

Generalized/Doubled/Nongeometric String Backgrounds

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Lower dimensional supergravity related to this issue

J. Maharana, J.H. Schwarz [hep-th/9207016](#)

L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré, T. Magri [hep-th/9605032](#)

N. Kaloper, R.C. Myers [hep-th/9901045](#)

T.W. Grimm [hep-th/0507153](#)

Generalized geometry

N.J. Hitchin [math/0209099](#)

M. Gualtieri [math/0401221](#)

M. Graña, R. Minasian, M. Petrini, A. Tomasiello [hep-th/0406137](#) [hep-th/0409176](#) [hep-th/0505212](#) [hep-th/0609124](#)

M. Graña, J. Louis, D. Waldram [hep-th/0505264](#) [hep-th/0612237](#)

M. Graña [hep-th/0509003](#)

I. Benmachiche, T.W. Grimm [hep-th/0602241](#)

[Workshop at DESY, Feb. 2007](#)

Doubled formalism

C.M. Hull [hep-th/0406102](#) [hep-th/0605149](#) [hep-th/0701203](#)

C.M. Hull, R.A. Reid-Edwards [hep-th/0503114](#) [arXiv:0711.4818](#)

A. Dabholkar, C.M. Hull [hep-th/0512005](#)

A. Lawrence, M.B. Schulz, B. Wecht [hep-th/0602025](#)

G. Dall'Agata, M. Prezas, H. Samtleben, M. Trigiante [arXiv:0712.1026](#)

and more...

Introduction

▶ $\mathcal{N} = 2$ supergravity

highly symmetric (controllable), dynamical (non-trivial), connectable to Seiberg-Witten, etc..
dictated by holomorphic functionals (prepotentials)

▶ $\mathcal{N} = 1$ supergravity

highly dynamical, less symmetric, connectable to (SUSY) GUTs, etc..
dictated by Kähler potential and superpotential

many ways to derive them from type II and heterotic string theories

Moduli in $\mathcal{N} = 2$ supergravity: [Appendix](#)

	vector multiplet	hypermultiplet
generic	coord. of Hodge-Kähler	coord. of quaternionic
IIA on Calabi-Yau	Kähler moduli	complex moduli + RR
IIB on Calabi-Yau	complex moduli	Kähler moduli + RR

Duality relations in $\mathcal{N} = 2$ theories:

$$\begin{array}{lll}
 \text{type IIA} & \longleftrightarrow & \text{type IIB} & \text{T-duality, mirror symmetry} \\
 \text{type II/CY}_3 & \longleftrightarrow & \text{heterotic}/[K3 \times T^2] & \text{S-duality}
 \end{array}$$

Reduction to $\mathcal{N} = 1$ supergravity is given in terms of orientifold planes

$$K^{\text{KS}} = -\log \left(\frac{4}{3} \int_{\text{CY}_3} J \wedge J \wedge J \right)$$

$$K^{\text{CS}} = -\log \left(i \int_{\text{CY}_3} \Omega \wedge \bar{\Omega} \right)$$

$$W_{\text{IIA,RR}} = i e^\phi \int_{\text{CY}_3} G_A \wedge e^{-B-iJ}$$

$$W_{\text{IIB,RR}} = i e^\phi \int_{\text{CY}_3} G_B \wedge \Omega$$

$$W_{H\text{-flux}} = \int_{\text{CY}_3} H_3 \wedge \Omega$$

$$F_n = dC_{n-1} - H_3 \wedge C_{n-3} \equiv e^B G$$

$$G_A = G_0 + G_2 + G_4 + G_6 \quad G_B = G_3$$

$$J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \bar{\Omega} \quad J \wedge \Omega = 0 = B \wedge \Omega$$

Question 1: Generic supersymmetric effective theory beyond Calabi-Yau geometry?

- ☹ condition on geometry from supersymmetry? \rightarrow $SU(3)$ -structure manifold Appendix
- ☹ identify “light” modes?
- ☹ generic form of Kähler potentials and superpotentials?

$$\begin{aligned}
 ds_{1,9}^2 &= e^{2A(y)} g_{\mu\nu} dx^\mu \otimes dx^\nu + g_{ij} dy^i \otimes dy^j \\
 \delta\psi_i &= \left(\partial_i + \frac{1}{4} \omega_{iab} \gamma^{ab} \right) \eta - \frac{1}{4} H_{ijk} \gamma^{jk} \eta + \dots \equiv 0 \\
 \delta\lambda &= -\frac{1}{4} \left(\gamma^i \partial_i \phi - \frac{1}{6} H_{ijk} \gamma^{ijk} \right) \eta + \dots \equiv 0
 \end{aligned}$$

$$(d - H_3 \wedge)(e^{4A} *_6 F) = 0 \qquad (d - H_3 \wedge)F = \delta(\text{source})$$

$$d(e^{4A-2\phi} *_6 H_3) = \mp e^{4A} F_n \wedge *_6 F_{n+2}$$

$$dH_3 = 0$$

Question 2: Modification of dualities among string theories by fluxes?

- ☹ T-duality (mirror symmetry) from (non-)Calabi-Yau to what?
- ☹ S-duality and U-duality symmetries?
- ☹ Find more non-trivial relations?

$$\begin{array}{ccc}
 \wedge^{\text{even}} T^* \mathcal{M}_6 & & \wedge^{\text{odd}} T^* \mathcal{M}_6 \\
 e^{-B-iJ} & \longleftrightarrow & \Omega \\
 G_A = G_0 + G_2 + G_4 + G_6 & & G_B = G_3
 \end{array}$$

Generically, a Calabi-Yau with non-trivial fluxes does **not** yield a supersymmetric solution...

How should we derive modified Kähler/superpotentials?

Extend geometrical information of compactified space

Decomposition of vector bundle on ten-dimensional spacetime:

$$T\mathcal{M}_{1,9} = T_{1,3} \oplus F$$

$$\begin{cases} T_{1,3} : & \text{a real } SO(1,3) \text{ vector bundle} \\ F : & \text{an } SO(6) \text{ vector bundle which admits a pair of } SU(3) \text{ structures} \end{cases}$$

10-dimensional spacetime itself is not decomposed yet, i.e., do not yet consider truncation of modes.

Decomposition of Lorentz symmetry:

$$Spin(1,9) \rightarrow Spin(1,3) \times Spin(6) = SL(2, \mathbb{C}) \times SU(4)$$

$$\mathbf{16}_1 = (\mathbf{2}, \mathbf{4})_1 \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}})_1 \quad \mathbf{16}_2 = (\mathbf{2}, \bar{\mathbf{4}})_2 \oplus (\bar{\mathbf{2}}, \mathbf{4})_2$$

Decomposition of supersymmetry parameters (with $a, b \in \mathbb{C}$):

$$\begin{cases} \epsilon_{\text{IIA}}^1 = \xi_+^1 \otimes (a\eta_+^1) + \xi_-^1 \otimes (\bar{a}\eta_-^1) \\ \epsilon_{\text{IIA}}^2 = \xi_+^2 \otimes (\bar{b}\eta_-^2) + \xi_-^2 \otimes (b\eta_+^2) \end{cases} \quad \begin{cases} \epsilon_{\text{IIB}}^1 = \xi_+^1 \otimes (a\eta_+^1) + \xi_-^1 \otimes (\bar{a}\eta_-^1) \\ \epsilon_{\text{IIB}}^2 = \xi_+^2 \otimes (b\eta_+^2) + \xi_-^2 \otimes (\bar{b}\eta_-^2) \end{cases}$$

Set $SU(3)$ invariant spinor η_+^A s.t. $D^{(T)}\eta_+^A = 0$ ($A = 1, 2$): [Appendix](#)

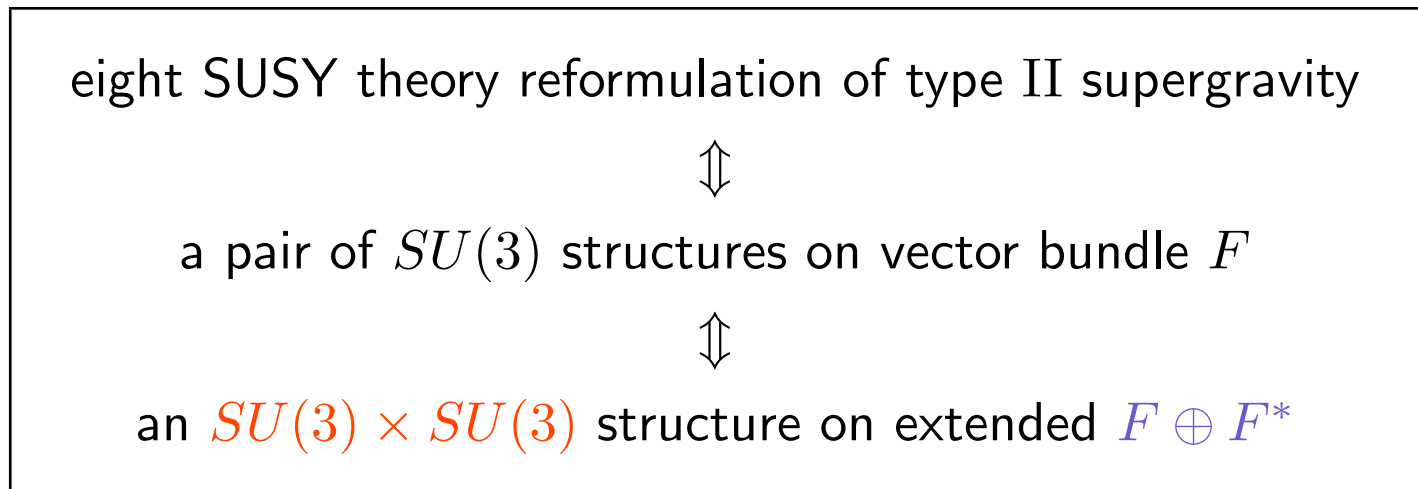
a pair of $SU(3)$ on F $(\eta_+^1, \eta_+^2) \iff$ a single $SU(3)$ on F $(\eta_+^1 = \eta_+^2 = \eta_+)$

Requirement that we have a pair of $SU(3)$ structures means there is a sub-supermanifold

$$\mathcal{N}^{1,9|4+4} \subset \mathcal{M}^{1,9|16+16}$$

((1,9): bosonic degrees
 4+4: eight Grassmann variables as spinors of $Spin(1,3)$ and singlet of $SU(3)$ s)

Equivalence such as



Entrance Gate to generalized geometry

i Information from Killing spinor eqs. with torsion $D^{(T)}\eta_{\pm} = 0$ (\exists complex Weyl η_{\pm})

► Invariant p -forms on $SU(3)$ -structure manifold:

a real two-form $J_{ij} = \mp 2i \eta_{\pm}^{\dagger} \gamma_{ij} \eta_{\pm}$

a holomorphic three-form $\Omega_{ijk} = -2i \eta_{-}^{\dagger} \gamma_{ijk} \eta_{+}$

$$dJ = \frac{3}{2} \text{Im}(\overline{\mathcal{W}}_1 \Omega) + \mathcal{W}_4 \wedge J + \mathcal{W}_3 \quad d\Omega = \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \overline{\mathcal{W}}_5 \wedge \Omega$$

► Five classes of (intrinsic) torsion are given as

component	interpretation	$SU(3)$ -representation
\mathcal{W}_1	$J \wedge d\Omega$ or $\Omega \wedge dJ$	$\mathbf{1} \oplus \mathbf{1}$
\mathcal{W}_2	$(d\Omega)_0^{2,2}$	$\mathbf{8} \oplus \mathbf{8}$
\mathcal{W}_3	$(dJ)_0^{2,1} + (dJ)_0^{1,2}$	$\mathbf{6} \oplus \overline{\mathbf{6}}$
\mathcal{W}_4	$J \wedge dJ$	$\mathbf{3} \oplus \overline{\mathbf{3}}$
\mathcal{W}_5	$(d\Omega)^{3,1}$	$\mathbf{3} \oplus \overline{\mathbf{3}}$

In case of heterotic string, see, for instance, K. Becker, M. Becker, J.-X. Fu, L.-S. Tseng, S.-T. Yau [hep-th/0604137](https://arxiv.org/abs/hep-th/0604137)

T. Kimura, P. Yi [hep-th/0605247](https://arxiv.org/abs/hep-th/0605247) etc.

► Vanishing torsion classes in special $SU(3)$ -structure manifolds:

complex	hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = 0$
	balanced	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = 0$
	special hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	Kähler	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	conformally Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = 3\mathcal{W}_4 + 2\mathcal{W}_5 = 0$
almost complex	symplectic	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	nearly Kähler	$\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	almost Kähler	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	quasi Kähler	$\mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	semi Kähler	$\mathcal{W}_4 = \mathcal{W}_5 = 0$
	half-flat	$\text{Im}\mathcal{W}_1 = \text{Im}\mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$

A configuration of **six-torus** T^6 in the presence of H -flux in **five-brane** solution:

$$\rightarrow \begin{cases} ds^2 = ds_{\mathbb{R}^{1,2}}^2 + (dx^1)^2 + (dx^2)^2 + (dy^3)^2 + V \left\{ d\xi^2 + (dx^3)^2 + (dy^1)^2 + (dy^2)^2 \right\} \\ H_3 = *_4 dV = \lambda dy^1 \wedge dy^2 \wedge dx^3 \\ e^{2\phi} = V = \lambda \xi \end{cases}$$

Perform **T-duality** along all x^i -directions with respect to the Buscher's rule:

$$ds^2 = ds_{\mathbb{R}^{1,2}}^2 + (d\tilde{x}^1)^2 + (d\tilde{x}^2)^2 + (dy^3)^2 + V^{-1}(d\tilde{x}^3 - \lambda y^1 dy^2)^2 + V \left\{ d\xi^2 + (dy^1)^2 + (dy^2)^2 \right\}$$

$$\tilde{H}_3 = 0 \quad e^{2\tilde{\phi}} = 1$$

Choose $e^1 = d\tilde{x}^1 + i\sqrt{V}dy^1$ $e^2 = d\tilde{x}^2 + i\sqrt{V}dy^2$ $e^3 = \frac{1}{\sqrt{V}}(d\tilde{x}^3 - \lambda y^1 dy^2) + idy^3$

Two- and three-forms: $J = -i\delta_{m\bar{n}} e^m \wedge \bar{e}^{\bar{n}}$ and $\Omega \equiv e^1 \wedge e^2 \wedge e^3$ with

$$dJ = -\frac{2\lambda}{\sqrt{V}} dy^1 \wedge dy^2 \wedge dy^3 \neq 0 \quad \text{and} \quad J \wedge dJ = 0$$

$$d\Omega = -\frac{\lambda}{\sqrt{V}} d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge dy^1 \wedge dy^2 \quad \text{i.e.,} \quad \text{Re} d\Omega \neq 0 \quad \text{and} \quad \text{Im} d\Omega = 0$$

This is a (torsionful) half-flat manifold \rightarrow Entrance Gate to doubled formalism



N.J. Hitchin

Generalized geometry

J on $T\mathcal{M}_d$, ω on $T^*\mathcal{M}_d \dashrightarrow \mathcal{J}_\pm$ on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$

“Cliff(6) pure spinor η_\pm ” on $T\mathcal{M}_6$

\dashrightarrow “Cliff(6,6) pure spinor Φ_\pm ” on $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$

Evaluate spaces of Φ_\pm to provide Kähler/superpotentials in supergravity



C.M. Hull

Doubled formalism

T^d with B-field $\dashrightarrow T^d \times T^d$ (with B-field)

Regard (T)-duality transformation as a part of transition function

Go beyond (non)-abelian gauged supergravity with B-field

and its duality transformation

Generalized geometry

Introduce a generalized almost complex structure \mathcal{J} on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$ s.t.

$$\mathcal{J} : T\mathcal{M}_d \oplus T^*\mathcal{M}_d \longrightarrow T\mathcal{M}_d \oplus T^*\mathcal{M}_d$$

$$\mathcal{J}^2 = -\mathbb{1}_{2d}$$

$$\exists O(d, d) \text{ invariant metric } L, \text{ s.t. } \mathcal{J}^T L \mathcal{J} = L$$

Structure group on $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$:

$\exists L$	$GL(2d)$	\dashrightarrow	$O(d, d)$
$\mathcal{J}^2 = -\mathbb{1}_{2d}$	$O(d, d)$	\dashrightarrow	$U(d/2, d/2)$
$\mathcal{J}_1, \mathcal{J}_2$	$U_1(d/2, d/2) \cap U_2(d/2, d/2)$	\dashrightarrow	$U(d/2) \times U(d/2)$
integrable $\mathcal{J}_{1,2}$	$U(d/2) \times U(d/2)$	\dashrightarrow	$SU(d/2) \times SU(d/2)$

► Integrability is discussed by “(0,1)” part of the complexified $T\mathcal{M}_d \oplus T^*\mathcal{M}_d$:

$$\Pi \equiv \frac{1}{2}(\mathbb{1}_{2d} - i\mathcal{J})$$

$$\Pi A = A \quad \text{where } A = v + \zeta \text{ is a section of } T\mathcal{M}_d \oplus T^*\mathcal{M}_d$$

We call this A *i-eigenbundle* $L_{\mathcal{J}}$, whose dimension is $\dim L_{\mathcal{J}} = d$.

Integrability condition of \mathcal{J} is

$$\bar{\Pi}[\Pi(v + \zeta), \Pi(w + \eta)]_{\mathbb{C}} = 0 \quad v, w \in T\mathcal{M}_d \quad \zeta, \eta \in T^*\mathcal{M}_d$$

$$[v + \zeta, w + \eta]_{\mathbb{C}} = [v, w] + \mathcal{L}_v\eta - \mathcal{L}_w\zeta - \frac{1}{2}d(\iota_v\eta - \iota_w\zeta) : \text{ Courant bracket}$$

► Two typical examples of generalized almost complex structures:

$$\mathcal{J}_1 = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I^T \end{pmatrix} \quad \text{w/ } I^2 = -\mathbb{1}_d: \text{ almost complex structure}$$

$$\mathcal{J}_2 = \begin{pmatrix} \mathbf{0} & -J^{-1} \\ J & \mathbf{0} \end{pmatrix} \quad \text{w/ } J: \text{ almost symplectic form}$$

integrable $\mathcal{J}_1 \iff$ integrable of I

integrable $\mathcal{J}_2 \iff$ integrable of J

On a usual geometry, J_{ij} can be given by an $SU(3)$ invariant (pure) spinor η_+ as

$$J_{ij} = -2i\eta_+^\dagger \gamma_{ij} \eta_+ \quad \gamma^m \eta_+ = 0 \quad \gamma^{\bar{n}} \eta_+ \neq 0$$

In a similar analogy, we want to find $\text{Cliff}(6, 6)$ pure spinor(s) Φ .

∴) Compared to almost complex structures, (pure) spinors can be easily utilized in supergravity framework.

On $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$, we can define Cliff(6,6) algebra and $Spin(6,6)$ spinor Φ :

$$\{\Gamma^i, \Gamma^j\} = 0 \quad \{\Gamma^i, \Gamma_j\} = \delta_j^i \quad \{\Gamma_i, \Gamma_j\} = 0$$

Irreducible repr. of $Spin(6,6)$ spinor is a Majorana-Weyl

→ a generic $Spin(6,6)$ spinor bundle S splits to S^\pm (Weyl)

Weyl spinor bundles S^\pm are isomorphic to bundles of forms on $T^*\mathcal{M}_6$:

$$S^+ \text{ on } T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \sim \wedge^{\text{even}} T^*\mathcal{M}_6$$

$$S^- \text{ on } T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \sim \wedge^{\text{odd}} T^*\mathcal{M}_6$$

Thus we often regard a Cliff(6,6) spinor as a form on $\wedge^{\text{even/odd}} T^*\mathcal{M}_6$

A form-valued representation of the algebra

$$\Gamma^i = dx^i \wedge \quad \Gamma_j = \iota_j$$

IF Φ is annihilated by half numbers of the Cliff(6,6) generators:

→ Φ is called a **pure spinor**

cf.) $SU(3)$ invariant spinor η_+ is a Cliff(6) pure spinor: $\gamma^m \eta_+ = 0$

An equivalent definition of a $\text{Cliff}(6, 6)$ pure spinor is given by “Clifford action”:

$$(v + \zeta) \cdot \Phi = v^i \iota_{\partial_i} \Phi + \zeta_i dx^i \wedge \Phi \quad \text{w/ } v: \text{ vector} \quad \zeta: \text{ one-form}$$

Define the annihilator of a spinor as

$$L_\Phi \equiv \{v + \zeta \in T\mathcal{M}_6 \oplus T^*\mathcal{M}_6 \mid (v + \zeta) \cdot \Phi = 0\}$$

$$\dim L_\Phi \leq d$$

If $\dim L_\Phi = 6$ (maximally isotropic) $\rightarrow \Phi$ is a **pure spinor**

Correspondence between pure spinors and generalized almost complex structures:

$$\mathcal{J} \leftrightarrow \Phi \quad \text{if } L_{\mathcal{J}} = L_{\Phi} \quad \text{with } \dim L_{\Phi} = 6$$

More precisely: $\mathcal{J} \leftrightarrow$ a line bundle of pure spinor Φ

\therefore) rescaling Φ does not change its annihilator L_{Φ}

Then, we can rewrite the generalized almost complex structure as

$$\mathcal{J}_{\pm\Pi\Sigma} = \langle \text{Re}\Phi_{\pm}, \Gamma_{\Pi\Sigma} \text{Re}\Phi_{\pm} \rangle$$

w/ Mukai pairing:

$$\text{even forms: } \langle \Psi_+, \Phi_+ \rangle = \Psi_6 \wedge \Phi_0 - \Psi_4 \wedge \Phi_2 + \Psi_2 \wedge \Phi_4 - \Psi_0 \wedge \Phi_6$$

$$\text{odd forms: } \langle \Psi_-, \Phi_- \rangle = \Psi_5 \wedge \Phi_1 - \Psi_3 \wedge \Phi_3 + \Psi_1 \wedge \Phi_5$$

$$\mathcal{J} \text{ is integrable} \quad \longleftrightarrow \quad \exists \text{ vector } v \text{ and one-form } \zeta \text{ s.t. } d\Phi = (v_{\perp} + \zeta \wedge)\Phi$$

$$\text{generalized CY} \quad \longleftrightarrow \quad \exists \Phi \text{ is pure s.t. } d\Phi = 0$$

$$\text{“twisted” GCY} \quad \longleftrightarrow \quad \exists \Phi \text{ is pure, and } H \text{ is closed s.t. } (d - H \wedge)\Phi = 0$$

A $\text{Cliff}(6, 6)$ spinor can also be mapped to a bispinor:

$$C \equiv \sum_k \frac{1}{k!} C_{i_1 \dots i_k}^{(k)} dx^{i_1} \wedge \dots \wedge dx^{i_k} \quad \longleftrightarrow \quad \mathcal{C} \equiv \sum_k \frac{1}{k!} C_{i_1 \dots i_k}^{(k)} \gamma_{\alpha\beta}^{i_1 \dots i_k}$$

On a geometry of a **single** $SU(3)$ -structure, the following two $SU(3, 3)$ spinors:

$$\begin{aligned} \Phi_{0+} &= \eta_+ \otimes \eta_+^\dagger = \frac{1}{4} \sum_{k=0}^6 \frac{1}{k!} \eta_+^\dagger \gamma_{i_k \dots i_1} \eta_+ \gamma^{i_1 \dots i_k} = \frac{1}{8} e^{-iJ} \\ \Phi_{0-} &= \eta_+ \otimes \eta_-^\dagger = \frac{1}{4} \sum_{k=0}^6 \frac{1}{k!} \eta_-^\dagger \gamma_{i_k \dots i_1} \eta_+ \gamma^{i_1 \dots i_k} = -\frac{i}{8} \Omega \end{aligned}$$

Check purity: $(\delta + iI)_i{}^j \gamma_j \eta_+ \otimes \eta_\pm^\dagger = 0 = \eta_+ \otimes \eta_\pm^\dagger \gamma_j (\delta \mp iI)^j{}_i$

One-to-one correspondence: $\Phi_{0-} \leftrightarrow \mathcal{J}_1, \quad \Phi_{0+} \leftrightarrow \mathcal{J}_2$

On a generic geometry of a **pair** of $SU(3)$ -structure defined by (η_+^1, η_+^2) : [Appendix](#)

$$\begin{aligned} \Phi_{0+} &= \eta_+^1 \otimes \eta_+^{2\dagger} = \frac{1}{8} (\bar{c}_\parallel e^{-ij} - i\bar{c}_\perp w) \wedge e^{-iv \wedge v'} \\ \Phi_{0-} &= \eta_+^1 \otimes \eta_-^{2\dagger} = -\frac{1}{8} (c_\perp e^{-ij} + ic_\parallel w) \wedge (v + iv') \\ &\quad \Phi_\pm = e^{-B} \Phi_{0\pm} \end{aligned} \quad |c_\parallel|^2 + |c_\perp|^2 = 1$$

Each Φ_{\pm} defines an $SU(3, 3)$ structure on E . Common structure is $SU(3) \times SU(3)$.

(F is extended to E by including $e^{-\mathcal{B}}$)

Compatibility requires

$$\begin{aligned}\langle \Phi_+, V \cdot \Phi_- \rangle &= \langle \bar{\Phi}_+, V \cdot \Phi_- \rangle = 0 \quad \text{for } \forall V = x + \xi \\ \langle \Phi_+, \bar{\Phi}_+ \rangle &= \langle \Phi_-, \bar{\Phi}_- \rangle\end{aligned}$$

Start with a real form $\chi_f \in \wedge^{\text{even/odd}} F^*$ (associated with a real $Spin(6, 6)$ spinor χ_s)

Regard χ_f as a stable form satisfying

$$q(\chi_f) = -\frac{1}{4} \langle \chi_f, \Gamma_{\Pi\Sigma} \chi_f \rangle \langle \chi_f, \Gamma^{\Pi\Sigma} \chi_f \rangle \in \wedge^6 F^* \otimes \wedge^6 F^*$$

$$U = \{ \chi_f \in \wedge^{\text{even/odd}} F^* : q(\chi_f) < 0 \}$$

Define a Hitchin function

$$H(\chi_f) \equiv \sqrt{-\frac{1}{3}q(\chi_f)} \in \wedge^6 F^*$$

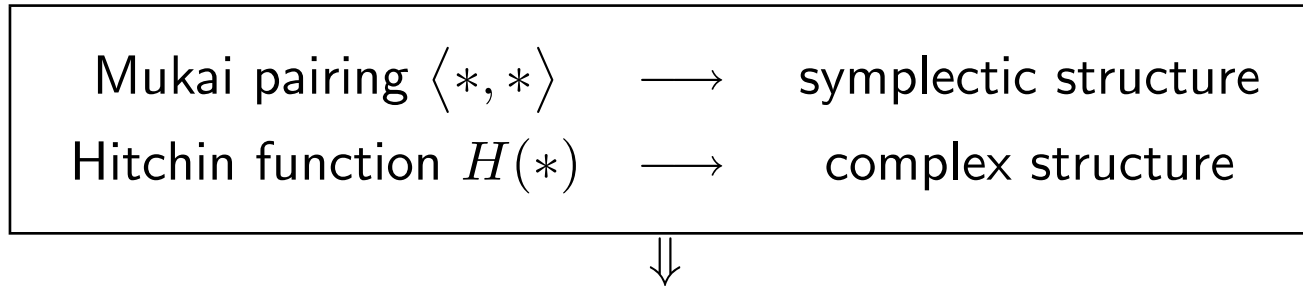
which gives an integrable complex structure on U

Then we can get another real form $\hat{\chi}_f$ and a complex form Φ_f by Mukai pairing

$$\begin{aligned} \langle \hat{\chi}_f, \chi_f \rangle &= -dH(\chi_f) \quad \text{i.e.,} \quad \hat{\chi}_f = -\frac{\partial H(\chi_f)}{\partial \chi_f} \\ \longrightarrow \quad \Phi_f &\equiv \frac{1}{2}(\chi_f + i\hat{\chi}_f) \quad H(\Phi_f) = i\langle \Phi_f, \bar{\Phi}_f \rangle \end{aligned}$$

Hitchin showed: Φ_f is a (form corresponding to) **pure spinor!**

Consider the space of pure spinors Φ ...



The space of pure spinor is Kähler (or, rather **rigid** special Kähler)!

Quotienting this space by the \mathbb{C}^* action $\Phi \rightarrow \lambda\Phi$ for $\lambda \in \mathbb{C}^*$

--> The space becomes a **local** special Kähler geometry with Kähler potential K :

$$e^{-K} = H(\Phi) = i\langle \Phi, \bar{\Phi} \rangle = i(\bar{z}^I \mathcal{F}_I - z^I \bar{\mathcal{F}}_I) \in \wedge^6 F^*$$

z^I : holomorphic homogeneous coordinates

\mathcal{F}_I : derivative of prepotential \mathcal{F} , i.e., $\mathcal{F}_I = \partial\mathcal{F}/\partial z^I$

These are nothing but objects which we want to introduce in $\mathcal{N} = 2$ supergravity!

Space of pure spinors Φ_{\pm} on $F \oplus F^*$ with $SU(3) \times SU(3)$ structure

||

special Kähler geometry of local type = Hodge-Kähler geometry

$$e^{-K_{\pm}} = H(\Phi_{\pm}) = i\langle \Phi_{\pm}, \bar{\Phi}_{\pm} \rangle = i(\bar{\mathcal{Z}}_{\pm}^I \mathcal{F}_{\pm I} - \mathcal{Z}_{\pm}^I \bar{\mathcal{F}}_{\pm I}) \in \wedge^6 F^*$$

For a single $SU(3)$ -structure case:

$$\begin{aligned} \Phi_+ &= -\frac{1}{8}e^{-\mathcal{B}-iJ} & K_+ &= -\log\left(\frac{1}{48}J \wedge J \wedge J\right) \\ \Phi_- &= -\frac{i}{8}e^{-\mathcal{B}}\Omega & K_- &= -\log\left(\frac{i}{64}\Omega \wedge \bar{\Omega}\right) \end{aligned}$$

Structure of forms is exactly same as the one in the case of Calabi-Yau compactification!

We should truncate Kaluza-Klein massive modes from these forms to obtain 4-dimensional supergravity.

As introduced, we want to obtain four-dimensional $\mathcal{N} = 1, 2$ supergravity theories

Type IIA/IIB supergravity theories have 32 supercharges w/ field multiplets

- 1 gravity multiplet
- 6 gravitino multiplets ← *should be truncated*
- 15 vector multiplets
- 9 hypermultiplets
- 1 tensor multiplet

in the language of “ $\mathcal{N} = 2$ ” multiplets

Consider truncation of 6 gravitino multiplets in terms of group theoretical descriptions

Let us discuss group-theoretical properties of massless fields

on a generalized tangent bundle $T_{3,1} \oplus F \oplus F^*$ with $SU(3) \times SU(3)$ structure

First, consider decomposition of $\mathfrak{8}_S$, $\mathfrak{8}_C$, $\mathfrak{8}_V$ of $SO(8)$ (i.e., light-cone gauge)

$$\begin{array}{rcccl}
 SO(8) & \rightarrow & SO(2) \times SO(6) & \rightarrow & SO(2) \times SU(3) \\
 \hline
 \mathfrak{8}_S & \rightarrow & \mathbf{4}_{\frac{1}{2}} \oplus \bar{\mathbf{4}}_{-\frac{1}{2}} & \rightarrow & \mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_{\frac{1}{2}} \oplus \bar{\mathbf{3}}_{-\frac{1}{2}} \\
 \mathfrak{8}_C & \rightarrow & \mathbf{4}_{-\frac{1}{2}} \oplus \bar{\mathbf{4}}_{\frac{1}{2}} & \rightarrow & \mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_{-\frac{1}{2}} \oplus \bar{\mathbf{3}}_{\frac{1}{2}} \\
 \mathfrak{8}_V & \rightarrow & \mathbf{1}_1 \oplus \mathbf{1}_{-1} \oplus \mathbf{6}_0 & \rightarrow & \mathbf{1}_{\frac{1}{2}} \oplus \mathbf{1}_{-\frac{1}{2}} \oplus \mathbf{3}_0 \oplus \bar{\mathbf{3}}_0
 \end{array}$$

Using this, consider the decompositions of (NS,R), (R,NS), (NS,NS) and (R,R) sectors...

\mathbf{a}_b denotes a field in the $SU(3)$ repr. \mathbf{a} and 4-dimensional helicity \mathbf{b} . \mathbf{T} denotes an antisymmetric tensor.

► Fermions: (R,NS) and (NS,R) sectors:

	$SO(8)_L \times SO(8)_R$	\rightarrow	$SO(2) \times SU(3)_L \times SU(3)_R$
IIA/IIB	$(\mathbf{8}_S, \mathbf{8}_V)$	\rightarrow	$(\mathbf{1}, \mathbf{1})_{\pm\frac{3}{2}, \pm\frac{1}{2}} \oplus (\mathbf{3}, \mathbf{1})_{\frac{3}{2}, -\frac{1}{2}} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-\frac{3}{2}, \frac{1}{2}} \oplus (\mathbf{1}, \mathbf{3})_{\pm\frac{1}{2}} \oplus (\mathbf{1}, \bar{\mathbf{3}})_{\pm\frac{1}{2}}$ $\oplus (\mathbf{3}, \mathbf{3})_{\frac{1}{2}} \oplus (\bar{\mathbf{3}}, \mathbf{3})_{-\frac{1}{2}} \oplus (\mathbf{3}, \bar{\mathbf{3}})_{\frac{1}{2}} \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}})_{-\frac{1}{2}}$
IIB	$(\mathbf{8}_V, \mathbf{8}_S)$	\rightarrow	$(\mathbf{1}, \mathbf{1})_{\pm\frac{3}{2}, \pm\frac{1}{2}} \oplus (\mathbf{3}, \mathbf{1})_{\pm\frac{1}{2}} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1}, \mathbf{3})_{\frac{3}{2}, -\frac{1}{2}} \oplus (\mathbf{1}, \bar{\mathbf{3}})_{-\frac{3}{2}, \frac{1}{2}}$ $\oplus (\mathbf{3}, \mathbf{3})_{\frac{1}{2}} \oplus (\bar{\mathbf{3}}, \mathbf{3})_{\frac{1}{2}} \oplus (\mathbf{3}, \bar{\mathbf{3}})_{-\frac{1}{2}} \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}})_{-\frac{1}{2}}$
IIA	$(\mathbf{8}_V, \mathbf{8}_C)$	\rightarrow	$(\mathbf{1}, \mathbf{1})_{\pm\frac{3}{2}, \pm\frac{1}{2}} \oplus (\mathbf{3}, \mathbf{1})_{\pm\frac{1}{2}} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{1}, \mathbf{3})_{-\frac{3}{2}, \frac{1}{2}} \oplus (\mathbf{1}, \bar{\mathbf{3}})_{\frac{3}{2}, -\frac{1}{2}}$ $\oplus (\mathbf{3}, \mathbf{3})_{-\frac{1}{2}} \oplus (\bar{\mathbf{3}}, \mathbf{3})_{-\frac{1}{2}} \oplus (\mathbf{3}, \bar{\mathbf{3}})_{\frac{1}{2}} \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}})_{\frac{1}{2}}$

$(\mathbf{1}, \mathbf{1})_{\pm\frac{3}{2}}$: 2 gravitinos in gravity multiplet

$(\mathbf{3}, \mathbf{1})_{\pm\frac{3}{2}}$ etc.: 6 gravitinos in gravitino multiplets

$(\mathbf{3}, \mathbf{1})_{\pm\frac{1}{2}}$ etc.: fermions in gravitino multiplets

⌋ (should not be included in $\mathcal{N} = 2$ theory)

► Bosons: (NS,NS) sector:

$$\mathbf{8}_V \times \mathbf{8}_V = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} = (\phi, \mathcal{B}_{MN}, \mathcal{G}_{MN})$$

$SO(8)_L \times SO(8)_R$	\rightarrow	$SO(2) \times SU(3)_L \times SU(3)_R$
	$\mathcal{E}_{\mu\nu}$	$(\mathbf{1}, \mathbf{1})_{\pm 2} \oplus (\mathbf{1}, \mathbf{1})_T$
	$\mathcal{E}_{\mu i}$	$(\mathbf{1}, \mathbf{3})_{\pm 1} \oplus (\mathbf{1}, \bar{\mathbf{3}})_{\pm 1}$
$\mathcal{E}_{MN} = \mathcal{G}_{MN} + \mathcal{B}_{MN}$	\rightarrow	
	$\mathcal{E}_{i\nu}$	$(\mathbf{3}, \mathbf{1})_{\pm 1} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{\pm 1}$
	\mathcal{E}_{ij}	$(\mathbf{3}, \mathbf{3})_0 \oplus (\mathbf{3}, \bar{\mathbf{3}})_0 \oplus (\bar{\mathbf{3}}, \mathbf{3})_0 \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}})_0$

► Bosons: (R,R) sector:

	$SO(8)_L \times SO(8)_R$	\rightarrow	$SO(2) \times SU(3)_L \times SU(3)_R$
IIA	$(\mathbf{8}_S, \mathbf{8}_C)$	\rightarrow	$(\mathbf{1}, \mathbf{1})_{\pm 1, 0} \oplus (\mathbf{3}, \mathbf{3})_0 \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}})_0 \oplus (\mathbf{3}, \bar{\mathbf{3}})_1 \oplus (\bar{\mathbf{3}}, \mathbf{3})_{-1}$
IIB	$(\mathbf{8}_S, \mathbf{8}_S)$	\rightarrow	$(\mathbf{1}, \mathbf{1})_{\pm 1, 0} \oplus (\mathbf{3}, \mathbf{3})_1 \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}})_{-1} \oplus (\mathbf{3}, \bar{\mathbf{3}})_0 \oplus (\bar{\mathbf{3}}, \mathbf{3})_0$

Field expressions:

IIA	$\mathcal{A}_0^- = \mathcal{A}_{(0,1)} + \mathcal{A}_{(0,3)} + \mathcal{A}_{(0,5)} \simeq (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{3}, \mathbf{3})_0 \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}})_0$
	$\mathcal{A}_1^+ = \mathcal{A}_{(1,0)} + \mathcal{A}_{(1,2)} + \mathcal{A}_{(1,4)} + \mathcal{A}_{(1,6)} \simeq (\mathbf{1}, \mathbf{1})_{\pm 1} \oplus (\mathbf{3}, \bar{\mathbf{3}})_1 \oplus (\bar{\mathbf{3}}, \mathbf{3})_{-1}$
IIB	$\mathcal{A}_0^+ = \mathcal{A}_{(0,0)} + \mathcal{A}_{(0,2)} + \mathcal{A}_{(0,4)} + \mathcal{A}_{(0,6)} \simeq (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{3}, \bar{\mathbf{3}})_0 \oplus (\bar{\mathbf{3}}, \mathbf{3})_0$
	$\mathcal{A}_1^- = \mathcal{A}_{(1,1)} + \mathcal{A}_{(1,3)} + \mathcal{A}_{(1,5)} \simeq (\mathbf{1}, \mathbf{1})_1 \oplus (\mathbf{3}, \mathbf{3})_1 \oplus (\bar{\mathbf{3}}, \bar{\mathbf{3}})_{-1}$

where $\mathcal{A}_{(p,q)}$ is a “4-dimensional” p -form and a “6-dimensional” q -form

RR field strength is $\mathcal{G}^\pm = d\mathcal{A}_0^\mp$, whose gauge potential is $\mathcal{C} = e^{\mathcal{B}}\mathcal{A}$ w/ $\mathcal{F} = d\mathcal{C} - \mathcal{H}_3 \wedge \mathcal{C} = e^{\mathcal{B}}\mathcal{G}$

► Reduction: effective theory with two gravitinos

→ all repr. of the form $(\mathbf{3}, \mathbf{1}), (\bar{\mathbf{3}}, \mathbf{1}), (\mathbf{1}, \mathbf{3}), (\mathbf{1}, \bar{\mathbf{3}})$ (6 gravitino multiplets) are **projected out!**

type IIA multiplet	$SU(3) \times SU(3)$ repr.	bosonic field content
gravity multiplet	$(\mathbf{1}, \mathbf{1})$	$g_{\mu\nu} \quad \mathcal{A}_1^+$
tensor multiplet	$(\mathbf{1}, \mathbf{1})$	$\mathcal{B}_{\mu\nu} \quad \phi \quad \mathcal{A}_0^-$
vector multiplet	$(\mathbf{3}, \bar{\mathbf{3}})$	$\mathcal{A}_1^+ \quad \delta\Phi^+$
hypermultiplet	$(\mathbf{3}, \mathbf{3})$	$\delta\Phi^- \quad \mathcal{A}_0^-$

type IIB multiplet	$SU(3) \times SU(3)$ repr.	bosonic field content
gravity multiplet	$(\mathbf{1}, \mathbf{1})$	$g_{\mu\nu} \quad \mathcal{A}_1^-$
tensor multiplet	$(\mathbf{1}, \mathbf{1})$	$\mathcal{B}_{\mu\nu} \quad \phi \quad \mathcal{A}_0^+$
vector multiplet	$(\mathbf{3}, \mathbf{3})$	$\mathcal{A}_1^- \quad \delta\Phi^-$
hypermultiplet	$(\mathbf{3}, \bar{\mathbf{3}})$	$\delta\Phi^+ \quad \mathcal{A}_0^+$

Notice that all fields are still living on 10-dimensional space, i.e., all KK modes are included.

In case of a tangent bundle $T_{3,1} \oplus F \oplus F^*$ w/ a **single** $SU(3)$ -structure (i.e., $\eta_+^1 = \eta_+^2$):

Ten-dimensional fields are decomposed as

\mathcal{G}_{MN}	$g_{\mu\nu}$	$\mathbf{1}_{\pm 2}$	Ψ_M	Ψ_μ	$\mathbf{1}_{\pm \frac{3}{2}} + \mathbf{3}_{\pm \frac{3}{2}}$
	$\mathcal{G}_{\mu i}$	$(\mathbf{3} + \bar{\mathbf{3}})_{\pm 1}$		Ψ_i	$\mathbf{1}_{\pm \frac{1}{2}} + \mathbf{3}_{\pm \frac{1}{2}} + 2 \times \bar{\mathbf{3}}_{\pm \frac{1}{2}} + \mathbf{6}_{\pm \frac{1}{2}} + \mathbf{8}_{\pm \frac{1}{2}}$
	\mathcal{G}_{ij}	$\mathbf{1}_0 + (\mathbf{6} + \bar{\mathbf{6}})_0 + \mathbf{8}_0$			
\mathcal{B}_{MN}	$\mathcal{B}_{\mu\nu}$	$\mathbf{1}_T$	λ	λ	$\mathbf{1}_{\pm \frac{1}{2}} + \mathbf{3}_{\pm \frac{1}{2}}$
	$\mathcal{B}_{\mu i}$	$(\mathbf{3} + \bar{\mathbf{3}})_{\pm 1}$			
	\mathcal{B}_{ij}	$\mathbf{1}_0 + (\mathbf{3} + \bar{\mathbf{3}})_0 + \mathbf{8}_0$			
ϕ	ϕ	$\mathbf{1}_0$			

\mathcal{C}_M	\mathcal{C}_μ	$\mathbf{1}_{\pm 1}$
	\mathcal{C}_i	$(\mathbf{3} + \bar{\mathbf{3}})_0$
\mathcal{C}_{MNP}	$\mathcal{C}_{\mu\nu k}$	$(\mathbf{3} + \bar{\mathbf{3}})_T$
	$\mathcal{C}_{\mu j k}$	$\mathbf{1}_T + (\mathbf{3} + \bar{\mathbf{3}})_{\pm 1} + \mathbf{8}_{\pm 1}$
	\mathcal{C}_{ijk}	$(\mathbf{1} + \mathbf{1})_0 + (\mathbf{3} + \bar{\mathbf{3}})_{\pm 1} + (\mathbf{6} + \bar{\mathbf{6}})_0$
\mathcal{C}_0	\mathcal{C}_0	$\mathbf{1}_0$
\mathcal{C}_{MN}	$\mathcal{C}_{\mu\nu}$	$\mathbf{1}_T$
	$\mathcal{C}_{\mu i}$	$(\mathbf{3} + \bar{\mathbf{3}})_{\pm 1}$
	\mathcal{C}_{ij}	$\mathbf{1}_0 + (\mathbf{3} + \bar{\mathbf{3}})_0 + \mathbf{8}_0$
\mathcal{C}_{MNPQ}	$\mathcal{C}_{\mu j k l}$	$\frac{1}{2}[(\mathbf{1} + \mathbf{1})_{\pm 1} + (\mathbf{3} + \bar{\mathbf{3}})_{\pm 1} + (\mathbf{6} + \bar{\mathbf{6}})_{\pm 1}]$
	$\mathcal{C}_{ijkl}/\mathcal{C}_{\mu\nu kl}$	$\mathbf{1}_0 + (\mathbf{3} + \bar{\mathbf{3}})_0 + \mathbf{8}_0$

Standard four-dimensional $\mathcal{N} = 2$ supergravity = “absence of 6 gravitino multiplets”

IIA multiplets	$SU(3)$ repr.	field contents
gravity multiplet	1	$g_{\mu\nu}$ \mathcal{C}_μ Ψ_μ
tensor multiplet	1	$\mathcal{B}_{\mu\nu}$ ϕ \mathcal{C}_{ijk} λ
vector multiplet	8 + 1	$\mathcal{C}_{\mu jk}$ \mathcal{G}_{ij} \mathcal{B}_{ij} Ψ_i
hypermultiplet	6	\mathcal{G}_{ij} \mathcal{C}_{ijk} Ψ_i

IIB multiplets	$SU(3)$ repr.	field contents
gravity multiplet	1	$g_{\mu\nu}$ $\mathcal{C}_{\mu jkl}$ Ψ_μ
tensor multiplet	1	$\mathcal{B}_{\mu\nu}$ $\mathcal{C}_{\mu\nu}$ ϕ \mathcal{C}_0 λ
vector multiplet	6	$\mathcal{C}_{\mu jkl}$ \mathcal{G}_{ij} Ψ_i
hypermultiplet	8 + 1	\mathcal{G}_{ij} \mathcal{B}_{ij} \mathcal{C}_{ij} \mathcal{C}_{ijkl} Ψ_i

Notice that all fields are still living on 10-dimensional space, i.e., all KK modes are included.

Analyze potential (interaction) terms:

given in the supersymmetry transformation of 4-dimensional $\mathcal{N} = 2$ gravitinos ψ_μ^A

$$\hat{\Psi}_\mu^A \equiv \Psi_\mu^A + \frac{1}{2} \gamma_\mu^i \Psi_i^A = \psi_{A\mu+} \otimes \eta_\pm^A + \psi_{A\mu-} \otimes \eta_\mp^A + \dots$$

$$\delta\psi_{A\mu} = D_\mu \xi_A + i \gamma_\mu S_{AB} \xi^B \quad A = 1, 2$$

$$S_{AB} = \frac{i}{2} e^{\frac{1}{2} K_V} \sigma_{AB}^x \mathcal{P}^x \quad \sigma_{AB}^x = \begin{pmatrix} \delta^{x1} - i\delta^{x2} & -\delta^{x3} \\ -\delta^{x3} & -\delta^{x1} - i\delta^{x2} \end{pmatrix} \quad x = 1, 2, 3$$

\mathcal{P}^x : $\mathcal{N} = 2$ Killing prepotentials, which yield $\mathcal{N} = 1$ superpotentials

To get S_{AB} , project the SUSY transformation $\delta\hat{\Psi}_\mu$ onto $SU(3)$ -singlet parts from

$$\begin{aligned}\delta\Psi_M &= D_M\epsilon - \frac{1}{96}e^{-\phi}\left(\gamma_M{}^{PQR}\mathcal{H}_{PQR} - 9\gamma^{PQ}\mathcal{H}_{MPQ}\right)\mathcal{P}\epsilon \\ &\quad - \sum_n \frac{1}{64n!}e^{\frac{5-n}{4}\phi}\left[(n-1)\gamma_M{}^{N_1\cdots N_n} - n(9-n)\delta_M{}^{N_1}\gamma^{N_2\cdots N_n}\right]\mathcal{F}_{N_1\cdots N_n}\mathcal{P}_n\epsilon\end{aligned}$$

In type IIB case (w/ $\mathcal{F}^- = \mathcal{F}_1 + \mathcal{F}_3 + \mathcal{F}_5$, $\sigma(\mathcal{F}^-) = -\mathcal{F}_1 + \mathcal{F}_3 - \mathcal{F}_5$):

$$\begin{aligned}\begin{pmatrix} \delta\psi_{\mu+}^1 \\ \delta\psi_{\mu+}^2 \end{pmatrix} &= \begin{pmatrix} D_\mu\xi_+^1 \\ D_\mu\xi_+^2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \gamma_\mu\xi_-^1 \bar{\eta}_-^1 \gamma^i D_i\eta_+^1 \\ \gamma_\mu\xi_-^2 \bar{\eta}_-^2 \gamma^i D_i\eta_+^2 \end{pmatrix} + \frac{1}{48}\begin{pmatrix} \gamma_\mu\xi_-^1 \mathcal{H}_{ijk} \bar{\eta}_-^1 \gamma^{ijk}\eta_+^1 \\ -\gamma_\mu\xi_-^2 \mathcal{H}_{ijk} \bar{\eta}_-^2 \gamma^{ijk}\eta_+^2 \end{pmatrix} \\ &\quad - \frac{1}{8}\begin{pmatrix} -\gamma_\mu\xi_-^2 e^\phi \frac{1}{n!}\mathcal{F}_{i_1\cdots i_n}^- \bar{\eta}_-^1 \gamma^{i_1\cdots i_n}\eta_+^2 \\ \gamma_\mu\xi_-^1 e^\phi \frac{1}{n!}\sigma(\mathcal{F}^-)_{i_1\cdots i_n} \bar{\eta}_-^2 \gamma^{i_1\cdots i_n}\eta_+^1 \end{pmatrix}\end{aligned}$$

Then we obtain

$$\begin{aligned}
S_{11} &= \frac{i}{2} \bar{\eta}_-^1 \gamma^i D_i \eta_+^1 - \frac{i}{48} \mathcal{H}_{ijk} \bar{\eta}_-^1 \gamma^{ijk} \eta_+^1 = -\frac{1}{8} \langle \Phi_-, d\Phi_+ \rangle \\
S_{22} &= \frac{i}{2} \bar{\eta}_-^2 \gamma^i D_i \eta_+^2 + \frac{i}{48} \mathcal{H}_{ijk} \bar{\eta}_-^2 \gamma^{ijk} \eta_+^2 = \frac{1}{8} \langle \Phi_-, d\bar{\Phi}_+ \rangle \\
S_{12} &= \frac{i}{8n!} e^\phi \mathcal{F}_{i_1 \dots i_n}^- \bar{\eta}_-^1 \gamma^{i_1 \dots i_n} \eta_+^2 = \frac{1}{8} \langle \Phi_-, \mathcal{G}^- \rangle \\
S_{21} &= \frac{i}{8n!} e^\phi \sigma(\mathcal{F})_{i_1 \dots i_n}^- \bar{\eta}_-^2 \gamma^{i_1 \dots i_n} \eta_+^1 = \frac{1}{8} \langle \Phi_-, \mathcal{G}^- \rangle \\
\mathcal{F} &= d\mathcal{C} - \mathcal{H}_3 \wedge \mathcal{C} = e^{\mathcal{B}} \mathcal{G} \quad \mathcal{C} = e^{\mathcal{B}} \mathcal{A} \quad \mathcal{G}^\pm = d\mathcal{A}_0^\mp
\end{aligned}$$

Summarizing information, we obtain (also for type IIA)

$$\begin{aligned}
S_{AB}^{(4)}(\text{IIB}) &= \frac{1}{8} e^{\frac{1}{2} K_-} \begin{pmatrix} -e^{\frac{1}{2} K_+ + \phi^{(4)}} \langle \Phi_-, d\Phi_+ \rangle & -e^{2\phi^{(4)}} \langle \Phi_-, \mathcal{G}^- \rangle \\ -e^{2\phi^{(4)}} \langle \Phi_-, \mathcal{G}^- \rangle & e^{\frac{1}{2} K_+ + \phi^{(4)}} \langle \Phi_-, d\bar{\Phi}_+ \rangle \end{pmatrix} \\
S_{AB}^{(4)}(\text{IIA}) &= \frac{1}{8} e^{\frac{1}{2} K_+} \begin{pmatrix} e^{\frac{1}{2} K_- + \phi^{(4)}} \langle \Phi_+, d\Phi_- \rangle & e^{2\phi^{(4)}} \langle \Phi_+, \mathcal{G}^+ \rangle \\ e^{2\phi^{(4)}} \langle \Phi_+, \mathcal{G}^+ \rangle & -e^{\frac{1}{2} K_- + \phi^{(4)}} \langle \Phi_+, d\bar{\Phi}_- \rangle \end{pmatrix} \\
g_{\mu\nu}^{(4)} &= e^{-2\phi^{(4)}} g_{\mu\nu} \quad \phi^{(4)} = \phi - \frac{1}{4} \log \det \mathcal{G}_{ij}
\end{aligned}$$

$\mathcal{N} = 1$ superpotentials and Kähler potentials can be read as

$$\delta\psi_\mu = D_\mu\xi + ie^{K/2}W\gamma_\mu\xi^c \quad K = K_+ + K_- + 2\phi^{(4)}$$

Most generic form of $\mathcal{N} = 1$ superpotentials on $SU(3) \times SU(3)$ structure:

$$W_{\text{IIA}} = \cos^2\alpha e^{i\beta}\langle\Phi_+, d\Phi_-\rangle - \sin^2\alpha e^{-i\beta}\langle\Phi_+, d\bar{\Phi}_-\rangle + \sin 2\alpha e^\phi\langle\Phi_+, \mathcal{G}^+\rangle$$

$$W_{\text{IIB}} = -\cos^2\alpha e^{i\beta}\langle\Phi_-, d\Phi_+\rangle + \sin^2\alpha e^{-i\beta}\langle\Phi_-, d\bar{\Phi}_+\rangle - \sin 2\alpha e^\phi\langle\Phi_-, \mathcal{G}^-\rangle$$

$$\mathcal{G}^+ = \mathcal{G}_0 + \mathcal{G}_2 + \mathcal{G}_4 + \mathcal{G}_6 \quad \mathcal{G}^- = \mathcal{G}_1 + \mathcal{G}_3 + \mathcal{G}_5$$

$$\mathcal{G}^\pm = d\mathcal{A}_0^\mp \quad \mathcal{C} = e^{\mathcal{B}}\mathcal{A} \quad \mathcal{F} = d\mathcal{C} - \mathcal{H}_3 \wedge \mathcal{C} = e^{\mathcal{B}}\mathcal{G}$$

Reducing to **single** $SU(3)$ -structure by $\eta_+^1 = \eta_+^2 \equiv \eta_+$, we obtain well-known forms:

$2\alpha = -\beta = \frac{\pi}{2} \text{ in } W_{\text{IIB}}$	$W_{\text{GVW}} = -ie^\phi\langle\mathcal{F}_3 - \tau\mathcal{H}_3, \Omega\rangle$
$\alpha = \frac{\pi}{4}, d\Phi_- = 0 \text{ in } W_{\text{IIA}}$	$W_{\text{IIA,RR}} = e^\phi\langle e^{-\mathcal{B}-iJ}, \mathcal{G}^+\rangle$
$\beta = \frac{\pi}{2}, \mathcal{G}^+ = 0 \text{ in } W_{\text{IIA}}$	$W_{\text{half-flat}} = i\langle e^{-\mathcal{B}-iJ}, d(\text{Re}\Omega)\rangle$
$a = \cos\alpha e^{-i\beta/2}, b = \sin\alpha e^{i\beta/2}, \tau = \mathcal{C}_0 + ie^{-\phi}$	

We have obtained Kähler potentials and superpotentials

which should appear in **four-dimensional** $\mathcal{N} = 1, 2$ supergravity theories

in the language of **ten-dimensional fields**:

$$e^{-K_{\pm}} = i \langle \Phi_{\pm}, \bar{\Phi}_{\pm} \rangle = i (\bar{\mathcal{Z}}_{\pm}^I \mathcal{F}_{\pm I} - \mathcal{Z}_{\pm}^I \bar{\mathcal{F}}_{\pm I})$$

$$W_{\text{IIA/IIB}} = \pm \cos^2 \alpha e^{i\beta} \langle \Phi_{\pm}, d\Phi_{\mp} \rangle \mp \sin^2 \alpha e^{-i\beta} \langle \Phi_{\pm}, d\bar{\Phi}_{\mp} \rangle \pm \sin 2\alpha e^{\phi} \langle \Phi_{\pm}, \mathcal{G}^{\pm} \rangle$$

Next task is to find **a suitable truncation** of massive modes

by decomposition $\mathcal{M}_{1,9} = \mathcal{M}_{1,3} \times_{\text{W}} \mathcal{M}_6$ with $T_{1,3} \equiv T\mathcal{M}_{1,3}$ and $F \equiv T\mathcal{M}_6$

We truncate ten-dimensional theory
with keeping only a **finite number of light modes** in the spectrum.

Generically, however, the distinction between heavy and light modes
in a Kaluza-Klein expansion on $\mathcal{M}_{1,9} = \mathcal{M}_{1,3} \times_W \mathcal{M}_6$ is **not** straightforward!

✓ If \mathcal{M}_6 is a Calabi-Yau

All the field deformations give **massless** modes in **four-dimensional** viewpoint

||

Corresponding fields on \mathcal{M}_6 are **harmonic** and are **finite** in number

✓ If \mathcal{M}_6 is a generic geometry (w/ torsion)

Existence of finite number of harmonic forms are **not guaranteed**..

Instead, we **assume** existence of a certain finite-dimensional subspace of $\wedge^* T^* \mathcal{M}_6$

If there exists harmonic forms on \mathcal{M}_6 , we can evaluate the dimensions of the forms via Index theorem: T. Kimura [arXiv:0704.2111](https://arxiv.org/abs/0704.2111)

Assumption the existence of finite-dimensional subset of p -forms:

$$\Lambda_{\text{finite}}^p \subset \Lambda^p T^* \mathcal{M}_6 \quad U^{\text{finite}} = U \cap \Lambda_{\text{finite}}^*$$

Note: the truncation should not break supersymmetry

--> special Kähler geometry on U should give special Kähler geometry on U^{finite}

i.e., we require $\left\{ \begin{array}{l} \text{Mukai pairing } \langle *, * \rangle \text{ is non-degenerate on } \Lambda_{\text{finite}}^p \\ \text{if } \chi \in U^{\text{finite}}, \text{ then } \hat{\chi} \in U^{\text{finite}} \end{array} \right.$

First we introduce a set of basis forms (w/ Mukai pairing as symplectic structure):

$$\text{even forms : } \Sigma_+ = \{\omega_A, \tilde{\omega}^B\}, \quad \int_{\mathcal{M}_6} \langle \omega_A, \tilde{\omega}^B \rangle = \delta_A^B, \quad A, B = 0, \dots, b^+$$

$$\text{odd forms : } \Sigma_- = \{\alpha_K, \beta^L\}, \quad \int_{\mathcal{M}_6} \langle \alpha_K, \beta^L \rangle = \delta_K^L, \quad K, L = 0, \dots, b^-$$

Using this, the pure spinors Φ_{\pm} are expanded

$$\Phi_+ = e^{-\mathcal{B}} \Phi_{0+} = \mathcal{X}^A \omega_A - \mathcal{G}_A \tilde{\omega}^A$$

$$\Phi_- = e^{-\mathcal{B}} \Phi_{0-} = \mathcal{Z}^K \alpha_K - \mathcal{F}_K \beta^K$$

The compatibility is read as (w/ using $\forall V = x + \xi \in E$)

$$\langle \omega_A, V \cdot \alpha_K \rangle = \langle \omega_A, V \cdot \beta^K \rangle = \langle \tilde{\omega}^A, V \cdot \alpha_K \rangle = \langle \tilde{\omega}^A, V \cdot \beta^K \rangle = 0$$

The truncated Kähler potentials by $\int_{\mathcal{M}_6} \langle \omega_A, \tilde{\omega}^B \rangle = \delta_A^B$ and $\int_{\mathcal{M}_6} \langle \alpha_K, \beta^L \rangle = \delta_K^L$ are

$$e^{-K_+} = i \int_{\mathcal{M}_6} \langle \Phi_+, \bar{\Phi}_+ \rangle = i \left(\bar{\chi}^A \mathcal{G}_A - \chi^A \bar{\mathcal{G}}_A \right)$$

$$e^{-K_-} = i \int_{\mathcal{M}_6} \langle \Phi_-, \bar{\Phi}_- \rangle = i \left(\bar{z}^K \mathcal{F}_K - z^K \bar{\mathcal{F}}_K \right)$$

RR fields are also expanded as

$$\begin{array}{l} \text{type IIA:} \\ \text{type IIB:} \end{array} \left\{ \begin{array}{l} \mathcal{A}_0^- = \xi^K \alpha_K + \tilde{\xi}_L \beta^L \\ \mathcal{A}_1^+ = A_1^A \omega_A + \tilde{A}_{1B} \tilde{\omega}^B \\ \mathcal{A}_0^+ = \xi^A \omega_A + \tilde{\xi}_B \tilde{\omega}^B \\ \mathcal{A}_1^- = A_1^K \alpha_K + \tilde{A}_{1L} \beta^L \end{array} \right. \quad \begin{array}{l} \text{w/} \\ \text{w/} \end{array} \left\{ \begin{array}{l} \xi^K, \tilde{\xi}_L : \text{scalars} \\ A_1^A, \tilde{A}_{1B} : \text{vectors} \\ \xi^A, \tilde{\xi}_B : \text{scalars} \\ A_1^K, \tilde{A}_{1L} : \text{vectors} \end{array} \right.$$

Convenient to define dual antisymmetric tensor fields of \mathcal{A}_0^- and \mathcal{A}_0^+ :

$$\begin{aligned} \mathcal{A}_0^- \leftrightarrow \mathcal{A}_2^- &\equiv \tilde{C}_2^K \alpha_K + C_{2L} \beta^L & \mathcal{A}_0^+ \leftrightarrow \mathcal{A}_2^+ &\equiv \tilde{C}_2^A \omega_A + C_{2B} \tilde{\omega}^B \\ \xi^K \leftrightarrow C_{2K} & \quad \tilde{\xi}_K \leftrightarrow \tilde{C}_2^K & \xi^A \leftrightarrow C_{2A} & \quad \tilde{\xi}_A \leftrightarrow \tilde{C}_2^A \end{aligned}$$

The most general differential conditions which can be imposed on basis forms are

$$\begin{aligned} d\alpha_K &\sim p_K^A \omega_A + e_{KA} \tilde{\omega}^A & d\beta^K &\sim q^{KA} \omega_A + m^K_A \tilde{\omega}^A \\ d\omega_A &\sim m^K_A \alpha_K - e_{KA} \beta^K & d\tilde{\omega}^A &\sim -q^{KA} \alpha_K + p_K^A \beta^K \end{aligned}$$

p_K^A , q^{KA} , e_{KA} and m^K_A are $(b^+ + 1) \times (b^- + 1)$ -dimensional constant matrices

Not necessary to be closed as in Calabi-Yau

Introduce a notation $\Sigma_+ = \begin{pmatrix} \omega_A \\ \tilde{\omega}^B \end{pmatrix}$, $\Sigma_- = \begin{pmatrix} \alpha_K \\ \beta^L \end{pmatrix}$ and $Q = \begin{pmatrix} p_K^A & e_{KB} \\ q^{LA} & m^L_B \end{pmatrix}$.

In terms of them the above differential condition is

$$d\Sigma_- \sim Q\Sigma_+ \quad d\Sigma_+ \sim \mathcal{S}_+ Q^T (\mathcal{S}_-)^{-1} \Sigma_-$$

\mathcal{S}_\pm : the symplectic structures on U^\pm

If we impose $d^2 = 0$ on the charged matrix as $Q\mathcal{S}_+Q^T = 0 = Q^T(\mathcal{S}_-)^{-1}Q$, we obtain

$$\begin{aligned} q^{KA}m_A^L - m^K_Aq^{AL} = 0 & \quad p_K^Ae_{AL} - e_{KA}p^A_L = 0 & \quad p_K^Am_A^L - e_{KA}q^{AL} = 0 \\ q^{AK}p_K^B - p^A_Kq^{KB} = 0 & \quad m_A^Ke_{KB} - e_{AK}m^K_B = 0 & \quad m_A^Kp_K^B - e_{AK}q^{KB} = 0 \end{aligned}$$

Kinetic terms $|\mathcal{G}_n|^2$ generate mass terms via truncation of fields:

► Type IIA:

$$\mathcal{G}_{2p} = d\mathcal{A}_{2p-1} \sim d_6\mathcal{A}_2^- + d_4\mathcal{A}_1^+ \equiv D_2^A \omega_A + \tilde{D}_{2A} \tilde{\omega}^A$$

$$D_2^A = d_4 A_1^A + \tilde{C}_2^K p_K^A + C_{2K} q^{AK}$$

$$\tilde{D}_{2A} = d_4 \tilde{A}_1^A + \tilde{C}_2^K e_{AK} + C_{2K} m^K{}_A$$

► Type IIB:

$$\mathcal{G}_{2p+1} = d\mathcal{A}_{2p} \sim d_6\mathcal{A}_2^+ + d_4\mathcal{A}_1^- \equiv D_2^K \alpha_K + \tilde{D}_{2K} \beta^K$$

$$D_2^K = d_4 A_1^K - \tilde{C}_2^A m^K{}_A + C_{2A} q^{AK}$$

$$\tilde{D}_{2K} = d_4 \tilde{A}_1^K + \tilde{C}_2^A e_{AK} - C_{2A} p_K^A$$

Then charge matrices give massive modes of RR fields:

	e_{AK}	$m^K{}_A$	p_K^A	q^{KA}
IIA	massive A_μ^A	massive A_μ^A	massive \tilde{C}_2^K	massive C_{2K}
IIB	massive A_μ^K	massive \tilde{C}_2^A	massive A_μ^K	massive C_{2A}

► Type IIA Killing prepotentials \mathcal{P}^x in S_{AB} w/ $\mathcal{G}^+ = d\mathcal{A}_0^- + G_{(RR)}^A \omega_A + \tilde{G}_{(RR)A} \tilde{\omega}^A$:

$$\begin{aligned} \mathcal{P}^1 + i\mathcal{P}^2 &= -2e^{\frac{1}{2}K_- + \phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_+, d\Phi_- \rangle \\ &= 2e^{\frac{1}{2}K_- + \phi^{(4)}} \left(-\chi^A e_{AK} \mathcal{Z}^K + \chi^A m_A{}^K \mathcal{F}_K - \mathcal{G}_{AP}{}^A{}_K \mathcal{Z}^K + \mathcal{G}_{Aq}{}^{AK} \mathcal{F}_K \right) \\ \mathcal{P}^3 &= e^{2\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_+, \mathcal{G}^+ \rangle \\ &= e^{2\phi^{(4)}} \left[\chi^A (\tilde{G}_{(RR)A} + e_{AK} \xi^K + m_A{}^K \tilde{\xi}_K) + \mathcal{G}_A (G_{(RR)}^A + p^A{}_K \xi^K + q^{AK} \tilde{\xi}_K) \right] \end{aligned}$$

$\mathcal{N} = 1$ superpotential W_{IIA} is given by

$$W_{\text{IIA}} = \cos^2 \alpha e^{i\beta} \int_{\mathcal{M}_6} \langle \Phi_+, d\Phi_- \rangle - \sin^2 \alpha e^{-i\beta} \int_{\mathcal{M}_6} \langle \Phi_+, d\bar{\Phi}_- \rangle + \sin 2\alpha e^{\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_+, \mathcal{G}^+ \rangle$$

► Type IIB Killing prepotentials \mathcal{P}^x in S_{AB} w/ $\mathcal{G}^- = d\mathcal{A}_0^+ + G_{(RR)}^K \alpha_K + \tilde{G}_{(RR)L} \beta^L$:

$$\begin{aligned}
 \mathcal{P}^1 - i\mathcal{P}^2 &= -2e^{\frac{1}{2}K_++\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_-, d\Phi_+ \rangle \\
 &= 2e^{\frac{1}{2}K_++\phi^{(4)}} \left(-z^K e_{KA} \mathcal{X}^A - z^K p_K{}^A \mathcal{G}_A + \mathcal{F}_K m^K{}_A \mathcal{X}^A + \mathcal{F}_K q^{KA} \mathcal{G}_A \right) \\
 \mathcal{P}^3 &= -e^{2\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_-, \mathcal{G}^- \rangle \\
 &= -e^{2\phi^{(4)}} \left[z^K (\tilde{G}_{(RR)K} - e_{KA} \xi^A + p_K{}^A \tilde{\xi}_A) + \mathcal{F}_K (G_{(RR)}^K + m^K{}_A \xi^A - q^{KA} \tilde{\xi}_A) \right]
 \end{aligned}$$

$\mathcal{N} = 1$ superpotential W_{IIB} is given by

$$W_{\text{IIB}} = -\cos^2 \alpha e^{i\beta} \int_{\mathcal{M}_6} \langle \Phi_-, d\Phi_+ \rangle + \sin^2 \alpha e^{-i\beta} \int_{\mathcal{M}_6} \langle \Phi_-, d\bar{\Phi}_+ \rangle - \sin 2\alpha e^{\phi^{(4)}} \int_{\mathcal{M}_6} \langle \Phi_-, \mathcal{G}^- \rangle$$

Generically, scalar potential V in four-dimensional theory is

$$V = e^K \left(g^{a\bar{b}} D_a W \overline{D_b W} - 3|W|^2 \right)$$

$$g_{a\bar{b}} = \partial_a \bar{\partial}_b (K_+ + K_- + 2\phi^{(4)}) \quad D_a W = (\partial_a + \partial_a K) W$$

Expanded the scalar potential V by “scalar fields” $\{\mathcal{X}^A, \xi^A, \tilde{\xi}_A, \mathcal{Z}^K, \xi^K, \tilde{\xi}_K\}$,

we would obtain non-trivial mass terms in $\mathcal{N} = 1$ theory

--> so-called **moduli stabilization**

- ▶ Introduce a pair of $SU(3)$ structures on $F \sim SU(3) \times SU(3)$ structure on $F \oplus F^*$
- ▶ Define generalized complex structures \mathcal{J}_i
- ▶ Construct $Spin(6, 6)$ pure spinors Φ_{\pm}
- ▶ Evaluate the space of pure spinors, and define Hitchin functional $H(\Phi_{\pm})$
- ▶ Derive Kähler potentials K_{\pm} and superpotentials $W_{\text{IIA/IIB}}$
- ▶ Truncation of ten-dimensional fields

Remaining problem of flux compactification in type IIA/IIB is...

to find concrete dimensions b^{\pm} of (non-)harmonic forms on compactified geometry \mathcal{M}_6 !

→ a (mathematical) future problem

Doubled geometry

Start from low energy effective field theory for ten-dimensional string theory including

$$S = \int d^{10}x \sqrt{-\mathcal{G}} e^{-\phi} \left\{ \mathcal{R} + (\nabla\phi)^2 - \frac{1}{12} \mathcal{H}_{MNP} \mathcal{H}^{MNP} \right\}$$
$$\mathcal{H} = d\mathcal{B}$$

Consider the field theory compactified on (twisted) torus in the presence of B-field

VEV of three-form \mathcal{H} gives rise to the structure constant of a certain Lie algebra

Decomposition of fields by Kaluza-Klein Compactification on a flat d -torus

$$ds^2 = g_{\mu\nu}(x, y)dx^\mu \otimes dx^\nu + \mathcal{G}_{ij}(x, y)(dy^i + \mathcal{V}^i_\mu(x, y)dx^\mu) \otimes (dy^j + \mathcal{V}^j_\nu(x, y)dx^\nu)$$

$$\mathcal{B} = \frac{1}{2}\mathcal{B}_{\mu\nu}(x, y)dx^\mu \wedge dx^\nu + \mathcal{B}_{\mu i}(x, y)dx^\mu \wedge dy^i + \frac{1}{2}\mathcal{B}_{ij}(x, y)dy^i \wedge dy^j$$

with **Ansatz** (truncation of massive KK modes)

$$g_{\mu\nu}(x, y) = g_{\mu\nu}(x) \quad \mathcal{G}_{ij}(x, y) = g_{ij}(x) \quad \mathcal{V}^i_\mu(x, y) = V^i_\mu(x)$$

$$\mathcal{B}_{\mu\nu}(x, y) = \mathcal{B}_{\mu\nu}(x) \quad \mathcal{B}_{\mu i}(x, y) = \mathcal{B}_{\mu i}(x) \quad \mathcal{B}_{ij}(x, y) = B_{ij}(x)$$

$$\phi(x, y) = \varphi(x) + \frac{1}{2} \log |\det g_{ij}(x)|$$

Reduced degrees of freedom to demonstrate manifest gauge invariance:

$$B_{\mu i} = \mathcal{B}_{\mu i} + B_{ij}V^j_\mu$$

$$B_{\mu\nu} = \mathcal{B}_{\mu\nu} + V^i_{[\mu}B_{\nu]i} - B_{ij}V^i_\mu V^j_\nu$$

The reduced action in D -dimensions (w/ setting $D = 10 - d$):

$$S = \int d^D x \sqrt{-g} e^{-\varphi} \left\{ R + (\nabla\varphi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{8} L_{IJ} \nabla_\mu \mathcal{M}^{JK} L_{KL} \nabla^\mu \mathcal{M}^{LI} - \frac{1}{4} F_{\mu\nu}^I L_{IJ} \mathcal{M}^{JK} L_{KL} F^{L\mu\nu} \right\}$$

This theory has $U(1)^{2d}$ gauge symmetry and a manifest global $O(d, d)$ symmetry with

$$\mathcal{M}_{IJ} = \begin{pmatrix} g_{ij} - B_{ik} g^{kl} B_{lj} & B_{ik} g^{kj} \\ -g^{ik} B_{kj} & g^{ij} \end{pmatrix} : \text{ scalar moduli matrix}$$

$$F^I = dA^I \quad A_\mu^I = \begin{pmatrix} V_\mu^i \\ B_{\mu i} \end{pmatrix} \quad H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2} A_\mu^I L_{IJ} F_{\nu\rho}^J + (\text{cyclic permutations})$$

$$L^{IJ} \equiv \begin{pmatrix} \mathbf{0}_d & \mathbb{1}_d \\ \mathbb{1}_d & \mathbf{0}_d \end{pmatrix} : \quad O(d, d) \text{ invariant metric, } \forall M^I{}_J \in O(d, d), \quad M^I{}_K L^{KL} M_L{}^J = L^{IJ}$$

Non-abelian gauge symmetry by considering a $2d$ -dimensional subgroup G of $O(d, d)$:

The fundamental repr. of $O(d, d)$ becomes the adjoint repr. of G under the embedding

$$[T_I, T_J] = f_{IJ}{}^K T_K \quad T_I = \frac{1}{2} \Theta_I{}^{JK} t_{JK} \quad \left\{ \begin{array}{l} T_I : \text{generators of } G \text{ with structure constant } f_{IJ}{}^K \\ t_{JK} : \text{generators of } O(d, d) \\ \Theta_I{}^{JK} : \text{embedding tensor} \end{array} \right.$$

Then, D -dimensional theory with gauge symmetry G is

$$S = \int d^D x \sqrt{-g} e^{-\varphi} \left\{ R + (\nabla\varphi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{1}{8} L_{IJ} \mathcal{D}_\mu \mathcal{M}^{JK} L_{KL} \mathcal{D}^\mu \mathcal{M}^{LI} - \frac{1}{4} F_{\mu\nu}^I \mathcal{M}^{JK} L_{KL} F^{L\mu\nu} - g^2 W(\mathcal{M}) \right\}$$

with covariantized form (w/ $f_{IJK} = f_{IJ}{}^L L_{KL}$)

$$\mathcal{D}_\mu \mathcal{M}^{IJ} = \partial_\mu \mathcal{M}^{IJ} - g f_{KL}{}^I A_\mu^K \mathcal{M}^{LJ} - g f_{KL}{}^J A_\mu^K \mathcal{M}^{IL}$$

$$F = dA + gA \wedge A, \quad H = dB - \frac{1}{2} \text{tr} \left(A \wedge F + \frac{2g}{3} A \wedge A \wedge A \right)$$

$$W(\mathcal{M}) = a \mathcal{M}^{II'} \mathcal{M}^{JJ'} \mathcal{M}^{KK'} f_{IJK} f_{I'J'K'} + b \mathcal{M}^{II'} \mathcal{M}^{JJ'} L^{KK'} f_{IJK} f_{I'J'K'} + c \mathcal{M}^{II'} L^{JJ'} L^{KK'} f_{IJK} f_{I'J'K'} + W_0$$

What is T_I ? \leftarrow non-abelianized generators of gauge fields $A_\mu^I = (V_\mu^i, B_{\mu i})^T$

$$T_I \ni \begin{cases} Z_i : & \text{generators corresponding to } V_\mu^i \\ X^i : & \text{generators corresponding to } B_{\mu i} \end{cases}$$

$$[T_I, T_J] = f_{IJ}^K T_K \quad \longrightarrow \quad \begin{cases} [Z_i, Z_j] = \tau_{ij}^k Z_k + h_{ijk} X^k \\ [X^i, X^j] = 0 \\ [X^i, Z_j] = \tau^i_{jk} X^k \end{cases}$$

τ_{ij}^k : structure constant of twisted torus

h_{ijk} : VEV of NS three-form H_{ijk}

$$\tau^l_{i'[i\tau_{jk}]^{i'}} = 0 \quad \text{Jacobi}$$

$$h_{i'[ij\tau_{kl}]^{i'}} = 0 \quad dH_3 = 0$$

$$\tau^i_{ij} = 0 \quad \text{invariance of } \sqrt{-g}$$

► twisted torus is introduced by vielbein $dy^i \rightarrow \eta^a = \eta^a_i(y) dy^i$:

$$\mathcal{G}_{ij}(x, y) = g_{ij}(x) \rightarrow g_{ab}(x) \eta^a_i(y) \eta^b_j(y)$$

$$g_{ij}(dy^i + V^i_\mu dx^\mu)(dy^j + V^j_\nu dx^\nu) \rightarrow g_{ab}(\eta^a + V^a_\mu dx^\mu)(\eta^b + V^b_\nu dx^\nu)$$

$$d\eta^a = -\frac{1}{2} \tau_{bc}^a \eta^b \wedge \eta^c$$

► Possibility of extension of the Lie algebra

$$\begin{aligned} [Z_a, Z_b] &= \tau_{ab}{}^c Z_c + h_{abc} X^c \\ [X^a, X^b] &= 0 \\ [X^a, Z_b] &= \tau^a{}_{bc} X^c \end{aligned}$$

↓

$$\begin{aligned} [Z_a, Z_b] &= \tau_{ab}{}^c Z_c + h_{abc} X^c \\ [X^a, X^b] &= Q^{ab}{}_c X^c + R^{abc} Z_c \\ [X^a, Z_b] &= \tau^a{}_{bc} X^c - Q^{ac}{}_b Z_c \end{aligned}$$

Why should we study the additional structure constants $Q^{ab}{}_c$ and R^{abc} ?

↓

Because they are related via T-duality transformations!

$$h_{abc} \xrightarrow{\text{T-dual}} \tau^a{}_{bc} \xrightarrow{\text{T-dual}} Q^{ab}{}_c \xrightarrow{\text{T-dual}} R^{abc}$$

J. Shelton, W. Taylor, B. Wecht [hep-th/0508133](https://arxiv.org/abs/hep-th/0508133) A. Dabholkar, C.M. Hull [hep-th/0512005](https://arxiv.org/abs/hep-th/0512005)

► Geometries generated by T-duality transformations:

h_{abc}	flat torus w/ three-form flux
\downarrow	T-duality transformation
$\tau^a{}_{bc}$	nilmanifold w/ non-trivial isometry group
\downarrow	T-duality transformation
$Q^{ab}{}_c$	T-fold: globally nongeometric, locally geometric, <i>stringy</i>
\downarrow	T-duality transformation
R^{abc}	doubled geometry: even locally nongeometric, <i>stringy</i>

So far, we discussed effective theory compactified on d -dimensional space in the presence of flux.

In order to include the above information, we double the “dimensions of compactified geometry” and study sigma model

→ doubled formalism

Glue two local patches of a conventional string background with transition function by

diffeomorphism

and

duality transformations

Let Y^i be fields in sigma model corresponding to coordinates y^i on a torus T^d .

In formulating CFT on T^d ,

extra d coordinates \tilde{Y}_i for a dual torus \tilde{T}^d are needed

These are conjugate to the winding number,

and are needed to write vertex operators such as $e^{ik_L \cdot Y_L}$ where $Y_L = Y - \tilde{Y}$.

→ Degrees of freedom of a sigma model are doubled from Y^i to $\{Y^i, \tilde{Y}_i\}$.

Start with a sigma model on a space \mathcal{M}_d with metric g_{ij} and B-field B_{ij} :

$$S_c = \int_{\Sigma} \left(\frac{1}{2} g_{ij} dY^i \wedge *dY^j + \frac{1}{2} B_{ij} dY^i \wedge dY^j \right)$$

This is extended to the action on a doubled space \mathcal{M}_{2d} w/ **scalar moduli matrix** \mathcal{M}_{IJ} :

$$S = \int_{\Sigma} \frac{1}{4} \mathcal{M}_{IJ} d\mathbb{Y}^I \wedge *d\mathbb{Y}^J = \int_{\Sigma} \frac{1}{4} \mathcal{M}_{AB} \eta^A \wedge *\eta^B$$

$$\mathcal{M}_{IJ} = \begin{pmatrix} g_{ij} - B_{ik} g^{kl} B_{lj} & B_{ik} g^{kj} \\ -g^{ik} B_{kj} & g^{ij} \end{pmatrix} \quad \mathbb{Y}^I = \begin{pmatrix} Y^i \\ \tilde{Y}_i \end{pmatrix}$$

$$\eta^A = \eta^A_I d\mathbb{Y}^I, \quad \mathcal{M}_{IJ} = \eta_I^A \delta_{AB} \eta^B_J, \quad \eta^A_I = \begin{pmatrix} e^a_i & \mathbf{0} \\ -e_a^j B_{ji} & e_a^i \end{pmatrix}$$

Bianchi identity (Maurer-Cartan eq.): $d\eta^A = -\frac{1}{2} f_{BC}^A \eta^B \wedge \eta^C$

Self-duality constraint (to go back to conventional system): $\eta^A = L^{AB} \mathcal{M}_{BC} * \eta^C$

w/ $O(d, d)$ invariant metric $L^{AB} \equiv \langle \eta^A, \eta^B \rangle$

$$\mathcal{M}_{IJ} \text{ takes value in coset } \frac{O(d, d)}{O(d) \times O(d)}$$

This sigma model on doubled space \mathcal{M}_{2d} has

- ▶ $O(d, d)$ global symmetry by $g \in O(d, d)$ w/ $g^A_C L^{CD} g_D^B = L^{AB}$:

$$\mathbb{Y}^I \rightarrow \mathbb{Y}'^I = g^I_J \mathbb{Y}^J$$

$$\eta^A_I(\mathbb{Y}) \rightarrow \eta'^A_I(\mathbb{Y}') = g^A_B \eta^B_J(\mathbb{Y}') g^J_I$$

$$\mathcal{M}_{IJ}(\mathbb{Y}) \rightarrow \mathcal{M}'_{IJ}(\mathbb{Y}') = g_I^K \mathcal{M}_{KL}(\mathbb{Y}') g^L_J$$

Basis vector is kept invariant under the transformation: so-called “active transformation”

- ▶ $O(d) \times O(d)$ local symmetry: $\eta^A_I(\mathbb{Y}) \rightarrow \eta'^A_I(\mathbb{Y}) = h^A_B(\mathbb{Y}) \eta^B_I(\mathbb{Y})$

A realization of fractional transformation of $M_{ij} = \mathcal{G}_{ij} + \mathcal{B}_{ij}$:

$$g = \begin{pmatrix} A & \beta \\ \Theta & D \end{pmatrix} : \quad M \rightarrow (DM + \Theta)(\beta M + A)^{-1}$$

$$\left\{ \begin{array}{l} \Theta : \text{ gauge transformation of B-field } B \rightarrow B + \Theta \\ D, A : \text{ ordinary diffeomorphism on } g_{ij} \\ \beta : \text{ non-local transformation w/ mixing } Y^i \text{ and } \tilde{Y}_i \leftarrow \text{ duality trsf.} \end{array} \right.$$

T-duality transformation (ex. $d = 3$ case):

$$\rho_i = \begin{pmatrix} \mathbb{1}_d - T_i & T_i \\ T_i & \mathbb{1}_d - T_i \end{pmatrix} \in O(d, d; \mathbb{Z})$$

$$T_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

This action exchanges physical coordinates Y^i with dual coordinates \tilde{Y}_i

A role of L : reduction of (co)tangent bundle of doubled space \mathcal{M}_{2d}

$$L^{AB} = \langle \eta^A, \eta^B \rangle = \begin{pmatrix} \mathbf{0}_d & \mathbb{1}_d \\ \mathbb{1}_d & \mathbf{0}_d \end{pmatrix}, \quad L^{IJ} \equiv \langle dY^I, dY^J \rangle = \eta^I{}_A L^{AB} \eta_B{}^J$$

This implies $T^*\mathcal{M}_{2d} = T\mathcal{M}_d \oplus T^*\mathcal{M}_d$ s.t.

$$\langle dY^i, d\tilde{Y}_j \rangle = \delta_j^i \quad \rightarrow \quad d\tilde{Y}_i = \frac{\partial}{\partial Y^i}$$

$$\eta^A = \eta^A{}_I dY^I = \begin{pmatrix} e^a{}_i dY^i \\ e_a{}^i (d\tilde{Y}_i - B_{ij} dY^j) \end{pmatrix} = \begin{pmatrix} e^a{}_i dY^i \\ e_a{}^i \left(\frac{\partial}{\partial Y^i} - B_{ij} dY^j \right) \end{pmatrix}$$

a connection to generalized geometry

Start from a flat three-torus T^3 with B-field given by the following forms:

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$H = dB = 3dx \wedge dy \wedge dz \quad B = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$$

Doubled vielbein and doubled metric are given as

$$\eta^A{}_I = \begin{pmatrix} e^a{}_i & \mathbf{0} \\ -e_a{}^j B_{ji} & e_a{}^i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 0 \\ \hline 0 & -z & y & | & 1 & 0 & 0 \\ z & 0 & -x & | & 0 & 1 & 0 \\ -y & x & 0 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{M}_{IJ} = \eta_I{}^A \delta_{AB} \eta^B{}_J = \begin{pmatrix} 1 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 0 \\ \hline 0 & -z & y & | & 1 & 0 & 0 \\ z & 0 & -x & | & 0 & 1 & 0 \\ -y & x & 0 & | & 0 & 0 & 1 \end{pmatrix}$$

► Maurer-Cartan eq. of doubled vielbein $\eta^A = \begin{pmatrix} \eta^a \\ \tilde{\eta}_a \end{pmatrix}$ gives a structure constant $f_{AB}{}^C$:

$$d\eta^1 = 0$$

$$d\eta^2 = 0$$

$$d\eta^3 = 0$$

$$d\tilde{\eta}_1 = 2\eta^2 \wedge \eta^3$$

$$d\tilde{\eta}_2 = 2\eta^3 \wedge \eta^1$$

$$d\tilde{\eta}_3 = 2\eta^2 \wedge \eta^3$$

$$\therefore d\eta^a = 0 \quad d\tilde{\eta}_a = -\frac{1}{2}h_{abc}\eta^b \wedge \eta^c$$

$$[Z_a, Z_b] = h_{abc}X^c \quad h_{123} = -2$$

► Periodicity of physical coordinates Y^i and dual ones \tilde{Y}_i :

$$\begin{aligned}
 (x, \tilde{y}, \tilde{z}) &\sim (x + 1, \tilde{y} + z, \tilde{z} - y) & \tilde{x} &\sim \tilde{x} + 1 \\
 (y, \tilde{z}, \tilde{x}) &\sim (y + 1, \tilde{z} + x, \tilde{x} - z) & \tilde{y} &\sim \tilde{y} + 1 \\
 (z, \tilde{x}, \tilde{y}) &\sim (z + 1, \tilde{x} + y, \tilde{y} - x) & \tilde{z} &\sim \tilde{z} + 1
 \end{aligned}$$

Existence of the non-trivial B-field is encoded in the redefinition of the dual coordinates by the actual coordinates, when the latter are shifted.

► Doubled vielbein by T-duality along z -direction:

$$(\eta_\tau)^A{}_I = (\rho_z)^A{}_B \eta^B{}_J (\rho_z)^J{}_I = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -y & x & 1 & 0 & 0 & 0 \\ \hline 0 & -\tilde{z} & 0 & 1 & 0 & y \\ \tilde{z} & 0 & 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$(\mathcal{M}_\tau)_{IJ} = (\rho_z)_I{}^K \mathcal{M}_{KL} (\rho_z)^L{}_J = (\eta_\tau)_I{}^A \delta_{AB} (\eta_\tau)^B{}_J$$

$$= \left(\begin{array}{ccc|ccc} 1 + y^2 + \tilde{z}^2 & -xy & -y & 0 & \tilde{z} & -x\tilde{z} \\ -xy & 1 + x^2 + \tilde{z}^2 & x & -\tilde{z} & 0 & -y\tilde{z} \\ -y & x & 1 & 0 & 0 & 0 \\ \hline 0 & -\tilde{z} & 0 & 1 & 0 & y \\ \tilde{z} & 0 & 0 & 0 & 1 & -x \\ -x\tilde{z} & -y\tilde{z} & 0 & y & -x & 1 + x^2 + y^2 \end{array} \right)$$

Metric and B-field can be read from the doubled metric as

$$ds^2 = (dx)^2 + (dy)^2 + (dz - ydx + xdy)^2$$

$$B = \tilde{z} dx \wedge dy \quad H = dx \wedge dy \wedge d\tilde{z}$$

- Maurer-Cartan eq. of doubled vielbein $\eta^A = \begin{pmatrix} \eta^a \\ \tilde{\eta}_a \end{pmatrix}$ gives a structure constant $f_{AB}{}^C$:

$$d\eta^1 = 0 \qquad d\eta^2 = 0 \qquad d\eta^3 = 2\eta^1 \wedge \eta^2$$

$$d\tilde{\eta}_1 = 2\eta^2 \wedge \tilde{\eta}_3 \qquad d\tilde{\eta}_2 = 2\tilde{\eta}_3 \wedge \eta^1 \qquad d\tilde{\eta}_3 = 0$$

$$\therefore d\eta^a = -\frac{1}{2}\tau^a{}_{bc}\eta^b \wedge \eta^c \qquad d\tilde{\eta}_a = -\frac{1}{2}\tau_{ab}{}^c\eta^b \wedge \eta^c$$

$$[Z_a, Z_b] = \tau_{ab}{}^c Z_c \qquad [X^a, Z_b] = \tau^a{}_{bc} X^c \qquad \tau^1{}_{23} = -2$$

a geometry with torsion, similar to half-flat manifold: a connection to generalized geometry

- Periodicity of physical coordinates Y^i and dual ones \tilde{Y}_i :

$$(x, \tilde{y}, z) \sim (x + 1, \tilde{y} + z, z - y)$$

$$\tilde{x} \sim \tilde{x} + 1$$

$$(y, z, \tilde{x}) \sim (y + 1, z + x, \tilde{x} - z)$$

$$\tilde{y} \sim \tilde{y} + 1$$

$$z \sim z + 1$$

$$(\tilde{z}, \tilde{x}, \tilde{y}) \sim (\tilde{z} + 1, \tilde{x} + \tilde{y}, \tilde{y} - \tilde{x})$$

- Doubled vielbein by T-duality along (y, z) -directions:

$$(\eta_Q)^A{}_I = (\rho_y \rho_z)^A{}_B \eta^B{}_J (\rho_z \rho_y)^J{}_I = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \tilde{z} & 1 & 0 & 0 & 0 & -x \\ -\tilde{y} & 0 & 1 & 0 & x & 0 \\ \hline 0 & 0 & 0 & 1 & -\tilde{z} & \tilde{y} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\begin{aligned} (\mathcal{M}_Q)_{IJ} &= (\rho_y \rho_z)_I{}^K \mathcal{M}_{KL} (\rho_z \rho_y)^L{}_J = (\eta_Q)_I{}^A \delta_{AB} (\eta_Q)^B{}_J \\ &= \left(\begin{array}{ccc|ccc} 1 + \tilde{y}^2 + \tilde{z}^2 & \tilde{z} & -\tilde{y} & 0 & -x\tilde{y} & -x\tilde{z} \\ \tilde{z} & 1 & 0 & 0 & 0 & -x \\ -\tilde{y} & 0 & 1 & 0 & x & 0 \\ \hline 0 & 0 & 0 & 1 & -\tilde{z} & \tilde{y} \\ -x\tilde{y} & 0 & x & -\tilde{z} & 1 + x^2 + \tilde{z}^2 & -\tilde{y}\tilde{z} \\ -x\tilde{z} & -x & 0 & \tilde{y} & -\tilde{y}\tilde{z} & 1 + x^2 + \tilde{y}^2 \end{array} \right) \end{aligned}$$

Doubled vielbein η_Q is not a lower block triangular form, even though correct metric and B-field can be read from \mathcal{M}_Q .

local $O(3) \times O(3)$ transformation to describe correct form of doubled vielbein

$$\begin{aligned}
 (\eta'_Q)^A{}_I &= h^A{}_B (\eta_Q)^B{}_I \equiv \begin{pmatrix} (e_Q)^a{}_i & \mathbf{0}_3 \\ -(e_Q)_a{}^j (B_Q)_{ji} & (e_Q)_a{}^i \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 & | & 0 & 0 & 0 \\ \frac{\tilde{z}}{\sqrt{1+x^2}} & \frac{1}{\sqrt{1+x^2}} & 0 & | & 0 & 0 & 0 \\ -\frac{\tilde{y}}{\sqrt{1+x^2}} & 0 & \frac{1}{\sqrt{1+x^2}} & | & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & | & 1 & -\tilde{z} & \tilde{y} \\ -\frac{x\tilde{y}}{\sqrt{1+x^2}} & 0 & \frac{x}{\sqrt{1+x^2}} & | & 0 & \sqrt{1+x^2} & 0 \\ -\frac{x\tilde{z}}{\sqrt{1+x^2}} & -\frac{x}{\sqrt{1+x^2}} & 0 & | & 0 & 0 & \sqrt{1+x^2} \end{pmatrix} \\
 h^A{}_B &= \begin{pmatrix} 1 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1+x^2}} & 0 & | & 0 & 0 & \frac{x}{\sqrt{1+x^2}} \\ 0 & 0 & \frac{1}{\sqrt{1+x^2}} & | & 0 & -\frac{x}{\sqrt{1+x^2}} & 0 \\ \hline 0 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 0 & \frac{x}{\sqrt{1+x^2}} & | & 0 & \frac{1}{\sqrt{1+x^2}} & 0 \\ 0 & -\frac{x}{\sqrt{1+x^2}} & 0 & | & 0 & 0 & \frac{1}{\sqrt{1+x^2}} \end{pmatrix}
 \end{aligned}$$

- ▶ Metric and B-field can be read from the doubled metric as

$$ds^2 = (dx)^2 + \frac{1}{1+x^2} \left[(dy + \tilde{z} dx)^2 + (dz - \tilde{y} dx)^2 \right]$$

$$B = \frac{x}{1+x^2} \left[-\tilde{y} dx \wedge dy - dy \wedge dz + \tilde{z} dz \wedge dx \right]$$

$$H = \frac{-1+x^2}{(1+x^2)^2} dx \wedge dy \wedge dz - \frac{x}{1+x^2} dx \wedge dy \wedge d\tilde{y} + \frac{x}{1+x^2} dz \wedge dx \wedge d\tilde{z}$$

Both g_{ij} and B_{ij} are not well-defined functions of x .

They give however a good global description upon using the identification.

- ▶ Maurer-Cartan eq. of doubled vielbein $\eta^A = \begin{pmatrix} \eta^a \\ \tilde{\eta}_a \end{pmatrix}$ gives a structure constant $f_{AB}{}^C$:

$$d\eta^1 = 0 \qquad d\eta^2 = 2\tilde{\eta}_3 \wedge \eta^1 \qquad d\eta^3 = 2\eta^1 \wedge \tilde{\eta}_2$$

$$d\tilde{\eta}_1 = 2\tilde{\eta}_2 \wedge \tilde{\eta}_3 \qquad d\tilde{\eta}_2 = 0 \qquad d\tilde{\eta}_3 = 0$$

$$\therefore d\eta^a = -\frac{1}{2} Q^{ab}{}_c \tilde{\eta}_b \wedge \eta^c \qquad d\tilde{\eta}_a = -\frac{1}{2} Q_a{}^{bc} \tilde{\eta}_b \wedge \tilde{\eta}_c$$

$$[X^a, X^b] = Q^{ab}{}_c X^c \qquad [Z_a, X^b] = Q_a{}^{bc} Z_c \qquad Q^{12}{}_3 = -2$$

- ▶ Periodicity of physical coordinates Y^i and dual ones \tilde{Y}_i :

$$(x, y, z) \sim (x + 1, y + \tilde{z}, z - \tilde{y})$$

$$\tilde{x} \sim \tilde{x} + 1$$

$$y \sim y + 1$$

$$(\tilde{y}, z, \tilde{x}) \sim (\tilde{y} + 1, z + x, \tilde{x} - \tilde{z})$$

$$z \sim z + 1$$

$$(\tilde{z}, \tilde{x}, y) \sim (\tilde{z} + 1, \tilde{x} + \tilde{y}, y - x)$$

Identification which shifts the ordinary coordinates Y^i by a dual one \tilde{Y}_i when identifying a base coordinate $x \sim x + 1$.

This means that one should identify properly the B-field B_{ij} and the metric g_{ij} .

- Doubled vielbein by T-duality along (x, y, z) -directions:

$$(\eta_R)^A{}_I = (\rho_z \rho_y \rho_x)^A{}_B \eta^B{}_J (\rho_x \rho_y \rho_z)^J{}_I = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\tilde{z} & \tilde{y} \\ 0 & 1 & 0 & \tilde{z} & 0 & -\tilde{x} \\ 0 & 0 & 1 & -\tilde{y} & \tilde{x} & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\begin{aligned} (\mathcal{M}_R)_{IJ} &= (\rho_y \rho_x)_I{}^K \mathcal{M}_{KL} (\rho_x \rho_y)^L{}_J = (\eta_R)_I{}^A \delta_{AB} (\eta_R)^B{}_J \\ &= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\tilde{z} & \tilde{y} \\ 0 & 1 & 0 & \tilde{z} & 0 & -\tilde{x} \\ 0 & 0 & 1 & -\tilde{y} & \tilde{x} & 0 \\ \hline 0 & \tilde{z} & -\tilde{y} & 1 + \tilde{y}^2 + \tilde{z}^2 & -\tilde{x}\tilde{y} & -\tilde{x}\tilde{z} \\ -\tilde{z} & 0 & \tilde{x} & -\tilde{x}\tilde{y} & 1 + \tilde{x}^2 + \tilde{z}^2 & -\tilde{y}\tilde{z} \\ \tilde{y} & -\tilde{x} & 0 & -\tilde{x}\tilde{z} & -\tilde{y}\tilde{z} & 1 + \tilde{x}^2 + \tilde{y}^2 \end{array} \right) \end{aligned}$$

Doubled vielbein η_R is not a lower block triangular form, even though correct metric and B-field can be read from \mathcal{M}_R .

local $O(3) \times O(3)$ transformation to describe correct form of doubled vielbein

$$\begin{aligned}
 (\eta'_R)^A{}_I &= h^A{}_B (\eta_R)^B{}_I \equiv \begin{pmatrix} (e_R)^{a_i} & \mathbf{0}_3 \\ -(e_R)_{a^j} (B_R)_{ji} & (e_R)_{a^i} \end{pmatrix} \\
 &= \begin{pmatrix} \chi(1 + \tilde{x}^2) & \chi(\tilde{x}\tilde{y} + \tilde{z}) & \chi(\tilde{z}\tilde{x} - \tilde{y}) & 0 & 0 & 0 \\ \chi(\tilde{x}\tilde{y} - \tilde{z}) & \chi(1 + \tilde{y}^2) & \chi(\tilde{y}\tilde{z} + \tilde{x}) & 0 & 0 & 0 \\ \chi(\tilde{z}\tilde{x} + \tilde{y}) & \chi(\tilde{y}\tilde{z} - \tilde{x}) & \chi(1 + \tilde{z}^2) & 0 & 0 & 0 \\ \hline -\chi(\tilde{y}^2 + \tilde{z}^2) & \chi(\tilde{x}\tilde{y} + \tilde{z}) & \chi(\tilde{z}\tilde{x} - \tilde{y}) & 1 & \tilde{z} & -\tilde{y} \\ \chi(\tilde{x}\tilde{y} - \tilde{z}) & -\chi(\tilde{x}^2 + \tilde{z}^2) & \chi(\tilde{x} + \tilde{y}\tilde{z}) & -\tilde{z} & 1 & \tilde{x} \\ \chi(\tilde{z}\tilde{x} + \tilde{y}) & \chi(\tilde{y}\tilde{z} - \tilde{x}) & -\chi(\tilde{x}^2 + \tilde{y}^2) & \tilde{y} & -\tilde{x} & 1 \end{pmatrix} \\
 h^A{}_B &= \chi \begin{pmatrix} 1 + \tilde{x}^2 & \tilde{x}\tilde{y} + \tilde{z} & \tilde{z}\tilde{x} - \tilde{y} & -\tilde{y}^2 - \tilde{z}^2 & \tilde{x}\tilde{y} + \tilde{z} & \tilde{z}\tilde{x} - \tilde{y} \\ \tilde{x}\tilde{y} - \tilde{z} & 1 + \tilde{y}^2 & \tilde{y}\tilde{z} + \tilde{x} & \tilde{x}\tilde{y} - \tilde{z} & -\tilde{z}^2 - \tilde{x}^2 & \tilde{y}\tilde{z} + \tilde{x} \\ \tilde{z}\tilde{x} + \tilde{y} & \tilde{y}\tilde{z} - \tilde{x} & 1 + \tilde{z}^2 & \tilde{z}\tilde{x} + \tilde{y} & \tilde{y}\tilde{z} - \tilde{x} & -\tilde{x}^2 - \tilde{y}^2 \\ \hline -\tilde{y}^2 - \tilde{z}^2 & \tilde{x}\tilde{y} + \tilde{z} & \tilde{z}\tilde{x} - \tilde{y} & 1 + \tilde{x}^2 & \tilde{x}\tilde{y} + \tilde{z} & \tilde{z}\tilde{x} - \tilde{y} \\ \tilde{x}\tilde{y} - \tilde{z} & -\tilde{z}^2 - \tilde{x}^2 & \tilde{y}\tilde{z} + \tilde{x} & \tilde{x}\tilde{y} - \tilde{z} & 1 + \tilde{y}^2 & \tilde{y}\tilde{z} + \tilde{x} \\ \tilde{z}\tilde{x} + \tilde{y} & \tilde{y}\tilde{z} - \tilde{x} & -\tilde{x}^2 - \tilde{y}^2 & \tilde{z}\tilde{x} + \tilde{y} & \tilde{y}\tilde{z} - \tilde{x} & 1 + \tilde{z}^2 \end{pmatrix} \\
 \chi &= \frac{1}{1 + \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2}
 \end{aligned}$$

$$ds^2 = \frac{1}{1 + \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2} \left[(dx)^2 + (dy)^2 + (dz)^2 + (\tilde{x} dx + \tilde{y} dy + \tilde{z} dz)^2 \right]$$

$$B = -\frac{1}{1 + \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2} \left[\tilde{z} dx \wedge dy + \tilde{x} dy \wedge dz + \tilde{y} dz \wedge dx \right]$$

$$H = \chi^2 \left[2\tilde{x}\tilde{z} dx \wedge dy \wedge d\tilde{x} - (\chi^{-1} - 2\tilde{x}^2) dy \wedge dz \wedge d\tilde{x} + 2\tilde{x}\tilde{y} dz \wedge dx \wedge d\tilde{x} \right. \\ \left. + 2\tilde{y}\tilde{z} dx \wedge dy \wedge d\tilde{y} + 2\tilde{x}\tilde{y} dy \wedge dz \wedge d\tilde{y} - (\chi^{-1} - 2\tilde{y}^2) dz \wedge dx \wedge d\tilde{y} \right. \\ \left. - (\chi^{-1} - 2\tilde{z}^2) dx \wedge dy \wedge d\tilde{z} + 2\tilde{x}\tilde{z} dy \wedge dz \wedge d\tilde{z} + 2\tilde{y}\tilde{z} dz \wedge dx \wedge d\tilde{z} \right]$$

$$\chi = \frac{1}{1 + \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2}$$

When the dual coordinates \tilde{Y}_i are projected out (i.e., $\tilde{Y}_i \equiv 0$),
the geometry looks reduced to a torus without B-field.

► Maurer-Cartan eq. of doubled vielbein $\eta^A = \begin{pmatrix} \eta^a \\ \tilde{\eta}_a \end{pmatrix}$ gives a structure constant $f_{AB}{}^C$:

$$d\eta^1 = 2\tilde{\eta}_2 \wedge \tilde{\eta}_3 \quad d\eta^2 = 2\tilde{\eta}_3 \wedge \tilde{\eta}_1 \quad d\eta^3 = 2\tilde{\eta}_1 \wedge \tilde{\eta}_2$$

$$d\tilde{\eta}_1 = 0 \quad d\tilde{\eta}_2 = 0 \quad d\tilde{\eta}_3 = 0$$

$$\therefore d\eta^a = -\frac{1}{2}R^{abc}\tilde{\eta}_b \wedge \tilde{\eta}_c \quad d\tilde{\eta}_a = 0$$

$$[X^a, X^b] = R^{abc}Z_c \quad R^{123} = -2$$

► Periodicity of physical coordinates Y^i and dual ones \tilde{Y}_i :

$$\begin{array}{ll}
 x \sim x + 1 & (\tilde{x}, y, z) \sim (\tilde{x} + 1, y + \tilde{z}, z - \tilde{y}) \\
 y \sim y + 1 & (\tilde{y}, z, x) \sim (\tilde{y} + 1, z + \tilde{x}, x - \tilde{z}) \\
 z \sim z + 1 & (\tilde{z}, x, y) \sim (\tilde{z} + 1, x + \tilde{y}, y - \tilde{x})
 \end{array}$$

Although the naive projection to the base space may seem to yield a flat torus with a trivial B-field, the identifications required on \tilde{Y}_i have an extreme effect on the field content.

These identifications involve β -transformations related to the shift of a base coordinate Y^i by \tilde{Y}_i .

These identifications entangle the metric and the B-field, without any relation to a geometric action on the base coordinates Y^i , i.e., we obtain a **fully nongeometric** background.

- ▶ Start from scalar moduli matrix
- ▶ Introduce dual coordinates induced by B-field
- ▶ Extend Lie algebra
- ▶ Perform T-duality transformations
- ▶ Evaluate geometries

Extend to U-fold endowed with U-duality transformation (hidden symmetry)

Investigate quantum aspects of the doubled sigma model

Discussions

- ▶ Find a way to analyze dimensions of moduli spaces
- ▶ Find relations between generalized geometry and doubled formalism
- ▶ Find application to moduli stabilization, landscape of flux vacua, etc.
- ▶ Study non-perturbative effects on generalized/doubled geometries
- ▶ Include D-branes (and orientifold planes) into generalized/doubled geometries

C. Albertsson, R.A. Reid-Edwards, TK “D-branes and doubled geometry” to appear...

Appendix

Compactification Ansatz for the ten-dimensional spacetime:

$$\mathcal{M}_{1,9} = \mathcal{M}_{1,3} \times_W \mathcal{M}_6$$

$$ds_{1,9}^2 = \mathcal{G}_{MN} dx^M dx^N = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + \mathcal{G}_{ij} dy^i dy^j$$

Maximal symmetry of $\mathcal{M}_{1,3} \rightarrow \langle \text{fermions} \rangle = 0$

Supersymmetric vacuum $\leftrightarrow \langle \delta(\text{fermions}) \rangle = 0$

$$\delta \begin{pmatrix} \Psi_M^1 \\ \Psi_M^2 \end{pmatrix} = D_M \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix} - \frac{1}{96} e^{-\phi} \left(\gamma_M^{PQR} \mathcal{H}_{PQR} - 9 \gamma^{PQ} \mathcal{H}_{MPQ} \right) \mathcal{P} \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix}$$

$$- \sum_n \frac{1}{64n!} e^{\frac{5-n}{4}\phi} \left[(n-1) \gamma_M^{N_1 \dots N_n} - n(9-n) \delta_M^{N_1} \gamma^{N_2 \dots N_n} \right] \mathcal{F}_{N_1 \dots N_n} \mathcal{P}_n \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix}$$

	n	\mathcal{P}	\mathcal{P}_n
IIA	0, 2, 4, 6, 8	γ_{11}	$(\gamma_{11})^{n/2} \sigma^1$
IIB	1, 5, 9	$-\sigma^3$	$i\sigma^2$
	3, 7		σ^1

Decomposition of Lorentz symmetry:

$$Spin(1, 9) \rightarrow Spin(1, 3) \times Spin(6) = SL(2, \mathbb{C}) \times SU(4)$$

$$\mathbf{16}_1 = (\mathbf{2}, \mathbf{4})_1 \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}})_1, \quad \mathbf{16}_2 = (\mathbf{2}, \bar{\mathbf{4}})_2 \oplus (\bar{\mathbf{2}}, \mathbf{4})_2$$

Decomposition of supersymmetry parameters (with $a, b \in \mathbb{C}$):

$$\left\{ \begin{array}{l} \epsilon_{\text{IIA}}^1 = \xi_+^1 \otimes (a\eta_+^1) + \xi_-^1 \otimes (\bar{a}\eta_-^1) \\ \epsilon_{\text{IIA}}^2 = \xi_+^2 \otimes (\bar{b}\eta_-^2) + \xi_-^2 \otimes (b\eta_+^2) \end{array} \right. \quad \left\{ \begin{array}{l} \epsilon_{\text{IIB}}^1 = \xi_+^1 \otimes (a\eta_+^1) + \xi_-^1 \otimes (\bar{a}\eta_-^1) \\ \epsilon_{\text{IIB}}^2 = \xi_+^2 \otimes (b\eta_+^2) + \xi_-^2 \otimes (\bar{b}\eta_-^2) \end{array} \right.$$

Set $SU(3)$ invariant spinor η_+^A s.t. $D^{(T)}\eta_+^A = 0$ ($A = 1, 2$):

$$\begin{array}{c} \text{spacetime } \mathcal{M}_{1,3} \\ \hline \mathcal{N} = 2: (\xi_+^1, \xi_+^2) \\ \downarrow \\ \mathcal{N} = 1: (\xi_+^1 = \xi_+^2 = \xi_+) \end{array}$$

$$\begin{array}{c} \text{compactified space } \mathcal{M}_6 \\ \hline \text{a pair of } SU(3) (\eta_+^1, \eta_+^2) \\ \downarrow \\ \text{a single } SU(3) (\eta_+^1 = \eta_+^2 = \eta_+) \end{array}$$

[back to spinor decompositions](#)

Four-dimensional $\mathcal{N} = 1$ Minkowski vacua in type IIA [hep-th/0509003](https://arxiv.org/abs/hep-th/0509003)

IIA	$a = 0$ or $b = 0$ (type A)	$a = be^{i\beta}$ (type BC)	
1	$\mathcal{W}_1 = H_3^{(1)} = 0$		
	$F_0^{(1)} = \mp F_2^{(1)} = F_4^{(1)} = \mp F_6^{(1)}$	$F_{2n}^{(1)} = 0$	
8	$\mathcal{W}_2 = F_2^{(8)} = F_4^{(8)} = 0$	generic β	$\beta = 0$
		$\text{Re}\mathcal{W}_2 = e^\phi F_2^{(8)}$ $\text{Im}\mathcal{W}_2 = 0$	$\text{Re}\mathcal{W}_2 = e^\phi F_2^{(8)} + e^\phi F_4^{(8)}$ $\text{Im}\mathcal{W}_2 = 0$
6	$\mathcal{W}_3 = \mp *_6 H_3^{(6)}$	$\mathcal{W}_3 = H_3^{(6)}$	
3	$\bar{\mathcal{W}}_5 = 2\mathcal{W}_4 = \mp 2iH_3^{(3)} = \bar{\partial}\phi$	$F_2^{(3)} = 2i\bar{\mathcal{W}}_5 = -2i\bar{\partial}A = \frac{2i}{3}\bar{\partial}\phi$	
	$\bar{\partial}A = \bar{\partial}a = 0$	$\mathcal{W}_4 = 0$	

type A	NS-flux only (common to IIA, IIB, heterotic) $\mathcal{W}_1 = \mathcal{W}_2 = 0, \mathcal{W}_3 \neq 0$: complex
type BC	RR-flux only $\mathcal{W}_1 = \text{Im}\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0, \text{Re}\mathcal{W}_2 \neq 0, \mathcal{W}_5 \neq 0$: symplectic

For $\mathcal{N} = 1$ AdS₄ vacua: [hep-th/0403049](https://arxiv.org/abs/hep-th/0403049) [hep-th/0407263](https://arxiv.org/abs/hep-th/0407263) [hep-th/0412250](https://arxiv.org/abs/hep-th/0412250) [hep-th/0502154](https://arxiv.org/abs/hep-th/0502154) [hep-th/0609124](https://arxiv.org/abs/hep-th/0609124) , etc..

IIB	$a = 0$ or $b = 0$ (type A)	$a = \pm ib$ (type B)	$a = \pm b$ (type C)	(type ABC)
1	$\mathcal{W}_1 = F_3^{(1)} = H_3^{(1)} = 0$			
8	$\mathcal{W}_2 = 0$			
6	$F_3^{(6)} = 0$ $\mathcal{W}_3 = \pm * H_3^{(6)}$	$\mathcal{W}_3 = 0$ $e^\phi F_3^{(6)} = \mp * H_3^{(6)}$	$H_3^{(6)} = 0$ $\mathcal{W}_3 = \pm e^\phi * F_3^{(6)}$	(***)
3	$\bar{\mathcal{W}}_5 = 2\mathcal{W}_4 = \mp 2iH_3^{(\bar{3})} = 2\bar{\partial}\phi$ $\bar{\partial}A = \bar{\partial}a = 0$	$e^\phi F_5^{(3)} = \frac{2i}{3}\bar{\mathcal{W}}_5 = i\mathcal{W}_4$ $= -2i\bar{\partial}A = -4i\bar{\partial}\log a$ $\bar{\partial}\phi = 0$	$e^\phi F_3^{(\bar{3})} = 2i\bar{\mathcal{W}}_5 = -2i\bar{\partial}A$ $= -4i\bar{\partial}\log a = -i\bar{\partial}\phi$	(***)
		F $e^\phi F_1^{(\bar{3})} = 2e^\phi F_5^{(\bar{3})}$ $= i\bar{\mathcal{W}}_5 = i\mathcal{W}_4 = i\bar{\partial}\phi$		

type A	NS-flux only (common to IIA, IIB, heterotic) $dJ \pm iH_3$ is (2,1)-primitive $\mathcal{W}_1 = \mathcal{W}_2 = 0$: complex
type B	both NS- and RR-flux $G_3 = F_3 - ie^{-\phi}H_3 = -i*_6 G_3$ is (2,1)-primitive $\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0, 2\mathcal{W}_5 = 3\mathcal{W}_4 = -6\bar{\partial}A$: conformally CY
type C	RR-flux only (S-dual of type A) $d(e^{-\phi}J) \pm iF_3$ is (2,1)-primitive $\mathcal{W}_1 = \mathcal{W}_2 = 0$: complex
type ABC	(skip detail...)

NS-NS fields in ten-dimensional spacetime are expanded as

$$\begin{aligned}\phi(x, y) &= \varphi(x) \\ \mathcal{G}_{m\bar{n}}(x, y) &= iv^a(x)(\omega_a)_{m\bar{n}}(y), \quad \mathcal{G}_{mn}(x, y) = i\bar{z}^k(x) \left(\frac{(\bar{\chi}_k)_{m\bar{p}\bar{q}} \Omega^{\bar{p}\bar{q}}}{\|\Omega\|^2} \right) (y) \\ \mathcal{B}_2(x, y) &= B_2(x) + b^a(x)\omega_a(y)\end{aligned}$$

R-R fields in type IIA are

$$\begin{aligned}\mathcal{C}_1(x, y) &= C_1^0(x) \\ \mathcal{C}_3(x, y) &= C_1^a(x)\omega_a(y) + \xi^K(x)\alpha_K(y) - \tilde{\xi}_K(x)\beta^K(y)\end{aligned}$$

R-R fields in type IIB are

$$\begin{aligned}\mathcal{C}_0(x, y) &= C_0(x) \\ \mathcal{C}_2(x, y) &= C_2(x) + c^a(x)\omega_a(y) \\ \mathcal{C}_4(x, y) &= V_1^K(x)\alpha_K(y) + \rho_a(x)\tilde{\omega}^a(y)\end{aligned}$$

cohomology class	basis	
$H^{(1,1)}$	ω_a	$a = 1, \dots, h^{(1,1)}$
$H^{(0)} \oplus H^{(1,1)}$	$\omega_A = (1, \omega_a)$	$A = 0, 1, \dots, h^{(1,1)}$
$H^{(2,2)}$	$\tilde{\omega}^a$	$a = 1, \dots, h^{(1,1)}$
$H^{(2,1)}$	χ_k	$k = 1, \dots, h^{(2,1)}$
$H^{(3)}$	(α_K, β^K)	$K = 0, 1, \dots, h^{(2,1)}$

Four-dimensional **type IIA** $\mathcal{N} = 2$ ungauged supergravity action of bosonic fields is

$$S_{\text{IIA}}^{(4)} = \int_{\mathcal{M}_{1,3}} \left(-\frac{1}{2} R * \mathbf{1} + \frac{1}{2} \text{Re} \mathcal{N}_{AB} F^A \wedge F^B + \frac{1}{2} \text{Im} \mathcal{N}_{AB} F^A \wedge * F^B - G_{a\bar{b}} dt^a \wedge * d\bar{t}^{\bar{b}} - h_{uv} dq^u \wedge * dq^v \right)$$

gravity multiplet	$(g_{\mu\nu}, C_1^0)$	1
vector multiplet	(C_1^a, v^a, b^a)	$a = 1, \dots, h^{(1,1)}$
hypermultiplet	$(z^k, \xi^k, \tilde{\xi}_k)$	$k = 1, \dots, h^{(2,1)}$
tensor multiplet	$(B_2, \varphi, \xi^0, \tilde{\xi}_0)$	1

Various functions in the actions:

$$\begin{aligned}
 B + iJ &= (b^a + iv^a) \omega_a = t^a \omega_a & K^{\text{KS}} &= -\log \left(\frac{4}{3} \int_{\mathcal{M}_6} J \wedge J \wedge J \right) \\
 \mathcal{K}_{abc} &= \int_{\mathcal{M}_6} \omega_a \wedge \omega_b \wedge \omega_c & \mathcal{K}_{ab} &= \int_{\mathcal{M}_6} \omega_a \wedge \omega_b \wedge J = \mathcal{K}_{abc} v^c \\
 \mathcal{K}_a &= \int_{\mathcal{M}_6} \omega_a \wedge J \wedge J = \mathcal{K}_{abc} v^b v^c & \mathcal{K} &= \int_{\mathcal{M}_6} J \wedge J \wedge J = \mathcal{K}_{abc} v^a v^b v^c \\
 \text{Re} \mathcal{N}_{AB} &= \begin{pmatrix} -\frac{1}{3} \mathcal{K}_{abc} b^a b^b b^c & \frac{1}{2} \mathcal{K}_{abc} b^b b^c \\ \frac{1}{2} \mathcal{K}_{abc} b^b b^c & -\mathcal{K}_{abc} b^c \end{pmatrix} & \text{Im} \mathcal{N}_{AB} &= -\frac{\mathcal{K}}{6} \begin{pmatrix} 1 + 4G_{ab} b^a b^b & -4G_{ab} b^b \\ -4G_{ab} b^b & 4G_{ab} \end{pmatrix} \\
 G_{a\bar{b}} &= \frac{3}{2} \frac{\int_{\mathcal{M}_6} \omega_a \wedge * \omega_b}{\int_{\mathcal{M}_6} J \wedge J \wedge J} = \partial_{t^a} \bar{\partial}_{\bar{t}^{\bar{b}}} K^{\text{KS}}
 \end{aligned}$$

Four-dimensional **type IIB** $\mathcal{N} = 2$ ungauged supergravity action of bosonic fields is

$$S_{\text{IIB}}^{(4)} = \int_{\mathcal{M}_{1,3}} \left(-\frac{1}{2} R * \mathbf{1} + \frac{1}{2} \text{Re} \mathcal{M}_{KL} F^K \wedge F^L + \frac{1}{2} \text{Im} \mathcal{M}_{KL} F^K \wedge *F^L - G_{k\bar{l}} dz^k \wedge *d\bar{z}^{\bar{l}} - h_{pq} d\tilde{q}^p \wedge *d\tilde{q}^q \right)$$

gravity multiplet	$(g_{\mu\nu}, V_1^0)$	1
vector multiplet	(V_1^k, z^k)	$k = 1, \dots, h^{(2,1)}$
hypermultiplet	(v^a, b^a, c^a, ρ_a)	$a = 1, \dots, h^{(1,1)}$
tensor multiplet	(B_2, C_2, φ, C_0)	1

Various functions in the actions:

$$\Omega = \mathcal{Z}^K \alpha_K - \mathcal{F}_K \beta^K \quad z^k = \mathcal{Z}^K / \mathcal{Z}^0 \quad \mathcal{F}_{KL} = \partial_L \mathcal{F}_K$$

$$K^{\text{CS}} = -\log \left(i \int_{\mathcal{M}_6} \Omega \wedge \bar{\Omega} \right) \quad G_{k\bar{l}} = -\frac{\int \chi_k \wedge \bar{\chi}_{\bar{l}}}{\int \Omega \wedge \bar{\Omega}} = \partial_{z^k} \bar{\partial}_{\bar{z}^{\bar{l}}} K^{\text{CS}}$$

$$\mathcal{M}_{KL} = \bar{\mathcal{F}}_{KL} + 2i \frac{(\text{Im} \mathcal{F})_{KM} \mathcal{Z}^M (\text{Im} \mathcal{F})_{LN} \mathcal{Z}^N}{\mathcal{Z}^N (\text{Im} \mathcal{F})_{NM} \mathcal{Z}^M}$$

▶ on a single $SU(3)$:	<div style="border-bottom: 1px dashed black; padding-bottom: 5px;">a real two-form</div> <div style="padding-bottom: 5px;">a complex three-form</div>	$J_{ij} = \mp 2i \eta_{\pm}^{\dagger} \gamma_{ij} \eta_{\pm}$ $\Omega_{ijk} = -2i \eta_{-}^{\dagger} \gamma_{ijk} \eta_{+}$
▶ on a pair of $SU(3)$:	<div style="border-bottom: 1px dashed black; padding-bottom: 5px;">two real vectors</div> <div style="padding-bottom: 5px;">(J^A, Ω^A)</div>	$(v - iv')^i = \eta_{+}^{1\dagger} \gamma^i \eta_{-}^2$ $J^1 = j + v \wedge v' \quad \Omega^1 = w \wedge (v + iv')$ $J^2 = j - v \wedge v' \quad \Omega^1 = w \wedge (v - iv')$ <p style="color: blue; margin-top: 10px;">(j, w): local $SU(2)$-invariant forms</p>

If $\eta_{+}^1 \neq \eta_{+}^2$ globally, a pair of $SU(3)$ is reduced to global single $SU(2)$ w/ (j, w, v, v')

If $\eta_{+}^1 = \eta_{+}^2$ globally, a pair of $SU(3)$ is reduced to a single global $SU(3)$ w/ (J, Ω)

$$\eta_{+}^2 = c_{\parallel} \eta_{+}^1 + c_{\perp} (v + iv')^i \gamma_i \eta_{-}^1 \quad |c_{\parallel}|^2 + |c_{\perp}|^2 = 1$$

cf.) a pair of $SU(3)$ on $T\mathcal{M}_6 \sim$ an $SU(3) \times SU(3)$ on $T\mathcal{M}_6 \oplus T^*\mathcal{M}_6$

back to pure spinors