

# AdS Vacua and Generalized Geometries

arXiv:0810.0937 [hep-th] (to appear in JHEP)




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Introduction

We are looking for the origin of 4D physics

Physical information

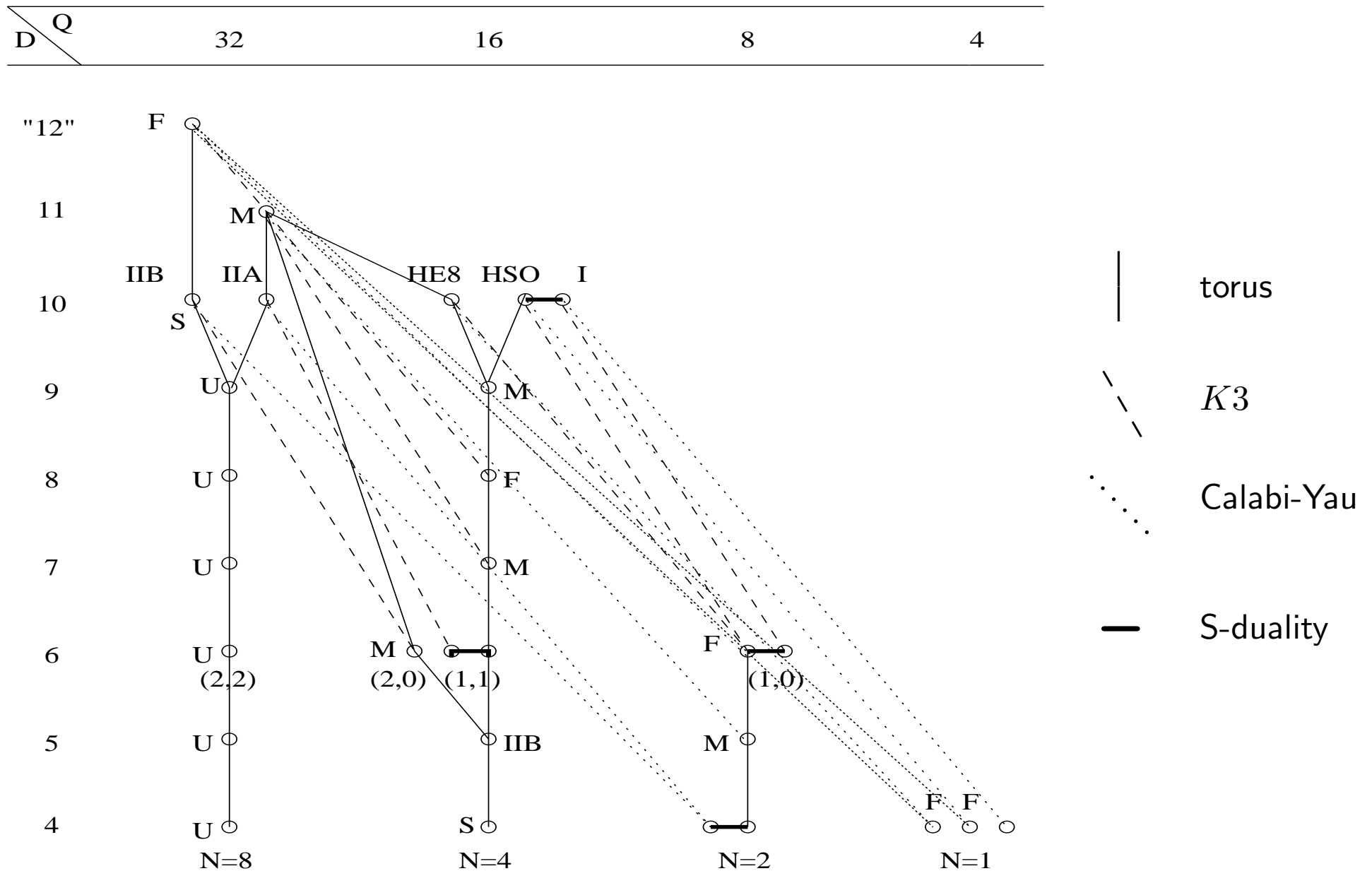
-  Particle contents and spectra
-  (Broken) symmetries
-  Potential, vacuum and cosmological constant

What kind of 4D models come from String Theories?



What kind of Compactifications?

$$4 = 10 - 6 = 11 - 7$$



B. de Wit and J. Louis, in the Proceedings "NATO Advanced Study Institute on Strings, Branes and Dualities (1997)" [hep-th/9801132](https://arxiv.org/abs/hep-th/9801132)

- Many **Abelian** Supergravities (SUGRA) in lower dimensions

Compactifications on Tori, Calabi-Yaus, etc.

Minkowski ground state, massless fields

Global  $E_7$  symmetry (4D  $\mathcal{N} = 8$  case)

- Many **Gauged** SUGRA in lower dimensions

Compactifications on group manifolds, torsionful manifolds, etc.

Scalar potential generating **masses** [**Moduli Stabilization**]

Non-trivial **cosmological constant**

There are various Gauged SUGRA  
which cannot be derived from String Theories  
compactified on **conventional** geometric backgrounds

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We want to derive all Gauged SUGRA from String Theories



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We want to derive all Gauged SUGRA from String Theories  
Compactify String Theories on **non-conventional** geometries:

There are various Gauged SUGRA  
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We want to derive all Gauged SUGRA from String Theories  
Compactify String Theories on **non-conventional** geometries:

Nongeometric String Backgrounds

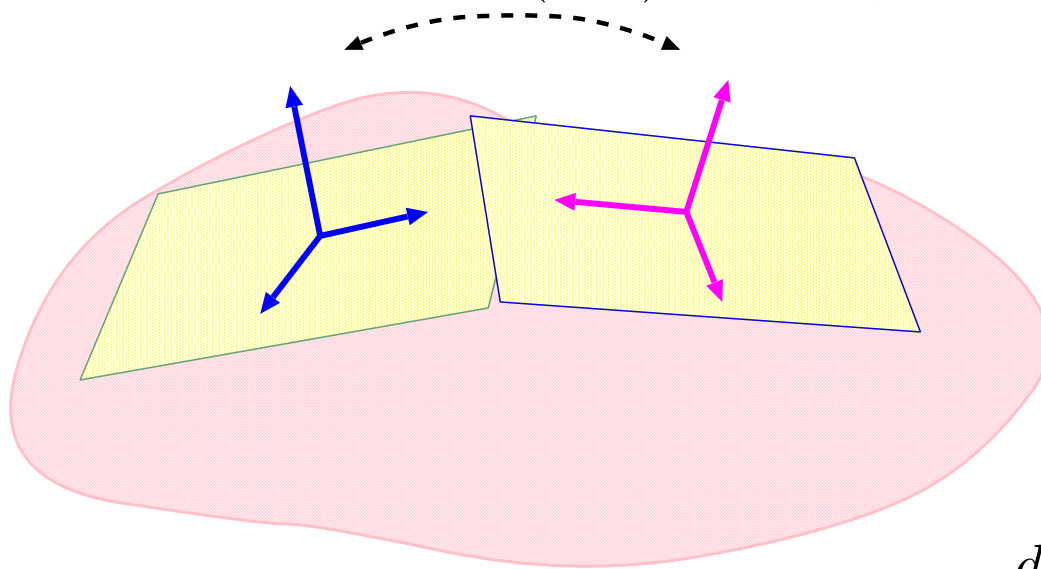
# What is a **Non**geometric String Background?

Structure group = Diffeo.  $(GL(d, \mathbb{R})) +$  Duality transf. groups



coming from String dualities

$GL(d, \mathbb{R}) +$  duality transf.



$d$ -dim. internal space  $\mathcal{M}_d \simeq$  monodifold

# SUGRA on Nongeometric String Backgrounds

cf.) Lower-dim. Gauged SUGRA compactified by Scherk-Schwarz mechanism

“Kaloper-Myers” algebra:

$$\begin{aligned} [Z_a, Z_b] &= f_{ab}{}^c Z_c + H_{abc} X^c \\ [X^a, X^b] &= Q^{ab}{}_c X^c + R^{abc} Z_c \\ [X^a, Z_b] &= f^a{}_{bc} X^c - Q^{ac}{}_b Z_c \end{aligned}$$

Various “fluxes” are involved

N. Kaloper, R.C. Myers [hep-th/9901045](#)

J. Shelton, W. Taylor, B. Wecht [hep-th/0508133](#), A. Dabholkar, C.M. Hull [hep-th/0512005](#)

M. Graña, R. Minasian, M. Petrini, D. Waldram [arXiv:0807.4527](#)

String Theories compactified on Nongeometric Backgrounds



All(?) Gauged SUGRA

Hitchin's **Generalized Geometries** to study vacua

Hull's **Doubled Formalism** to find gauge symmetries

► Calabi-Yau three-folds  $\rightarrow$  Fluxes are highly restricted

$$\left\{ \begin{array}{ll} \text{type IIA :} & \text{No fluxes} \\ \text{type IIB :} & F_3 - \tau H \quad (\text{warped Calabi-Yau}) \\ \text{heterotic :} & \text{No fluxes} \end{array} \right.$$

►  $SU(3)$ -structure manifolds  $\rightarrow$  Some components of fluxes can be interpreted as torsion

1: Piljin Yi, TK “*Comments on heterotic flux compactifications*” JHEP 0607 (2006) 030, [hep-th/0605247](https://arxiv.org/abs/hep-th/0605247)

2: TK “*Index theorems on torsional geometries*” JHEP 0708 (2007) 048, [arXiv:0704.2111](https://arxiv.org/abs/0704.2111)

► Generalized geometries  $\rightarrow$  All (non)geometric fluxes can be introduced

“Complete” classification of  $\mathcal{N} = 1$  SUSY solutions

Search 4D SUSY vacua in type IIA theory compactified on generalized geometries

- **Moduli stabilization**

We find SUSY AdS (or Minkowski) vacua

- **Mathematical feature**

We obtain a powerful rule to evaluate vacua:

**Discriminants** of superpotentials governing the cosmological constant

- **Stringy effects**

We see  $\alpha'$  **corrections** in certain configurations

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4D  $\mathcal{N} = 1$  supergravity:

$$S = \int \left( \frac{1}{2} R * \mathbf{1} - \frac{1}{2} F^a \wedge * F^a - K_{\mathcal{M}\bar{\mathcal{N}}} \nabla \phi^{\mathcal{M}} \wedge * \nabla \bar{\phi}^{\bar{\mathcal{N}}} - V * \mathbf{1} \right)$$

$$V = e^K \left( K^{\mathcal{M}\bar{\mathcal{N}}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\bar{\mathcal{N}}} \mathcal{W}} - 3 |\mathcal{W}|^2 \right) + \frac{1}{2} |D^a|^2$$

$K$  : Kähler potential

$\mathcal{W}$  : superpotential ←--  $\delta\psi_{\mu} = \nabla_{\mu} \varepsilon - e^{\frac{K}{2}} \mathcal{W} \gamma_{\mu} \varepsilon^c$

$D^a$  : D-term ←--  $\delta\chi^a = \text{Im} F_{\mu\nu}^a \gamma^{\mu\nu} \varepsilon + i D^a \varepsilon$

10D string theories provide  $K, \mathcal{W}, D^a$

via compactifications:  $10 = 4 + 6$

Decompose 10D type IIA SUSY parameters:

$$\epsilon^1 = \varepsilon_1 \otimes (\bar{a} \eta_-^1) + \varepsilon_1^c \otimes (a \eta_+^1), \quad \epsilon^2 = \varepsilon_2 \otimes (b \eta_+^2) + \varepsilon_2^c \otimes (\bar{b} \eta_-^2)$$

$\delta(\text{fermions}) = 0$  provide **spinor equations** on the 6D internal space  $\mathcal{M}$ :

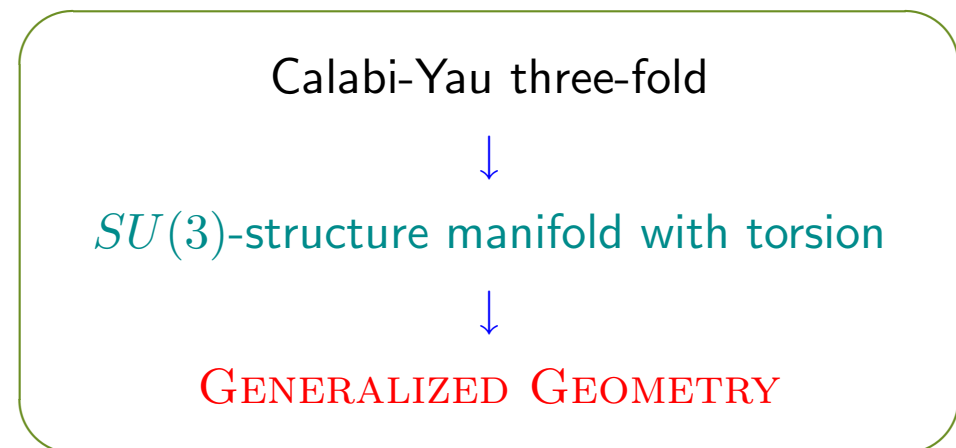
$$\delta\psi_m^{\mathcal{A}} = \left( \partial_m + \frac{1}{4} \omega_{mab} \gamma^{ab} \right) \eta^{\mathcal{A}} + (\text{3-form fluxes} \cdot \eta)^{\mathcal{A}} + (\text{other fluxes} \cdot \eta)^{\mathcal{A}} = 0$$

with a pair of  $SU(3)$  invariant Weyl spinors  $\eta_+^1, \eta_+^2$ :

$$\eta_+^2 = c_{\parallel}(y) \eta_+^1 + c_{\perp}(y) (v + iv')^m \gamma_m \eta_-^1, \quad (v - iv')^m \equiv \eta_+^{1\dagger} \gamma^m \eta_-^2$$

Information of

6D  $SU(3)$  Killing spinors  $\eta_+^1, \eta_+^2$ :



$$10\text{D} = 4\text{D} (\Lambda_{\text{cosmo.}} = -|\mu|^2) + 6\text{D}: \quad ds_{10}^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + ds_6^2$$

Consider polyforms  $\Phi_{\pm}^0$  on the internal space  $\mathcal{M}$  which satisfy

$$\begin{aligned} e^{-2A+\phi}(d - H\wedge)(e^{2A-\phi}\Phi_+^0) &= -2\bar{\mu} \text{Re}\Phi_-^0 \\ e^{-2A+\phi}(d - H\wedge)(e^{2A-\phi}\Phi_-^0) &= -3i \text{Im}(\mu\Phi_+^0) + dA \wedge \overline{\Phi_-^0} \\ &\quad + \frac{1}{16}e^\phi \left[ (|a|^2 - |b|^2)F + i(|a|^2 + |b|^2) * \lambda(F) \right] \end{aligned}$$

M. Graña, R. Minasian, M. Petrini, A. Tomasiello [hep-th/0609124](#) + D. Cassani, A. Bilal [arXiv:0707:3125](#)

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► On Calabi-Yau ( $d\Phi_{\pm}^0 = 0$ ) or  $SU(3)$ -structure manifolds ( $d\Phi_{\pm}^0 \neq 0$ ) ( $\eta_+^1 = \eta_+^2$ ):

$$\begin{aligned} \Phi_+^0 &= e^{-iJ}, & \Phi_-^0 &= -\Omega \\ J_{mn} &= -2i \eta_+^\dagger \gamma_{mn} \eta_+, & \Omega_{mnp} &= -2i \eta_-^\dagger \gamma_{mnp} \eta_+ \end{aligned}$$

$$10D = 4D (\Lambda_{\text{cosmo.}} = -|\mu|^2) + 6D: \quad ds_{10}^2 = e^{2A(y)} g_{\mu\nu} dx^\mu dx^\nu + ds_6^2$$

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► On  $SU(3) \times SU(3)$  generalized geometries ( $\eta_+^1 \neq \eta_+^2$  at some points  $y$ ):

$$\begin{aligned} \Phi_+^0 &= (\bar{c}_{\parallel} e^{-ij} - i\bar{c}_{\perp} w) \wedge e^{-iv \wedge v'}, & \Phi_-^0 &= (c_{\parallel} e^{-ij} + ic_{\perp} w) \wedge (v + iv') \\ J^A &= j \pm v \wedge v', & \Omega^A &= w \wedge (v \pm iv') \end{aligned}$$

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$\Phi_{\pm} (= e^{-B} \Phi_{\pm}^0)$  are even/odd-forms *isomorphic* to  $SU(3,3)$  invariant Weyl spinors



The compactified space  $\mathcal{M}$  admits Hitchin's generalized complex structures

$\mathcal{J}^{\Lambda}_{\Sigma} : T\mathcal{M} \oplus T^*\mathcal{M} \rightarrow T\mathcal{M} \oplus T^*\mathcal{M}$  with basis  $\{dx^m \wedge, \iota_{\partial_n}\}$  and (6,6)-signature

$$\mathcal{J}^2 = -\mathbf{1}_{12}, \quad \mathcal{J}^{\Lambda}_{\Sigma} = \langle \text{Re } \Phi_{\pm}, \Gamma^{\Lambda}_{\Sigma} \text{Re } \Phi_{\pm} \rangle$$

$\Phi_{\pm}$ :  $SU(3,3)$  invariant (pure) spinors

$\Gamma^{\Lambda}$ :  $\text{Cliff}(6,6)$  gamma matrices

cf.) ordinary complex structure  $J^m_n : T\mathcal{M} \rightarrow T\mathcal{M}$

$$J^2 = -\mathbf{1}_6, \quad J^m_n = -2i \eta_+^{\dagger} \gamma^m_n \eta_+$$

$\eta_+$ :  $SU(3)$  invariant (pure) spinor



Moduli spaces of  $\mathcal{M}$  are the special Kähler geometries of local type  
 Kähler potentials, prepotentials, projective coordinates

$$K_+ = -\log i \int_{\mathcal{M}} \langle \Phi_+, \bar{\Phi}_+ \rangle = -\log i (\bar{X}^A \mathcal{F}_A - X^A \bar{\mathcal{F}}_A)$$

$$K_- = -\log i \int_{\mathcal{M}} \langle \Phi_-, \bar{\Phi}_- \rangle = -\log i (\bar{Z}^I \mathcal{G}_I - Z^I \bar{\mathcal{G}}_I)$$

Expand the even/odd-forms  $\Phi_{\pm}$  by the basis forms:

$$\Phi_+ = X^A \omega_A - \mathcal{F}_A \tilde{\omega}^A, \quad \omega_A = (1, \omega_a), \quad \tilde{\omega}^A = (\tilde{\omega}^a, \text{vol}(\mathcal{M})) \quad : \quad 0, 2, 4, 6\text{-forms}$$

$$\Phi_- = Z^I \alpha_I - \mathcal{G}_I \beta^I, \quad \alpha_I = (\alpha_0, \alpha_i), \quad \beta^I = (\beta^i, \beta^0) \quad : \quad 1, 3, 5\text{-forms}$$

(if  $\eta_+^1 \neq \eta_+^2$  at some points)

On generalized geometries with a **single  $SU(3)$** -structure ( $\eta_+^1 = \eta_+^2$ ):

$$\begin{aligned} d_H \omega_A &= m_A^I \alpha_I - e_{IA} \beta^I & d_H \tilde{\omega}^A &= 0 \\ d_H \alpha_I &= e_{IA} \tilde{\omega}^A & d_H \beta^I &= m_A^I \tilde{\omega}^A \end{aligned}$$

where NS three-form  $H$  deforms the differential operator:

$$dH = 0, \quad H = H^{\text{fl}} + dB, \quad H^{\text{fl}} = m_0^I \alpha_I - e_{I0} \beta^I$$

$$d_H \equiv d - H^{\text{fl}} \wedge$$

background	charges	
NS three-form flux	$e_{I0}$	$m_0^I$
torsion	$e_{Ia}$	$m_a^I$

On generalized geometries with  $SU(3) \times SU(3)$  structures ( $\eta_+^1 \neq \eta_+^2$  at some points):

Extend to the generalized differential operator  $\mathcal{D}$ :

$$d_H = d - H^{\text{fl}} \wedge \quad \rightarrow \quad \mathcal{D} \equiv d - H^{\text{fl}} \wedge - f \cdot -Q \cdot -R \lrcorner$$

$$\mathcal{D}\omega_A \sim m_A^I \alpha_I - e_{IA} \beta^I \quad \mathcal{D}\tilde{\omega}^A \sim -q^{IA} \alpha_I + p_I^A \beta^I$$

$$\mathcal{D}\alpha_I \sim p_I^A \omega_A + e_{IA} \tilde{\omega}^A \quad \mathcal{D}\beta^I \sim q^{IA} \omega_A + m_A^I \tilde{\omega}^A$$

On generalized geometries with  $SU(3) \times SU(3)$  structures ( $\eta_+^1 \neq \eta_+^2$  at some points):

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The internal space becomes **nongeometric**:

$(f \cdot C)_{m_1 \dots m_{k+1}} \equiv f^a {}_{[m_1 m_2} C_{ a  m_3 \dots m_{k+1}}$	(part of) structure const. in Gauged SUGRA
$(Q \cdot C)_{m_1 \dots m_{k-1}} \equiv Q^{ab} {}_{[m_1} C_{ ab  m_2 \dots m_{k-1}}$	T-fold
$(R \lrcorner C)_{m_1 \dots m_{k-3}} \equiv R^{abc} C_{abcm_1 \dots m_{k-3}}$	locally nongeometric background

Structure group = Diffeo. + **duality trsf.**  $\dashrightarrow$  *Hull's Doubled formalism to study gauge symmetries*

backgrounds	flux charges			
Calabi-Yau	—			
Calabi-Yau with $H$	$e_{I0}$	$m_0^I$		
generalized geometry w/ $SU(3)$	$e_{IA}$	$m_A^I$		
generalized geometry w/ $SU(3) \times SU(3)$	$e_{IA}$	$m_A^I$	$p_I^A$	$q^{IA}$

RR-fluxes on  $SU(3) \times SU(3)$  generalized geometries:

$$G = G^{\text{fl}} + \mathcal{D}A, \quad \mathcal{D}G = 0$$

$$G^{\text{fl}} = m_{\text{RR}}^A \omega_A - e_{\text{RR}A} \tilde{\omega}^A, \quad A = \xi^I \alpha_I - \tilde{\xi}_I \beta^I$$



$$G \sim G^A \omega_A - \tilde{G}_A \tilde{\omega}^A$$

$$G^A \sim m_{\text{RR}}^A + \xi^I p_I^A - \tilde{\xi}_I q^{IA}, \quad \tilde{G}_A \sim e_{\text{RR}A} - \xi^I e_{IA} + \tilde{\xi}_I m_A^I$$

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Functionals are given by two Kähler potentials on two Hodge-Kähler geometries of  $\Phi_{\pm}$ :

$$K = K_+ + 4\varphi$$

$$K_+ = -\log i \int_{\mathcal{M}} \langle \Phi_+, \bar{\Phi}_+ \rangle = -\log i (\bar{X}^A \mathcal{F}_A - X^A \bar{\mathcal{F}}_A)$$

$$K_- = -\log i \int_{\mathcal{M}} \langle \Phi_-, \bar{\Phi}_- \rangle = -\log i (\bar{Z}^I \mathcal{G}_I - Z^I \bar{\mathcal{G}}_I)$$

$$\int_{\mathcal{M}} \text{vol}_6 = \frac{1}{8} e^{-K_{\pm}} = e^{-2\varphi + 2\phi^{(10)}}$$

Introduce  $\mathcal{C} = \sqrt{2}ab e^{-\phi^{(10)}} = 4ab e^{\frac{K_-}{2} - \varphi}$

$$\begin{aligned} \therefore e^{-2\varphi} &= \frac{|\mathcal{C}|^2}{16|a|^2|b|^2} e^{-K_-} = \frac{i}{16|a|^2|b|^2} \int_{\mathcal{M}} \langle \mathcal{C}\Phi_-, \bar{\mathcal{C}}\bar{\Phi}_- \rangle \\ &= \frac{1}{8|a|^2|b|^2} \left[ \text{Im}(\mathcal{C}Z^I) \text{Re}(\mathcal{C}\mathcal{G}_I) - \text{Re}(\mathcal{C}Z^I) \text{Im}(\mathcal{C}\mathcal{G}_I) \right] \end{aligned}$$



See the SUSY variation of 4D  $\mathcal{N} = 2$  gravitinos:

$$\delta\psi_{\mathcal{A}\mu} = \nabla_{\mu}\varepsilon_{\mathcal{A}} - S_{\mathcal{A}\mathcal{B}}\gamma_{\mu}\varepsilon^{\mathcal{B}} + \dots$$

$$S_{\mathcal{A}\mathcal{B}} = \frac{i}{2}e^{\frac{K_{+}}{2}} \begin{pmatrix} \mathcal{P}^1 - i\mathcal{P}^2 & -\mathcal{P}^3 \\ -\mathcal{P}^3 & -\mathcal{P}^1 - i\mathcal{P}^2 \end{pmatrix}_{\mathcal{A}\mathcal{B}}$$

The components are also written by  $\Phi_{\pm}$ :

$$\mathcal{P}^1 - i\mathcal{P}^2 = 2e^{\frac{K_{-}}{2} + \varphi} \int_{\mathcal{M}} \langle \Phi_{+}, \mathcal{D}\Phi_{-} \rangle, \quad \mathcal{P}^1 + i\mathcal{P}^2 = 2e^{\frac{K_{-}}{2} + \varphi} \int_{\mathcal{M}} \langle \Phi_{+}, \mathcal{D}\bar{\Phi}_{-} \rangle$$

$$\mathcal{P}^3 = -\frac{1}{\sqrt{2}}e^{2\varphi} \int_{\mathcal{M}} \langle \Phi_{+}, G \rangle$$

Note:  $\hat{\Psi}_{\mathcal{A}\mu} = \Psi_{\mathcal{A}\mu} + \frac{1}{2}\Gamma_{\mu}{}^m\Psi_m^{\mathcal{A}} = \psi_{\mathcal{A}\mu\pm} \otimes \eta_{+} + \psi_{\mathcal{A}\mu\mp} \otimes \eta_{-} + \dots$

4D  $\mathcal{N} = 1$  fermions given by the SUSY truncation from 4D  $\mathcal{N} = 2$  system:

SUSY parameter:  $\varepsilon \equiv \bar{n}^{\mathcal{A}} \varepsilon_{\mathcal{A}} = a \varepsilon_1 + \bar{b} \varepsilon_2$

gravitino:  $\psi_{\mu} \equiv \bar{n}^{\mathcal{A}} \psi_{\mathcal{A}\mu} = a \psi_{1\mu} + \bar{b} \psi_{2\mu}, \quad \tilde{\psi}_{\mu} \equiv (b \psi_{1\mu} - \bar{a} \psi_{2\mu})$

gauginos:  $\chi^A \equiv -2 e^{\frac{K_{\pm}}{2}} D_b X^A (\bar{n}^{\mathcal{C}} \epsilon_{\mathcal{C}\mathcal{E}} \chi^{\mathcal{E}b})$

where  $\bar{n}^{\mathcal{A}} = (a, \bar{b}), \quad \epsilon_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

4D  $\mathcal{N} = 1$  fermions given by the SUSY truncation from 4D  $\mathcal{N} = 2$  system:

SUSY parameter:  $\varepsilon \equiv \bar{n}^{\mathcal{A}} \varepsilon_{\mathcal{A}} = a \varepsilon_1 + \bar{b} \varepsilon_2$

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where  $\bar{n}^{\mathcal{A}} = (a, \bar{b}), \quad \epsilon_{\mathcal{A}\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

SUSY variations yield the superpotential and the D-term:

$$\delta\psi_{\mu} = \nabla_{\mu}\varepsilon - \bar{n}^{\mathcal{A}} S_{\mathcal{A}\mathcal{B}} n^{*\mathcal{B}} \gamma_{\mu} \varepsilon^{\mathcal{C}} \equiv \nabla_{\mu}\varepsilon - e^{\frac{K}{2}} \mathcal{W} \gamma_{\mu} \varepsilon^{\mathcal{C}}$$

$$\delta\tilde{\psi}_{\mu} = 0$$

$$\delta\chi^A = \text{Im}F_{\mu\nu}^A \gamma^{\mu\nu} \varepsilon + i D^A \varepsilon$$

$$\mathcal{W} = \frac{i}{4\bar{a}b} \left[ 4i e^{\frac{K_-}{2} - \varphi} \int_{\mathcal{M}} \langle \Phi_+, \mathcal{D}\text{Im}(ab\Phi_-) \rangle + \frac{1}{\sqrt{2}} \int_{\mathcal{M}} \langle \Phi_+, G \rangle \right]$$

$$\equiv \mathcal{W}^{\text{RR}} + U^I \mathcal{W}_I^{\text{Q}} + \tilde{U}_I \tilde{\mathcal{W}}_Q^I$$

$$\mathcal{W}^{\text{RR}} = -\frac{i}{4\bar{a}b} \left[ X^A e_{\text{RRA}} - \mathcal{F}_A m_{\text{RR}}^A \right]$$

$$\mathcal{W}_I^{\text{Q}} = \frac{i}{4\bar{a}b} \left[ X^A e_{IA} + \mathcal{F}_A p_I^A \right], \quad \tilde{\mathcal{W}}_Q^I = -\frac{i}{4\bar{a}b} \left[ X^A m_A^I + \mathcal{F}_A q^{IA} \right]$$

$$U^I = \xi^I + i \text{Im}(\mathcal{C}Z^I), \quad \tilde{U}_I = \tilde{\xi}_I + i \text{Im}(\mathcal{C}\mathcal{G}_I)$$

$$D^A = 2e^{K_+} (K_+)^{c\bar{d}} D_c X^A \overline{D_d X^B} \left[ \bar{n}^c (\sigma_x) c^B n_B \right] \left( \mathcal{P}_B^x - \mathcal{N}_{BC} \tilde{\mathcal{P}}^{xC} \right)$$

$\mathcal{N} = 2$  multiplets:

$(t^a = X^a/X^0, z^i = Z^i/Z^0)$

gravity multiplet	$g_{\mu\nu}, A_\mu^0$	
vector multiplets	$A_\mu^a, t^a = b^a + iv^a$	$a = 1, \dots, b^+$
hypermultiplets	$z^i, \xi^i, \tilde{\xi}_i$	$i = 1, \dots, b^-$
tensor multiplet	$B_{\mu\nu}, \varphi, \xi^0, \tilde{\xi}_0$	


 O6 orientifold projection:  $\mathcal{O} \equiv \Omega_{WS} (-1)^{F_L} \sigma$ 
 $\mathcal{N} = 1$  multiplets:

gravity multiplet	$g_{\mu\nu}$	
vector multiplets	$A_\mu^{\hat{a}}$	$\hat{a} = 1, \dots, \hat{n}_v = b^+ - n_{ch}$
chiral multiplets	$t^{\check{a}} = b^{\check{a}} + iv^{\check{a}}$	$\check{a} = 1, \dots, n_{ch}$
chiral/linear multiplets	$U^{\check{I}} = \xi^{\check{I}} + i \text{Im}(\mathcal{C}Z^{\check{I}})$ $\tilde{U}_{\hat{I}} = \tilde{\xi}_{\hat{I}} + i \text{Im}(\mathcal{C}\mathcal{G}_{\hat{I}})$	$I = (\check{I}, \hat{I}) = 0, 1, \dots, b^-$
(projected out)	$B_{\mu\nu}, A_\mu^0, A_\mu^{\check{a}}, t^{\hat{a}}, U^{\hat{I}}, \tilde{U}_{\check{I}}$	

 Parameters are restricted as  $a = \bar{b} e^{i\theta}$  and  $|a|^2 = |b|^2 = \frac{1}{2}$

We are ready to search SUSY vacua in 4D  $\mathcal{N} = 1$  theory given by  $K, \mathcal{W}, D^a$

**NEXT:** Consider **two** situations

- generalized geometry with RR-flux charges

$$e_{IA}, m_A^I, p_I^A, q^{IA}, e_{\text{RRA}}, m_{\text{RR}}^A$$

- $SU(3)$ -structure manifold

$$e_{IA}, m_A^I$$

# Contents

- Introduction
- Compactifications and Fluxes
- Generalized Geometries
- Setup in  $\mathcal{N} = 1$  Theory
- **My Work: Search of SUSY AdS Vacua**
- Summary and Discussions

$$V = e^K \left( K^{\mathcal{M}\bar{\mathcal{N}}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\bar{\mathcal{N}}} \mathcal{W}} - 3|\mathcal{W}|^2 \right) + \frac{1}{2} |D^a|^2$$

$$\equiv V_{\mathcal{W}} + V_D$$

Search of vacua  $\partial_{\mathcal{P}} V|_* = 0$

$V_* > 0$  : de Sitter space (non-SUSY)

$V_* = 0$  : Minkowski space

$V_* < 0$  : Anti-de Sitter space



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$$0 = \partial_{\mathcal{P}} V_{\mathcal{W}} = e^K \left\{ K^{\mathcal{M}\bar{\mathcal{N}}} D_{\mathcal{P}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\bar{\mathcal{N}}} \mathcal{W}} + \partial_{\mathcal{P}} K^{\mathcal{M}\bar{\mathcal{N}}} D_{\mathcal{M}} \mathcal{W} \overline{D_{\bar{\mathcal{N}}} \mathcal{W}} - 2\overline{\mathcal{W}} D_{\mathcal{P}} \mathcal{W} \right\}$$

$$0 = \partial_{\mathcal{P}} V_D \quad \dashrightarrow \quad D^a = 0$$

Consider the SUSY condition  $D_{\mathcal{P}} \mathcal{W} \equiv (\partial_{\mathcal{P}} + \partial_{\mathcal{P}} K) \mathcal{W} = 0$  in various cases.

1. Set a simple prepotential:  $\mathcal{F} = D_{abc} \frac{X^a X^b X^c}{X^0}$
2. Consider the simplest model: single modulus  $t$  of  $\Phi_+$  (and  $U$  of  $\Phi_-$ )

1. Set a simple prepotential:  $\mathcal{F} = D_{abc} \frac{X^a X^b X^c}{X^0}$
2. Consider the simplest model: single modulus  $t$  of  $\Phi_+$  (and  $U$  of  $\Phi_-$ )

The superpotential is reduced to

$$\mathcal{W} = \mathcal{W}^{\text{RR}} + U \mathcal{W}^{\text{Q}}$$

$$\mathcal{W}^{\text{RR}} = m_{\text{RR}}^0 t^3 - 3 m_{\text{RR}} t^2 + e_{\text{RR}} t + e_{\text{RR}0}$$

$$\mathcal{W}^{\text{Q}} = p_0^0 t^3 - 3 p_0 t^2 - e_0 t - e_{00}$$

Consider the SUSY condition:

$$D_t \mathcal{W} = 0 \quad \rightarrow \quad 0 = D_t \mathcal{W}^{\text{RR}} + U D_t \mathcal{W}^{\text{Q}}$$

$$D_U \mathcal{W} = 0 \quad \rightarrow \quad 0 = \frac{i}{\text{Im}U} \left( \mathcal{W}^{\text{RR}} + \text{Re}U \mathcal{W}^{\text{Q}} \right)$$

The discriminant of the superpotential  $\mathcal{W}^{\text{RR}}$  (and  $\mathcal{W}^{\text{Q}}$ ) governs the SUSY solutions.

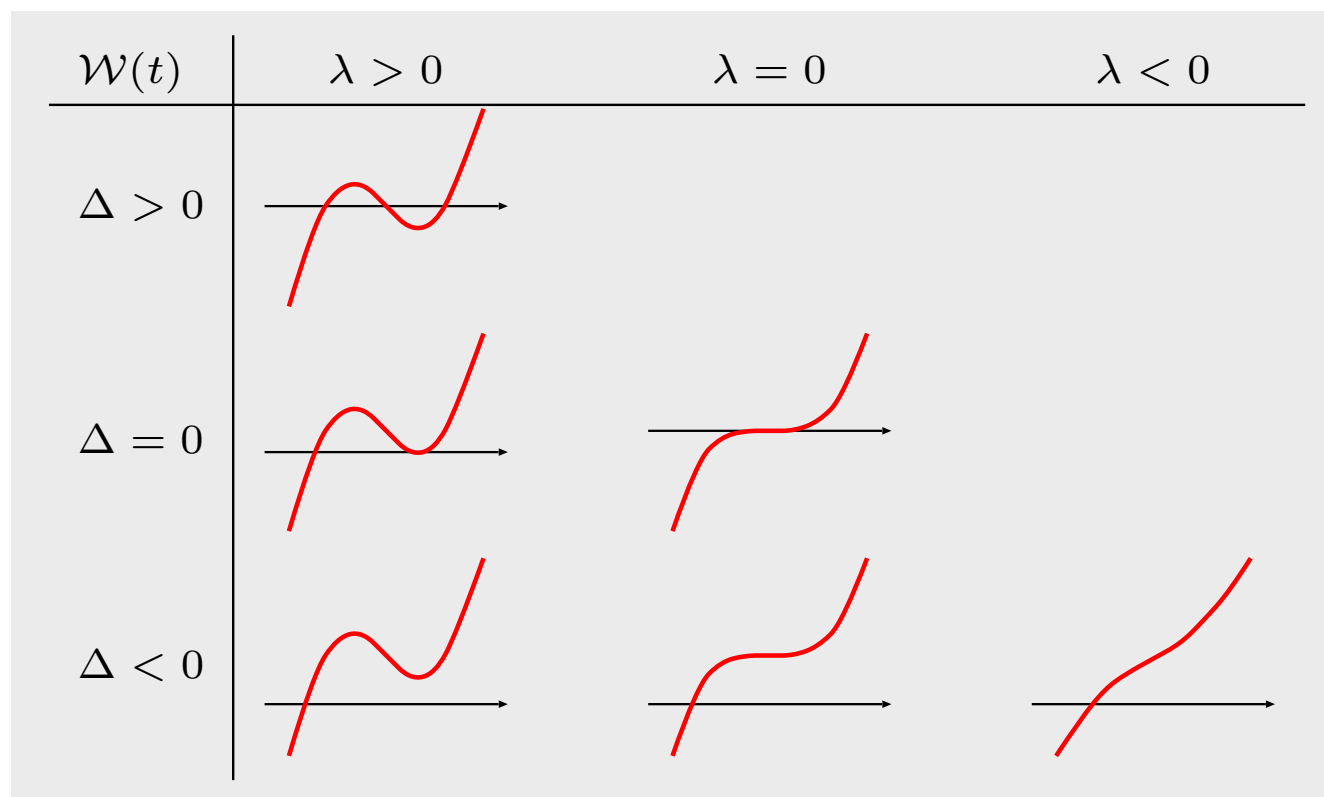
► Discriminant of cubic equation

Consider a cubic function and its derivative: 
$$\begin{cases} \mathcal{W}(t) = at^3 + bt^2 + ct + d \\ \partial_t \mathcal{W}(t) = 3at^2 + 2bt + c \end{cases}$$

Discriminants  $\Delta(\mathcal{W})$  and  $\Delta(\partial_t \mathcal{W})$  are

$$\Delta(\mathcal{W}) \equiv \Delta = -4b^3d + b^2c^2 - 4ac^3 + 18abcd - 27a^2d^2$$

$$\Delta(\partial_t \mathcal{W}) \equiv \lambda = 4(b^2 - 3ac)$$



$\Delta^{\text{RR}} > 0$  case: always  $\lambda^{\text{RR}} > 0$ , and exists a zero point:  $D_t \mathcal{W}^{\text{RR}} = 0$

$$D_t \mathcal{W}^{\text{RR}}|_* = 0$$

$$t_*^{\text{RR}} = \frac{6(3m_{\text{RR}}^0 e_{\text{RR}0} + m_{\text{RR}} e_{\text{RR}})}{\lambda^{\text{RR}}} - 2i \frac{\sqrt{3\Delta^{\text{RR}}}}{\lambda^{\text{RR}}}$$

$$\mathcal{W}_*^{\text{RR}} = -\frac{24\Delta^{\text{RR}}}{(\lambda^{\text{RR}})^3} \left( 36(m_{\text{RR}})^3 + 36(m_{\text{RR}}^0)^2 e_{\text{RR}0} - 3m_{\text{RR}}\lambda^{\text{RR}} - 4i m_{\text{RR}}^0 \sqrt{3\Delta^{\text{RR}}} \right)$$

$\Delta^{\text{RR}} > 0$  case: always  $\lambda^{\text{RR}} > 0$ , and exists a zero point:  $D_t \mathcal{W}^{\text{RR}} = 0$

$$\begin{aligned}
 D_t \mathcal{W}^{\text{RR}}|_* &= 0 \\
 t_*^{\text{RR}} &= \frac{6(3m_{\text{RR}}^0 e_{\text{RR}0} + m_{\text{RR}} e_{\text{RR}})}{\lambda^{\text{RR}}} - 2i \frac{\sqrt{3\Delta^{\text{RR}}}}{\lambda^{\text{RR}}} \\
 \mathcal{W}_*^{\text{RR}} &= -\frac{24\Delta^{\text{RR}}}{(\lambda^{\text{RR}})^3} \left( 36(m_{\text{RR}})^3 + 36(m_{\text{RR}}^0)^2 e_{\text{RR}0} - 3m_{\text{RR}} \lambda^{\text{RR}} - 4i m_{\text{RR}}^0 \sqrt{3\Delta^{\text{RR}}} \right)
 \end{aligned}$$

$\Delta^{\text{RR}} < 0$  case: only  $\lambda^{\text{RR}} < 0$  is physically allowed, and exists a zero point:  $\mathcal{W}^{\text{RR}} = 0$

$$\begin{aligned}
 \mathcal{W}_*^{\text{RR}} &= m_{\text{RR}}^0 (t_* - e)(t_* - \alpha)(t_* - \bar{\alpha}) = 0, \quad t_* = \alpha^{\text{RR}} = \alpha_1 + i\alpha_2 \\
 \alpha_1 &= \frac{\lambda^{\text{RR}} + F^{2/3} + 12m_{\text{RR}} F^{1/3}}{12m_{\text{RR}}^0 F^{1/3}} \\
 (\alpha_2)^2 &= \frac{1}{m_{\text{RR}}^0} \left( e_{\text{RR}} - 6m_{\text{RR}} \alpha_1 + 3m_{\text{RR}}^0 (\alpha_1)^2 \right) \\
 e &= -\frac{1}{m_{\text{RR}}^0} \left( -3m_{\text{RR}} + 2m_{\text{RR}}^0 \alpha_1 \right) \\
 F &= 108(m_{\text{RR}}^0)^2 e_{\text{RR}0} + 12m_{\text{RR}}^0 \sqrt{-3\Delta^{\text{RR}}} + 108(m_{\text{RR}})^3 - 9m_{\text{RR}} \lambda^{\text{RR}} \\
 D_t \mathcal{W}^{\text{RR}}|_* &= 2i m_{\text{RR}}^0 (e - \alpha^{\text{RR}}) \alpha_2
 \end{aligned}$$

... Analysis of  $\mathcal{W}^{\text{Q}}$  is also discussed.

Three types of solutions to satisfy  $0 = D_t \mathcal{W}^{\text{RR}} + U D_t \mathcal{W}^{\text{Q}}$  and  $0 = \mathcal{W}^{\text{RR}} + \text{Re}U \mathcal{W}^{\text{Q}}$ :

Three types of solutions to satisfy  $0 = D_t \mathcal{W}^{\text{RR}} + U D_t \mathcal{W}^{\text{Q}}$  and  $0 = \mathcal{W}^{\text{RR}} + \text{Re} U \mathcal{W}^{\text{Q}}$ :

- SUSY AdS vacuum: moduli are (almost) stabilized

$$\Delta^{\text{RR}} > 0, \quad \Delta^{\text{Q}} > 0; \quad D_t \mathcal{W}^{\text{RR}}|_* = 0 = D_t \mathcal{W}^{\text{Q}}|_*$$

$$t_*^{\text{RR}} = t_*^{\text{Q}}, \quad \text{Re} U_* = -\frac{\mathcal{W}_*^{\text{RR}}}{\mathcal{W}_*^{\text{Q}}}$$

$$V_* = -3e^K |\mathcal{W}_*|^2 = -\frac{4}{[\text{Re}(\mathcal{C}\mathcal{G}_0)]^2} \sqrt{\frac{\Delta^{\text{Q}}}{3}} \ll \mathcal{O}(1)$$



Three types of solutions to satisfy  $0 = D_t \mathcal{W}^{\text{RR}} + U D_t \mathcal{W}^{\text{Q}}$  and  $0 = \mathcal{W}^{\text{RR}} + \text{Re} U \mathcal{W}^{\text{Q}}$ :

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- SUSY Minkowski vacuum: moduli are stabilized

$$\Delta^{\text{RR}} < 0, \quad \Delta^{\text{Q}} < 0; \quad \mathcal{W}_*^{\text{RR}} = 0 = \mathcal{W}_*^{\text{Q}}$$

---


$$\alpha^{\text{RR}} = \alpha^{\text{Q}}, \quad U_* = -\frac{D_t \mathcal{W}^{\text{RR}}|_*}{D_t \mathcal{W}^{\text{Q}}|_*} \neq 0$$

$$V_* = 0$$

Three types of solutions to satisfy  $0 = D_t \mathcal{W}^{\text{RR}} + U D_t \mathcal{W}^{\text{Q}}$  and  $0 = \mathcal{W}^{\text{RR}} + \text{Re}U \mathcal{W}^{\text{Q}}$ :

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$$\Delta^{\text{RR}} < 0, \quad \Delta^{\text{Q}} < 0; \quad \mathcal{W}_*^{\text{RR}} = 0 = \mathcal{W}_*^{\text{Q}}$$

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$$\alpha^{\text{RR}} = \alpha^{\text{Q}}, \quad U_* = -\frac{D_t \mathcal{W}^{\text{RR}}|_*}{D_t \mathcal{W}^{\text{Q}}|_*} \neq 0$$

$$V_* = 0$$

- SUSY AdS vacua, but moduli  $t$  and  $U$  are not fixed: non-stabilized point

$$U = -\frac{D_t \mathcal{W}^{\text{RR}}(t)}{D_t \mathcal{W}^{\text{Q}}(t)}, \quad \text{Re}U = -\frac{\mathcal{W}^{\text{RR}}(t)}{\mathcal{W}^{\text{Q}}(t)}$$

1. Set  $e_{\text{RRA}} = 0 = m_{\text{RR}}^A$ ,  $p_I^A = 0 = q^{IA}$ , and single modulus  $t$  of  $\Phi_+$  (and  $U$  of  $\Phi_-$ )
2. Set a deformed prepotential:  $\mathcal{F} = \frac{(X^t)^3}{X^0}$

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Superpotential  $\mathcal{W} = U\mathcal{W}^{\mathbb{Q}}$  with a simple setting  $N_1 \neq 0$ ,  $N_n = 0$ :

$$D_t \mathcal{W}^{\mathbb{Q}} = -e_{00} + \frac{3(t - \bar{t})^2 - \partial_t P}{(t - \bar{t})^3 - P} (e_{00} + e_0 t)$$

$$P \equiv -2(N_1 t^4 - \bar{N}_1 \bar{t}^4 - 2N_1 t^3 \bar{t} + 2\bar{N}_1 t \bar{t}^3)$$

SUSY condition

$$D_t \mathcal{W} = D_U \mathcal{W} = 0$$

has a solution

$$t_*^{\mathbb{Q}} = -\frac{2e_{00}}{e_0}, \quad \text{Re } U_* = 0$$

$$\mathcal{W}_*^{\mathbb{Q}} = e_{00}, \quad \text{Im } N_1 < 0$$

$$V_* = -3e^K |\mathcal{W}_*|^2 = \frac{1}{[\text{Re}(\mathcal{C}\mathcal{G}_0)]^2} \frac{3(e_0)^4}{16(e_{00})^2 \text{Im } N_1}$$

Also heterotic string on  $SU(3)$ -structure manifolds with torsion which carries  $\alpha'$  corrections

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- Introduction
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## Summary

- Studied generalized geometries and their applications to string compactifications
- Obtained a powerful rule to discuss SUSY vacua: **Discriminants**
- Exhibited that  $\alpha'$  **corrections** are included in certain configurations

## Discussions

- More generic configurations
- Gauge symmetries
- Understanding the physical interpretation of **nongeometric fluxes**

THANK YOU



Appendices

# Contents

- Differential Forms: Geometric Objects
- Generalized (Complex) Geometries

Decomposition of vector bundle on 10D spacetime:

$$T\mathcal{M}_{9,1} = T_{3,1} \oplus F$$
$$\left\{ \begin{array}{l} T_{3,1} : \text{ a real } SO(3,1) \text{ vector bundle} \\ F : \text{ an } SO(6) \text{ vector bundle which admits a pair of } SU(3) \text{ structures} \end{array} \right.$$

Decomposition of vector bundle on 10D spacetime:

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Decomposition of Lorentz symmetry:

$$Spin(9,1) \rightarrow Spin(3,1) \times Spin(6) = SL(2, \mathbb{C}) \times SU(4)$$

$$\mathbf{16} = (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}}) \quad \mathbf{16} = (\mathbf{2}, \bar{\mathbf{4}}) \oplus (\bar{\mathbf{2}}, \mathbf{4})$$

Decomposition of supersymmetry parameters (with  $a, b \in \mathbb{C}$ ):

$$\epsilon_{\text{IIA}}^1 = \varepsilon_1 \otimes (\bar{a} \eta_-^1) + \varepsilon_1^c \otimes (a \eta_+^1)$$

$$\epsilon_{\text{IIA}}^2 = \varepsilon_2 \otimes (b \eta_+^2) + \varepsilon_2^c \otimes (\bar{b} \eta_-^2)$$

$$\epsilon_{\text{IIB}}^1 = \varepsilon_1 \otimes (\bar{a} \eta_-^1) + \varepsilon_1^c \otimes (a \eta_+^1)$$

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Set  $SU(3)$  invariant spinor  $\eta_+^{\mathcal{A}}$  s.t.  $\nabla^{(T)} \eta_+^{\mathcal{A}} = 0$  ( $\mathcal{A} = 1, 2$ )

a pair of  $SU(3)$  on  $F$   $(\eta_+^1, \eta_+^2)$   $\longleftrightarrow$  a single  $SU(3)$  on  $F$   $(\eta_+^1 = \eta_+^2 = \eta_+)$

Requirement that we have a pair of  $SU(3)$  structures means there is a sub-supermanifold

$$\mathcal{N}^{9,1|4+4} \subset \mathcal{M}^{9,1|16+16}$$

$$\left( \begin{array}{l} (9, 1) : \text{ bosonic degrees} \\ 4 + 4 : \text{ eight Grassmann variables as spinors of } Spin(3, 1) \text{ and singlet of } SU(3)\text{s} \end{array} \right)$$

Equivalence such as

eight SUSY theory reformulation of type II strings



a pair of  $SU(3)$  structures on vector bundle  $F$



$SU(3) \times SU(3)$  structures on extended  $F \oplus F^*$

▶ with a single  $SU(3)$ :

a real two-form

$$J_{mn} = \mp 2i \eta_{\pm}^{\dagger} \gamma_{mn} \eta_{\pm}$$

a complex three-form

$$\Omega_{mnp} = -2i \eta_{-}^{\dagger} \gamma_{mnp} \eta_{+}$$

▶ with a single  $SU(3)$ :

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▶ with a pair of  $SU(3)$ :

two real vectors

$$(v - iv')^m = \eta_{+}^{1\dagger} \gamma^m \eta_{-}^2$$

$(J^A, \Omega^A)$

$$J^1 = j + v \wedge v', \quad \Omega^1 = w \wedge (v + iv')$$

$$J^2 = j - v \wedge v', \quad \Omega^2 = w \wedge (v - iv')$$

$(j, w)$ : locally  $SU(2)$ -invariant two-forms



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$(j, w)$ : locally  $SU(2)$ -invariant two-forms

$$\eta_{+}^2 = c_{\parallel} \eta_{+}^1 + c_{\perp} (v + iv')^m \gamma_m \eta_{-}^1, \quad |c_{\parallel}|^2 + |c_{\perp}|^2 = 1$$

If  $\eta_{+}^1 \neq \eta_{+}^2$  globally: a single  $SU(2)$  w/  $(j, w, v, v')$

If  $\eta_{+}^1 = \eta_{+}^2$  globally: a single  $SU(3)$  w/  $(J, \Omega)$

a pair of  $SU(3)$  on  $F \sim SU(3) \times SU(3)$  on  $F \oplus F^*$

**i** Information from Killing spinor eqs. with torsion  $\nabla^{(T)}\eta_{\pm} = 0$  ( $\exists$  complex Weyl  $\eta_{\pm}$ )

► Invariant  $p$ -forms on  $SU(3)$ -structure manifold:

a real two-form  $J_{mn} = \mp 2i \eta_{\pm}^{\dagger} \gamma_{mn} \eta_{\pm}$

a holomorphic three-form  $\Omega_{mnp} = -2i \eta_{-}^{\dagger} \gamma_{mnp} \eta_{+}$

$$dJ = \frac{3}{2} \text{Im}(\overline{\mathcal{W}}_1 \Omega) + \mathcal{W}_4 \wedge J + \mathcal{W}_3 \quad d\Omega = \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \overline{\mathcal{W}}_5 \wedge \Omega$$

► Five classes of (intrinsic) torsion are given as

components	interpretations	$SU(3)$ -representations
$\mathcal{W}_1$	$J \wedge d\Omega$ or $\Omega \wedge dJ$	$\mathbf{1} \oplus \mathbf{1}$
$\mathcal{W}_2$	$(d\Omega)_0^{2,2}$	$\mathbf{8} \oplus \mathbf{8}$
$\mathcal{W}_3$	$(dJ)_0^{2,1} + (dJ)_0^{1,2}$	$\mathbf{6} \oplus \overline{\mathbf{6}}$
$\mathcal{W}_4$	$J \wedge dJ$	$\mathbf{3} \oplus \overline{\mathbf{3}}$
$\mathcal{W}_5$	$(d\Omega)^{3,1}$	$\mathbf{3} \oplus \overline{\mathbf{3}}$

► Classification of  $SU(3)$ -structure manifolds:

complex	hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = 0$
	balanced	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = 0$
	special hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	Kähler	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	conformally Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = 3\mathcal{W}_4 + 2\mathcal{W}_5 = 0$
almost complex	symplectic	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	nearly Kähler	$\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	almost Kähler	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	quasi Kähler	$\mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	semi Kähler	$\mathcal{W}_4 = \mathcal{W}_5 = 0$
	half-flat	$\text{Im}\mathcal{W}_1 = \text{Im}\mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$

# Contents

- Differential Forms: Geometric Objects
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Introduce a generalized almost complex structure  $\mathcal{J}$  on  $F \oplus F^*$  s.t.

$$\mathcal{J} : F \oplus F^* \longrightarrow F \oplus F^*$$

$$\mathcal{J}^2 = -\mathbf{1}_{2d}$$

$$\exists O(d, d) \text{ invariant metric } L, \text{ s.t. } \mathcal{J}^T L \mathcal{J} = L$$

Structure group on $F \oplus F^*$			
$\exists L$	$GL(2d)$	$\dashrightarrow$	$O(d, d)$
$\mathcal{J}^2 = -\mathbf{1}_{2d}$	$O(d, d)$	$\dashrightarrow$	$U(d/2, d/2)$
$\mathcal{J}_1, \mathcal{J}_2$	$U_1(d/2, d/2) \cap U_2(d/2, d/2)$	$\dashrightarrow$	$U(d/2) \times U(d/2)$
integrable $\mathcal{J}_{1,2}$	$U(d/2) \times U(d/2)$	$\dashrightarrow$	$SU(d/2) \times SU(d/2)$

- ▶ Integrability is discussed by “(0, 1)” part of the complexified  $F \oplus F^*$ :

$$\Pi \equiv \frac{1}{2}(\mathbf{1}_{2d} - i\mathcal{J})$$

$$\Pi A = A \quad \text{where } A = v + \zeta \text{ is a section of } F \oplus F^*$$

We call this  $A$  **i-eigenbundle**  $L_{\mathcal{J}}$  whose dimension is  $\dim L_{\mathcal{J}} = d$ .

► Integrability is discussed by “(0, 1)” part of the complexified  $F \oplus F^*$ :

$$\Pi \equiv \frac{1}{2}(\mathbf{1}_{2d} - i\mathcal{J})$$

$$\Pi A = A \quad \text{where } A = v + \zeta \text{ is a section of } F \oplus F^*$$

We call this  $A$  **i-eigenbundle**  $L_{\mathcal{J}}$  whose dimension is  $\dim L_{\mathcal{J}} = d$ .

Integrability condition of  $\mathcal{J}$  is

$$\overline{\Pi}[\Pi(v + \zeta), \Pi(w + \eta)]_{\text{Courant}} = 0; \quad v, w \in \text{section of } F; \quad \zeta, \eta \in \text{section of } F^*$$

$$[v + \zeta, w + \eta]_{\text{Courant}} = [v, w]_{\text{Lie}} + \mathcal{L}_v \eta - \mathcal{L}_w \zeta - \frac{1}{2}d(\iota_v \eta - \iota_w \zeta) \quad \text{Courant bracket}$$

- ▶ Two examples of generalized almost complex structures:

$$\mathcal{J}_- = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I^T \end{pmatrix} \quad \text{w/ } I^2 = -\mathbf{1}_d: \text{ almost complex structure}$$

$$\mathcal{J}_+ = \begin{pmatrix} \mathbf{0} & -J^{-1} \\ J & \mathbf{0} \end{pmatrix} \quad \text{w/ } J: \text{ almost symplectic form}$$

$$\text{integrable } \mathcal{J}_- \quad \leftrightarrow \quad \text{integrable } I$$

$$\text{integrable } \mathcal{J}_+ \quad \leftrightarrow \quad \text{integrable } J$$

On a usual geometry,  $J_{mn} = g_{mp} I^p_n$  is given by an  $SU(3)$  invariant (pure) spinor  $\eta_+$  as

$$J_{mn} = -2i \eta_+^\dagger \gamma_{mn} \eta_+ \quad \gamma^i \eta_+ = 0 \quad \gamma^{\bar{i}} \eta_+ \neq 0$$

In a similar analogy, we want to find pure spinor(s)  $\Phi$  on generalized geometry.



On  $F \oplus F^*$ , we can define Cliff(6, 6) algebra and  $Spin(6, 6)$  spinor  $\Phi$ :

$$\{\Gamma^m, \Gamma^n\} = 0 \quad \{\Gamma^m, \tilde{\Gamma}_n\} = \delta_n^m \quad \{\tilde{\Gamma}_m, \tilde{\Gamma}_n\} = 0$$

Irreducible repr. of  $Spin(6, 6)$  spinor is a Majorana-**Weyl**

→ a generic  $Spin(6, 6)$  spinor bundle  $S$  splits to  $S^\pm$  (Weyl)

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Weyl spinor bundles  $S^\pm$  are isomorphic to bundles of forms  $F^*$ :

$$\Phi_+ \in S^+ \quad \sim \quad \text{section of } \wedge^{\text{even}} F^*$$

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IF  $\Phi$  is annihilated by half numbers of the Cliff(6, 6) generators:

→  $\Phi$  is called a **pure spinor**

cf.)  $SU(3)$  invariant spinor  $\eta_+$  is a pure spinor:  $\gamma^i \eta_+ = 0$

An equivalent definition of a pure spinor  $\Phi$  is given by “Clifford action”:

$$(v + \zeta) \cdot \Phi = v^m \iota_{\partial_m} \Phi + \zeta_n dx^n \wedge \Phi \quad \text{w/ } v: \text{ vector} \quad \zeta: \text{ one-form}$$

Define the annihilator of spinors as

$$L_\Phi \equiv \{v + \zeta \in F \oplus F^* \mid (v + \zeta) \cdot \Phi = 0\}$$
$$\dim L_\Phi \leq 6$$

If  $\dim L_\Phi = 6$  (maximally isotropic)  $\rightarrow \Phi$  is a **pure spinor**

Correspondence between pure spinors and generalized almost complex structures:

$$\mathcal{J} \leftrightarrow \Phi \quad \text{if } L_{\mathcal{J}} = L_{\Phi} \quad \text{with } \dim L_{\Phi} = 6$$

More precisely:  $\mathcal{J} \leftrightarrow$  a line bundle of pure spinor  $\Phi$

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Then, we can rewrite the generalized almost complex structure as

$$\mathcal{J}_{\pm\Pi\Sigma} = \langle \text{Re}\Phi_{\pm}, \Gamma_{\Pi\Sigma} \text{Re}\Phi_{\pm} \rangle$$

w/ Mukai pairing:

$$\text{even forms: } \langle \Psi_+, \Phi_+ \rangle = \Psi_6 \wedge \Phi_0 - \Psi_4 \wedge \Phi_2 + \Psi_2 \wedge \Phi_4 - \Psi_0 \wedge \Phi_6$$

$$\text{odd forms: } \langle \Psi_-, \Phi_- \rangle = \Psi_5 \wedge \Phi_1 - \Psi_3 \wedge \Phi_3 + \Psi_1 \wedge \Phi_5$$

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$$\mathcal{J} \text{ is integrable} \iff \exists \text{ vector } v \text{ and one-form } \zeta \text{ s.t. } d\Phi = (v_{\perp} + \zeta \wedge)\Phi$$

$$\text{generalized CY} \iff \exists \Phi \text{ is pure s.t. } d\Phi = 0$$

$$\text{"twisted" GCY} \iff \exists \Phi \text{ is pure, and } H \text{ is closed s.t. } (d - H \wedge)\Phi = 0$$

A spinor  $\Phi$  can also be mapped to a bispinor by using

$$C \equiv \sum_k \frac{1}{k!} C_{m_1 \dots m_k}^{(k)} dx^{m_1} \wedge \dots \wedge dx^{m_k} \quad \longleftrightarrow \quad \mathcal{C} \equiv \sum_k \frac{1}{k!} C_{m_1 \dots m_k}^{(k)} \gamma_{\alpha\beta}^{m_1 \dots m_k}$$



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On a geometry of a **single**  $SU(3)$ -structure, the following two  $SU(3,3)$  spinors:

$$\begin{aligned} \Phi_{0+} &= \eta_+ \otimes \eta_+^\dagger = \frac{1}{4} \sum_{k=0}^6 \frac{1}{k!} \eta_+^\dagger \gamma_{m_k \dots m_1} \eta_+ \gamma^{m_1 \dots m_k} = \frac{1}{8} e^{-iJ} \\ \Phi_{0-} &= \eta_+ \otimes \eta_-^\dagger = \frac{1}{4} \sum_{k=0}^6 \frac{1}{k!} \eta_-^\dagger \gamma_{m_k \dots m_1} \eta_+ \gamma^{m_1 \dots m_k} = -\frac{i}{8} \Omega \end{aligned}$$

Check purity:  $(\delta + iJ)_m^n \gamma_n \eta_+ \otimes \eta_\pm^\dagger = 0 = \eta_+ \otimes \eta_\pm^\dagger \gamma_n (\delta \mp iJ)^n_m$

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On a generic geometry of a **pair** of  $SU(3)$ -structures defined by  $(\eta_+^1, \eta_+^2)$

$$\begin{aligned} \Phi_{0+} &= \eta_+^1 \otimes \eta_+^{2\dagger} = \frac{1}{8} (\bar{c}_\parallel e^{-ij} - i\bar{c}_\perp w) \wedge e^{-iv \wedge v'} \\ \Phi_{0-} &= \eta_+^1 \otimes \eta_-^{2\dagger} = -\frac{1}{8} (c_\perp e^{-ij} + ic_\parallel w) \wedge (v + iv') \end{aligned} \quad |c_\parallel|^2 + |c_\perp|^2 = 1$$

$$\Phi_{\pm} \equiv e^{-B} \Phi_{0\pm}$$

Each  $\Phi_{\pm}$  defines an  $SU(3,3)$  structure on  $E$ . Common structure is  $SU(3) \times SU(3)$ .

$F$  is extended to  $E$  by including  $e^{-B}$

Compatibility requires

$$\begin{aligned} \langle \Phi_+, V \cdot \Phi_- \rangle &= \langle \bar{\Phi}_+, V \cdot \Phi_- \rangle = 0 \quad \text{for } \forall V = x + \xi \\ \langle \Phi_+, \bar{\Phi}_+ \rangle &= \langle \Phi_-, \bar{\Phi}_- \rangle \end{aligned}$$

Start with a real form  $\chi_f \in \wedge^{\text{even/odd}} F^*$  (associated with a real  $Spin(6,6)$  spinor  $\chi_s$ )

Regard  $\chi_f$  as a stable form satisfying

$$q(\chi_f) = -\frac{1}{4} \langle \chi_f, \Gamma_{\Pi\Sigma} \chi_f \rangle \langle \chi_f, \Gamma^{\Pi\Sigma} \chi_f \rangle \in \wedge^6 F^* \otimes \wedge^6 F^*$$

$$U = \{ \chi_f \in \wedge^{\text{even/odd}} F^* \mid q(\chi_f) < 0 \}$$

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Define a Hitchin function

$$H(\chi_f) \equiv \sqrt{-\frac{1}{3}q(\chi_f)} \in \wedge^6 F^*$$

which gives an integrable complex structure on  $U$

Start with a real form  $\chi_f \in \Lambda^{\text{even/odd}} F^*$  (associated with a real  $Spin(6,6)$  spinor  $\chi_s$ )

Regard  $\chi_f$  as a stable form satisfying

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Then we can get another real form  $\hat{\chi}_f$  and a complex form  $\Phi_f$  by Mukai pairing

$$\begin{aligned} \langle \hat{\chi}_f, \chi_f \rangle &= -dH(\chi_f) \quad \text{i.e.,} \quad \hat{\chi}_f = -\frac{\partial H(\chi_f)}{\partial \chi_f} \\ \dashrightarrow \quad \Phi_f &\equiv \frac{1}{2}(\chi_f + i\hat{\chi}_f) \quad H(\Phi_f) = i\langle \Phi_f, \bar{\Phi}_f \rangle \end{aligned}$$

Hitchin showed:  $\Phi_f$  is a (form corresponding to) **pure spinor!**

Consider the space of pure spinors  $\Phi$  ...

Mukai pairing $\langle *, * \rangle$	$\longrightarrow$	symplectic structure
Hitchin function $H(*)$	$\longrightarrow$	complex structure



The space of pure spinor is Kähler

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The space of pure spinor is Kähler

Compatible with  $\Phi \rightarrow \lambda\Phi$  w/  $\lambda \in \mathbb{C}^*$

--> The space becomes a **local** special Kähler geometry with Kähler potential  $K$ :

$$\exp(-K) = H(\Phi) = i\langle \Phi, \bar{\Phi} \rangle = i(\bar{X}^A \mathcal{F}_A - X^A \bar{\mathcal{F}}_A) \in \wedge^6 F^*$$

$X^A$  : holomorphic projective coordinates

$\mathcal{F}_A$  : derivative of prepotential  $\mathcal{F}$  ( $\mathcal{F}_A = \partial\mathcal{F}/\partial X^A$ )



THE END