.

(F is extended to E by including e^{-B})

Compatibility requires

$$\begin{aligned}\langle \Phi_+, V \cdot \Phi_- \rangle &= \langle \bar{\Phi}_+, V \cdot \Phi_- \rangle = 0 \quad \text{for } \forall V = x + \xi \\ \langle \Phi_+, \bar{\Phi}_+ \rangle &= \langle \Phi_-, \bar{\Phi}_- \rangle\end{aligned}$$

[go back to SECTION 2](#)

Start with a real form $\chi_f \in \wedge^{\text{even/odd}} F^*$ (associated with a real $Spin(6,6)$ spinor χ_s)

Regard χ_f as a stable form satisfying

$$q(\chi_f) = -\frac{1}{4} \langle \chi_f, \Gamma_{\Pi\Sigma} \chi_f \rangle \langle \chi_f, \Gamma^{\Pi\Sigma} \chi_f \rangle \in \wedge^6 F^* \otimes \wedge^6 F^*$$

$$U = \{ \chi_f \in \wedge^{\text{even/odd}} F^* : q(\chi_f) < 0 \}$$

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Define a Hitchin function

$$H(\chi_f) \equiv \sqrt{-\frac{1}{3}q(\chi_f)} \in \wedge^6 F^*$$

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Then we can get another real form $\hat{\chi}_f$ and a complex form Φ_f by Mukai pairing

$$\begin{aligned} \langle \hat{\chi}_f, \chi_f \rangle &= -dH(\chi_f) \quad \text{i.e.,} \quad \hat{\chi}_f = -\frac{\partial H(\chi_f)}{\partial \chi_f} \\ \dashrightarrow \quad \Phi_f &\equiv \frac{1}{2}(\chi_f + i\hat{\chi}_f) \quad H(\Phi_f) = i\langle \Phi_f, \bar{\Phi}_f \rangle \end{aligned}$$

Hitchin showed: Φ_f is a (form corresponding to) **pure spinor!**

Consider the space of pure spinors Φ ...

Mukai pairing $\langle *, * \rangle$	\longrightarrow	symplectic structure
Hitchin function $H(*)$	\longrightarrow	complex structure



The space of pure spinor is Kähler

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The space of pure spinor is Kähler

Quotienting this space by the \mathbb{C}^* action $\Phi \rightarrow \lambda\Phi$ for $\lambda \in \mathbb{C}^*$

--> The space becomes a **local** special Kähler geometry with Kähler potential K :

$$e^{-K} = H(\Phi) = i\langle \Phi, \bar{\Phi} \rangle = i(\bar{X}^A \mathcal{F}_A - X^A \bar{\mathcal{F}}_A) \in \wedge^6 F^*$$

X^A : holomorphic projective coordinates

\mathcal{F}_A : derivative of prepotential \mathcal{F} , i.e., $\mathcal{F}_A = \partial\mathcal{F}/\partial X^A$

These are nothing but objects which we want to introduce in $\mathcal{N} = 2$ supergravity!

Spaces of pure spinors Φ_{\pm} on $F \oplus F^*$ with $SU(3) \times SU(3)$ structures

||

special Kähler geometries of local type = Hodge-Kähler geometries

For the single $SU(3)$ -structure case:

$$\begin{aligned}\Phi_+ &= \frac{1}{8} e^{-B-iJ} & K_+ &= -\log \left(\frac{1}{48} J \wedge J \wedge J \right) \\ \Phi_- &= -\frac{i}{8} e^{-B} \Omega & K_- &= -\log \left(\frac{i}{64} \Omega \wedge \bar{\Omega} \right)\end{aligned}$$

Structure of forms is exactly same as the one in the case of Calabi-Yau compactification!

We should truncate Kaluza-Klein massive modes from these forms to obtain 4D supergravity.

Appendix: setup in $\mathcal{N} = 1$ theory

Functionals are given by two Kähler potentials on two Hodge-Kähler geometries of Φ_{\pm} :

$$\begin{aligned}
 K &= K_+ + 4\varphi \\
 K_+ &= -\log i \int_{\mathcal{M}} \langle \Phi_+, \bar{\Phi}_+ \rangle = -\log i (\bar{X}^A \mathcal{F}_A - X^A \bar{\mathcal{F}}_A) \\
 K_- &= -\log i \int_{\mathcal{M}} \langle \Phi_-, \bar{\Phi}_- \rangle = -\log i (\bar{Z}^I \mathcal{G}_I - Z^I \bar{\mathcal{G}}_I) \\
 \int_{\mathcal{M}} \text{vol}_6 &= \frac{1}{8} e^{-K_{\pm}} = e^{-2\varphi + 2\phi^{(10)}}
 \end{aligned}$$

Introduce $\mathcal{C} = \sqrt{2}ab e^{-\phi^{(10)}} = 4ab e^{\frac{K_-}{2} - \varphi}$

$$\begin{aligned}
 \therefore e^{-2\varphi} &= \frac{|\mathcal{C}|^2}{16|a|^2|b|^2} e^{-K_-} = \frac{i}{16|a|^2|b|^2} \int_{\mathcal{M}} \langle \mathcal{C}\Phi_-, \bar{\mathcal{C}}\bar{\Phi}_- \rangle \\
 &= \frac{1}{8|a|^2|b|^2} \left[\text{Im}(\mathcal{C}Z^I) \text{Re}(\mathcal{C}\mathcal{G}_I) - \text{Re}(\mathcal{C}Z^I) \text{Im}(\mathcal{C}\mathcal{G}_I) \right]
 \end{aligned}$$

See the SUSY variation of 4D $\mathcal{N} = 2$ gravitinos:

$$\delta\psi_{\mathcal{A}\mu} = \nabla_{\mu}\varepsilon_{\mathcal{A}} - S_{\mathcal{A}\mathcal{B}}\gamma_{\mu}\varepsilon^{\mathcal{B}} + \dots$$

$$S_{\mathcal{A}\mathcal{B}} = \frac{i}{2}e^{\frac{K_{+}}{2}} \begin{pmatrix} \mathcal{P}^1 - i\mathcal{P}^2 & -\mathcal{P}^3 \\ -\mathcal{P}^3 & -\mathcal{P}^1 - i\mathcal{P}^2 \end{pmatrix}_{\mathcal{A}\mathcal{B}}$$

The components are also written by Φ_{\pm} :

$$\mathcal{P}^1 - i\mathcal{P}^2 = 2e^{\frac{K_{-}}{2} + \varphi} \int_{\mathcal{M}} \langle \Phi_{+}, \mathcal{D}\Phi_{-} \rangle, \quad \mathcal{P}^1 + i\mathcal{P}^2 = 2e^{\frac{K_{-}}{2} + \varphi} \int_{\mathcal{M}} \langle \Phi_{+}, \mathcal{D}\bar{\Phi}_{-} \rangle$$

$$\mathcal{P}^3 = -\frac{1}{\sqrt{2}}e^{2\varphi} \int_{\mathcal{M}} \langle \Phi_{+}, G \rangle$$

Note: $\hat{\Psi}_{\mathcal{A}\mu} = \Psi_{\mathcal{A}\mu} + \frac{1}{2}\Gamma_{\mu}{}^m\Psi_m^{\mathcal{A}} = \psi_{\mathcal{A}\mu\pm} \otimes \eta_{+} + \psi_{\mathcal{A}\mu\mp} \otimes \eta_{-} + \dots$

4D $\mathcal{N} = 1$ fermions given by the SUSY truncation from 4D $\mathcal{N} = 2$ system:

SUSY parameter : $\varepsilon \equiv \bar{n}^{\mathcal{A}} \varepsilon_{\mathcal{A}} = a \varepsilon_1 + \bar{b} \varepsilon_2$

gravitino : $\psi_{\mu} \equiv \bar{n}^{\mathcal{A}} \psi_{\mathcal{A}\mu} = a \psi_{1\mu} + \bar{b} \psi_{2\mu}, \quad \tilde{\psi}_{\mu} \equiv (b \psi_{1\mu} - \bar{a} \psi_{2\mu})$

gauginos : $\chi^A \equiv -2 e^{\frac{K_+}{2}} D_b X^A (\bar{n}^{\mathcal{C}} \epsilon_{\mathcal{C}\mathcal{E}} \chi^{\mathcal{E}b})$

where $\bar{n}^{\mathcal{A}} = (a, \bar{b}), \quad \epsilon_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

SUSY variations yield the superpotential and the D-term:

$$\delta\psi_\mu = \nabla_\mu \varepsilon - \bar{n}^A S_{AB} n^{*B} \gamma_\mu \varepsilon^c \equiv \nabla_\mu \varepsilon - e^{\frac{K}{2}} \mathcal{W} \gamma_\mu \varepsilon^c$$

$$\delta\tilde{\psi}_\mu = 0$$

$$\delta\chi^A = \text{Im}F_{\mu\nu}^A \gamma^{\mu\nu} \varepsilon + i D^A \varepsilon$$

$$\mathcal{W} = \frac{i}{4\bar{a}b} \left[4i e^{\frac{K_-}{2} - \varphi} \int_{\mathcal{M}} \langle \Phi_+, \mathcal{D}\text{Im}(ab\Phi_-) \rangle + \frac{1}{\sqrt{2}} \int_{\mathcal{M}} \langle \Phi_+, G \rangle \right]$$

$$\equiv \mathcal{W}^{\text{RR}} + U^I \mathcal{W}_I^{\text{Q}} + \tilde{U}_I \tilde{\mathcal{W}}_I^{\text{Q}}$$

$$\mathcal{W}^{\text{RR}} = -\frac{i}{4\bar{a}b} \left[X^A e_{\text{RRA}} - \mathcal{F}_A m_{\text{RR}}^A \right]$$

$$\mathcal{W}_I^{\text{Q}} = \frac{i}{4\bar{a}b} \left[X^A e_{IA} + \mathcal{F}_A p_I^A \right], \quad \tilde{\mathcal{W}}_I^{\text{Q}} = -\frac{i}{4\bar{a}b} \left[X^A m_A^I + \mathcal{F}_A q^{IA} \right]$$

$$D^A = 2e^{K_+} (K_+)^{cd} D_c X^A \overline{D_d X^B} [\bar{n}^C (\sigma_x)_C^B n_B] \left(\mathcal{P}_B^x - \mathcal{N}_{BC} \tilde{\mathcal{P}}^{xC} \right)$$

Classification of SUSY solutions on the $SU(3)$ generalized geometries ($\eta_+^1 = \eta_+^2$):

M. Graña, R. Minasian, M. Petrini, A. Tomasiello [hep-th/0407249](https://arxiv.org/abs/hep-th/0407249)

IIA	$a = 0$ or $b = 0$ (type A)	$a = b e^{i\beta}$ (type BC)
1	$\mathcal{W}_1 = H_3^{(1)} = 0$	
	$F_0^{(1)} = \mp F_2^{(1)} = F_4^{(1)} = \mp F_6^{(1)}$	$F_{2n}^{(1)} = 0$
8	$\mathcal{W}_2 = F_2^{(8)} = F_4^{(8)} = 0$	generic β
		$\beta = 0$
		$\text{Re}\mathcal{W}_2 = e^\phi F_2^{(8)}$
		$\text{Re}\mathcal{W}_2 = e^\phi F_2^{(8)} + e^\phi F_4^{(8)}$
		$\text{Im}\mathcal{W}_2 = 0$
		$\text{Im}\mathcal{W}_2 = 0$
6	$\mathcal{W}_3 = \mp *_6 H_3^{(6)}$	$\mathcal{W}_3 = H_3^{(6)} = 0$
3	$\bar{\mathcal{W}}_5 = 2\mathcal{W}_4 = \mp 2iH_3^{(3)} = \bar{\partial}\phi$	$F_2^{(3)} = 2i\bar{\mathcal{W}}_5 = -2i\bar{\partial}A = \frac{2i}{3}\bar{\partial}\phi$
	$\bar{\partial}A = \bar{\partial}a = 0$	$\mathcal{W}_4 = 0$

type A	NS-flux only (common to IIA, IIB, heterotic) $\mathcal{W}_1 = \mathcal{W}_2 = 0, \mathcal{W}_3 \neq 0$: complex
type BC	RR-flux only $\mathcal{W}_1 = \text{Im}\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0, \text{Re}\mathcal{W}_2 \neq 0, \mathcal{W}_5 \neq 0$: symplectic

For $\mathcal{N} = 1$ AdS₄ vacua: [hep-th/0403049](https://arxiv.org/abs/hep-th/0403049), [hep-th/0407263](https://arxiv.org/abs/hep-th/0407263), [hep-th/0412250](https://arxiv.org/abs/hep-th/0412250), [hep-th/0502154](https://arxiv.org/abs/hep-th/0502154), [hep-th/0609124](https://arxiv.org/abs/hep-th/0609124), etc.

$SU(3) \times SU(3)$ generalized geometries ($\eta_+^1 \neq \eta_+^2$ at some points)

would complete the classification. (But, it's quite hard to find all solutions.)

Appendix: generalization of the differential operator

Recall that Φ_{\pm} are expanded in terms of truncation bases Σ_+ and Σ_- .

Whenever $c_{\parallel} \neq 0$, the structure Φ_+ contains a scalar. This implies that at least one of the forms in the basis Σ_+ contains a *scalar*. Let us call this element Σ_+^1 , and take the simple case where the only non-zero elements of \mathbb{Q} are those of the form $\mathbb{Q}_{\hat{I}}^1$ (where $\hat{I} = 1, \dots, 2b^- + 2$).

Thus $d(\Sigma_-)_{\hat{I}} = \mathbb{Q}_{\hat{I}}^1 \Sigma_+^1$ and so if $\mathbb{Q}_{\hat{I}}^1 \neq 0$ then $(d\Sigma_-)_{\hat{I}}$ contains a *scalar*.

But this is *not possible* if d is an honest exterior derivative, acting as $d : \Lambda^p \rightarrow \Lambda^{p+1}$.

The same is true if c_{\parallel} is zero. In this case, there may be no scalars in any of the even forms Σ_- , and for an “honest” d operator, there should be then *no one-forms* in $d\Sigma_-$. But we again see from that Φ_- contains a *one-form*, and as a consequence so do some of the elements in Σ_- .

One way to generate a completely general charge matrix \mathbb{Q} in this picture is to consider a modified operator \mathfrak{d} which is now a generic map $\mathfrak{d} : U^+ \rightarrow U^-$ which satisfies $\mathfrak{d}^2 = 0$ but does not transform the degree of a form properly.

In particular, the operator \mathfrak{d} can map a p -form to a $(p - 1)$ -form.
 Of course, this \mathfrak{d} does *not* act this way in *conventional* geometrical compactifications.

One is thus led to conjecture that to obtain a generic \mathbb{Q} we must consider non-geometrical compactifications. One can still use the structures

$$d\Sigma_- \sim \mathbb{Q}\Sigma_+, \quad d\Sigma_+ \sim \mathcal{S}_+ \mathbb{Q}^T (\mathcal{S}_-)^{-1} \Sigma_-$$

to derive sensible effective actions, expanding in bases Σ_+ and Σ_- with a generalised \mathfrak{d} operator, but there is of course now *no interpretation* in terms of differential forms and the exterior derivative.

--> introduce generalized fluxes
 (not only geometrical H - and f -fluxes, but also Q - and R -fluxes)

For a geometrical background it is natural to consider forms of the type

$$\omega = e^{-B} \omega_{m_1 \dots m_p} e^{m_1} \wedge \dots \wedge e^{m_p} \quad \text{w/ } \omega_{m_1 \dots m_p} \text{ constant}$$

Action of d on ω is

$$d\omega = -H^{\text{fl}} \wedge \omega + f \cdot \omega, \quad (f \cdot \omega)_{m_1 \dots m_{p+1}} = f^a_{[m_1 m_2} \omega_a | m_3 \dots m_{p+1}]$$

The natural **nongeometric extension** is then to an operator \mathcal{D} such that

$$\mathcal{D} := d - H^{\text{fl}} \wedge -f \cdot -Q \cdot -R \lrcorner$$

$$(Q \cdot \omega)_{m_1 \dots m_{p-1}} = Q^{ab}{}_{[m_1} \omega_{|ab| m_2 \dots m_{p-1}]}, \quad (R \lrcorner \omega)_{m_1 \dots m_{p-3}} = R^{abc} \omega_{abcm_1 \dots m_{p-3}}$$

Requiring $\mathcal{D}^2 = 0$ implies that same conditions on fluxes as arose from the Jacobi identities for the extended Lie algebra

$$\begin{aligned} [Z_a, Z_b] &= f_{ab}{}^c Z_c + H_{abc} X^c \\ [X^a, X^b] &= Q^{ab}{}_c X^c + R^{abc} Z_c \\ [X^a, Z_b] &= f^a{}_{bc} X^c - Q^{ac}{}_b Z_c \end{aligned}$$

We can see nongeometric information in \mathbb{Q} as contribution from Q and R .

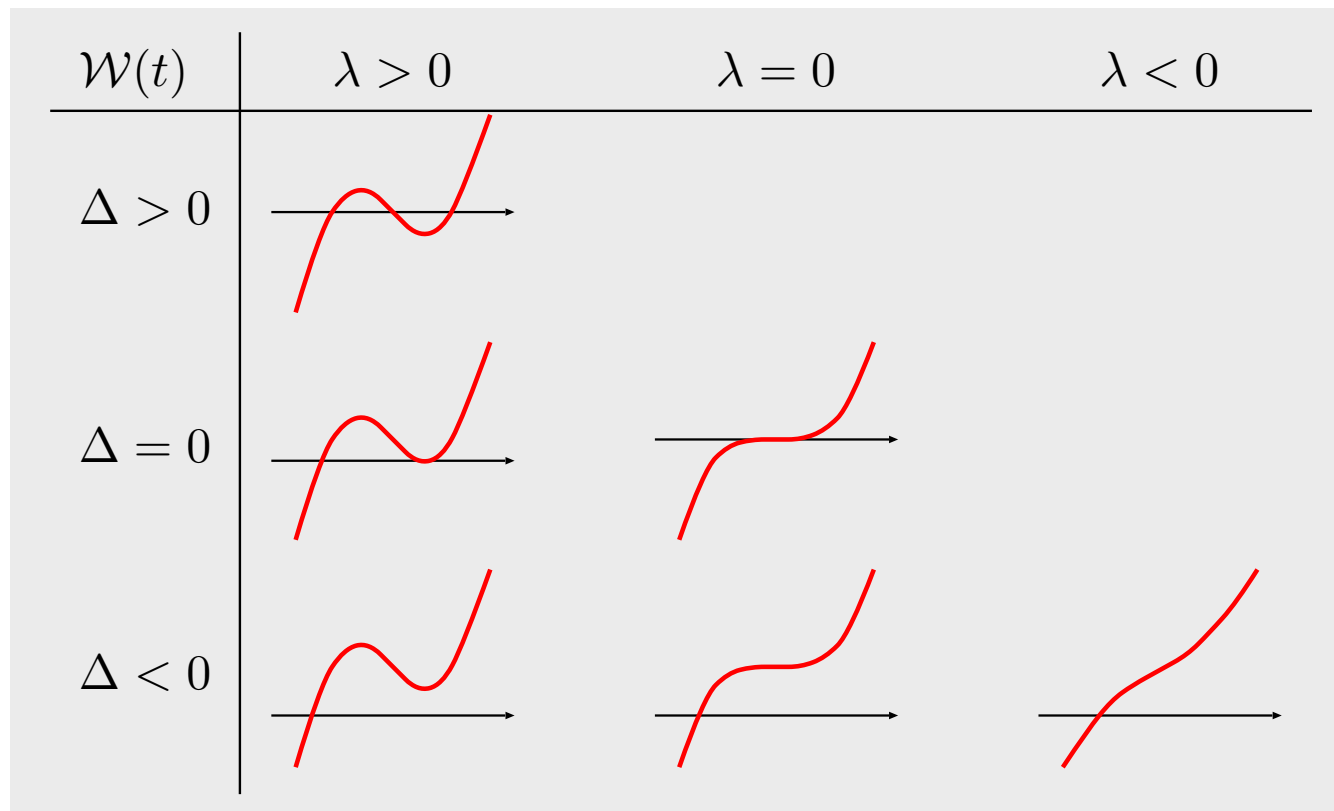
Appendix: discriminant of cubic equation

Consider a cubic function and its derivative:
$$\begin{cases} \mathcal{W}(t) = at^3 + bt^2 + ct + d \\ \partial_t \mathcal{W}(t) = 3at^2 + 2bt + c \end{cases}$$

Discriminants $\Delta(\mathcal{W})$ and $\Delta(\partial_t \mathcal{W})$ are

$$\Delta(\mathcal{W}) \equiv \Delta = -4b^3d + b^2c^2 - 4ac^3 + 18abcd - 27a^2d^2$$

$$\Delta(\partial_t \mathcal{W}) \equiv \lambda = 4(b^2 - 3ac)$$



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