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Supergravity and Doubled Geometry

based on arXiv:0806.1783 [hep-th]

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Introduction

Start from low energy effective field theory for ten-dimensional string theory including

$$S = \int d^{10}x \sqrt{-\mathcal{G}} e^{-2\Phi} \left\{ \mathcal{R} + 4(\nabla\Phi)^2 - \frac{1}{12} \mathcal{H}_{MNP} \mathcal{H}^{MNP} \right\}$$

$$\mathcal{H} = d\mathcal{B}$$

Consider the field theory compactified on (twisted) torus in the **presence** of B-field.

motivation

flux compactifications
duality relations among flux vacua

Reduced D -dim. action compactified on a flat d -torus ($D = 10 - d$):

$$S = \int d^D x \sqrt{-g} e^{-2\phi} \left\{ R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right. \\ \left. + \frac{1}{8} L_{IJ} \nabla_\mu \mathcal{M}^{JK} L_{KL} \nabla^\mu \mathcal{M}^{LI} - \frac{1}{4} F_{\mu\nu}^I L_{IJ} \mathcal{M}^{JK} L_{KL} F^{L\mu\nu} \right\}$$

This theory has $U(1)^{2d}$ gauge symmetry and a manifest global $O(d, d)$ symmetry with

$$\mathcal{M}_{IJ} = \begin{pmatrix} g_{ij} - B_{ik} g^{kl} B_{lj} & B_{ik} g^{kj} \\ -g^{ik} B_{kj} & g^{ij} \end{pmatrix} : \text{moduli, taking values in } \frac{O(d, d)}{O(d) \times O(d)}$$

$$F^I = dA^I, \quad A_\mu^I = \begin{pmatrix} V_\mu^i \\ B_{\mu i} \end{pmatrix}, \quad H_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]} - \frac{3}{2} A_{[\mu}^I L_{|IJ|} F_{\nu\rho]}^J$$

$$L^{IJ} \equiv \begin{pmatrix} \mathbf{0}_d & \mathbb{1}_d \\ \mathbb{1}_d & \mathbf{0}_d \end{pmatrix} : \quad \begin{array}{l} O(d, d) \text{ invariant metric s.t.} \\ \forall M \in O(d, d), \quad MLM^T = L \end{array}$$

Non-abelian gauge symmetry from a **2d-dimensional** subgroup G of $O(d, d)$:

Fundamental repr. of $O(d, d)$ becomes adjoint repr. of G under embedding

$$[T_I, T_J] = t_{IJ}{}^K T_K, \quad T_I = \frac{1}{2} \Theta_I{}^{JK} \mathfrak{m}_{JK}$$

$$\left\{ \begin{array}{ll} T_I : & \text{generators of } G \text{ with structure constant } t_{IJ}{}^K \\ \mathfrak{m}_{JK} : & \text{generators of } O(d, d) \\ \Theta_I{}^{JK} : & \text{embedding tensor} \end{array} \right.$$

T_I are (non-)abelian generators for gauge fields $A_\mu^I = (V_\mu^i, B_{\mu i})^T$:

$$T_I \ni \begin{cases} Z_i : & \text{generators for } V_\mu^i \\ X^i : & \text{generators for } B_{\mu i} \end{cases} \quad \dashrightarrow \quad \begin{aligned} [Z_i, Z_j] &= f_{ij}^k Z_k + h_{ijk} X^k \\ [X^i, X^j] &= 0 \\ [X^i, Z_j] &= f^i_{jk} X^k \end{aligned}$$

f_{ij}^k : structure constant of twisted torus

h_{ijk} : (minus) VEV of three-form H_{ijk}

► Twisted torus is introduced by vielbein $dy^i \rightarrow e^m = e^m_i(y) dy^i$:

$$g_{ij}(x) \rightarrow G_{ij}(x, y) = g_{mn}(x) e_i^m(y) e_j^n(y)$$

$$g_{ij}(x) (dy^i + V^i_\mu dx^\mu) (dy^j + V^j_\nu dx^\nu) \rightarrow g_{mn}(x) (e^m(y) + V^m_\mu dx^\mu) (e^n(y) + V^n_\nu dx^\nu)$$

We often switch off 4-dim. fluctuations: $g_{mn}(x) \rightarrow \delta_{mn}$, $B_{mn}(x) \rightarrow 0$, $G_{ij}(x, y) \rightarrow G_{ij}(y)$.

$$\begin{aligned}[Z_m, Z_n] &= f_{mn}{}^p Z_p + h_{mnp} X^p \\ [X^m, X^n] &= 0 \\ [X^m, Z_n] &= f^m{}_{np} X^p\end{aligned}$$

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$$\Downarrow$$

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[Z_m, Z_n] &= f_{mn}{}^p Z_p + h_{mnp} X^p \\
[X^m, X^n] &= Q^{mn}{}_p X^p + R^{mnp} Z_p \\
[X^m, Z_n] &= f^m{}_{np} X^p - Q^{mp}{}_n Z_p
\end{aligned}$$

Why should we study additional structure constants $Q^{mn}{}_p$ and R^{mnp} ?

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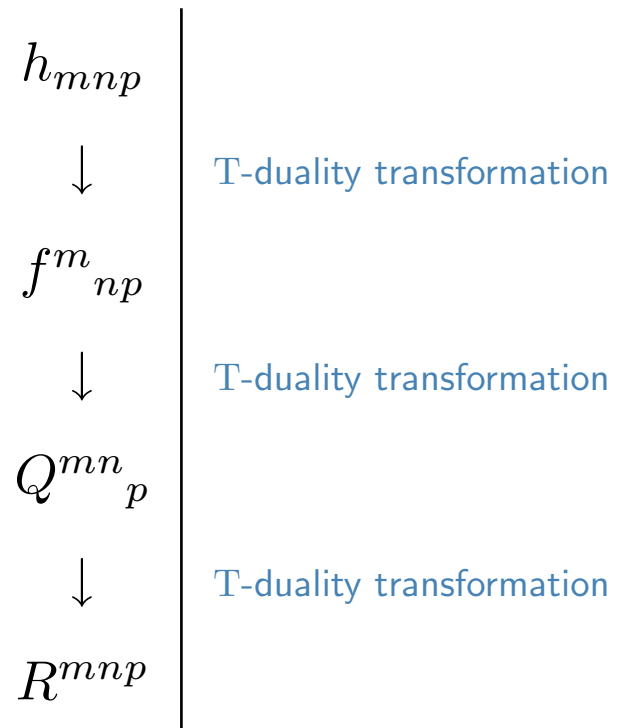
$$\Downarrow$$

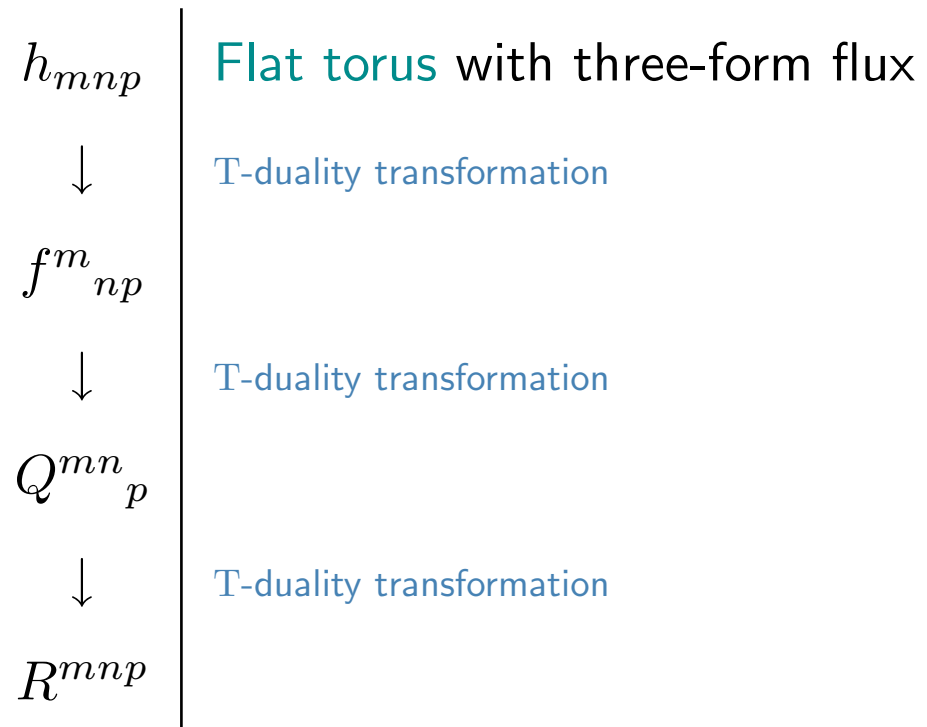
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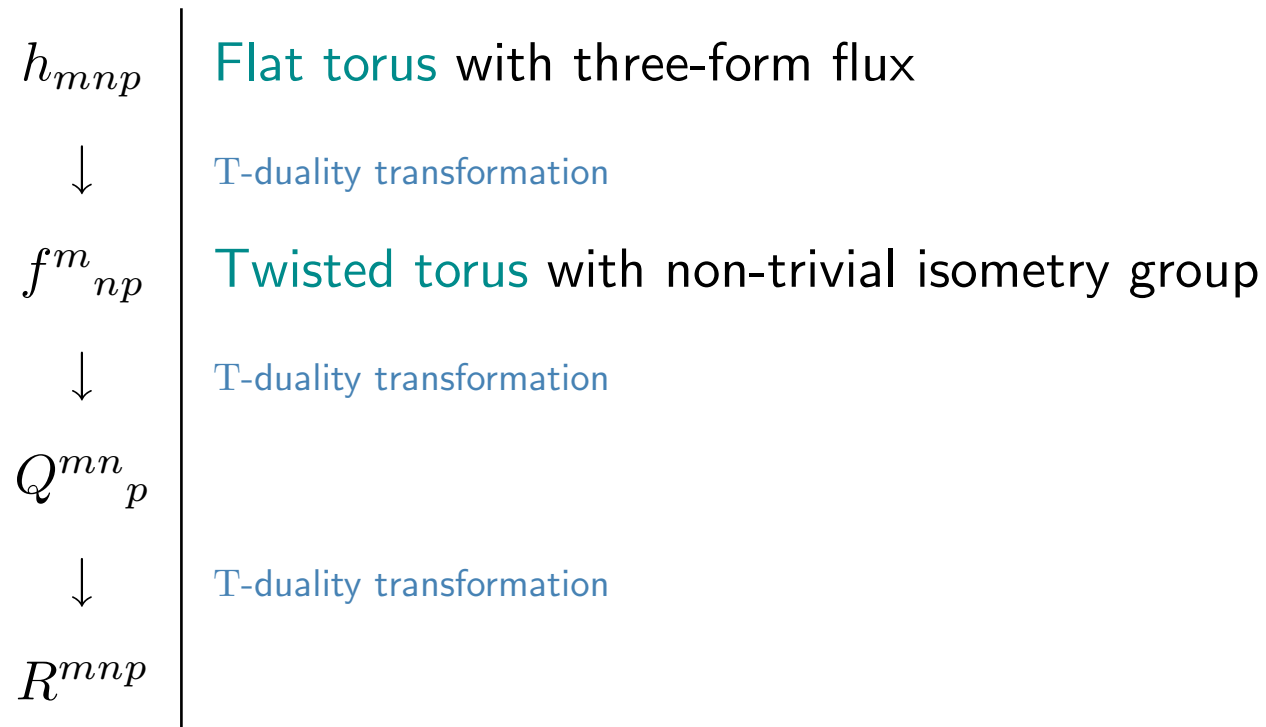
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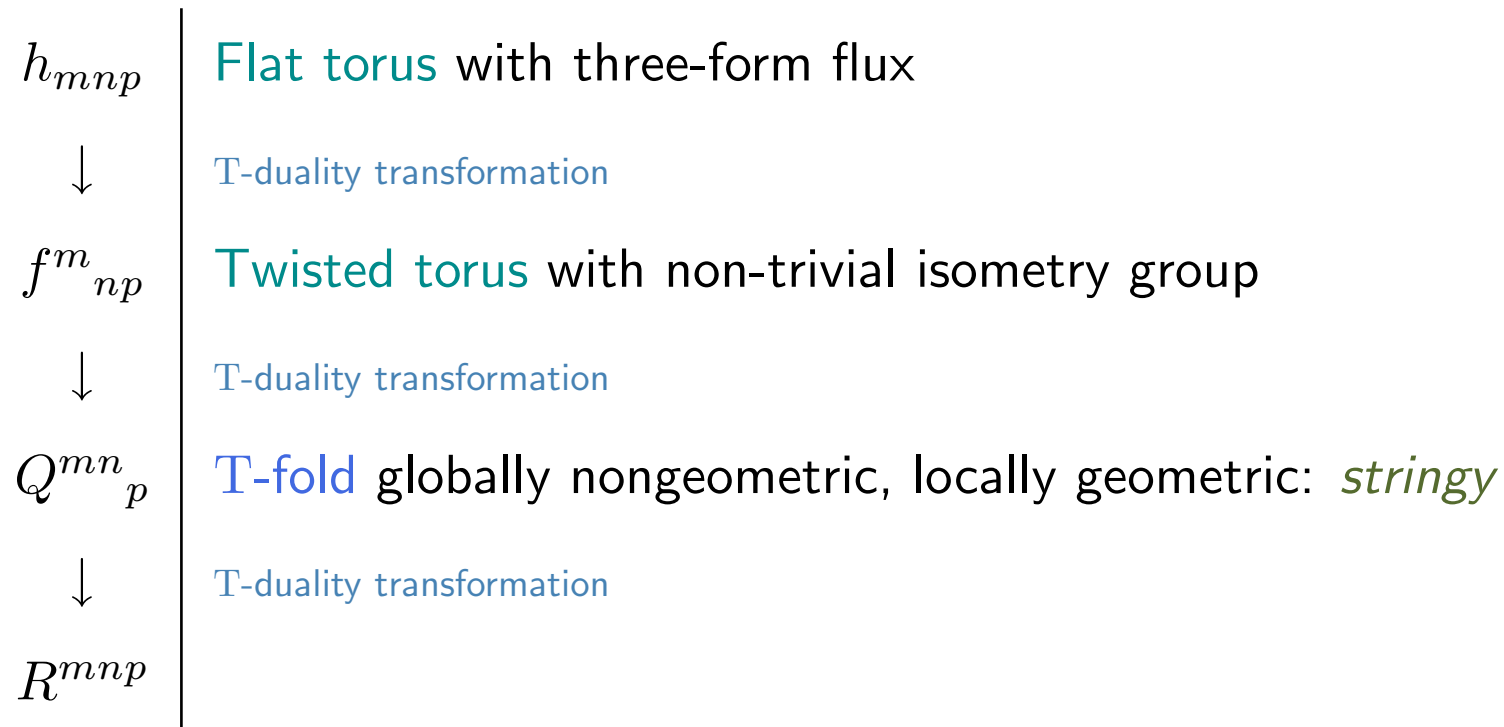
$$\downarrow$$

Because they are related via T-duality transformations









h_{mnp}	Flat torus with three-form flux
$\downarrow$	T-duality transformation
f^m_{np}	Twisted torus with non-trivial isometry group
$\downarrow$	T-duality transformation
Q^{mn}_p	T-fold globally nongeometric, locally geometric: <i>stringy</i>
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C.M. Hull

In order to include the above information,

we double the *compactified geometry* $\mathcal{M}_d$ to $\mathcal{M}_{2d} = \mathcal{M}_d \times \tilde{\mathcal{M}}_d$

and study an extended sigma model on it. $\rightarrow$ Doubled Formalism

C.M. Hull [hep-th/0406102](#) [hep-th/0605149](#)

C.M. Hull, R.A. Reid-Edwards [hep-th/0503114](#) [arXiv:0711.4818](#)

Doubled formalism

Start with a sigma model on a space $\mathcal{M}_d$ with metric $G_{ij}(y)$ and B-field $B_{ij}(y)$:

$$S_c = \frac{1}{2} \int_{\Sigma} \left(G_{ij} dY^i \wedge * dY^j + B_{ij} dY^i \wedge dY^j \right)$$

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Extend to the action with [the Wess-Zumino term](#) on a doubled space $\mathcal{M}_{2d} (= G/\Gamma)$

$$S = \frac{1}{4} \int_{\Sigma} \mathcal{M}_{MN} \mathcal{P}^M \wedge * \mathcal{P}^N + \frac{1}{12} \int_V t_{MNP} \mathcal{P}^M \wedge \mathcal{P}^N \wedge \mathcal{P}^P$$

Σ : string worldsheet (without boundary)

V : an extension of Σ s.t. $\partial V = \Sigma$

G : $2d$ -dim. (non-)compact Lie group w/ $[T_M, T_N] = t_{MN}{}^P T_P$

Γ : a discrete subgroup of G chosen s.t. $\mathcal{M}_{2d}$ is compact

for doubled sigma model with **BOUNDARY**: C.Albertsson's POSTER

Constituents of the action $S = \frac{1}{4} \int_{\Sigma} \mathcal{M}_{MN} \mathcal{P}^M \wedge * \mathcal{P}^N + \frac{1}{12} \int_V t_{MNP} \mathcal{P}^M \wedge \mathcal{P}^N \wedge \mathcal{P}^P$

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✓ Scalar fields of doubled coordinates and doubled vielbeins:

$$\mathbb{Y}^I ; \quad \mathcal{P} = \mathfrak{g}^{-1} d\mathfrak{g} = \mathcal{P}^M{}_I (r T_M) d\mathbb{Y}^I , \quad \text{w/ } \mathfrak{g} \in G \subset O(d, d)$$

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- ✓ Doubled metric from doubled vielbeins:

$$\mathcal{M}_{MN} = \mathcal{P}_M{}^I \mathcal{M}_{IJ} \mathcal{P}^J{}_N, \quad \mathcal{M}_{IJ} \text{ takes values in a coset } \frac{O(d, d)}{O(d) \times O(d)}$$

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- ✓ Self-duality constraint (to go back to conventional system):

$$\mathcal{P}^M = L^{MN} \mathcal{M}_{NP} * \mathcal{P}^P$$

Generators of the Lie algebra T_M given by $2d \times 2d$ matrix projectors $\Pi^M_N, \tilde{\Pi}^M_N$:

$$\Pi^M_N \Pi^N_P = \Pi^M_P, \quad \Pi^M_N \tilde{\Pi}^N_P = 0, \quad \Pi^M_N + \tilde{\Pi}^M_N = \delta^M_N$$

$$\Pi^M_N \equiv \begin{pmatrix} \Pi^m_N \\ 0 \end{pmatrix}, \quad \tilde{\Pi}^M_N \equiv \begin{pmatrix} 0 \\ \tilde{\Pi}_{mN} \end{pmatrix}$$

$$X^m = \Pi^m_M L^{MN} T_N, \quad Z_m = \tilde{\Pi}_{mM} L^{MN} T_N$$

Then doubled coordinates, vielbeins and metric are polarized as

$$Y^I \equiv \Pi^I_J \mathbb{Y}^J = \begin{pmatrix} Y^i \\ 0 \end{pmatrix}, \quad \tilde{Y}^I \equiv \tilde{\Pi}^I_J \mathbb{Y}^J = \begin{pmatrix} 0 \\ \tilde{Y}_i \end{pmatrix}$$

$$\mathcal{M}_{IJ} = \begin{pmatrix} G_{ij} - B_{ik} G^{kl} B_{lj} & B_{ik} G^{kj} \\ -G^{ik} B_{kj} & G^{ij} \end{pmatrix}$$

$$\mathcal{P}^M_I = \begin{pmatrix} e^m_i & 0 \\ -e_m^j B_{ji} & e_m^i \end{pmatrix}$$

$$\rho \in O(d, d); \quad \mathbb{Y}^I \rightarrow \mathbb{Y}'^I = \rho^I_J \mathbb{Y}^J, \quad \mathcal{P}^M_I(\mathbb{Y}) \rightarrow \mathcal{P}'^M_I(\mathbb{Y}') = \rho^P_Q \mathcal{P}^Q_J(\mathbb{Y}') \rho^J_I$$

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A realization of fractional transformation of $M_{ij} = G_{ij} + B_{ij}$

$$\rho = \begin{pmatrix} A & \beta \\ \Theta & D \end{pmatrix} : \quad M \rightarrow (DM + \Theta)(\beta M + A)^{-1}$$

$$\left\{ \begin{array}{ll} \Theta : & \text{gauge transformation of B-field } B \rightarrow B + \Theta \\ D, A : & \text{diffeomorphism} \\ \beta : & \text{duality transformation with mixing } Y^i \text{ and } \tilde{Y}_i \end{array} \right.$$

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T-duality transformation (ex. $d = 3$ case):

$$\rho_i = \begin{pmatrix} \mathbb{1}_3 - T_i & T_i \\ T_i & \mathbb{1}_3 - T_i \end{pmatrix} \in O(3, 3; \mathbb{Z})$$

$$T_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

This action exchanges physical coordinates Y^i with dual coordinates $\tilde{Y}_i$

Example

Start from a flat three-torus T^3 with a three-form flux H given by the following forms:

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2, \quad H = dB = m dx \wedge dy \wedge dz$$

with a symmetric gauge $B = k \left(x dy \wedge dz + y dz \wedge dx + z dx \wedge dy \right), \quad k = \frac{m}{3}$

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Doubled vielbein $\mathcal{P}^M_I$

$$\mathcal{P}^M_I = \begin{pmatrix} e^m_i & \mathbf{0} \\ -e_m^j B_{ji} & e_m^i \end{pmatrix} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & -kz & ky & 1 & 0 & 0 \\ kz & 0 & -kx & 0 & 1 & 0 \\ -ky & kx & 0 & 0 & 0 & 1 \end{array} \right)$$

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Doubled vielbein $\mathcal{P}^M_I$ and doubled metric $\mathcal{M}_{IJ} = \mathcal{P}_I^M \delta_{MN} \mathcal{P}^N_J$ are given as

$$\mathcal{P}^M_I = \begin{pmatrix} e^m_i & 0 \\ -e_m^j B_{ji} & e_m^i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 0 \\ \hline 0 & -kz & ky & | & 1 & 0 & 0 \\ kz & 0 & -kx & | & 0 & 1 & 0 \\ -ky & kx & 0 & | & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{M}_{IJ} = \begin{pmatrix} 1 + k^2 y^2 + k^2 z^2 & -k^2 xy & -k^2 zx & | & 0 & kz & -ky \\ -k^2 xy & 1 + k^2 x^2 + k^2 z^2 & -k^2 yz & | & -kz & 0 & kx \\ -k^2 zx & -k^2 yz & 1 + k^2 x^2 + k^2 y^2 & | & ky & -kx & 0 \\ \hline 0 & -kz & ky & | & 1 & 0 & 0 \\ kz & 0 & -kx & | & 0 & 1 & 0 \\ -ky & kx & 0 & | & 0 & 0 & 1 \end{pmatrix}$$

► Bianchi identity of doubled vielbein $\mathcal{P}^M = \begin{pmatrix} \mathcal{P}^m \\ \tilde{\mathcal{P}}_m \end{pmatrix}$ gives a structure constant $t_{MN}{}^P$:

$$\begin{aligned} d\tilde{\mathcal{P}}_1 &= \frac{2m}{3} \mathcal{P}^2 \wedge \mathcal{P}^3, & d\tilde{\mathcal{P}}_2 &= \frac{2m}{3} \mathcal{P}^3 \wedge \mathcal{P}^1, & d\tilde{\mathcal{P}}_3 &= \frac{2m}{3} \mathcal{P}^1 \wedge \mathcal{P}^2 \\ \therefore d\mathcal{P}^m &= 0, & d\tilde{\mathcal{P}}_m &= -\frac{r}{2} t_{mnp} \mathcal{P}^n \wedge \mathcal{P}^p \end{aligned}$$

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Then we can read the structure constant $t_{mnp} \equiv h_{mnp}$ of the Lie algebra as

$$[Z_m, Z_n] = h_{mnp} X^p, \quad h_{123} = -\frac{2m}{3r} \equiv -H_{123}$$

We can also fix the scaling factor in the Bianchi identity:

$$r = \frac{2}{3}, \quad d\mathcal{P}^M = -\frac{1}{3} t_{NP}{}^M \mathcal{P}^N \wedge \mathcal{P}^P$$

► Periodicity of physical coordinates Y^i and dual coordinates $\tilde{Y}_i$:

$$(x, \tilde{y}, \tilde{z}) \sim (x + 1, \tilde{y} + kz, \tilde{z} - ky) \qquad \tilde{x} \sim \tilde{x} + 1$$

$$(y, \tilde{z}, \tilde{x}) \sim (y + 1, \tilde{z} + kx, \tilde{x} - kz) \qquad \tilde{y} \sim \tilde{y} + 1$$

$$(z, \tilde{x}, \tilde{y}) \sim (z + 1, \tilde{x} + ky, \tilde{y} - kx) \qquad \tilde{z} \sim \tilde{z} + 1$$

This does not change the metric G_{ij} and the B-field B_{ij} .

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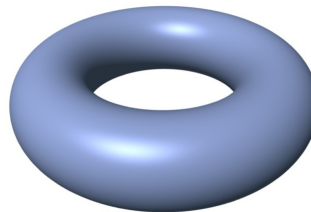
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- Self-duality constraint: $Y^i = (x, y, z)$, $\tilde{Y}_i = (\tilde{x}, \tilde{y}, \tilde{z})$ and $\sigma^\pm \equiv \sigma^0 \pm \sigma^1$

$$\partial_\pm \tilde{Y}_i = (B_{ij}(Y) \mp \delta_{ij}) \partial_\pm Y^j$$

We can completely take the projection onto the physical space

--> geometric background



- Doubled vielbein by T-duality along z -direction:

$$(\mathcal{P}_f)^M{}_I = (\rho z)^M{}_N \mathcal{P}^N{}_J (\rho z)^J{}_I = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -ky & kx & 1 & 0 & 0 & 0 \\ \hline 0 & -k\tilde{z} & 0 & 1 & 0 & ky \\ k\tilde{z} & 0 & 0 & 0 & 1 & -kx \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$(\mathcal{M}_f)_{IJ} = (\rho z)_I{}^K \mathcal{M}_{KL} (\rho z)^L{}_J \equiv \left(\begin{array}{cc} G_f - B_f G_f^{-1} B_f & B_f G_f^{-1} \\ -G_f^{-1} B_f & G_f^{-1} \end{array} \right)$$

“Metric” G_f and “B-field” B_f can be read from the doubled metric as

$$(G_f)_{ij} = \begin{pmatrix} 1 + k^2 y^2 & -k^2 xy & -ky \\ -k^2 xy & 1 + k^2 x^2 & kx \\ -ky & kx & 1 \end{pmatrix}, \quad (B_f)_{ij} = \begin{pmatrix} 0 & k\tilde{z} & 0 \\ -k\tilde{z} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

► Bianchi identity of doubled vielbein $\mathcal{P}^M = \begin{pmatrix} \mathcal{P}^m \\ \tilde{\mathcal{P}}_m \end{pmatrix}$ gives a structure constant $t_{MN}{}^P$:

$$\begin{aligned} d\tilde{\mathcal{P}}_1 &= \frac{2m}{3} \mathcal{P}^2 \wedge \tilde{\mathcal{P}}_3, & d\tilde{\mathcal{P}}_2 &= \frac{2m}{3} \tilde{\mathcal{P}}_3 \wedge \mathcal{P}^1, & d\mathcal{P}^3 &= \frac{2m}{3} \mathcal{P}^1 \wedge \mathcal{P}^2 \\ \therefore d\mathcal{P}^m &= -\frac{1}{3} t^m{}_{np} \mathcal{P}^n \wedge \mathcal{P}^p, & d\tilde{\mathcal{P}}_m &= -\frac{1}{3} t_{mn}{}^p \mathcal{P}^n \wedge \tilde{\mathcal{P}}_p \end{aligned}$$

Then we can read the structure constant $t_{mn}{}^p \equiv f_{mn}{}^p$ as

$$[Z_m, Z_n] = f_{mn}{}^p Z_p, \quad [X^m, Z_n] = f^m{}_{np} X^p, \quad f^1{}_{23} = -m$$

► Periodicity of physical coordinates Y^i and dual coordinates $\tilde{Y}_i$:

$$(x, \tilde{y}, z) \sim (x + 1, \tilde{y} + kz, z - ky)$$

$$\tilde{x} \sim \tilde{x} + 1$$

$$(y, z, \tilde{x}) \sim (y + 1, z + kx, \tilde{x} - kz)$$

$$\tilde{y} \sim \tilde{y} + 1$$

$$z \sim z + 1$$

$$(\tilde{z}, \tilde{x}, \tilde{y}) \sim (\tilde{z} + 1, \tilde{x} + k\tilde{y}, \tilde{y} - k\tilde{x})$$

$$ds^2 = (dx)^2 + (dy)^2 + (dz - ky dx + kx dy)^2, \quad B = k\tilde{z} dx \wedge dy$$

The metric is invariant and the B-field is shifted via this periodic shift.

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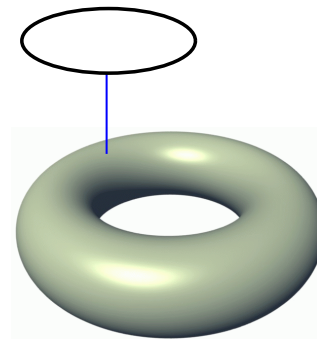
$$ds^2 = (dx)^2 + (dy)^2 + (dz - ky dx + kx dy)^2, \quad B = k\tilde{z} dx \wedge dy$$

The metric is invariant and the B-field is shifted via this periodic shift.

- Self-duality constraint is $\partial_{\pm} \tilde{Y}_i = (B_{ij}(\tilde{Y}) \mp G_{ij}(Y)) \partial_{\pm} Y^j$

We can completely take the projection on the physical space

--> geometric background



- Doubled vielbein by T-duality along (y, z) -directions:

$$(\mathcal{P}_Q)^M{}_I = (\rho_y \rho_z)^M{}_N \mathcal{P}^N{}_J (\rho_z \rho_y)^J{}_I = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ k\tilde{z} & 1 & 0 & 0 & 0 & -kx \\ -k\tilde{y} & 0 & 1 & 0 & kx & 0 \\ \hline 0 & 0 & 0 & 1 & -k\tilde{z} & k\tilde{y} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$(\mathcal{M}_Q)_{IJ} = (\rho_y \rho_z)_I{}^K \mathcal{M}_{KL} (\rho_z \rho_y)^L{}_J \equiv \left(\begin{array}{cc} G_Q - B_Q G_Q^{-1} B_Q & B_Q G_Q^{-1} \\ -G_Q^{-1} B_Q & G_Q^{-1} \end{array} \right)$$

The “metric” G_Q and “B-field” B_Q are

$$(G_Q)_{ij} = \frac{1}{1 + k^2 x^2} \left(\begin{array}{ccc} 1 + k^2(x^2 + \tilde{y}^2 + \tilde{z}^2) & k\tilde{z} & -k\tilde{y} \\ k\tilde{z} & 1 & 0 \\ -k\tilde{y} & 0 & 1 \end{array} \right)$$

$$(B_Q)_{ij} = \frac{1}{1 + k^2 x^2} \left(\begin{array}{ccc} 0 & -k^2 x \tilde{y} & -k^2 x \tilde{z} \\ k^2 x \tilde{y} & 0 & -kx \\ k^2 x \tilde{z} & kx & 0 \end{array} \right)$$

► Bianchi identity of doubled vielbein $\mathcal{P}^M = \begin{pmatrix} \mathcal{P}^m \\ \tilde{\mathcal{P}}_m \end{pmatrix}$ gives a structure constant $t_{MN}{}^P$:

$$\begin{aligned} d\tilde{\mathcal{P}}_1 &= \frac{2m}{3} \tilde{\mathcal{P}}_2 \wedge \tilde{\mathcal{P}}_3, & d\mathcal{P}^2 &= \frac{2m}{3} \tilde{\mathcal{P}}_3 \wedge \mathcal{P}^1, & d\mathcal{P}^3 &= \frac{2m}{3} \mathcal{P}^1 \wedge \tilde{\mathcal{P}}_2 \\ \therefore d\mathcal{P}^m &= -\frac{1}{3} t^{mn}{}_p \tilde{\mathcal{P}}_n \wedge \mathcal{P}^p, & d\tilde{\mathcal{P}}_m &= -\frac{1}{3} t_m{}^{np} \tilde{\mathcal{P}}_n \wedge \tilde{\mathcal{P}}_p \end{aligned}$$

Then we can read the structure constant $t^{mn}{}_p \equiv Q^{mn}{}_p$ as

$$[X^m, X^n] = Q^{mn}{}_p X^p, \quad [Z_m, X^n] = Q_m{}^{np} Z_p, \quad Q^{12}{}_3 = -m$$

► Periodicity of physical coordinates Y^i and dual ones $\tilde{Y}_i$:

$$(x, y, z) \sim (x + 1, y + k\tilde{z}, z - k\tilde{y})$$

$$\tilde{x} \sim \tilde{x} + 1$$

$$y \sim y + 1$$

$$(\tilde{y}, z, \tilde{x}) \sim (\tilde{y} + 1, z + kx, \tilde{x} - k\tilde{z})$$

$$z \sim z + 1$$

$$(\tilde{z}, \tilde{x}, y) \sim (\tilde{z} + 1, \tilde{x} + k\tilde{y}, y - kx)$$

$$ds^2 = (dx)^2 + \frac{1}{1 + k^2 x^2} \left[(dy + k\tilde{z} dx)^2 + (dz - k\tilde{y} dx)^2 \right]$$

This periodic shift yields a β -trsf: duality trsf. $\dashrightarrow$ globally nongeometric

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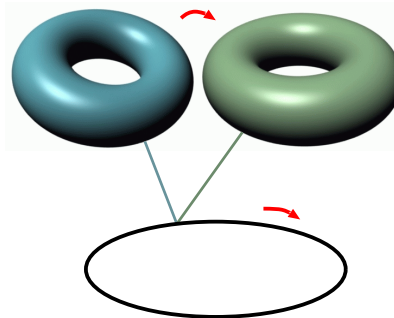
$$ds^2 = (dx)^2 + \frac{1}{1 + k^2 x^2} \left[(dy + k\tilde{z} dx)^2 + (dz - k\tilde{y} dx)^2 \right]$$

This periodic shift yields a β -trsf: duality trsf. $\rightarrow$ globally nongeometric

However, imposing the self-duality constraint,

we see that this duality transformation is interpreted as T-duality on fibred T^2

in terms of only the physical coordinate objects $\rightarrow$ locally geometric



- Doubled vielbein by T-duality along (x, y, z) -directions:

$$(\mathcal{P}_R)^M{}_I = (\rho_x \rho_y \rho_z)^M{}_N \mathcal{P}^N{}_J (\rho_z \rho_y \rho_x)^J{}_I = \begin{pmatrix} 1 & 0 & 0 & | & 0 & -k\tilde{z} & k\tilde{y} \\ 0 & 1 & 0 & | & k\tilde{z} & 0 & -k\tilde{x} \\ 0 & 0 & 1 & | & -k\tilde{y} & k\tilde{x} & 0 \\ \hline 0 & 0 & 0 & | & 1 & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 1 \end{pmatrix}$$

$$(\mathcal{M}_R)_{IJ} = (\rho_x \rho_y \rho_z)_I{}^K \mathcal{M}_{KL} (\rho_z \rho_y \rho_x)^L{}_J \equiv \begin{pmatrix} G_R - B_R G_R^{-1} B_R & B_R G_R^{-1} \\ -G_R^{-1} B_R & G_R^{-1} \end{pmatrix}$$

The “metric” G_R and “B-field” B_R can be read from the doubled metric as

$$(G_R)_{ij} = \frac{1}{1 + k^2 \tilde{x}^2 + k^2 \tilde{y}^2 + k^2 \tilde{z}^2} \begin{pmatrix} 1 + k^2 \tilde{x}^2 & k^2 \tilde{x} \tilde{y} & k^2 \tilde{z} \tilde{x} \\ k^2 \tilde{x} \tilde{y} & 1 + k^2 \tilde{y}^2 & k^2 \tilde{y} \tilde{z} \\ k^2 \tilde{z} \tilde{x} & k^2 \tilde{y} \tilde{z} & 1 + k^2 \tilde{z}^2 \end{pmatrix}$$

$$(B_R)_{ij} = \frac{1}{1 + k^2 \tilde{x}^2 + k^2 \tilde{y}^2 + k^2 \tilde{z}^2} \begin{pmatrix} 0 & -k\tilde{z} & k\tilde{y} \\ k\tilde{z} & 0 & -k\tilde{x} \\ -k\tilde{y} & k\tilde{x} & 0 \end{pmatrix}$$

► Bianchi identity of doubled vielbein $\mathcal{P}^M = \begin{pmatrix} \mathcal{P}^m \\ \tilde{\mathcal{P}}_m \end{pmatrix}$ gives a structure constant $t_{MN}{}^P$:

$$\begin{aligned} d\mathcal{P}^1 &= \frac{2m}{3} \tilde{\mathcal{P}}_2 \wedge \tilde{\mathcal{P}}_3, & d\mathcal{P}^2 &= \frac{2m}{3} \tilde{\mathcal{P}}_3 \wedge \tilde{\mathcal{P}}_1, & d\mathcal{P}^3 &= \frac{2m}{3} \tilde{\mathcal{P}}_1 \wedge \tilde{\mathcal{P}}_2 \\ \therefore d\mathcal{P}^m &= -\frac{m}{3} t^{mnp} \tilde{\mathcal{P}}_n \wedge \tilde{\mathcal{P}}_p, & d\tilde{\mathcal{P}}_m &= 0 \end{aligned}$$

Then we can read the structure constant $t^{mnp} \equiv R^{mnp}$ as

$$[X^m, X^n] = R^{mnp} Z_p, \quad R^{123} = -m$$

► Periodicity of physical coordinates Y^i and dual coordinates $\tilde{Y}_i$:

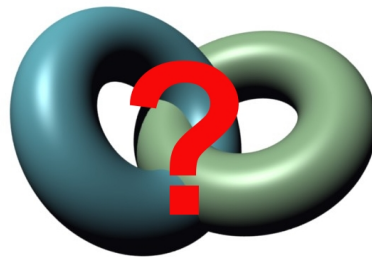
$$\begin{aligned} x &\sim x + 1 & (\tilde{x}, y, z) &\sim (\tilde{x} + 1, y + k\tilde{z}, z - k\tilde{y}) \\ y &\sim y + 1 & (\tilde{y}, z, x) &\sim (\tilde{y} + 1, z + k\tilde{x}, x - k\tilde{z}) \\ z &\sim z + 1 & (\tilde{z}, x, y) &\sim (\tilde{z} + 1, x + k\tilde{y}, y - k\tilde{x}) \end{aligned}$$

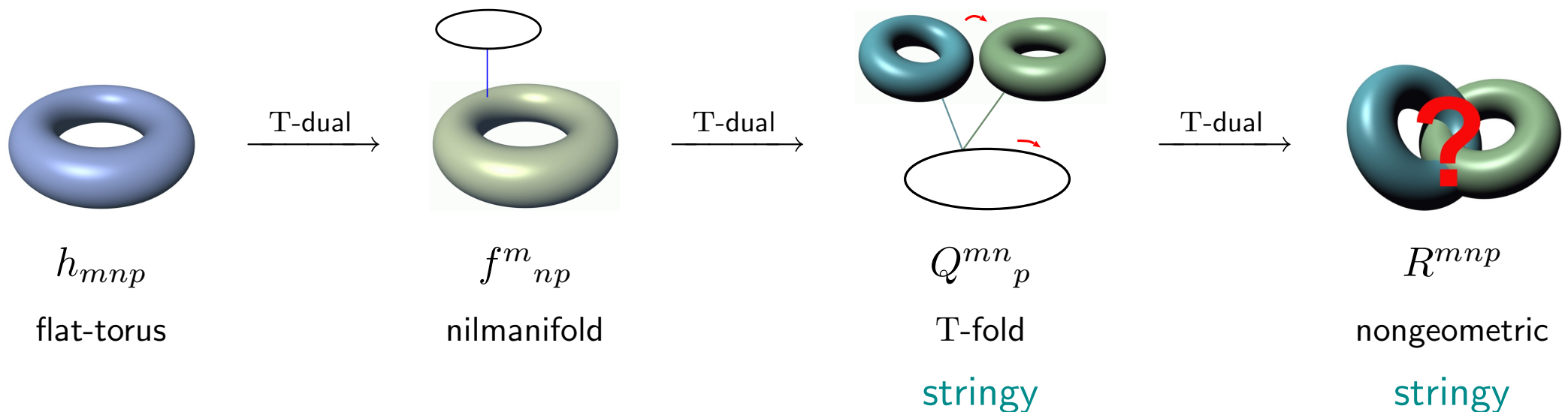
$$ds^2 = \frac{1}{1 + k^2\tilde{x}^2 + k^2\tilde{y}^2 + k^2\tilde{z}^2} \left[(dx)^2 + (dy)^2 + (dz)^2 + k^2(\tilde{x} dx + \tilde{y} dy + \tilde{z} dz)^2 \right]$$

This periodic shift yields a β -trsf: duality trsf. $\dashrightarrow$ **globally nongeometric**

The self-duality constraint does **not** yield a well-defined projection

$\dashrightarrow$ **locally nongeometric**





T-duality in the presence of B-field generates geometric/nongeometric backgrounds.

They also have to be investigated as low energy **stringy** geometries.

Extended formalism can proceed the analysis.

Generalized geometry would also know the existence of Q - and R -fluxes.

M. Graña, J. Louis, D. Waldram [hep-th/0612237](#)

Discussions

- ▶ Start from scalar moduli matrix in supergravity on $\mathcal{M}_d$
- ▶ Introduce doubled space $\mathcal{M}_{2d}$ induced by B-field
- ▶ Perform T-duality transformations
- ▶ Evaluate Lie algebra and geometries

Introduction of D-branes in the doubled geometry
C. Albertsson's POSTER

Extend to U-fold endowed with U-duality transformation (hidden symmetry)

? Supersymmetry on doubled geometry ?

? Investigate quantum aspects of the doubled sigma model ?

Appendix

Decomposition of fields by Kaluza-Klein compactification on a flat d -torus

$$ds^2 = \mathcal{G}_{\mu\nu}(x, y)dx^\mu \otimes dx^\nu + \mathcal{G}_{ij}(x, y)(dy^i + \mathcal{V}^i_\mu(x, y)dx^\mu) \otimes (dy^j + \mathcal{V}^j_\nu(x, y)dx^\nu)$$

$$\mathcal{B} = \frac{1}{2}\mathcal{B}_{\mu\nu}(x, y)dx^\mu \wedge dx^\nu + \mathcal{B}_{\mu i}(x, y)dx^\mu \wedge dy^i + \frac{1}{2}\mathcal{B}_{ij}(x, y)dy^i \wedge dy^j$$

with **Ansatz** (truncation of massive Kaluza-Klein modes)

$$\mathcal{G}_{\mu\nu}(x, y) = g_{\mu\nu}(x), \quad \mathcal{G}_{ij}(x, y) = g_{ij}(x), \quad \mathcal{V}^i_\mu(x, y) = V^i_\mu(x)$$

$$\mathcal{B}_{\mu\nu}(x, y) = \mathcal{B}_{\mu\nu}(x), \quad \mathcal{B}_{\mu i}(x, y) = \mathcal{B}_{\mu i}(x), \quad \mathcal{B}_{ij}(x, y) = B_{ij}(x)$$

$$\Phi(x, y) = \phi(x) + \frac{1}{4} \log |\det g_{ij}(x)|$$

Reduced degrees of freedom to demonstrate manifest gauge invariance:

$$B_{\mu i} = \mathcal{B}_{\mu i} + B_{ij}V^j_\mu$$

$$B_{\mu\nu} = \mathcal{B}_{\mu\nu} + V^i_{[\mu}B_{\nu]i} - B_{ij}V^i_\mu V^j_\nu$$

$$\mathcal{M}_{IJ} \text{ takes values in a coset } \frac{O(d, d)}{O(d) \times O(d)}$$

This sigma model on the doubled space $\mathcal{M}_{2d}$ has

- ▶ $O(d, d)$ global symmetry by $\rho \in O(d, d)$ w/ $\rho^M{}_P L^{PQ} \rho_Q{}^N = L^{MN}$:

$$\mathbb{Y}^I \rightarrow \mathbb{Y}'^I = \rho^I{}_J \mathbb{Y}^J$$

$$\mathcal{P}^M{}_I(\mathbb{Y}) \rightarrow \mathcal{P}'^M{}_I(\mathbb{Y}') = \rho^P{}_Q \mathcal{P}^Q{}_J(\mathbb{Y}') \rho^J{}_I$$

$$\mathcal{M}_{IJ}(\mathbb{Y}) \rightarrow \mathcal{M}'_{IJ}(\mathbb{Y}') = \rho_I{}^K \mathcal{M}_{KL}(\mathbb{Y}') \rho^L{}_J$$

- ▶ $O(d) \times O(d)$ local symmetry: $\mathcal{P}^M{}_I(\mathbb{Y}) \rightarrow \mathcal{P}'^M{}_I(\mathbb{Y}) = h^M{}_N(\mathbb{Y}) \mathcal{P}^N{}_I(\mathbb{Y})$

Using the worldsheet coordinates σ^a , we see the self-duality constraint as

$$\mathcal{P}^M = L^{MN} \mathcal{M}_{NP} * \mathcal{P}^P \quad \longleftrightarrow \quad d\mathbb{Y}^I = L^{IJ} \mathcal{M}_{JK} * d\mathbb{Y}^K$$

$$\therefore \quad \left(\partial_a \mathbb{Y}^I - \sqrt{-\eta} \varepsilon_a{}^b L^{IJ} \mathcal{M}_{JK} \partial_b \mathbb{Y}^K \right) d\sigma^a = 0 \quad \text{w/} \quad \begin{cases} \eta_{ab} = \text{diag.}(+, -) \\ \varepsilon_{01} = 1 = \varepsilon^{10} \end{cases}$$

Taking the polarization, we obtain a set of non-trivial equations:

$$(\partial_0 \pm \partial_1) \tilde{Y}_i = \left(B_{ij}(Y, \tilde{Y}) \mp G_{ij}(Y, \tilde{Y}) \right) (\partial_0 \pm \partial_1) Y^j$$

Then the dual coordinates $\tilde{Y}_i$ are related to the physical coordinates Y^i .

Reduction of (co)tangent bundle of doubled space $\mathcal{M}_{2d}$

$$L^{MN} = \langle \mathcal{P}^M, \mathcal{P}^N \rangle = \begin{pmatrix} \mathbf{0}_d & \mathbb{1}_d \\ \mathbb{1}_d & \mathbf{0}_d \end{pmatrix}, \quad L^{IJ} \equiv \langle d\mathbb{Y}^I, d\mathbb{Y}^J \rangle = \mathcal{P}^I{}_M L^{MN} \mathcal{P}_N{}^J$$

This implies $T^*\mathcal{M}_{2d} = T\mathcal{M}_d \oplus T^*\mathcal{M}_d$ s.t.

$$\langle dY^i, d\tilde{Y}_j \rangle = \delta_j^i \quad \rightarrow \quad d\tilde{Y}_i = \frac{\partial}{\partial Y^i}$$

$$\mathcal{P}^M = \mathcal{P}^M{}_I d\mathbb{Y}^I = \begin{pmatrix} e^m{}_i dY^i \\ e_m{}^i (d\tilde{Y}_i - B_{ij} dY^j) \end{pmatrix} = \begin{pmatrix} e^m{}_i dY^i \\ e_m{}^i \left(\frac{\partial}{\partial Y^i} - B_{ij} dY^j \right) \end{pmatrix}$$

a connection to Generalized Geometry

- ✓ a flat three-torus: *introduction*

$$H = dB \equiv m dx \wedge dy \wedge dz$$

- ✓ a nilmanifold given by T-duality along z -direction of the three-torus

$$H = k dx \wedge dy \wedge d\tilde{z}, \quad k = \frac{m}{3}$$

- ✓ a T-fold given by T-duality along (y, z) -directions of the three-torus

$$H = \frac{k(-1 + k^2 x^2)}{(1 + k^2 x^2)^2} dx \wedge dy \wedge dz - \frac{k^2 x}{1 + k^2 x^2} dx \wedge dy \wedge d\tilde{y} + \frac{k^2 x}{1 + k^2 x^2} dz \wedge dx \wedge d\tilde{z}$$

- ✓ a nongeometric space given by T-duality along (x, y, z) -directions of the three-torus

$$\begin{aligned} H = \chi^2 \Big\{ & 2k^2 \tilde{x} \tilde{z} dx \wedge dy \wedge d\tilde{x} - (\chi^{-1} - 2k^2 \tilde{x}^2) dy \wedge dz \wedge d\tilde{x} + 2k^2 \tilde{x} \tilde{y} dz \wedge dx \wedge d\tilde{x} \\ & + 2k^2 \tilde{y} \tilde{z} dx \wedge dy \wedge d\tilde{y} + 2k^2 \tilde{x} \tilde{y} dy \wedge dz \wedge d\tilde{y} - (\chi^{-1} - 2k^2 \tilde{y}^2) dz \wedge dx \wedge d\tilde{y} \\ & - (\chi^{-1} - 2k^2 \tilde{z}^2) dx \wedge dy \wedge d\tilde{z} + 2k^2 \tilde{x} \tilde{z} dy \wedge dz \wedge d\tilde{z} + 2k^2 \tilde{y} \tilde{z} dz \wedge dx \wedge d\tilde{z} \Big\} \end{aligned}$$

$$\chi^{-1} = 1 + k^2 (\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)$$