東京工業大学
セミナー (2012年5月16日)

Flux Compactifications, and Static Charged Black Holes in Massive Type IIA

based on arXiv:1108.1113, arXiv:1203.1544

木村 哲士 (立教大学)

以下の流れでお話をします

- 動機

  RN-AdS ブラックホールとストリング理論のコンパクト化

- 設定

  (non)-CY コンパクト化とゲージ化された超重力理論

- AdS ブラックホール

  nearly-Kähler coset space $G_2/SU(3)$

  AdS 真空

  静的条件と解

動機

ストリング理論のコンパクト化において

Reissner-Nordström AdS ブラックホール解を考えたい

Reissner-Nordström AdS ブラックホール解を考えたい

理由

- フラックスコンパクト化での重力解の調査
- ハイパー多重項がある $\mathcal{N} = 2$ gauged SUGRA の探索
- 「AdS/CMT」の設定を小耳に挟む：

荷電粒子が AdS ブラックホールに落ちていく過程をみる

Einstein + $\Lambda_{\text{c.c.}}$ + Maxwell + charged matters

この設定 (に近いもの) をストリング理論で与えるには？

設定を超重力理論に埋め込むときの障害

- 宇宙項 vs 物質場 vs 超対称性 (4D)

  物質場がある系で負の宇宙「定数」をそのまま超対称作用積分に導入することはできない

- 宇宙項 vs コンパクト化 (10D)

  負の宇宙項を持つ真空解を与えるには Romans′ mass が 必要 (type IIA)

- Maxwell 場 vs フラックス (4D)

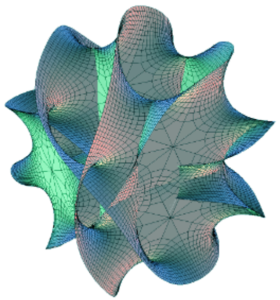
  field strength は Romans′ mass のために変形

AdS/CMT の起源を type IIA ストリング理論に求めないならば、
上記の障害は気にならない (?)

設定

ストリング理論のコンパクト化

Calabi-Yau 多様体 $\mathcal{M}_{\rm CY}$



Ricci 平坦な Kähler 多様体

トーションなし

ホロノミー群は $SU(3) \subset SU(4) \sim SO(6)$

Levi-Civita 接続の共変微分について共変定数な 2形式 (J) と正則3形式 (Ω)：

$$\mathrm{d}J \;=\; \nabla_{[m}J_{np]} \;=\; 0 \qquad \mathrm{d}\Omega \;=\; \nabla_{[m}\Omega_{npq]} \;=\; 0$$

NS-NS 場の展開：

$$\phi(x,y) = \varphi(x)$$

$$g_{\mathfrak{m}\bar{\mathfrak{n}}}(x,y) = \mathrm{i} v^a(x)\,(\omega_a)_{\mathfrak{m}\bar{\mathfrak{n}}}(y), \quad g_{\mathfrak{m}\mathfrak{n}}(x,y) = \mathrm{i}\,\overline{z}^{\bar{\jmath}}(x) \left(\frac{(\overline{\chi}_{\bar{\jmath}})_{\mathfrak{m}\bar{\mathfrak{p}}\bar{\mathfrak{q}}} \Omega^{\bar{\mathfrak{p}}\bar{\mathfrak{q}}}{}_{\mathfrak{n}}}{||\Omega||^2} \right)(y)$$

$$B_2(x,y) = B_2(x) + b^a(x)\omega_a(y)$$

$$\mathfrak{t}^a(x) \equiv b^a(x) + \mathrm{i} v^a(x)$$

R-R 場の展開：

$$C_1(x,y) = A_1^0(x)$$

$$C_3(x,y) = A_1^a(x) \wedge \omega_a(y) + \xi^I(x)\alpha_I(y) - \widetilde{\xi}_I(x)\beta^I(y)$$

コホモロジー	基底	自由度
$H^{(1,1)}$	ω_a	$a = 1, \ldots, h^{(1,1)}$
$H^{(0)} \oplus H^{(1,1)}$	$\omega_\Lambda = (1, \omega_a)$	$\Lambda = 0, 1, \ldots, h^{(1,1)}$
$H^{(2,2)} \oplus H^{(6)}$	$\widetilde{\omega}^\Lambda = (\widetilde{\omega}^a, \frac{\text{vol.}}{\lvert\text{vol.}\rvert})$	
$H^{(2,1)}$	χ_i	$i = 1, \ldots, h^{(2,1)}$
$H^{(3)}$	(α_I, β^I)	$I = 0, 1, \ldots, h^{(2,1)}$

$$\mathrm{d}\omega_\Lambda = 0 = \mathrm{d}\widetilde{\omega}^\Lambda$$

$$\mathrm{d}\alpha_I = 0 = \mathrm{d}\beta^I$$

non-CY manifold $\mathcal{M}_6$

Ricci 2-form がゼロのまま、トーションを許す

($SU(3)$-structure manifold)

$\mathrm{d}J \neq 0$ and/or $\mathrm{d}\Omega \neq 0$

CY からのズレ：

$$\mathrm{d}J = \frac{3}{2}\,\mathrm{Im}(\overline{\mathcal{W}}_1 \Omega) + \mathcal{W}_4 \wedge J + \mathcal{W}_3\,, \qquad \mathrm{d}\Omega = \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \overline{\mathcal{W}}_5 \wedge \Omega$$

$\mathcal{W}_1\,,\ \mathcal{W}_2\,,\ \mathcal{W}_3\,,\ \mathcal{W}_4\,,\ \mathcal{W}_5$: intrinsic torsion classes

$$\mathrm{d}J = \frac{3}{2}\,\mathrm{Im}(\overline{\mathcal{W}}_1\Omega) + \mathcal{W}_4 \wedge J + \mathcal{W}_3\,, \qquad \mathrm{d}\Omega = \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \overline{\mathcal{W}}_5 \wedge \Omega$$

complex	hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = 0$
	balanced	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = 0$
	special hermitian	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	Kähler	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	Calabi-Yau	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	conformally CY	$\mathcal{W}_1 = \mathcal{W}_2 = \mathcal{W}_3 = 3\mathcal{W}_4 + 2\mathcal{W}_5 = 0$
almost complex	symplectic	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = 0$
	nearly Kähler	$\mathcal{W}_2 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	almost Kähler	$\mathcal{W}_1 = \mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	quasi Kähler	$\mathcal{W}_3 = \mathcal{W}_4 = \mathcal{W}_5 = 0$
	semi Kähler	$\mathcal{W}_4 = \mathcal{W}_5 = 0$
	half-flat	$\mathrm{Im}\mathcal{W}_1 = \mathrm{Im}\mathcal{W}_2 = \mathcal{W}_4 = \mathcal{W}_5 = 0$

閉形式でない $(\mathrm{d}J, \mathrm{d}\Omega)$ を、基底形式の外微分の性質に翻訳する：

NS-NS

$$\mathrm{d}\begin{pmatrix} \beta^I \\ \alpha_I \end{pmatrix} \sim \begin{pmatrix} e_{\Lambda}{}^I & m^{\Lambda I} \\ e_{\Lambda I} & m^{\Lambda}{}_I \end{pmatrix} \begin{pmatrix} \widetilde{\omega}^{\Lambda} \\ \omega_{\Lambda} \end{pmatrix}$$

Σ_- $\quad Q^{\mathrm{T}}$ $\quad \Sigma_+$

$e_0{}^I, e_{0I}$:	H-flux charges ($H^{\mathrm{fl}} = -e_0{}^I \alpha_I + e_{0I} \beta^I$)
$e_a{}^I, e_{aI}$:	geometric flux charges (トーション)
$m^{\Lambda I}, m^{\Lambda}{}_I$:	nongeometric flux charges ($e_{\Lambda}{}^I, e_{\Lambda I}$ の"磁気的"双対)

R-R

$$\mathbf{F} \equiv F_0 + F_2 + \ldots + F_{10} \equiv \mathrm{e}^B \mathbf{G} \quad (\text{自己双対条件 } \mathbf{F} = \lambda(*\mathbf{F}), \quad \lambda(F_{(k)}) \equiv (-)^{[\frac{k+1}{2}]} F_{(k)})$$

$$\frac{1}{\sqrt{2}}\mathbf{G} = (G_0^{\Lambda} + G_2^{\Lambda} + G_4^{\Lambda})\,\omega_{\Lambda} - (\widetilde{G}_{0\Lambda} + \widetilde{G}_{2\Lambda} + \widetilde{G}_{4\Lambda})\,\widetilde{\omega}^{\Lambda} + (G_1^I + G_3^I)\,\alpha_I - (\widetilde{G}_{1I} + \widetilde{G}_{3I})\,\beta^I$$

$$G_0^{\Lambda} \equiv p^{\Lambda}, \quad \widetilde{G}_{0\Lambda} \equiv q_{\Lambda} - \xi^I e_{\Lambda I} + \widetilde{\xi}_I e_{\Lambda}{}^I$$

$c \equiv (p^{\Lambda}, q_{\Lambda})^{\mathrm{T}}$: R-R flux charges

(p^0: Romans' mass)

10次元IIA型作用 (democratic formulation) $S^{(10D)}_{IIA} = S_{NS} + \widetilde{S}_{R}$:

$$S_{NS} + \widetilde{S}_{R} = \frac{1}{2}\int e^{-2\phi}\Big\{R * \mathbb{1} + 4d\phi \wedge *d\phi - \frac{1}{2}H_3 \wedge *H_3\Big\} - \frac{1}{8}\int \big[\mathbf{F} \wedge *\mathbf{F}\big]_{10}$$

"自己双対 $\mathbf{F} = \lambda(*\mathbf{F})$" と "場の方程式 $(d + H\wedge) * \mathbf{F} = 0 \Leftrightarrow (d - H\wedge)\mathbf{F} = 0$"

↓

4次元 $\mathcal{N}=2$ 可換ゲージ群を持つ超重力理論 (非自明なポテンシャル項付き)

(非可換ゲージ群の実現は今のところ難しい)

3種類のゲージ化された超重力理論

ゲージ化：物質場の住む空間の isometry group をゲージ化すること

ベクトル多重項： special Kähler geometry (SKG)

ハイパー多重項： quaternionic geometry (QG)

Appendix

3種類のゲージ化された超重力理論

$0=m^{\Lambda I}=m^{\Lambda}{}_I=p^{\Lambda}$	$0=m^{\Lambda I}=m^{\Lambda}{}_I$	generic
$n_{\rm V}$ 個のベクトル多重項	$n_{\rm V}$ 個のベクトル多重項	$n_{\rm V}$ 個のベクトル多重項
$n_{\rm H}$ 個のハイパー多重項	$n_{\rm H}$ 個のハイパー多重項	$\widetilde{n}_{\rm H}$ 個のハイパー多重項
普遍ハイパー多重項	1 個のテンソル多重項	$n_{\rm T}$ 個のテンソル多重項
[hep-th/9605032]	[hep-th/0312210]	[hep-th/0409097]

スカラー場 $\{a, \xi^I, \widetilde{\xi}_I\}$ のいくつかが "磁荷" $\{p^{\Lambda}, m^{\Lambda}{}_I, m^{\Lambda I}\}$ によってテンソル場に双対変換

[hep-th/0701247], [arXiv:0804.0595]

- $\mathcal{N}=2$ ゲージ化されていない超重力理論では $\longrightarrow$ ポテンシャル項がない

  漸近平坦な (極限) 荷電ブラックホールの解析などではこの理論でも十分だった

- $\mathcal{N}=2$ ゲージ化された超重力理論では $\longrightarrow$ ポテンシャル項が登場する

  ポテンシャル項の極致が超対称性を (部分的に) 破り、宇宙項を与える

AdSブラックホール

in 4D $\mathcal{N} = 2$ gauged SUGRA with B-field

from massive type IIA on a nearly-Kähler coset space $G_2/SU(3)$

TK, arXiv:1108.1113, arXiv:1203.1544

D. Cassani and A.K. Kashani-Poor [arXiv:0901.4251]

✓ NSNS-sector : torsion and H-flux

✓ RR-sector : 2-, 4-form and Romans' mass (0-form)

$$\mathrm{d}J = \frac{3}{2}\,\mathrm{Im}(\overline{\mathcal{W}}_1\Omega)\,, \quad \mathrm{d}\Omega = \mathcal{W}_1\, J \wedge J$$

$$\mathrm{d}\omega_\Lambda = e_\Lambda\,\alpha\,, \quad \mathrm{d}\alpha = 0\,, \quad \mathrm{d}\beta = e_\Lambda\,\widetilde{\omega}^\Lambda\,, \quad \mathrm{d}\widetilde{\omega}^\Lambda = 0$$

$$e_\Lambda m_{\mathrm{R}}^\Lambda = 0$$

✓ 1 vector multiplet with cubic prepotential $\mathcal{F} = \dfrac{X^1 X^1 X^1}{X^0}$

✓ 1 universal hypermultiplet (no other HMs) $\rightarrow$ hyper-tensor multiplet

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4D multiplets	from NSNS	from RR	fermions
supergravity multiplet	$g_{\mu\nu}$	A^0_μ	ψ^i_μ
1 vector multiplet	$\mathfrak{t}$	A^1_μ	λ_i
1 hyper-tensor multiplet	φ, $B_{\mu\nu}$	ξ^0, $\widetilde{\xi}_0$	ζ^α

A^0_μ : from RR one-form C_1

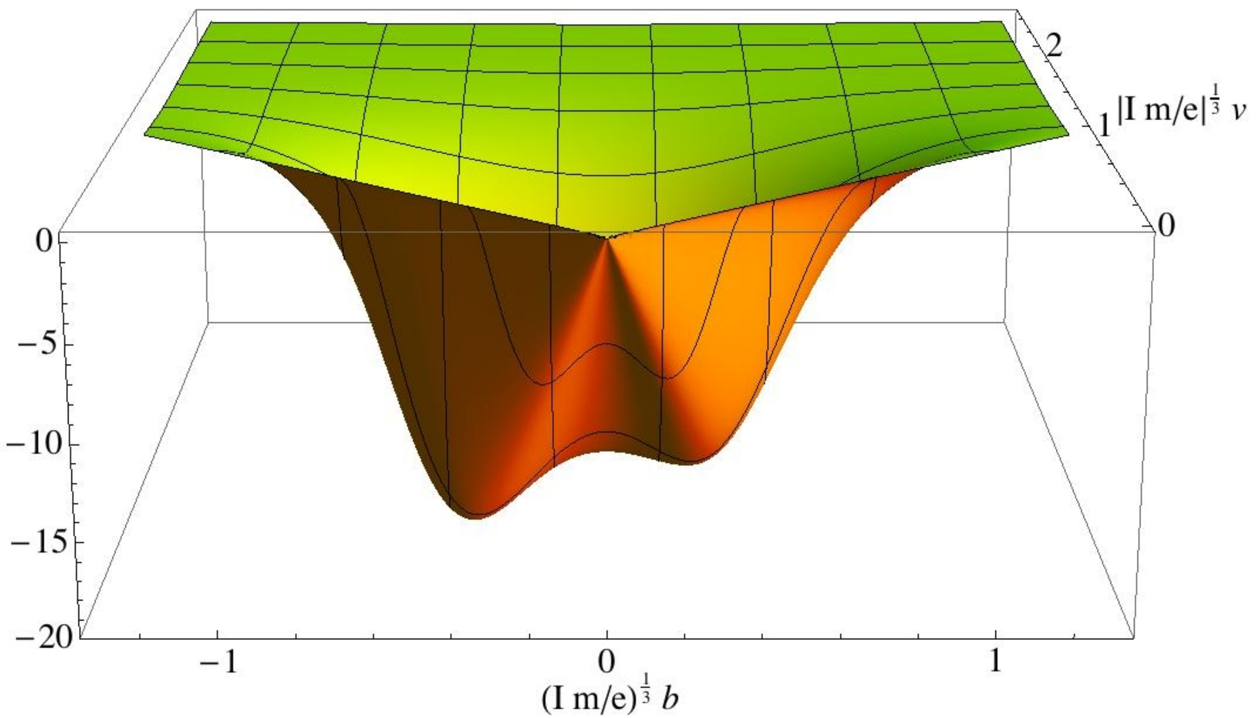
$\mathfrak{t}$: from complexified "Kähler" modulus $\mathfrak{t} = X^1/X^0 = b + \mathrm{i}v$

A^1_μ : from RR three-form C_3

ξ^0, $\widetilde{\xi}_0$: from RR three-form C_3

Expansion Lagrangian

$$F^\Lambda_2 = \mathrm{d}A^\Lambda_1 + m^\Lambda_{\mathrm{R}} B_2$$: Stückelberg-type deformation



in arXiv:0901.4251

ひとつの $\mathcal{N}=1$ AdS 真空 と ふたつの $\mathcal{N}=0$ AdS 真空 Appendix

$$\Lambda_{\text{c.c.}}^{\mathcal{N}=0} < \Lambda_{\text{c.c.}}^{\mathcal{N}=1} < \Lambda_{\text{c.c.}}^{\mathcal{N}=0}$$

[NOTE]

type IIA 弦理論から 4D $\mathcal{N}=1$ AdS vacua を導出するには Romans′ mass が必要

D. Lüst and D. Tsimpis, hep-th/0412250

Reissner-Nordström AdS ブラックホールの下、物質場の配位を調べる：

$$\mathrm{d}s^2 = -\mathrm{e}^{2A(r)}\mathrm{d}t^2 + \mathrm{e}^{-2A(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$\mathrm{e}^{2A(r)} = 1 - \frac{2\eta}{r} + \frac{\mathcal{Z}^2}{r^2} + \frac{r^2}{\ell^2}\,, \quad \mathcal{Z}^2 = Q^2 + P^2\,, \quad \Lambda_{\text{c.c.}} = -\frac{3}{\ell^2}$$

vector fields A_μ^Λ の電荷・磁荷： $p^\Lambda = \int F_2^\Lambda\,, \quad q_\Lambda = \int \widetilde{F}_{2\Lambda}$

$$\downarrow$$

$$F_{\theta\phi}^\Lambda \equiv f^\Lambda(\theta,\phi)\sin\theta\,, \quad F_{tr}^\Lambda \equiv \frac{\mathrm{e}^{-2C(r)}}{r^2} g^\Lambda(\theta,\phi)$$

時間依存性がない解を探そうとすると、

A_μ^Λ, $B_{\mu\nu}$, $g_{\mu\nu}$ の運動方程式から「すべての場の共変定数性」が示される：

$$0 = \partial_\mu \mathfrak{t} = \partial_\mu\varphi = D_\mu\xi^0 = D_\mu\widetilde{\xi}_0 = \partial_{[\mu}B_{\nu\rho]} = F_{\mu\nu}^\Lambda$$

Appendix

ブラックホール電荷 $\mathscr{Z}^2 = Q^2 + P^2$ は

次の様に記述される：

$$\mathscr{Z}^2 \;=\; -\frac{1}{2}\Big[p^\Lambda \mu_{\Lambda\Sigma}\, p^\Sigma + (q_\Lambda - \nu_{\Lambda\Gamma}\, p^\Gamma)(\mu^{-1})^{\Lambda\Sigma}(q_\Sigma - \nu_{\Sigma\Delta}\, p^\Delta)\Big]$$

$$e^{-1}\mathscr{L} = \tfrac{1}{2}R + \tfrac{1}{2}\mu_{\Lambda\Sigma}F^\Lambda \wedge *F^\Sigma + \tfrac{1}{2}\nu_{\Lambda\Sigma}F^\Lambda \wedge F^\Sigma + \ldots$$

一方で、ゲージ場と電荷 (p^Λ, q_Λ) の関係は

次の様に与えられる：

$$0 \;=\; F^\Lambda_{\theta\phi} \;=\; p^\Lambda \sin\theta\,, \qquad 0 \;=\; F^\Lambda_{tr} \;=\; -\frac{1}{r^2}(\mu^{-1})^{\Lambda\Sigma}(q_\Sigma - \nu_{\Sigma\Gamma}p^\Gamma)$$

つまり、

$$p^\Lambda \;=\; 0 \;=\; q_\Lambda \quad\to\quad \mathscr{Z}^2 \;=\; 0$$

結果

Massive type IIA on $G_2/SU(3)$ から得られる

4D $\mathcal{N}=2$ gauged SUGRA with B-field は、

Reissner-Nordström AdS ブラックホール解を持ち得ない ($\mathcal{Z}^2=0$ に退化)。

ブラックホール質量・電荷・宇宙項などをコンパクト化の情報で記述するために、
もう少し非自明なブラックホール解を考える必要がある？

(そろそろ違うテーマをやりたい...)

おわりに

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- 設定

  (non)-CY コンパクト化とゲージ化された超重力理論

- AdS ブラックホール

  nearly-Kähler coset space $G_2/SU(3)$

  AdS 真空

  静的条件と共変定数解

Appendix

4D $\mathcal{N} = 2$ gauged SUGRA

Coset spaces

Gauged SUGRA from type IIA on $G_2/SU(3)$

Analysis on static AdS black holes

10D type IIA action $S^{(10D)}_{IIA} = S_{NS} + \widetilde{S}_R = S_{NS} + S_R + S_{CS}$: (democratic form)

$$S_{NS} = \frac{1}{2}\int e^{-2\phi}\Big\{R * \mathbb{1} + 4d\phi \wedge *d\phi - \frac{1}{2}H_3 \wedge *H_3\Big\}, \quad \widetilde{S}_R = -\frac{1}{8}\int [\mathbf{F} \wedge *\mathbf{F}]_{10}$$

with "constraint $\mathbf{F} = \lambda(*\mathbf{F})$" and "EoM (Bianchi) $(d + H\wedge) * \mathbf{F} = 0 \Leftrightarrow (d - H\wedge)\mathbf{F} = 0$"

↓ $SU(3)$-structure with $m_R^{\Lambda} = 0$

4D $\mathcal{N} = 2$ abelian gauged SUGRA (with $\xi^{\boldsymbol{I}} \equiv (\xi^I, \widetilde{\xi}_I)^T$):

$$S^{(4D)} = \int d^4x\sqrt{-g}\,\Big[\frac{1}{2}R + \frac{1}{4}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} - \frac{\epsilon^{\mu\nu\rho\sigma}}{8\sqrt{-g}}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma} - g_{a\bar{b}}\,\partial_\mu t^a\partial^\mu\bar{t}^{\bar{b}} - g_{i\bar{j}}\,\partial_\mu z^i\partial^\mu\bar{z}^{\bar{j}}$$
$$-\partial_\mu\varphi\partial^\mu\varphi + \frac{e^{2\varphi}}{2}(\mathbb{M}_H)_{\boldsymbol{IJ}}D_\mu\xi^{\boldsymbol{I}}D^\mu\xi^{\boldsymbol{J}} - \frac{e^{2\varphi}}{4}\big(D_\mu a - \xi^{\boldsymbol{I}}(\mathbb{C}_H)_{\boldsymbol{IJ}}D_\mu\xi^{\boldsymbol{J}}\big)^2 - V(t,\bar{t},q)\Big]$$

- $(e_\Lambda{}^I, e_{\Lambda I})$: geometric flux charges & $e_{R\Lambda}$: RR-flux charges ⟵ non-CY data
  (with constraints $e_\Lambda{}^I e_{\Sigma I} - e_{\Lambda I}e_\Sigma{}^I = 0$)
- $t^a \in SKG_V$ and $z^i \in SKG_H \subset \mathcal{HM}$ are ungauged (in general)
- $D_\mu\xi^I = \partial_\mu\xi^I - e_\Lambda{}^I A^\Lambda_\mu$ & $D_\mu\widetilde{\xi}_I = \partial_\mu\widetilde{\xi}_I - e_{\Lambda I}A^\Lambda_\mu$
- $D_\mu a = \partial_\mu a - (2e_{R\Lambda} - \xi^I e_{\Lambda I} + \widetilde{\xi}_I e_\Lambda{}^I)A^\Lambda_\mu$
- $V(t,\bar{t},q)$: scalar potential

D. Cassani, arXiv:0804.0595

Non-vanishing $m_{\rm R}^{\Lambda}$ dualizes the axion field a in standard SUGRA to B-field.

4D gauged action is different from the standard one:

$$
\begin{aligned}
S^{(4\mathrm{D})} \;=\; \int \Big[& \frac{1}{2} R(*\mathbb{1}) + \frac{1}{2} \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} F_2^{\Lambda} \wedge * F_2^{\Sigma} + \frac{1}{2} \mathrm{Re}\mathcal{N}_{\Lambda\Sigma} F_2^{\Lambda} \wedge F_2^{\Sigma} - g_{a\bar{b}}\, \mathrm{d}t^a \wedge * \mathrm{d}\bar{t}^{\bar{b}} - g_{i\bar{\jmath}}\, \mathrm{d}z^i \wedge * \mathrm{d}\bar{z}^{\bar{\jmath}} \\
& - \mathrm{d}\varphi \wedge * \mathrm{d}\varphi - \frac{\mathrm{e}^{-4\varphi}}{4} H_3 \wedge * H_3 - \frac{\mathrm{e}^{2\varphi}}{2} (\mathbb{M}_{\rm H})_{\boldsymbol{IJ}} D\xi^{\boldsymbol{I}} \wedge * D\xi^{\boldsymbol{J}} - V(*\mathbb{1}) \\
& + \frac{1}{2} \mathrm{d}B \wedge \Big[\xi^{\boldsymbol{I}} (\mathbb{C}_{\rm H})_{\boldsymbol{IJ}} D\xi^{\boldsymbol{J}} + \big(2 e_{{\rm R}\Lambda} - \xi^I e_{\Lambda I} + \widetilde{\xi}_I e_{\Lambda}{}^I \big) A_1^{\Lambda} \Big] - \frac{1}{2} m_{\rm R}^{\Lambda} e_{{\rm R}\Lambda} B_2 \wedge B_2 \Big]
\end{aligned}
$$

Constraints among flux charges:

$$
e_{\Lambda}{}^I e_{\Sigma I} - e_{\Lambda I} e_{\Sigma}{}^I \;=\; 0, \quad m_{\rm R}^{\Lambda} e_{\Lambda}{}^I \;=\; 0 \;=\; m_{\rm R}^{\Lambda} e_{\Lambda I}
$$

Scalar potential from (non)geometric flux compactifications:

$$V = \mathrm{g}^2 \Big[4 h_{uv} k^u \overline{k}^v + \sum_{x=1}^{3} \Big(g^{a\overline{b}} D_a \mathcal{P}_x D_{\overline{b}} \overline{\mathcal{P}}_x - 3 |\mathcal{P}_x|^2 \Big) \Big] = \ldots \equiv V_{\mathrm{NS}} + V_{\mathrm{R}} \quad \text{(abelian: } k_\Lambda^a = 0\text{)}$$

$$V_{\mathrm{NS}} = g^{a\overline{b}} D_a \mathcal{P}_+ D_{\overline{b}} \overline{\mathcal{P}}_+ + g^{i\overline{j}} D_i \mathcal{P}_+ D_{\overline{j}} \overline{\mathcal{P}}_+ - 2 |\mathcal{P}_+|^2$$

$$= -2\, \mathrm{g}^2 \mathrm{e}^{2\varphi} \Big[\overline{\Pi}_{\mathrm{H}}^{\mathrm{T}} \widetilde{Q}^{\mathrm{T}} \mathbb{M}_{\mathrm{V}} \widetilde{Q} \, \Pi_{\mathrm{H}} + \overline{\Pi}_{\mathrm{V}}^{\mathrm{T}} Q \, \mathbb{M}_{\mathrm{H}} Q^{\mathrm{T}} \Pi_{\mathrm{V}} + 4 \overline{\Pi}_{\mathrm{H}}^{\mathrm{T}} \mathbb{C}_{\mathrm{H}}^{\mathrm{T}} Q^{\mathrm{T}} (\Pi_{\mathrm{V}} \overline{\Pi}_{\mathrm{V}}^{\mathrm{T}} + \overline{\Pi}_{\mathrm{V}} \Pi_{\mathrm{V}}^{\mathrm{T}}) Q \, \mathbb{C}_{\mathrm{H}} \Pi_{\mathrm{H}} \Big]$$

$$V_{\mathrm{R}} = g^{a\overline{b}} D_a \mathcal{P}_3 D_{\overline{b}} \overline{\mathcal{P}}_3 + |\mathcal{P}_3|^2$$

$$= -\frac{1}{2} \mathrm{g}^2 \mathrm{e}^{4\varphi} (e_{\mathrm{R}\Lambda} - e_{\Lambda I} \xi^I + e_\Lambda{}^I \widetilde{\xi}_I) (\mathrm{Im}\mathcal{N})^{-1|\Lambda\Sigma} (e_{\mathrm{R}\Sigma} - e_{\Sigma I} \xi^I + e_\Sigma{}^I \widetilde{\xi}_I)$$

$\Pi_{\mathrm{V}} = \mathrm{e}^{\mathcal{K}_{\mathrm{V}}/2} (X^\Lambda, \mathcal{F}_\Lambda)^{\mathrm{T}}$
$t^a = X^a / X^0$
$a = 1, \ldots, n_{\mathrm{V}}$
SKG$_{\mathrm{V}}$ of vector-moduli

$$\mathcal{P}_+ \equiv \mathcal{P}_1 + \mathrm{i}\mathcal{P}_2 = 2\mathrm{e}^\varphi \, \Pi_{\mathrm{V}}^{\mathrm{T}} Q \, \mathbb{C}_{\mathrm{H}} \Pi_{\mathrm{H}}$$
$$\mathcal{P}_- \equiv \mathcal{P}_1 - \mathrm{i}\mathcal{P}_2 = 2\mathrm{e}^\varphi \, \Pi_{\mathrm{V}}^{\mathrm{T}} Q \, \mathbb{C}_{\mathrm{H}} \overline{\Pi}_{\mathrm{H}}$$
$$\mathcal{P}_3 = \mathrm{e}^{2\varphi} \, \Pi_{\mathrm{V}}^{\mathrm{T}} \mathbb{C}_{\mathrm{V}} (c_{\mathrm{R}} + \widetilde{Q} \xi)$$

$\Pi_{\mathrm{H}} = \mathrm{e}^{\mathcal{K}_{\mathrm{H}}/2} (Z^I, \mathcal{G}_I)^{\mathrm{T}}$
$z^i = Z^i / Z^0$
$i = 1, \ldots, n_{\mathrm{H}}$
SKG$_{\mathrm{H}}$ of hyper-moduli

$$\mathbb{C}_{\mathrm{V,H}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} ; \quad Q = \begin{pmatrix} e_\Lambda{}^I & e_{\Lambda I} \\ m^{\Lambda I} & m^\Lambda{}_I \end{pmatrix} , \quad \widetilde{Q} = \mathbb{C}_{\mathrm{H}}^{\mathrm{T}} Q \, \mathbb{C}_{\mathrm{V}} \quad c_{\mathrm{R}} = \begin{pmatrix} m_{\mathrm{R}}^\Lambda \\ e_{\mathrm{R}\Lambda} \end{pmatrix}$$

Cassani et.al., arXiv:0804.0595, arXiv:0911.2708 App.top

D. Cassani and A.K. Kashani-Poor, arXiv:0901.4251 App.top

$\mathcal{M}_6$	$\frac{G_2}{SU(3)} = S^6$	$\frac{Sp(2)}{S(U(2)\times U(1))} = \mathbb{C}P^3$	$\frac{SU(3)}{U(1)\times U(1)} = \mathbb{F}(1,2;3)$
$\mathcal{SM} = \text{SKG}_\text{V}$	$\frac{SU(1,1)}{U(1)} : \mathfrak{t}^3$	$\left(\frac{SU(1,1)}{U(1)}\right)^2 : \mathfrak{s}\mathfrak{t}^2$	$\left(\frac{SU(1,1)}{U(1)}\right)^3 : \mathfrak{stu}$
$\mathcal{HM} = \text{SQG}$	$\frac{SU(2,1)}{U(2)}$: UHM	$\frac{SU(2,1)}{U(2)}$: UHM	$\frac{SU(2,1)}{U(2)}$: UHM
$\text{SKG}_\text{H} \subset \mathcal{HM}$	—	—	—
matters	1 VM + 1 UHM	2 VM + 1 UHM	3 VM + 1 UHM

Each SKG_V has a cubic prepotential: $\mathcal{F} = \frac{1}{3!} d_{abc} \frac{X^a X^b X^c}{X^0}$

nilmanifolds and solvmanifolds: M. Graña, R. Minasian, M. Petrini and A. Tomasiello, hep-th/0609124
coset spaces with $SU(3)$- or $SU(2)$-structure: P. Koerber, D. Lüst and D. Tsimpis, arXiv:0804.0614
a pair of $SU(3)$-structures with $(m^{\Lambda I}, m^{\Lambda}{}_I)$: D. Gaiotto and A. Tomasiello, arXiv:0904.3959

10D type IIA on $\dfrac{G_2}{SU(3)}$ with fluxes

↓

4D $\mathcal{N}=2$ abelian gauged SUGRA with B-field ($\Lambda = 0, 1$ and $\boldsymbol{\xi^0} \equiv (\xi^0, \widetilde{\xi}_0)^{\mathrm{T}}$)

$$S = \int \Big[\frac{1}{2} R\,(*1) + \frac{1}{2} \mu_{\Lambda\Sigma}\, F^{\Lambda} \wedge * F^{\Sigma} + \frac{1}{2} \nu_{\Lambda\Sigma}\, F^{\Lambda} \wedge F^{\Sigma} - g_{\mathfrak{t}\bar{\mathfrak{t}}}\, \mathrm{d}\mathfrak{t} \wedge * \mathrm{d}\bar{\mathfrak{t}}$$
$$- \mathrm{d}\varphi \wedge * \mathrm{d}\varphi - \frac{\mathrm{e}^{-4\varphi}}{4} \mathrm{d}B \wedge * \mathrm{d}B - \frac{\mathrm{e}^{2\varphi}}{2} \Big(D\xi^0 \wedge * D\xi^0 + D\widetilde{\xi}_0 \wedge * D\widetilde{\xi}_0 \Big) + \mathrm{d}B \wedge \xi^0\, \mathrm{d}\widetilde{\xi}_0$$
$$+ \mathrm{d}B \wedge \big(e_{\mathrm{R}\Lambda} - e_{\Lambda 0}\, \xi^0 \big) A^{\Lambda} - \frac{1}{2} m_{\mathrm{R}}^{\Lambda}\, e_{\mathrm{R}\Lambda}\, B \wedge B - V\,(*1) \Big]$$

- $g_{\mu\nu}$, $\mathfrak{t}$, $B_{\mu\nu}$, φ; $({e_{\Lambda}}^0, e_{\Lambda 0})$: NS-NS sector
- A_{μ}^{Λ}, ξ^0, $\widetilde{\xi}_0$; $(m_{\mathrm{R}}^{\Lambda}, e_{\mathrm{R}\Lambda})$: R-R sector
- GM : $(g_{\mu\nu}, A_{\mu}^0)$, VM : $(A_{\mu}^a, \mathfrak{t})$, UHM → TM : $(\varphi, B_{\mu\nu}, \xi^0, \widetilde{\xi}_0)$
- $D\xi^0 = \mathrm{d}\xi^0 - {e_{\Lambda}}^0 A_1^{\Lambda}$, $D\widetilde{\xi}_0 = \mathrm{d}\widetilde{\xi}_0 - e_{\Lambda 0} A_1^{\Lambda}$
- $F_2^{\Sigma} = \mathrm{d}A_1^{\Sigma} + m_{\mathrm{R}}^{\Sigma} B_2$
- $V(\mathfrak{t}, \varphi, \xi^0) = V_{\mathrm{NS}}(\mathfrak{t}, \varphi) + V_{\mathrm{R}}(\mathfrak{t}, \varphi, \xi^0)$

Precise data on $\dfrac{G_2}{SU(3)}$:

$$e_{10} \neq 0,\; m_{\mathrm{R}}^0 \neq 0,\; e_{\mathrm{R}0} \neq 0$$
$${e_{\Lambda}}^0 = 0 = e_{00}$$
$$m_{\mathrm{R}}^1 = 0 = e_{\mathrm{R}1}$$

$\mu_{\Lambda\Sigma} \equiv \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}$, $\nu_{\Lambda\Sigma} \equiv \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}$

D. Cassani, arXiv:0804.0595 Main

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = \frac{1}{4}g_{\mu\nu}\, \mu_{\Lambda\Sigma}F^{\Lambda}_{\rho\sigma}F^{\Sigma\rho\sigma} - \mu_{\Lambda\Sigma}F^{\Lambda}_{\mu\rho}F^{\Sigma}_{\nu\sigma}\, g^{\rho\sigma} - g_{\mu\nu}\, g_{\mathrm{t}\bar{\mathrm{t}}}\partial_\rho \mathrm{t}\partial^\rho\bar{\mathrm{t}} + 2g_{\mathrm{t}\bar{\mathrm{t}}}\, \partial_\mu \mathrm{t}\partial_\nu\bar{\mathrm{t}}$$

$$-g_{\mu\nu}\, \partial_\rho\varphi\partial^\rho\varphi + 2\partial_\mu\varphi\partial_\nu\varphi - \frac{\mathrm{e}^{-4\varphi}}{24}g_{\mu\nu}\, H_{\rho\sigma\lambda}H^{\rho\sigma\lambda} + \frac{\mathrm{e}^{-4\varphi}}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} \qquad (\delta g_{\mu\nu})$$

$$-\frac{\mathrm{e}^{2\varphi}}{2}g_{\mu\nu}\Big(D_\rho\xi^0 D^\rho\xi^0 + D_\rho\widetilde{\xi}_0 D^\rho\widetilde{\xi}_0\Big) + \mathrm{e}^{2\varphi}\Big(D_\mu\xi^0 D_\nu\xi^0 + D_\mu\widetilde{\xi}_0 D_\nu\widetilde{\xi}_0\Big) - g_{\mu\nu}V\,,$$

$$0 = \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}\,\mu_{\Lambda\Sigma}F^{\Sigma\mu\sigma}\Big) - \frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}\partial_\mu\Big(\nu_{\Lambda\Sigma}F^{\Sigma}_{\nu\rho}\Big) + \frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}\partial_\mu B_{\nu\rho}(e_{\mathrm{R}\Lambda} - \xi^0 e_{\Lambda 0}) - \mathrm{e}^{2\varphi}Q_{\Lambda\mathbf{0}}D^\sigma\xi^{\mathbf{0}}\,, \qquad (\delta A^{\Lambda}_{\mu})$$

$$0 = \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}\, g_{\mathrm{t}\bar{\mathrm{t}}}\, g^{\mu\nu}\partial_\nu\bar{\mathrm{t}}\Big) + \frac{1}{4}\partial_{\mathrm{t}}(\mu_{\Lambda\Sigma})F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} - \frac{\epsilon^{\mu\nu\rho\sigma}}{8\sqrt{-g}}\partial_{\mathrm{t}}(\nu_{\Lambda\Sigma})F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma} - \partial_{\mathrm{t}}g_{\mathrm{t}\bar{\mathrm{t}}}\,\partial_\mu \mathrm{t}\partial^\mu\bar{\mathrm{t}} - \partial_{\mathrm{t}}V\,, \qquad (\delta \mathrm{t})$$

$$0 = \frac{2}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\varphi\Big) + \frac{\mathrm{e}^{4\varphi}}{6}H_{\mu\nu\rho}H^{\mu\nu\rho} - \mathrm{e}^{2\varphi}\Big(D_\mu\xi^0 D^\mu\xi^0 + D_\mu\widetilde{\xi}_0 D^\mu\widetilde{\xi}_0\Big) - \partial_\varphi V\,, \qquad (\delta\varphi)$$

$$0 = \frac{1}{\sqrt{-g}}\partial_\mu\Big(\mathrm{e}^{-4\varphi}\sqrt{-g}H^{\mu\rho\sigma}\Big) + \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\Big[D_\mu\xi^{\mathbf{0}}(\mathbb{C}_{\mathrm{H}})_{\mathbf{00}}D_\nu\xi^{\mathbf{0}} + (e_{\mathrm{R}\Lambda} - \xi^0 e_{\Lambda 0})F^{\Lambda}_{\mu\nu}\Big]$$

$$+2m^{\Lambda}_{\mathrm{R}}\mu_{\Lambda\Sigma}F^{\Sigma\rho\sigma} - \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}m^{\Lambda}_{\mathrm{R}}\nu_{\Lambda\Sigma}F^{\Sigma}_{\mu\nu}\,, \qquad (\delta B_{\mu\nu})$$

$$0 = -\frac{2}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}\,\mathrm{e}^{2\varphi}g^{\mu\nu}D_\nu\xi^{\mathbf{0}}\Big) + \frac{\partial V}{\partial\xi^{\mathbf{0}}} - \frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}\partial_\mu B_{\nu\rho}D_\sigma\xi^{\mathbf{0}}(\mathbb{C}_{\mathrm{H}})_{\mathbf{00}}\,. \qquad (\delta\xi^{\mathbf{0}})$$

Vacuum I : $\mathcal{N} = 1$

$$t_* = -\frac{\pm 1 + i\sqrt{15}}{2} \left[\frac{3}{5\,(e_{10})^2} \left| \frac{e_{R0}}{m_R^0} \right| \right]^{1/3} , \quad \xi_*^0 = -\frac{2}{5} \left[\frac{2\sqrt{3}\, m_R^0 (e_{R0})^2}{5\, e_{10}} \right]^{1/3} , \quad \exp(\varphi_*) = \frac{4}{3} \left[\frac{\sqrt{5}\, e_{10}}{\sqrt{3}\, m_R^0 (e_{R0})^2} \right]^{1/3}$$

$$V_* = -\frac{5\sqrt{5}}{2} \left[\frac{5\,(e_{10})^4}{2\sqrt{3}\, |m_R^0 (e_{R0})^5|} \right]^{1/3} \equiv \Lambda_{\rm c.c.}^{\rm I} < 0$$

Vacuum II : $\mathcal{N} = 0$

$$t_* = (\pm 1 - i\sqrt{3}) \left[\frac{3}{5\,(e_{10})^2} \left| \frac{e_{R0}}{m_R^0} \right| \right]^{1/3} , \quad \xi_*^0 = \left[\frac{9\, m_R^0 (e_{R0})^2}{25\, e_{10}} \right]^{1/3} , \quad \exp(\varphi_*) = \frac{2}{3} \left[\frac{25\, e_{10}}{\sqrt{3}\, m_R^0 (e_{R0})^2} \right]^{1/3}$$

$$V_* = -\frac{80}{27} \left[\frac{25\,(e_{10})^4}{\sqrt{3}\, |m_R^0 (e_{R0})^5|} \right]^{1/3} \equiv \Lambda_{\rm c.c.}^{\rm II} < 0$$

Vacuum III : $\mathcal{N} = 0$

$$t_* = -i \left[\frac{12}{\sqrt{5}\,(e_{10})^2} \left| \frac{e_{R0}}{m_R^0} \right| \right]^{1/3} , \quad \xi_*^0 = 0 , \quad \exp(\varphi_*) = \sqrt{5} \left[\frac{5\, e_{10}}{18\, m_R^0 (e_{R0})^2} \right]^{1/3}$$

$$V_* = -\frac{25\sqrt{5}}{6} \left[\frac{5\,(e_{10})^4}{18\, |m_R^0 (e_{R0})^5|} \right]^{1/3} \equiv \Lambda_{\rm c.c.}^{\rm III} < 0$$

Note: $m_R^0 > 0$; $\widetilde{\xi}_0$ is not fixed ; $\Lambda_{\rm c.c.}^{\rm II} < \Lambda_{\rm c.c.}^{\rm I} < \Lambda_{\rm c.c.}^{\rm III}$

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静的：

$$0 = \partial_t(\text{任意の場})$$

計量：

$$\mathrm{d}s^2 = -\mathrm{e}^{2A(r)}\mathrm{d}t^2 + \mathrm{e}^{-2A(r)}\mathrm{d}r^2 + \mathrm{e}^{2C(r)}r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\big)$$

電荷・磁荷：

$$p^\Lambda = \frac{1}{4\pi}\int F^\Lambda_{\mathit{2}}\,, \qquad q_\Lambda = \frac{1}{4\pi}\int \widetilde{F}_{\Lambda\mathit{2}} \quad \text{with } \widetilde{F}_{\mu\nu\Lambda} = \frac{\sqrt{-g}}{2}\epsilon_{\mu\nu\rho\sigma}\frac{\partial\mathscr{L}}{\partial F^\Lambda_{\mu\nu}}$$

$$F^\Lambda_{\theta\phi} \equiv f^\Lambda(\theta,\phi)\sin\theta\,, \qquad F^\Lambda_{tr} \equiv \frac{\mathrm{e}^{-2C}}{r^2}g^\Lambda(\theta,\phi)$$

Equation of motion for $A_\mu^\Lambda \quad (F_{\mu\nu}^\Lambda = 2\partial_{[\mu} A_{\nu]}^\Lambda + m_{\rm R}^\Lambda B_{\mu\nu})$:

$$0 = \frac{\epsilon^{\sigma\mu\nu\rho}}{2\sqrt{-g}} \partial_\mu \widetilde{F}_{\Lambda\nu\rho} - \frac{\epsilon^{\sigma\mu\nu\rho}}{2\sqrt{-g}} \partial_\mu B_{\nu\rho} (e_{\rm R\Lambda} - e_{\Lambda 0} \xi^0) - {\rm e}^{2\varphi} e_{\Lambda 0} D^\sigma \widetilde{\xi}_0$$

$$\begin{aligned}
H_{r\theta\phi} &= 0 \,, \quad H_{\theta\phi t} = 0 \,, \\
H_{\phi t r} &= \Big(\partial_\phi B_{tr} + \partial_r B_{\phi t} \Big) = \frac{1}{m_{\rm R}^\Sigma e_{\rm R\Sigma}} \frac{{\rm e}^{-2C}}{r^2} \partial_\phi \Big[e_{\rm R\Lambda} g^\Lambda(\theta,\phi) \Big] \,, \\
0 &= \partial_r \Big[\nu_{\Lambda\Sigma} f^\Sigma(\theta,\phi) - \mu_{\Lambda\Sigma}\, g^\Sigma(\theta,\phi) \Big] \,, \\
0 &= \partial_{\phi,\theta} \Big[(m_{\rm R}^\Lambda \mu_{\Lambda\Sigma}) f^\Sigma(\theta,\phi) + \big(m_{\rm R}^\Lambda \nu_{\Lambda\Sigma} - e_{\rm R\Sigma} \big) g^\Sigma(\theta,\phi) \Big] \,.
\end{aligned}$$

$$\rightarrow \boxed{\begin{aligned}
0 &= D_t \widetilde{\xi}_0 = -e_{\Lambda 0} A_t^\Lambda \quad \rightarrow \quad e_{\Lambda 0} F_{tr}^\Lambda = 0 \quad \rightarrow \quad e_{\Lambda 0} g^\Lambda(\theta,\phi) = 0 \,, \\
0 &= D_r \widetilde{\xi}_0 = \partial_r \widetilde{\xi}_0 - e_{\Lambda 0} A_r^\Lambda \,, \\
0 &= \frac{{\rm e}^{-2C}}{r^2} \partial_\phi \Big[\mu_{\Lambda\Sigma} f^\Sigma(\theta,\phi) + \Big(\nu_{\Lambda\Sigma} - \frac{e_{\rm R\Lambda} - e_{\Lambda 0}\xi^0}{m_{\rm R}^\Gamma e_{\rm R\Gamma}} e_{\rm R\Sigma} \Big) g^\Sigma(\theta,\phi) \Big] + e_{\Lambda 0}\, {\rm e}^{2\varphi} \sin\theta\, D_\theta \widetilde{\xi}_0 \,, \\
0 &= \frac{{\rm e}^{-2C}}{r^2} \partial_\theta \Big[\mu_{\Lambda\Sigma} f^\Sigma(\theta,\phi) + \Big(\nu_{\Lambda\Sigma} - \frac{e_{\rm R\Lambda} - e_{\Lambda 0}\xi^0}{m_{\rm R}^\Gamma e_{\rm R\Gamma}} e_{\rm R\Sigma} \Big) g^\Sigma(\theta,\phi) \Big] - e_{\Lambda 0} \frac{{\rm e}^{2\varphi}}{\sin\theta} D_\phi \widetilde{\xi}_0 \,.
\end{aligned}}$$

Equation of motion for $B_{\mu\nu}$:

$$0 = \frac{1}{\sqrt{-g}}\partial_\mu\Big(\mathrm{e}^{-4\varphi}\sqrt{-g}H^{\mu\rho\sigma}\Big) + \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\Big[D_\mu\xi^0 D_\nu\widetilde{\xi}_0 - D_\mu\widetilde{\xi}_0 D_\nu\xi^0 + (e_{\mathrm{R}\Lambda} - e_{\Lambda 0}\xi^0)F^\Lambda_{\mu\nu}\Big]$$
$$+2m^\Lambda_{\mathrm{R}}\mu_{\Lambda\Sigma}F^{\Sigma\rho\sigma} - \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}m^\Lambda_{\mathrm{R}}\nu_{\Lambda\Sigma}F^\Sigma_{\mu\nu}$$

$\rightarrow$

$$\begin{aligned}
0 &= -\frac{1}{m^\Sigma_{\mathrm{R}}e_{\mathrm{R}\Sigma}}\frac{\mathrm{e}^{-2C}}{r^2}\partial_\theta\Big[\mathrm{e}^{-4\varphi}\sin\theta\,\partial_\theta\Big(e_{\mathrm{R}\Lambda}g^\Lambda(\theta,\phi)\Big)\Big] - \frac{1}{m^\Sigma_{\mathrm{R}}e_{\mathrm{R}\Sigma}}\frac{\mathrm{e}^{-2C}}{r^2\sin\theta}\partial_\phi\Big[\mathrm{e}^{-4\varphi}\partial_\phi\Big(e_{\mathrm{R}\Lambda}g^\Lambda(\theta,\phi)\Big)\Big] \\
&\quad +2\Big[\Big(m^\Lambda_{\mathrm{R}}\nu_{\Lambda\Sigma} - (e_{\mathrm{R}\Sigma} - e_{\Sigma 0}\,\xi^0)\Big)f^\Sigma(\theta,\phi) - (m^\Lambda_{\mathrm{R}}\mu_{\Lambda\Sigma})g^\Sigma(\theta,\phi)\Big]\sin\theta \\
&\quad -2\Big(D_\theta\xi^0 D_\phi\widetilde{\xi}_0 - D_\theta\widetilde{\xi}_0 D_\phi\xi^0\Big)\,, \\
0 &= \frac{\sin\theta}{m^\Sigma_{\mathrm{R}}e_{\mathrm{R}\Sigma}}\partial_\theta\Big(e_{\mathrm{R}\Lambda}g^\Lambda(\theta,\phi)\Big)\partial_r\Big(\frac{\mathrm{e}^{-4\varphi-2C}}{r^2}\Big) + 2D_r\xi^0 D_\phi\widetilde{\xi}_0\,, \\
0 &= \frac{1}{m^\Sigma_{\mathrm{R}}e_{\mathrm{R}\Sigma}}\frac{1}{\sin\theta}\partial_\phi\Big(e_{\mathrm{R}\Lambda}g^\Lambda(\theta,\phi)\Big)\partial_r\Big(\frac{\mathrm{e}^{-4\varphi-2C}}{r^2}\Big) - 2D_r\xi^0 D_\theta\widetilde{\xi}_0\,, \\
0 &= \frac{2\mathrm{e}^{-4C}}{r^4\sin\theta}\Big[(m^\Lambda_{\mathrm{R}}\mu_{\Lambda\Sigma})f^\Sigma(\theta,\phi) + (m^\Lambda_{\mathrm{R}}\nu_{\Lambda\Sigma} - e_{\mathrm{R}\Sigma})g^\Sigma(\theta,\phi)\Big]
\end{aligned}$$

Equation of motion for $g_{\mu\nu}$:

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} \ &= \ \frac{1}{4} g_{\mu\nu}\, \mu_{\Lambda\Sigma} F^{\Lambda}_{\rho\sigma} F^{\Sigma\rho\sigma} - \mu_{\Lambda\Sigma} F^{\Lambda}_{\mu\rho} F^{\Sigma}_{\nu\sigma}\, g^{\rho\sigma} - g_{\mu\nu}\, g_{\mathfrak{t}\bar{\mathfrak{t}}} \partial_{\rho} \mathfrak{t} \partial^{\rho} \bar{\mathfrak{t}} + 2 g_{\mathfrak{t}\bar{\mathfrak{t}}}\, \partial_{\mu} \mathfrak{t} \partial_{\nu} \bar{\mathfrak{t}} \\
&\quad - g_{\mu\nu}\, \partial_{\rho} \varphi \partial^{\rho} \varphi + 2 \partial_{\mu} \varphi \partial_{\nu} \varphi - \frac{\mathrm{e}^{-4\varphi}}{24} g_{\mu\nu}\, H_{\rho\sigma\lambda} H^{\rho\sigma\lambda} + \frac{\mathrm{e}^{-4\varphi}}{4} H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma} \\
&\quad - \frac{\mathrm{e}^{2\varphi}}{2} g_{\mu\nu} \Big(D_{\rho} \xi^0 D^{\rho} \xi^0 + D_{\rho} \widetilde{\xi}_0 D^{\rho} \widetilde{\xi}_0 \Big) + \mathrm{e}^{2\varphi} \Big(D_{\mu} \xi^0 D_{\nu} \xi^0 + D_{\mu} \widetilde{\xi}_0 D_{\nu} \widetilde{\xi}_0 \Big) \\
&\quad - g_{\mu\nu} V
\end{aligned}
$$

From now on we focus on

$$
\mathrm{e}^{2A(r)} \ = \ 1 - \frac{2\eta}{r} + \frac{\mathcal{Z}^2}{r^2} + \frac{r^2}{\ell^2} \, , \qquad \mathrm{e}^{2C(r)} \ = \ 1
$$

$$g^{tt}E_{tt} - g^{rr}E_{rr} \;=\; 0 \;=\; -2\mathrm{e}^{2A(r)}\Big[g_{\mathrm{t}\bar{\mathrm{t}}}|\partial_r \mathrm{t}|^2 + (\partial_r\varphi)^2 + \frac{\mathrm{e}^{2\varphi}}{2}(D_r\xi^0)^2\Big]$$

$$g^{rr}E_{rr} + g^{\theta\theta}E_{\theta\theta} \;=\; \frac{6}{\ell^2} \;=\; -\frac{2}{r^2\sin^2\theta}\Big[g_{\mathrm{t}\bar{\mathrm{t}}}|\partial_\phi \mathrm{t}|^2 + (\partial_\phi\varphi)^2 + \frac{\mathrm{e}^{2\varphi}}{2}(D_\phi\xi^0)^2 + \frac{\mathrm{e}^{2\varphi}}{2}(D_\phi\widetilde{\xi}_0)^2\Big] - 2V$$

$$g^{rr}E_{rr} - g^{\theta\theta}E_{\theta\theta} \;=\; -\frac{2\mathcal{Z}^2}{r^4} \;=\; \frac{1}{r^4}\mu_{\Lambda\Sigma}\Big[f^\Lambda(\theta,\phi)f^\Sigma(\theta,\phi) + g^\Lambda g^\Sigma\Big]$$
$$-\frac{2}{r^2}\Big[g_{\mathrm{t}\bar{\mathrm{t}}}|\partial_\theta \mathrm{t}|^2 + (\partial_\theta\varphi)^2 + \frac{\mathrm{e}^{2\varphi}}{2}(D_\theta\xi^0)^2 + \frac{\mathrm{e}^{2\varphi}}{2}(D_\theta\widetilde{\xi}_0)^2\Big]$$

$$g^{\theta\theta}E_{\theta\theta} - g^{\phi\phi}E_{\phi\phi} \;=\; 0 \;=\; \frac{1}{r^2}\Big[g_{\mathrm{t}\bar{\mathrm{t}}}|\partial_\theta \mathrm{t}|^2 + (\partial_\theta\varphi)^2 + \frac{\mathrm{e}^{2\varphi}}{2}(D_\theta\xi^0)^2 + \frac{\mathrm{e}^{2\varphi}}{2}(D_\theta\widetilde{\xi}_0)^2\Big]$$
$$-\frac{1}{r^2\sin^2\theta}\Big[g_{\mathrm{t}\bar{\mathrm{t}}}|\partial_\phi \mathrm{t}|^2 + (\partial_\phi\varphi)^2 + \frac{\mathrm{e}^{2\varphi}}{2}(D_\phi\xi^0)^2 + \frac{\mathrm{e}^{2\varphi}}{2}(D_\phi\widetilde{\xi}_0)^2\Big]$$

$\rightarrow$

$$0 \;=\; \partial_r\mathrm{t} \;=\; \partial_\theta\mathrm{t} \;=\; \partial_\phi\mathrm{t}\,, \qquad 0 \;=\; \partial_r\varphi \;=\; \partial_\theta\varphi \;=\; \partial_\phi\varphi$$
$$0 \;=\; D_r\xi^0 \;=\; D_\theta\xi^0 \;=\; D_\phi\xi^0\,, \qquad 0 \;=\; D_\theta\widetilde{\xi}_0 \;=\; D_\phi\widetilde{\xi}_0$$
$$0 \;=\; H_{tr\theta} \;=\; H_{tr\phi}$$
$$\text{and} \quad f^\Lambda \;=\; p^\Lambda\,, \qquad g^\Lambda \;=\; -(\mu^{-1})^{\Lambda\Sigma}\big(q_\Sigma - \nu_{\Sigma\Gamma}\,p^\Gamma\big)$$

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