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A Technical Note on 4D $\mathcal{N} = 2$ Gauged Supergravity

— based on hep-th/9605032 —

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Abstract

In this note we study four-dimensional $\mathcal{N} = 2$ (electrically) gauged supergravity based on hep-th/9605032 [1]. We explicitly exhibit the gauge coupling constant g in the Lagrangian. If possible, we try to include the embedding tensor formalism to extend the system which has the symplectic covariance under the electro-magnetic duality in the presence of gauge coupling constant, whilst it has not been established yet (May 7, 2011).

Contents

1 4D $\mathcal{N}=2$ gauged supergravity in hep-th/9605032 [1]

1.1 Basic convention

Definition of the differential forms in a curved spacetime with the normalization $\varepsilon_{0123} \equiv +1$:[1]

$$\begin{aligned}
A^\Lambda &\equiv A^\Lambda_\mu \, \mathrm{d}x^\mu \,, && (1.1a)\\
F^\Lambda &\equiv \mathrm{d}A^\Lambda \equiv \mathcal{F}^\Lambda_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,, \qquad \mathcal{F}^\Lambda_{\mu\nu} \equiv \frac{1}{2}\big(\partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu\big) \,, && (1.1b)\\
*F^\Lambda &\equiv \widetilde{\mathcal{F}}^\Lambda_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,, \qquad \widetilde{\mathcal{F}}^\Lambda_{\mu\nu} \equiv \frac{\sqrt{-g}}{2!} \varepsilon_{\mu\nu\rho\sigma} \, \mathcal{F}^{\Lambda|\rho\sigma} \,, && (1.1c)\\
\mathrm{d}^4 x &\equiv -\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = -\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \,, && (1.1d)\\
\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\Big(\sqrt{-g}\,\mathrm{d}^4 x\Big) &\equiv \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \,, && (1.1e)\\
F^\Lambda \wedge F^\Sigma &= \sqrt{-g}\,\varepsilon^{\mu\nu\rho\sigma} \, \mathcal{F}^\Lambda_{\mu\nu}\mathcal{F}^\Sigma_{\rho\sigma}\big(\sqrt{-g}\,\mathrm{d}^4 x\big) \,, \qquad F^\Lambda \wedge *F^\Sigma = -2\,\mathcal{F}^\Lambda_{\mu\nu}\mathcal{F}^{\Sigma|\mu\nu}\big(\sqrt{-g}\,\mathrm{d}^4 x\big) \,. && (1.1f)
\end{aligned}$$

Definition of the Riemann curvature:[2]

$$\begin{aligned}
\Gamma^\mu{}_\nu &\equiv \Gamma^\mu{}_{\rho\nu} \, \mathrm{d}x^\rho \,, && (1.2a)\\
R^\mu{}_\nu &= \mathrm{d}\Gamma^\mu{}_\nu + \Gamma^\mu{}_\lambda \wedge \Gamma^\lambda{}_\nu = \Big(\partial_\rho \Gamma^\mu{}_{\sigma\nu} + \Gamma^\mu{}_{\rho\lambda}\Gamma^\lambda{}_{\sigma\nu}\Big)\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \equiv -\frac{1}{2}R^\mu{}_{\nu\rho\sigma} \, \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \,, && (1.2b)\\
R^\mu{}_{\nu\rho\sigma} &= -\Big(\partial_\rho \Gamma^\mu{}_{\sigma\nu} - \partial_\sigma \Gamma^\mu{}_{\rho\nu} + \Gamma^\mu{}_{\rho\lambda}\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\mu{}_{\sigma\lambda}\Gamma^\lambda{}_{\rho\nu}\Big) \,. && (1.2c)
\end{aligned}$$

There is no explicit expression of the affine connection one-form, while the one on the Kähler geometry is given as $\Gamma^i{}_j = \Gamma^i_{kj}\,\mathrm{d}z^k$ as in (C.56) in [1], where the Kähler geometry is torsion free: $\Gamma^i_{kj} = \Gamma^i_{jk}$. Here we also apply this expansion to the one of the affine connection one-form as above.

Decomposition of a generic tensor of degree two $\mathcal{T}_{\mu\nu}$ into a self-dual$(-)$ and a anti-self-dual$(+)$ parts in a curved spacetime in terms of $\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}$:[3]

$$\mathcal{T}^{(\mp)}_{\mu\nu} \equiv \frac{1}{2}\big(\mathcal{T}_{\mu\nu} \mp \frac{\mathrm{i}}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}\mathcal{T}^{\rho\sigma}\big) \quad \to \quad -\frac{\mathrm{i}}{2}\sqrt{-g}\,\varepsilon_{\mu\nu}{}^{\rho\sigma}\mathcal{T}^{(\mp)}_{\rho\sigma} = \pm\mathcal{T}^{(\mp)}_{\mu\nu} \,. \qquad (1.3)$$

This indicates that the "self-dual" tensor in [1] has an eigenvalue $+1$ when we act $-\frac{\mathrm{i}}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}$ on it, rather than $+\frac{\mathrm{i}}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}$. (The name "(anti-)self-duality"[4] is different from the other in my exprériences.)

Here let us introduce the Clifford algebra given by Dirac gamma matrices:

$$\begin{aligned}
\{\gamma_a, \gamma_b\} &= 2\eta_{ab} \,, \qquad [\gamma_a, \gamma_b] = 2\gamma_{ab} \,, \qquad \eta_{ab} \equiv (+1,-1,-1,-1) \,, && (1.4a)\\
\gamma_5 &\equiv -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 \,, && (1.4b)\\
(\gamma_0)^\dagger &= \gamma_0 \,, \qquad \gamma_0(\gamma_i)^\dagger\gamma_0 = \gamma_i \quad (i=1,2,3) \,, \qquad (\gamma_5)^\dagger = \gamma_5 \,, && (1.4c)\\
\epsilon_{abcd}\gamma^{cd} &= 2\mathrm{i}\,\gamma_{ab}\gamma_5 \,. && (1.4d)
\end{aligned}$$

Chirality of fermions:

$$\gamma_5 \begin{pmatrix} \lambda^{iA} \\ \zeta_\alpha \\ \psi_{A\mu} \end{pmatrix} = \begin{pmatrix} \lambda^{iA} \\ \zeta_\alpha \\ \psi_{A\mu} \end{pmatrix} , \qquad \gamma_5 \begin{pmatrix} \lambda^{\bar{i}}_A \\ \zeta^\alpha \\ \psi^A_\mu \end{pmatrix} = -\begin{pmatrix} \lambda^{\bar{i}}_A \\ \zeta^\alpha \\ \psi^A_\mu \end{pmatrix} . \qquad (1.5)$$

Dirac conjugate and Majorana condition on Dirac fermions:

$$\overline{\chi} \equiv \chi^\dagger\gamma_0 = \chi^{\mathrm{T}} C \,, \qquad (1.6)$$

[1] $\mathcal{F}_{\mu\nu} = \frac{1}{2!}F_{\mu\nu}$ if we define $F = \frac{1}{2!}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$.
[2] The overall sign of the curvature is opposite to the usual one.
[3] This is the generalization of the (anti-)self-dual tensors in the flat spacetime in [1].
[4] See below eq.(8.22) in [1].

with charge conjugation matrix:

$$C^2 \;=\; -1\,, \qquad C^{\rm T} \;=\; C^{-1} \;=\; -C\,, \qquad (C\gamma^a)^{\rm T} \;=\; C\gamma^a\,, \qquad (C\gamma^{ab})^{\rm T} \;=\; C\gamma^{ab}\,. \tag{1.7}$$

Fierz identities with a lower or upper dot implying right or left chirality respectively:

$$\chi_{\bullet}\overline{\xi}_{\bullet} \;=\; -\frac{1}{2}\overline{\xi}_{\bullet}\chi_{\bullet} + \frac{1}{8}\gamma_{ab}\,\overline{\xi}_{\bullet}\gamma^{ab}\chi_{\bullet}\,, \tag{1.8a}$$

$$\chi_{\bullet}\overline{\xi}^{\bullet} \;=\; -\frac{1}{2}\gamma_a\,\overline{\xi}^{\bullet}\gamma^a\chi_{\bullet}\,, \tag{1.8b}$$

$$\psi_A\overline{\psi}_B \;=\; \frac{1}{2}\overline{\psi}_B\psi_A - \frac{1}{8}\gamma_{ab}\,\overline{\psi}_B\gamma^{ab}\psi_A\,, \tag{1.8c}$$

$$\psi_A\overline{\psi}^B \;=\; \frac{1}{2}\gamma_a\,\overline{\psi}^B\gamma^a\psi_A\,. \tag{1.8d}$$

Hermiticity of bilinear forms of fermions:

$$(\overline{\chi}_{\bullet}\xi_{\bullet})^{\dagger} \;=\; \overline{\xi}^{\bullet}\chi^{\bullet} \;=\; \overline{\chi}^{\bullet}\xi^{\bullet}\,, \tag{1.9a}$$

$$(\overline{\chi}_{\bullet}\gamma^a\xi^{\bullet})^{\dagger} \;=\; \overline{\xi}_{\bullet}\gamma^a\chi^{\bullet} \;=\; -\overline{\chi}^{\bullet}\gamma^a\xi_{\bullet}\,, \tag{1.9b}$$

$$(\overline{\chi}_{\bullet}\gamma^{ab}\xi_{\bullet})^{\dagger} \;=\; -\overline{\xi}^{\bullet}\gamma^{ab}\chi^{\bullet} \;=\; \overline{\chi}^{\bullet}\gamma^{ab}\xi^{\bullet}\,, \tag{1.9c}$$

$$(\overline{\psi}_A\psi_B)^{\dagger} \;=\; -\overline{\psi}^B\psi^A \;=\; \overline{\psi}^A\psi^B\,, \tag{1.9d}$$

$$(\overline{\psi}^A\gamma^a\psi_B)^{\dagger} \;=\; -\overline{\psi}^B\gamma^a\psi_A \;=\; -\overline{\psi}_A\gamma^a\psi^B\,, \tag{1.9e}$$

$$(\overline{\psi}^A\gamma^{ab}\psi^B)^{\dagger} \;=\; \overline{\psi}_B\gamma^{ab}\psi_A \;=\; \overline{\psi}_A\gamma^{ab}\psi_B\,. \tag{1.9f}$$

$SU(2)$ and $Sp(2m)$ metrics:

$$\epsilon^{AB}\epsilon_{BC} \;=\; -\delta^A_C\,, \qquad \epsilon^{AB} \;=\; -\epsilon^{BA}\,, \tag{1.10a}$$

$$\mathbb{C}^{\alpha\beta}\mathbb{C}_{\beta\gamma} \;=\; -\gamma^{\alpha}_{\gamma}\,, \qquad \mathbb{C}^{\alpha\beta} \;=\; -\mathbb{C}^{\beta\alpha}\,, \tag{1.10b}$$

with any $SU(2)$ vectors P_A and any $Sp(2m)$ vectors P_{α}:

$$\epsilon_{AB}P^B \;=\; P_A\,, \qquad \epsilon^{AB}P_B \;=\; -P^A\,, \tag{1.11a}$$

$$\mathbb{C}_{\alpha\beta}P^{\beta} \;=\; P_{\alpha}\,, \qquad \mathbb{C}^{\alpha\beta}P_{\beta} \;=\; -P^{\alpha}\,. \tag{1.11b}$$

1.2 Mass dimensions and gravitational coupling constant

The complete Lagrangian in [1] carries many contracted terms by the following metrics and their inverse: $g_{\mu\nu}$, $g_{i\overline{j}}$, h_{uv}, ϵ_{AB}, $\mathbb{C}_{\alpha\beta}$, and the period matrix $\mathcal{N}_{\Lambda\Sigma}$. For simplicity, we want to define that they are dimensionless

$$[g_{\mu\nu}] \;=\; 0 \;=\; [g_{i\overline{j}}] \;=\; [h_{uv}] \;=\; [\epsilon_{AB}] \;=\; [\mathbb{C}_{\alpha\beta}] \;=\; [\mathcal{N}_{\Lambda\Sigma}]\,. \tag{1.12a}$$

Under this assumption, we can fix the mass dimensions as follows:

$$[z^i] \;=\; [q^u] \;=\; [A^{\Lambda}_{\mu}] \;=\; 1\,, \qquad [\psi_{A\mu}] \;=\; [\lambda^{iA}] \;=\; [\zeta_{\alpha}] \;=\; \frac{3}{2}\,, \qquad [\epsilon_A] \;=\; -\frac{1}{2}\,, \tag{1.12b}$$

$$[\mathcal{K}] \;=\; 2\,, \tag{1.12c}$$

In four-dimensional spacetime, the gauge coupling constant g is dimensionless: $[g] = 0$. On the other hand, the gravitational coupling constant κ has mass dimension $[\kappa] = -1$ with normalization (see (1.167) in [17])

$$\mathscr{L}_{\sf EH} \;\equiv\; -\frac{1}{2\kappa^2}R([1])\,. \tag{1.13}$$

Let us introduce the dimensionful coupling constant into the action in [1] with fixing mass dimensions all the bosonic fields and the fermionic fields, including supersymmetry parameters ϵ_A. However, it is a hard task to fix normalization factors in front of the gravitational coupling constant in [1] via the equations of motion, the Bianchi identity, and the invariance of the system under the local supersymmetry. But it is so hard to fix all the coefficients via equations. So for a while I just introduce the symbol "$\langle\kappa\rangle$" which represents the existence of the gravitational coupling constant up to coefficients. We refer to [17] in order to consider the insertion rule of $\langle\kappa\rangle$ into the Lagrangian in [1], i.e., we set "$\langle\kappa\rangle = 1$" if we want to go back from the later description (3.1) to [1].

If we do not mind recovering the dimensionful coupling constant in an explicit way, we do not have to be anxious about such the normalization fixing. For a while (since Jan 16, 2011) we do not touch such insertions into the Lagrangian.

1.3 Special Kähler geometry

This part in [1] is completely consistent with the ones in [18].

$$\mathcal{K} \;=\; -\log\big(\mathrm{i}\langle\,\Omega\,|\,\overline{\Omega}\,\rangle\big) \;=\; -\log\big\{\mathrm{i}\big(\overline{X}^{\Lambda}F_{\Lambda} - X^{\Lambda}\overline{F}_{\Lambda}\big)\big\}\,, \qquad z^i \;\equiv\; \frac{X^i}{X^0}\,, \tag{1.14a}$$

$$K \;=\; \frac{\mathrm{i}}{2\pi}g_{i\overline{j}}\,\mathrm{d}z^i\wedge\mathrm{d}\overline{z}^{\overline{j}} \;=\; \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}\log\big(\mathrm{i}\langle\,\Omega\,|\,\overline{\Omega}\,\rangle\big)\,, \qquad g_{i\overline{j}} \;=\; \partial_i\partial_{\overline{j}}\mathcal{K}(z,\overline{z})\,. \tag{1.14b}$$

Let now $\Phi(z,\overline{z})$ be a section of $\mathcal{U}^p$. The Kähler covariant derivative with the Kähler connection $\mathcal{Q}$ is defined as

$$\mathcal{Q} \;\equiv\; -\frac{\mathrm{i}}{2}\big(\partial_i\mathcal{K}\,\mathrm{d}z^i - \partial_{\overline{i}}\mathcal{K}\,\mathrm{d}\overline{z}^{\overline{i}}\big)\,, \tag{1.15a}$$

$$\nabla\Phi(z,\overline{z}) \;=\; \big(\mathrm{d}+\mathrm{i}p\mathcal{Q}\big)\Phi\,, \qquad \text{i.e.,}\quad \nabla_i\Phi \;=\; \Big(\partial_i + \frac{p}{2}\partial_i\mathcal{K}\Big)\Phi\,, \qquad \nabla_{\overline{i}}\Phi \;=\; \Big(\partial_{\overline{i}} - \frac{p}{2}\partial_{\overline{i}}\mathcal{K}\Big)\Phi\,. \tag{1.15b}$$

If we introduce $\widetilde{\Phi} = \mathrm{e}^{-p\mathcal{K}/2}\Phi$, its Kähler covariant derivative becomes

$$\nabla_i\widetilde{\Phi} \;=\; \big(\partial_i + p\partial_i\mathcal{K}\big)\widetilde{\Phi}\,, \qquad \nabla_{\overline{i}}\widetilde{\Phi} \;=\; \partial_{\overline{i}}\widetilde{\Phi}\,. \tag{1.15c}$$

Let us define a holomorphic section on a special Kähler geometry, which is nothing but the scalar field space of the vector multiplets (for detail computations, see chapter 3 in [18]):

$$V \;\equiv\; \left(\begin{array}{c} L^{\Lambda} \\ M_{\Sigma}\end{array}\right) \;\equiv\; \mathrm{e}^{\mathcal{K}/2}\Omega \;=\; \mathrm{e}^{\mathcal{K}/2}\left(\begin{array}{c} X^{\Lambda} \\ F_{\Sigma}\end{array}\right)\,, \qquad 1 \;=\; \mathrm{i}\langle\,V\,|\,\overline{V}\,\rangle \;=\; \mathrm{i}\big(\overline{L}^{\Lambda}M_{\Lambda} - L^{\Lambda}\overline{M}_{\Lambda}\big)\,, \tag{1.16a}$$

$$U_i \;\equiv\; \nabla_iV \;=\; \Big(\partial_i + \frac{1}{2}\partial_i\mathcal{K}\Big)V \;\equiv\; \left(\begin{array}{c} f_i^{\Lambda} \\ h_{\Sigma i}\end{array}\right)\,, \qquad \nabla_{\overline{i}}V \;=\; \Big(\partial_{\overline{i}} - \frac{1}{2}\partial_{\overline{i}}\mathcal{K}\Big)V \;=\; 0\,, \tag{1.16b}$$

$$\nabla_iU_j \;=\; \mathrm{i}C_{ijk}\,g^{k\overline{\ell}}\overline{U}_{\overline{\ell}}\,, \qquad \nabla_{\overline{i}}U_j \;=\; g_{i\overline{j}}V\,, \tag{1.16c}$$

$$\partial_{\overline{\ell}}C_{ijk} \;=\; \partial_{\ell}C_{\overline{ijk}} \;=\; 0\,, \qquad \nabla_{[\ell}C_{i]jk} \;=\; \nabla_{[\overline{\ell}}C_{\overline{i}]\overline{jk}} \;=\; 0\,, \tag{1.16d}$$

$$\nabla_mC_{ijk} \;=\; \Big(\partial_mC_{ijk} + (\partial_m\mathcal{K})C_{ijk}\Big) - \Gamma^{\ell}_{mi}C_{\ell jk} - \Gamma^{\ell}_{mj}C_{i\ell k} - \Gamma^{\ell}_{mk}C_{ij\ell}\,. \tag{1.16e}$$

Notice that if F_{Λ} is given by a function $F(X)$ at least locally, we can further describe as follows:

$$F \;\equiv\; \frac{1}{2}X^{\Lambda}F_{\Lambda}\,, \qquad F_{\Lambda} \;=\; \frac{\partial F}{\partial X^{\Lambda}} \;=\; X^{\Sigma}F_{\Sigma\Lambda}\,, \qquad C_{ijk} \;\equiv\; \mathrm{e}^{\mathcal{K}}(\partial_iX^{\Lambda})(\partial_jX^{\Sigma})(\partial_kX^{\Gamma})\frac{\partial^3F}{\partial X^{\Lambda}\partial X^{\Sigma}\partial X^{\Gamma}}\,. \tag{1.17}$$

There exist many useful equations on the special Kähler geometry:

$$\overline{\mathcal{N}}_{\Lambda\Sigma} \;\equiv\; \left(\begin{array}{c} \nabla_iF_{\Lambda} \\ \overline{F}_{\Lambda}\end{array}\right)\cdot\left(\begin{array}{c} \nabla_iX^{\Sigma} \\ \overline{X}^{\Sigma}\end{array}\right)^{-1} \quad \text{or} \quad \mathcal{N}_{\Lambda\Sigma} \;=\; \overline{F}_{\Lambda\Sigma} + 2\mathrm{i}\frac{(\mathrm{Im}F)_{\Lambda\Gamma}X^{\Gamma}(\mathrm{Im}F)_{\Sigma\Delta}X^{\Delta}}{X^{\Pi}(\mathrm{Im}F)_{\Pi\Xi}X^{\Xi}}\,, \tag{1.18a}$$

$$M_{\Lambda} \;=\; \mathcal{N}_{\Lambda\Sigma}L^{\Sigma}\,, \qquad h_{\Lambda i} \;=\; \overline{\mathcal{N}}_{\Lambda\Sigma}f_i^{\Sigma}\,, \qquad \langle V,U_i\rangle \;=\; \langle V,\overline{U}_{\overline{i}}\rangle \;=\; 0\,, \tag{1.18b}$$

$$(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\,L^{\Lambda}\overline{L}^{\Sigma} \;=\; -\frac{1}{2}\,, \qquad U^{\Lambda\Sigma} \;\equiv\; g^{i\overline{j}}f_i^{\Lambda}\overline{f}_{\overline{j}}^{\Sigma} \;=\; -\frac{1}{2}[(\mathrm{Im}\mathcal{N})^{-1}]^{\Lambda\Sigma} - \overline{L}^{\Lambda}L^{\Sigma}\,, \tag{1.18c}$$

$$g_{i\overline{j}} \;=\; -\mathrm{i}\langle\,U_i\,|\,\overline{U}_{\overline{j}}\,\rangle \;=\; -2(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\,f_i^{\Lambda}\overline{f}_{\overline{j}}^{\Sigma}\,, \tag{1.18d}$$

$$C_{ijk} \;=\; \langle\,\nabla_iU_j\,|\,U_k\,\rangle \;=\; f_i^{\Lambda}\partial_j\overline{\mathcal{N}}_{\Lambda\Sigma}\,f_k^{\Sigma} \;=\; (\mathcal{N}-\overline{\mathcal{N}})_{\Lambda\Sigma}\,f_i^{\Lambda}\partial_jf_k^{\Sigma}\,. \tag{1.18e}$$

The momentum map $\mathcal{P}_{\Lambda}^0$ under the gauging with variation $z^i \to z^i + \epsilon^{\Lambda}k_{\Lambda}^i(z)$ is defined as

$$\mathrm{i}\mathcal{P}_{\Lambda}^0 \;\equiv\; k_{\Lambda}^i\partial_i\mathcal{K} \;=\; -k_{\Lambda}^{\overline{i}}\partial_{\overline{i}}\mathcal{K}\,, \tag{1.19}$$

where $k_{\Lambda}^i(z)$ is a Killing vector of the gauged (sub)group of the isometry group on the special Kähler geometry. In terms of the holomorphic sections of the special Kähler geometry, we can describe it as

$$\mathcal{P}_{\Lambda}^0 \;=\; \mathrm{e}^{\mathcal{K}}\Big(F_{\Delta}f^{\Delta}{}_{\Lambda\Sigma}\overline{X}^{\Sigma} + \overline{F}_{\Delta}f^{\Delta}{}_{\Lambda\Sigma}X^{\Sigma}\Big)\,, \tag{1.20}$$

which implies $\mathcal{P}_{\Lambda}^0$ vanishes if the gauge symmetry from the isometry group is abelian.

1.4 Quaternionic geometry

Hypergeometry (both quaternionic and hyper-Kähler) is a $4m$-dimensional real manifold endowed with the metric

$$\mathrm{d}s^2 = h_{uv}(q)\,\mathrm{d}q^u \otimes \mathrm{d}q^v\,, \qquad h_{uv} = \epsilon_{AB}\,\mathbb{C}_{\alpha\beta}\,\mathcal{U}_u^{\alpha A}\,\mathcal{U}_v^{\beta B}\,, \tag{1.21a}$$

$$u,v = 1,\ldots,4m\,, \qquad A,B = 1,2\,, \qquad \alpha,\beta = 1,2,\ldots,2m\,, \tag{1.21b}$$

where $\mathcal{U}_u^{\alpha A}$ is a vielbein on the hypergeometry. Before gauging the system, we introduce the vielbein one-form:

$$\mathcal{U}^{\alpha A} \equiv \mathcal{U}_u^{\alpha A}(q)\,\mathrm{d}q^u\,. \tag{1.21c}$$

There exist various equations on the vielbein one-forms:

$$\mathcal{U}_{\alpha A} \equiv (\mathcal{U}^{\alpha A})^* = \epsilon_{AB}\,\mathbb{C}_{\alpha\beta}\,\mathcal{U}^{\beta B}\,, \qquad \text{(reality condition)} \tag{1.22a}$$

$$\mathcal{U}_{\alpha A}^u\,\mathcal{U}_v^{\alpha A} = \delta_v^u\,, \qquad \text{(inverse vielbein)} \tag{1.22b}$$

$$h_{uv}\,\epsilon^{AB} = \Big(\mathcal{U}_u^{\alpha A}\,\mathcal{U}_v^{\beta B} + \mathcal{U}_v^{\alpha A}\,\mathcal{U}_u^{\beta B}\Big)\mathbb{C}_{\alpha\beta}\,, \qquad ((1.21)+(1.22a)) \tag{1.22c}$$

$$\frac{1}{m}h_{uv}\,\mathbb{C}^{\alpha\beta} = \Big(\mathcal{U}_u^{\alpha A}\,\mathcal{U}_v^{\beta B} + \mathcal{U}_v^{\alpha A}\,\mathcal{U}_u^{\beta B}\Big)\epsilon_{AB}\,, \qquad ((1.21)+(1.22a)) \tag{1.22d}$$

$$\nabla\mathcal{U}^{\alpha A} \equiv \mathrm{d}\mathcal{U}^{\alpha A} + \frac{\mathrm{i}}{2}\omega^x(\epsilon\sigma_x\epsilon^{-1})^A{}_B \wedge \mathcal{U}^{\alpha B} + \Delta^{\alpha\beta} \wedge \mathcal{U}^{\gamma A}\,\mathbb{C}_{\beta\gamma} = 0\,, \qquad \text{(covariantly closed)} \tag{1.22e}$$

where $(\sigma^x)_A{}^B = (\sigma_x)_A{}^B$ are the standard Pauli matrices described as

$$(\sigma^x)_A{}^B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (\sigma^y)_A{}^B = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad (\sigma^z)_A{}^B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.23a}$$

$$(\epsilon\sigma_x\epsilon^{-1})^A{}_B = \epsilon^{AC}(\sigma_x)_C{}^D(\epsilon^{-1})_{DB} = -\epsilon^{AC}(\sigma_x)_C{}^D\epsilon_{BD}\,, \tag{1.23b}$$

Quaternionic algebra: three complex structures J^x, J^y and J^z, and their hyper-Kähler forms:

$$J^x J^y = -\delta^{xy}\mathbb{1} + \epsilon^{xyz}J^z\,, \tag{1.24a}$$

$$K^x \equiv K_{uv}^x\,\mathrm{d}q^u \wedge \mathrm{d}q^v\,, \qquad K_{uv}^x = h_{uw}(J^x)^w{}_v\,, \qquad \nabla K^x \equiv \mathrm{d}K^x + \epsilon^{xyz}\omega^y \wedge K^z \equiv 0\,. \tag{1.24b}$$

The curvature of the $SU(2)$-bundle of the connection ω^x is defined by K^x with a certain constant λ:

$$\Omega^x \equiv \mathrm{d}\omega^x + \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z = \lambda K^x\,, \qquad \Omega^x \equiv \Omega_{uv}^x\,\mathrm{d}q^u \wedge \mathrm{d}q^v\,. \tag{1.25}$$

The constant λ appears in the kinetic term of the scalar fields of the hypermultiplets (with $\lambda = -1$) [1]:

$$\mathscr{L}_{\mathsf{kin}}^{\mathsf{hyper}} = -\lambda\,g^{\mu\nu}\,h_{uv}\,\partial_\mu q^u \partial_\nu q^v = +h_{uv}\,\partial_\mu q^u \partial^\mu q^v\,. \tag{1.26}$$

Identities among the hyper-Kähler metric, the curvature of the $SU(2)$-bundle, and the constant λ derived from the quaternionic algebra (1.24):

$$h^{st}K_{us}^x K_{tv}^y = -\delta^{xy}h_{uv} + \epsilon^{xyz}K_{uv}^z\,, \qquad h^{st}\Omega_{us}^x\Omega_{tv}^y = -\lambda^2\delta^{xy}h_{uv} + \lambda\epsilon^{xyz}\Omega_{uv}^z\,. \tag{1.27}$$

The curvature of the $Sp(2m)$-bundle of the connection $\Delta^{\alpha\beta}$ is

$$\mathbb{R}^{\alpha\beta} \equiv \mathrm{d}\Delta^{\alpha\beta} + \Delta^{\alpha\gamma} \wedge \Delta^{\delta\beta}\,\mathbb{C}_{\gamma\beta} \equiv \mathbb{R}_{uv}^{\alpha\beta}\,\mathrm{d}q^u \wedge \mathrm{d}q^v\,. \tag{1.28}$$

The triholomorphic momentum map $\mathcal{P}_\Lambda^x$ under the gauging with variation $q^u \to q^u + \epsilon^\Lambda k_\Lambda^u(q)$ is defined via the hyper-Kähler two-forms in the following way:

$$\iota_\Lambda K^x = 2K_{uv}^x\,k_\Lambda^u\,\mathrm{d}q^v \equiv -\nabla\mathcal{P}_\Lambda^x = -\Big(\mathrm{d}\mathcal{P}_\Lambda^x + \epsilon^{xyz}\omega^y\mathcal{P}_\Lambda^z\Big)\,, \tag{1.29a}$$

$$\nabla_v\mathcal{P}_\Lambda^x = \partial_v\mathcal{P}_\Lambda^x + \epsilon^{xyz}\omega_v^y\mathcal{P}_\Lambda^z = -\frac{2}{\lambda}\Omega_{uv}^x\,k_\Lambda^u\,, \tag{1.29b}$$

where ι_Λ is the interior product [16] in terms of the Killing vector $\vec{k}_\Lambda = k_\Lambda^u\frac{\partial}{\partial q^u}$. In terms of the triholomorphic Poisson bracket $\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x \equiv 2K^x(\Lambda,\Sigma) - \lambda\epsilon^{xyz}\mathcal{P}_\Lambda^y\mathcal{P}_\Sigma^z = f^\Delta{}_{\Lambda\Sigma}\mathcal{P}_\Delta^x$ [1], we can see[5]

$$K_{uv}^x\,k_\Lambda^u k_\Sigma^v - \frac{\lambda}{2}\epsilon^{xyz}\,\mathcal{P}_\Lambda^y\mathcal{P}_\Sigma^z = \frac{1}{2}f^\Delta{}_{\Lambda\Sigma}\,\mathcal{P}_\Delta^x\,. \tag{1.30}$$

[5] The definition $K^x(\Lambda,\Sigma)$ in [1] seems to be $K_{uv}^x k_\Lambda^u k_\Sigma^v$, i.e., $\iota_\Lambda\iota_\Sigma K^x \equiv 2K^x(\Lambda,\Sigma)$. Note that the eq.(A.8) of [21] brings a wrong sign in front of Ω_{uv}^x.

2 Computation rules in [1]

2.1 Expansion of two-forms

In the literature [1], because of the definition of the gauge field strength $F^\Lambda = \mathcal{F}^\Lambda_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, I guess that all the (real) two-forms are expanded as $\mathcal{T} \equiv \mathcal{T}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, i.e., as in the following ways:

$$
\begin{aligned}
F^\Lambda &\equiv \mathcal{F}^\Lambda_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, & *F^\Lambda &\equiv \widetilde{\mathcal{F}}^\Lambda_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, && \text{(gauge field strength)}\,, && \text{(2.1a)}\\
\widehat{F}^\Lambda &\equiv \widehat{F}^\Lambda_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, & & && \text{(supercovariantized } F^\Lambda)\,, && \text{(2.1b)}\\
T^\pm &\equiv T^\pm_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, & U^\pm &\equiv U^\pm_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, && \text{(two-forms in } \delta\psi_{A\mu})\,, && \text{(2.1c)}\\
G^{i-} &\equiv G^{i-}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, & G^{\bar{i}+} &\equiv G^{\bar{i}+}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, && \text{(two-forms in } \delta\lambda^{iA},\ \delta\lambda^{\bar{i}}_A)\,, && \text{(2.1d)}\\
K^x &\equiv K^x_{uv}\,\mathrm{d}q^u \wedge \mathrm{d}q^v\,, & & && \text{(hyper-Kähler form)}\,, && \text{(2.1e)}\\
\Omega^x &\equiv \Omega^x_{uv}\,\mathrm{d}q^u \wedge \mathrm{d}q^v\,, & \Omega^x &= \lambda K^x\,, && \text{(curvature of } SU(2)\text{-bundle)}\,, && \text{(2.1f)}\\
\mathbb{R}^{\alpha\beta} &\equiv \mathbb{R}^{\alpha\beta}_{uv}\,\mathrm{d}q^u \wedge \mathrm{d}q^v\,, & & && \text{(curvature of } Sp(2m)\text{-bundle)}\,. && \text{(2.1g)}
\end{aligned}
$$

The spacetime curvature two-form (1.2) is the only one exception.

2.2 Two totally antisymmetric symbols $\varepsilon_{\mu\nu\rho\sigma}$ and $\epsilon_{\mu\nu\rho\sigma}$ in [1]

The literature [1] often confuses us with the usage of two totally antisymmetric symbols $\varepsilon_{\mu\nu\rho\sigma}$ and $\epsilon_{\mu\nu\rho\sigma}$ if we follow appendix A.4, which is strictly defined in [16]. Indeed, the former only appeared in the definition of the Hodge dual of a tensor of degree two and in the integration volume form in the flat spacetime, while the latter appeared in the definition of the discussion of the Lagrangian density in a curved spacetime. We cannot use $\varepsilon_{\mu\nu\rho\sigma}$ in a curved spacetime unless we attach the weight $\sqrt{-g}$.

Now let us check a discrepancy if the form (2.2) in [1] were still applicable in a curved spacetime: First, from the defintion of the Hodge dual in (2.2) of [1], we assume that $\varepsilon^{[1]}_{\mu\nu\rho\sigma}$ involves $\sqrt{-g}$ and the symbol $\varepsilon^{\mathsf{TK}}_{\mu\nu\rho\sigma}$ in appendix A.4 such as[6]:

$$
\begin{aligned}
F &= \mathcal{F}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \qquad *F = \widetilde{\mathcal{F}}_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \qquad \widetilde{\mathcal{F}}_{\mu\nu} = \frac{1}{2}\varepsilon^{[1]}_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma} = \frac{\sqrt{-g}}{2}\varepsilon^{\mathsf{TK}}_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma}\,, && \text{(2.2a)}\\
&\therefore \quad \varepsilon^{[1]}_{\mu\nu\rho\sigma} = \sqrt{-g}\,\varepsilon^{\mathsf{TK}}_{\mu\nu\rho\sigma}\,, \qquad \Big(\varepsilon^{\mathsf{TK}}_{0123} \equiv 1\Big)\,. && \text{(2.2b)}
\end{aligned}
$$

Now we compare the following two representations from (2.3) and (B.6) in [1] under a relation $\epsilon^{\mu\nu\rho\sigma}_{[1]} \equiv \alpha_1 \varepsilon^{\mu\nu\rho\sigma}_{[1]}$:

$$
\begin{aligned}
\mathrm{d}^4x &= -\frac{1}{4!}\varepsilon^{[1]}_{\mu\nu\rho\sigma}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma\,, && \text{(2.3a)}\\
\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma &= \frac{\epsilon^{\mu\nu\rho\sigma}_{[1]}}{\sqrt{-g}}\Big(\sqrt{-g}\,\mathrm{d}^4x\Big) = \frac{\alpha_1\varepsilon^{\mu\nu\rho\sigma}_{[1]}}{\sqrt{-g}}\Big(\sqrt{-g}\,\mathrm{d}^4x\Big)\,. && \text{(2.3b)}
\end{aligned}
$$

Multiplying $\varepsilon^{[1]}_{\mu\nu\rho\sigma}$ to the second line and applying the first line to it, we can determine the constant α_1:

$$
\begin{aligned}
\varepsilon^{[1]}_{\mu\nu\rho\sigma}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma &= -4!\,\mathrm{d}^4x\,, && \text{(2.4a)}\\
\varepsilon^{[1]}_{\mu\nu\rho\sigma}\frac{\alpha_1\varepsilon^{\mu\nu\rho\sigma}_{[1]}}{\sqrt{-g}}\Big(\sqrt{-g}\,\mathrm{d}^4x\Big) &= \alpha_1\varepsilon^{[1]}_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}_{[1]}\,\mathrm{d}^4x = \alpha_1(\sqrt{-g})^2\varepsilon^{\mathsf{TK}}_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}_{\mathsf{TK}}\,\mathrm{d}^4x = -4!\alpha_1\,\mathrm{d}^4x\,, && \text{(2.4b)}\\
\therefore \quad \alpha_1 &= 1\,, \qquad \text{i.e.,} \quad \varepsilon^{[1]}_{\mu\nu\rho\sigma} = \epsilon^{[1]}_{\mu\nu\rho\sigma}\,. && \text{(2.4c)}
\end{aligned}
$$

In appendix D of [1] the normalization is given as $\epsilon^{[1]}_{0123} = 1$, which is inconsistent with (2.2b) via (2.4c) unless the spacetime is flat:

$$
1 \equiv \epsilon^{[1]}_{0123} = \varepsilon^{[1]}_{0123} = \sqrt{-g}\,\varepsilon^{\mathsf{TK}}_{0123} = \sqrt{-g}\,. \qquad \text{(2.5)}
$$

Then we conclude that (2.2) of [1] is only applicable in the flat spacetime. This is the reason why I emphasized the explicit forms in a generic curved spacetime as in section 1.1.

[6] In this subsection I explicitly recall $\varepsilon^{\mathrm{TK}}_{\mu\nu\rho\sigma}$ as the epsilon tensor in appendix A.4, while the one in [1] is given as $\varepsilon^{[1]}_{\mu\nu\rho\sigma}$

2.3 "Four-form" and wedge products in a curved spacetime

Following [1], we used the definition of the orientation of the "four-form" d^4x in a curved spacetime as[7]

$$\mathrm{d}^4x \equiv -\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma = -\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\,. \tag{2.6}$$

Let us compute a wedge product of a two-form F_2 and itself, and a product of F_2 and its Hodge dual:

$$F \equiv \mathcal{F}_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,, \qquad *F \equiv \widetilde{\mathcal{F}}_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\,, \qquad \widetilde{\mathcal{F}}_{\mu\nu} \equiv \frac{\sqrt{-g}}{2!}\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma}\,, \tag{2.7a}$$

$$F\wedge F = \mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\,, \tag{2.7b}$$

$$F\wedge *F = \mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}_{\rho\sigma}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma = \frac{\sqrt{-g}}{2!}\mathcal{F}_{\mu\nu}\mathcal{F}^{\lambda\gamma}\,\varepsilon_{\rho\sigma\lambda\gamma}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\,. \tag{2.7c}$$

Now let us rewrite $\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma$ in terms of d^4x:

$$\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma \equiv \alpha_2\,\varepsilon^{\mu\nu\rho\sigma}\mathrm{d}^4x\,, \tag{2.8}$$

where α_2 is a constant which can be fixed by multiplying $\varepsilon_{\mu\nu\rho\sigma}$ to the above equation. By definition, the left-hand-side is

$$\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma = -4!\,\mathrm{d}^4x\,. \tag{2.9a}$$

On the other hand the right-hand-side is

$$\alpha_2\,\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{d}^4x = \frac{4!}{g}\alpha\,\mathrm{d}^4x\,, \tag{2.9b}$$

Comparing these two equations, we fix the constant $\alpha_2 = -g$ and

$$\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma = -g\,\varepsilon^{\mu\nu\rho\sigma}\mathrm{d}^4x \equiv \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\Big(\sqrt{-g}\,\mathrm{d}^4x\Big)\,. \tag{2.9c}$$

Because of this computation we also find the relation between $\epsilon^{\mu\nu\rho\sigma}$ in (3.1) and $\varepsilon^{\mu\nu\rho\sigma}$ from the following definition:

$$\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma \equiv \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\Big(\sqrt{-g}\,\mathrm{d}^4x\Big) = -g\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{d}^4x\,, \tag{2.10a}$$

$$\therefore \quad \epsilon^{\mu\nu\rho\sigma} = -g\,\varepsilon^{\mu\nu\rho\sigma}\,, \qquad \epsilon^{0123} = -g\,\varepsilon^{0123} = -1\,. \tag{2.10b}$$

Here let us newly define $\epsilon_{\mu\nu\rho\sigma}$ which does not appear in [1]:

$$\epsilon_{\mu\nu\rho\sigma} \equiv g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta}\,\epsilon^{\alpha\beta\gamma\delta} = g\,\epsilon^{\mu\nu\rho\sigma} = g(-g)\,\varepsilon^{\mu\nu\rho\sigma} = -g\,\varepsilon_{\mu\nu\rho\sigma}\,, \tag{2.11a}$$

$$\epsilon_{0123} = -g\,. \tag{2.11b}$$

Applying this result to the wedge products $F\wedge F$ and $F\wedge *F$, we obtain

$$\begin{aligned} F\wedge F &= \mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma = \mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\Big(\sqrt{-g}\,\mathrm{d}^4x\Big) = \mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\sqrt{-g}\,\varepsilon^{\mu\nu\rho\sigma}\big(\sqrt{-g}\,\mathrm{d}^4x\big) \\ &= 2\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}^{\mu\nu}\big(\sqrt{-g}\,\mathrm{d}^4x\big)\,, \end{aligned} \tag{2.12a}$$

$$\begin{aligned} F\wedge *F &= \frac{(-g)^{3/2}}{2!}\mathcal{F}_{\mu\nu}\mathcal{F}^{\lambda\gamma}\,\varepsilon_{\rho\sigma\lambda\gamma}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{d}^4x = \frac{(-g)^{3/2}}{2!}\mathcal{F}_{\mu\nu}\mathcal{F}^{\lambda\gamma}\Big\{\frac{2!}{g}(\delta^\mu_\lambda\delta^\nu_\gamma - \delta^\nu_\lambda\delta^\mu_\gamma)\Big\}\mathrm{d}^4x \\ &= -2\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\big(\sqrt{-g}\,\mathrm{d}^4x\big)\,. \end{aligned} \tag{2.12b}$$

In the last computation we used the following identities:

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\rho\sigma} = \frac{4!}{g}\,, \qquad \varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\nu\rho\sigma} = \frac{3!}{g}\delta^\alpha_\mu\,, \qquad \varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\rho\sigma} = \frac{2!}{g}\big(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu\big)\,. \tag{2.13}$$

Since $\epsilon^{\mu\nu\rho\sigma}$ is a number, the expression $\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma} = 8(\mathcal{F}_{01}\mathcal{F}_{23} + \mathcal{F}_{02}\mathcal{F}_{31} + \mathcal{F}_{03}\mathcal{F}_{12})$ does not depend on the metric at all. Then we can regard the wedge product $F\wedge F = 2\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}^{\mu\nu}(\sqrt{-g}\,\mathrm{d}^4x)$ as a topological term.

[7] As we discussed in appendix A.4, I defined the (natural) volume form as (vol.) $\equiv \sqrt{-g}\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3$.

2.4 Kinetic term and topological of gauge fields

Here let us again write down the Hodge dual, the (anti-)self-dual forms and the contractions of the epsilon tensors:

$$*F^{\Lambda} \ \equiv \ \widetilde{\mathcal{F}}^{\Lambda}_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu} \,, \qquad \widetilde{\mathcal{F}}^{\Lambda}_{\mu\nu} \ \equiv \ \frac{\sqrt{-g}}{2} \varepsilon_{\mu\nu\rho\sigma}\, \mathcal{F}^{\Lambda\rho\sigma} \,, \tag{2.14a}$$

$$\mathcal{T}^{(\mp)}_{\mu\nu} \ \equiv \ \frac{1}{2}\Big(\mathcal{T}_{\mu\nu} \mp \frac{\mathrm{i}}{2}\sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma} \mathcal{T}^{\rho\sigma} \Big) \quad \to \quad -\frac{\mathrm{i}}{2}\sqrt{-g}\, \varepsilon_{\mu\nu}{}^{\rho\sigma} \mathcal{T}^{(\mp)}_{\rho\sigma} \ = \ \pm \mathcal{T}^{(\mp)}_{\mu\nu} \,, \tag{2.14b}$$

$$\varepsilon_{\mu\nu\rho\sigma}\, \varepsilon^{\mu\nu\rho\sigma} \ = \ \frac{4!}{g} \,, \qquad \varepsilon_{\mu\nu\rho\sigma}\, \varepsilon^{\mu\nu\lambda\gamma} \ = \ \frac{2!}{g} \Big(\delta^{\lambda}_{\rho} \delta^{\gamma}_{\sigma} - \delta^{\gamma}_{\rho} \delta^{\lambda}_{\sigma} \Big) \,. \tag{2.14c}$$

Then the kinetic term and the topological term of the gauge fields in [1] is described as the following forms:

$$\mathscr{L}^{\mathsf{gauge}}_{\mathsf{kin}} \ \equiv \ \mathrm{i}\Big(\overline{\mathcal{N}}_{\Lambda\Sigma}\, \mathcal{F}^{\Lambda-}_{\mu\nu}\, \mathcal{F}^{\Sigma-|\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\, \mathcal{F}^{\Lambda+}_{\mu\nu} \mathcal{F}^{\Sigma+|\mu\nu} \Big) \,, \tag{2.15a}$$

$$\begin{aligned}
\mathrm{i}\overline{\mathcal{N}}_{\Lambda\Sigma}\, \mathcal{F}^{\Lambda-}_{\mu\nu}\, \mathcal{F}^{\Sigma-|\mu\nu} \ &= \ \frac{\mathrm{i}}{4} \overline{\mathcal{N}}_{\Lambda\Sigma} \Big(\mathcal{F}^{\Lambda}_{\mu\nu} - \frac{\mathrm{i}}{2}\sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma}\, \mathcal{F}^{\Lambda\rho\sigma} \Big) \Big(\mathcal{F}^{\Sigma\mu\nu} - \frac{\mathrm{i}}{2}\sqrt{-g}\, \varepsilon^{\mu\nu\lambda\gamma}\, \mathcal{F}^{\Sigma}_{\lambda\gamma} \Big) \\
&= \ \frac{\mathrm{i}}{4} \overline{\mathcal{N}}_{\Lambda\Sigma} \Big(\mathcal{F}^{\Lambda}_{\mu\nu} \mathcal{F}^{\Sigma\mu\nu} - \mathrm{i}\sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma}\, \mathcal{F}^{\Lambda\mu\nu} \mathcal{F}^{\Sigma\rho\sigma} + \frac{g}{4} \varepsilon_{\mu\nu\rho\sigma}\, \varepsilon^{\mu\nu\lambda\gamma}\, \mathcal{F}^{\Lambda\rho\sigma} \mathcal{F}^{\Sigma\lambda\gamma} \Big) \\
&= \ \frac{\mathrm{i}}{2} \overline{\mathcal{N}}_{\Lambda\Sigma} \Big(\mathcal{F}^{\Lambda}_{\mu\nu} \mathcal{F}^{\Sigma\mu\nu} - \frac{\mathrm{i}}{2}\sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma}\, \mathcal{F}^{\Lambda\mu\nu} \mathcal{F}^{\Lambda\rho\sigma} \Big) \,,
\end{aligned} \tag{2.15b}$$

$$\begin{aligned}
-\mathrm{i}\mathcal{N}_{\Lambda\Sigma} \mathcal{F}^{\Lambda+}_{\mu\nu} \mathcal{F}^{\Sigma+|\mu\nu} \ &= \ -\frac{\mathrm{i}}{4} \mathcal{N}_{\Lambda\Sigma} \Big(\mathcal{F}^{\Lambda} + \frac{\mathrm{i}}{2}\sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma}\, \mathcal{F}^{\Lambda\rho\sigma} \Big) \Big(\mathcal{F}^{\Sigma\mu\nu} + \frac{\mathrm{i}}{2}\sqrt{-g}\, \varepsilon^{\mu\nu\lambda\gamma}\, \mathcal{F}^{\Sigma}_{\lambda\gamma} \Big) \\
&= \ -\frac{\mathrm{i}}{4} \mathcal{N}_{\Lambda\Sigma} \Big(\mathcal{F}^{\Lambda}_{\mu\nu} \mathcal{F}^{\Sigma\mu\nu} + \mathrm{i}\sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma}\, \mathcal{F}^{\Lambda\mu\nu} \mathcal{F}^{\Sigma\rho\sigma} + \frac{g}{4} \varepsilon_{\mu\nu\rho\sigma}\, \varepsilon^{\mu\nu\lambda\gamma}\, \mathcal{F}^{\Lambda\rho\sigma}\, \mathcal{F}^{\Sigma\lambda\gamma} \Big) \\
&= \ -\frac{\mathrm{i}}{2} \mathcal{N}_{\Lambda\Sigma} \Big(\mathcal{F}^{\Lambda}_{\mu\nu} \mathcal{F}^{\Sigma\mu\nu} + \frac{\mathrm{i}}{2}\sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma}\, \mathcal{F}^{\Lambda\mu\nu} \mathcal{F}^{\Lambda\rho\sigma} \Big) \,,
\end{aligned} \tag{2.15c}$$

$$\begin{aligned}
\therefore \quad \mathscr{L}^{\mathsf{gauge}}_{\mathsf{kin}} \ &= \ \frac{\mathrm{i}}{2} \big(\overline{\mathcal{N}}_{\Lambda\Sigma} - \mathcal{N}_{\Lambda\Sigma} \big) \mathcal{F}^{\Lambda}_{\mu\nu} \mathcal{F}^{\Sigma\mu\nu} + \frac{1}{4} \big(\overline{\mathcal{N}}_{\Lambda\Sigma} + \mathcal{N}_{\Lambda\Sigma} \big) \sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma}\, \mathcal{F}^{\Lambda\mu\nu} \mathcal{F}^{\Sigma\rho\sigma} \\
&= \ (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\, \mathcal{F}^{\Lambda}_{\mu\nu} \mathcal{F}^{\Sigma\mu\nu} + (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\, \mathcal{F}^{\Lambda}_{\mu\nu} \widetilde{\mathcal{F}}^{\Sigma\mu\nu} \,.
\end{aligned} \tag{2.15d}$$

Note that we used the symmetry of the period matrix $\mathcal{N}_{\Lambda\Sigma} = \mathcal{N}_{\Sigma\Lambda}$ and the definition $\widetilde{\mathcal{F}}_{\mu\nu} = \frac{\sqrt{-g}}{2!} \varepsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\rho\sigma}$.

2.5 Magnetic dual of the gauge field strength

It is worth introducing the following definition:[8]

$$\mathcal{G}^{\pm}_{\Lambda\mu\nu} \ = \ \frac{1}{2}\Big(\mathcal{G}_{\Lambda\mu\nu} \pm \frac{\mathrm{i}}{2}\sqrt{-g}\, \varepsilon_{\mu\nu\rho\sigma} \mathcal{G}^{\rho\sigma}_{\Lambda} \Big) \ \equiv \ \pm \frac{\mathrm{i}}{2} \frac{\delta \mathscr{L}}{\delta \mathcal{F}^{\Lambda\pm|\mu\nu}} \,, \tag{2.16a}$$

$$\mathcal{G}_{\Lambda\mu\nu} \ = \ \mathcal{G}^{+}_{\Lambda\mu\nu} + \mathcal{G}^{-}_{\Lambda\mu\nu} \ \equiv \ \frac{1}{2} \frac{\delta \mathscr{L}}{\delta \widetilde{\mathcal{F}}^{\Lambda\mu\nu}} \,, \qquad \widetilde{\mathcal{G}}_{\Lambda\mu\nu} \ = \ \frac{1}{\mathrm{i}} \big(\mathcal{G}^{+}_{\Lambda\mu\nu} - \mathcal{G}^{-}_{\Lambda\mu\nu} \big) \ \equiv \ \frac{1}{2} \frac{\delta \mathscr{L}}{\delta \mathcal{F}^{\Lambda\mu\nu}} \,. \tag{2.16b}$$

The $\mathcal{G}^{\pm}_{\Lambda\mu\nu}$ is the magnetic dual of the gauge field strength, a generalization of the Hodge dual of the gauge field strength $\mathcal{F}^{\Lambda}_{\mu\nu}$. This form is more applicable when the topological term and charged matter fields are coupled to the system. Indeed if the magnetic sources are present, the naive Hodge dual is not able to follow physical degrees. If there are no matter fields, the Lagrangian is only $\mathscr{L}^{\mathsf{gauge}}_{\mathsf{kin}}$, which gives explicit expressions of $\mathcal{G}^{\pm}_{\Lambda\mu\nu}$ (or the set of $\mathcal{G}_{\Lambda\mu\nu}$ and $\widetilde{\mathcal{G}}_{\Lambda\mu\nu}$) in such a way as

$$\mathcal{G}^{+}_{\Lambda\mu\nu} \ = \ \frac{\mathrm{i}}{2} \frac{\delta \mathscr{L}^{\mathsf{gauge}}_{\mathsf{kin}}}{\delta \mathcal{F}^{\Lambda+|\mu\nu}} \ = \ \mathcal{N}_{\Lambda\Sigma} \mathcal{F}^{\Sigma+}_{\mu\nu} \,, \qquad \mathcal{G}^{-}_{\Lambda\mu\nu} \ = \ -\frac{\mathrm{i}}{2} \frac{\delta \mathscr{L}^{\mathsf{gauge}}_{\mathsf{kin}}}{\delta \mathcal{F}^{\Lambda-|\mu\nu}} \ = \ \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{F}^{\Sigma-}_{\mu\nu} \,, \tag{2.17a}$$

$$\begin{aligned}
\mathcal{G}_{\Lambda\mu\nu} \ &= \ \mathcal{N}_{\Lambda\Sigma} \mathcal{F}^{\Sigma+}_{\mu\nu} + \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{F}^{\Sigma-}_{\mu\nu} \ = \ \frac{1}{2} \mathcal{N}_{\Lambda\Sigma} \Big(\mathcal{F}^{\Sigma}_{\mu\nu} + \mathrm{i}\widetilde{\mathcal{F}}^{\Sigma}_{\mu\nu} \Big) + \frac{1}{2} \overline{\mathcal{N}}_{\Lambda\Sigma} \Big(\mathcal{F}^{\Sigma}_{\mu\nu} - \mathrm{i}\widetilde{\mathcal{F}}^{\Sigma}_{\mu\nu} \Big) \\
&= \ (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} \mathcal{F}^{\Sigma}_{\mu\nu} - (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} \widetilde{\mathcal{F}}^{\Sigma}_{\mu\nu} \,,
\end{aligned} \tag{2.17b}$$

$$\begin{aligned}
\widetilde{\mathcal{G}}_{\Lambda\mu\nu} \ &= \ \frac{1}{\mathrm{i}} \Big(\mathcal{N}_{\Lambda\Sigma} \mathcal{F}^{\Sigma+}_{\mu\nu} - \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{F}^{\Sigma-}_{\mu\nu} \Big) \ = \ \frac{1}{2\mathrm{i}} \mathcal{N}_{\Lambda\Sigma} \Big(\mathcal{F}^{\Sigma}_{\mu\nu} + \mathrm{i}\widetilde{\mathcal{F}}^{\Sigma}_{\mu\nu} \Big) - \frac{1}{2\mathrm{i}} \overline{\mathcal{N}}_{\Lambda\Sigma} \Big(\mathcal{F}^{\Sigma}_{\mu\nu} - \mathrm{i}\widetilde{\mathcal{F}}^{\Sigma}_{\mu\nu} \Big) \\
&= \ (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} \mathcal{F}^{\Sigma}_{\mu\nu} + (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} \widetilde{\mathcal{F}}^{\Sigma}_{\mu\nu} \,.
\end{aligned} \tag{2.17c}$$

[8] The definition $\widetilde{\mathcal{G}}^{\Lambda}_{\mu\nu} \equiv \frac{1}{2} \frac{\partial \mathscr{L}}{\partial \mathcal{F}^{\Lambda}_{\mu\nu}}$ in (2.12) of [1] should be corrected to $\widetilde{\mathcal{G}}_{\Lambda\mu\nu} \equiv \frac{1}{2} \frac{\partial \mathscr{L}}{\partial \mathcal{F}^{\Lambda\mu\nu}}$.

Let us check the relation between the two expressions $\mathcal{G}_{\Lambda\mu\nu}$ and $\widetilde{\mathcal{G}}_{\Lambda\mu\nu}$ in terms of the (anti-)self-duality (1.3):

$$\frac{\sqrt{-g}}{2}\,\varepsilon_{\mu\nu\rho\sigma}\mathcal{G}_{\Lambda}^{\rho\sigma} \ = \ \frac{\sqrt{-g}}{2}\,\varepsilon_{\mu\nu\rho\sigma}\Big(\mathcal{G}_{\Lambda}^{+\rho\sigma}+\mathcal{G}_{\Lambda}^{-\rho\sigma}\Big) \ = \ \frac{1}{\mathrm{i}}\Big(\mathcal{G}^{+}_{\Lambda\mu\nu}-\mathcal{G}^{-}_{\Lambda\mu\nu}\Big) \ = \ \widetilde{\mathcal{G}}_{\Lambda\mu\nu}\,, \tag{2.18a}$$

$$\frac{\sqrt{-g}}{2}\,\varepsilon_{\mu\nu\rho\sigma}\widetilde{\mathcal{G}}_{\Lambda}^{\rho\sigma} \ = \ \frac{\sqrt{-g}}{2\mathrm{i}}\,\varepsilon_{\mu\nu\rho\sigma}\Big(\mathcal{G}_{\Lambda}^{+\rho\sigma}-\mathcal{G}_{\Lambda}^{-\rho\sigma}\Big) \ = \ -\Big(\mathcal{G}^{+}_{\Lambda\mu\nu}+\mathcal{G}^{-}_{\Lambda\mu\nu}\Big) \ = \ -\mathcal{G}_{\Lambda\mu\nu}\,. \tag{2.18b}$$

Notice that, by definition in terms of $\varepsilon_{\mu\nu\rho\sigma}$, the Hodge dual is anti-projective. The above equation realizes the anti-projection of the Hodge dual. In a particular case as in (2.17), we can see more an explicit relation:

$$\begin{aligned}\frac{\sqrt{-g}}{2}\,\varepsilon_{\mu\nu\rho\sigma}\mathcal{G}_{\Lambda}^{\rho\sigma} \ &= \ \frac{\sqrt{-g}}{2}\,\varepsilon_{\mu\nu\rho\sigma}\Big[(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}^{\Sigma}_{\mu\nu}-(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\widetilde{\mathcal{F}}^{\Sigma}_{\mu\nu}\Big] \ = \ (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\widetilde{\mathcal{F}}^{\Sigma}_{\mu\nu}+(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}^{\Sigma}_{\mu\nu} \\ &= \ \widetilde{\mathcal{G}}_{\Lambda\mu\nu}\,,\end{aligned} \tag{2.19a}$$

$$\begin{aligned}\frac{\sqrt{-g}}{2}\,\varepsilon_{\mu\nu\rho\sigma}\widetilde{\mathcal{G}}_{\Lambda}^{\rho\sigma} \ &= \ \frac{\sqrt{-g}}{2}\,\varepsilon_{\mu\nu\rho\sigma}\Big[(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}^{\Sigma}_{\mu\nu}+(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\widetilde{\mathcal{F}}^{\Sigma}_{\mu\nu}\Big] \ = \ (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\widetilde{\mathcal{F}}^{\Sigma}_{\mu\nu}-(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}^{\Sigma}_{\mu\nu} \\ &= \ -\mathcal{G}_{\Lambda\mu\nu}\,.\end{aligned} \tag{2.19b}$$

We have to notice that $\mathcal{F}^{\Lambda}_{\mu\nu}$ is not independent of $\widetilde{\mathcal{F}}^{\Lambda}_{\mu\nu}$, while $\mathcal{F}^{\Lambda\pm}_{\mu\nu}$ is independent of $\mathcal{F}^{\Lambda\mp}_{\mu\nu}$:

$$\widetilde{\mathcal{F}}^{\Lambda}_{\mu\nu} \ = \ \frac{\sqrt{-g}}{2}\,\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda\rho\sigma}\,, \qquad \frac{\sqrt{-g}}{2}\,\varepsilon_{\mu\nu\rho\sigma}\widetilde{\mathcal{F}}^{\Lambda\rho\sigma} \ = \ \frac{-g}{4}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\rho\sigma\lambda\gamma}\mathcal{F}^{\Lambda}_{\lambda\gamma} \ = \ -\mathcal{F}^{\Lambda}_{\mu\nu}\,, \tag{2.20a}$$

$$\widetilde{\mathcal{F}}^{\Lambda}_{\mu\nu}\widetilde{\mathcal{F}}^{\Sigma\mu\nu} \ = \ \frac{-g}{4}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\mu\nu\lambda\gamma}\mathcal{F}^{\Lambda\rho\sigma}\mathcal{F}^{\Sigma}_{\lambda\gamma} \ = \ -\mathcal{F}^{\Lambda}_{\mu\nu}\mathcal{F}^{\Sigma\mu\nu}\,. \tag{2.20b}$$

By using (2.17) and (2.20), let us confirm the definition (2.16b):

$$\begin{aligned}\mathscr{L}^{\text{gauge}}_{\text{kin}} \ &= \ (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}\mathcal{F}^{\Sigma\mu\nu}+(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}\widetilde{\mathcal{F}}^{\Sigma\mu\nu} \\ &= \ (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}\mathcal{F}^{\Sigma\mu\nu}+(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\frac{\sqrt{-g}}{2}\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\Lambda\mu\nu}\mathcal{F}^{\Sigma\rho\sigma} \\ &= \ -(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\widetilde{\mathcal{F}}^{\Lambda}_{\mu\nu}\widetilde{\mathcal{F}}^{\Sigma\mu\nu}-(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\frac{\sqrt{-g}}{2}\varepsilon_{\mu\nu\rho\sigma}\widetilde{\mathcal{F}}^{\Lambda\mu\nu}\widetilde{\mathcal{F}}^{\Sigma\rho\sigma}\,,\end{aligned} \tag{2.21a}$$

$$\begin{aligned}\therefore \qquad \frac{\delta\mathscr{L}^{\text{gauge}}_{\text{kin}}}{\delta\mathcal{F}^{\Lambda\mu\nu}} \ &= \ 2(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}^{\Sigma}_{\mu\nu}+(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\Sigma\rho\sigma} \ = \ 2(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}^{\Sigma}_{\mu\nu}+2(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\widetilde{\mathcal{F}}^{\Sigma}_{\rho\sigma} \\ &= \ 2\widetilde{\mathcal{G}}_{\Lambda\mu\nu}\,,\end{aligned} \tag{2.21b}$$

$$\begin{aligned}\frac{\delta\mathscr{L}^{\text{gauge}}_{\text{kin}}}{\delta\widetilde{\mathcal{F}}^{\Lambda\mu\nu}} \ &= \ -2(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\widetilde{\mathcal{F}}^{\Sigma}_{\mu\nu}-(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}\widetilde{\mathcal{F}}^{\Sigma\rho\sigma} \ = \ -2(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\widetilde{\mathcal{F}}^{\Sigma}_{\mu\nu}+(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}^{\Sigma}_{\mu\nu} \\ &= \ 2\mathcal{G}_{\Lambda\mu\nu}\,.\end{aligned} \tag{2.21c}$$

2.6 Chiral spinors

A spinor with lower (or upper) dot is defined as a right (or left) chiral spinor, which can be described in terms of the chirality operator γ_a:

$$\chi_{\bullet} \ \equiv \ \frac{1-\gamma_5}{2}\chi\,, \qquad \chi^{\bullet} \ \equiv \ \frac{1+\gamma_5}{2}\chi\,. \tag{2.22}$$

In order for non-trivial identities on fermions, we have to regard a Dirac conjugates with lower (or upper) dot symbol as:

$$\overline{\chi}_{\bullet} \ \equiv \ (\overline{\chi})_{\bullet} \ = \ \overline{\chi}\frac{1-\gamma_5}{2} \ = \ \overline{(\chi^{\bullet})}\,, \qquad \text{i.e.,} \quad \overline{\chi}_{\bullet} \ \neq \ \overline{(\chi_{\bullet})}\,, \tag{2.23a}$$

$$\overline{\chi}^{\bullet} \ \equiv \ (\overline{\chi})^{\bullet} \ = \ \overline{\chi}\frac{1+\gamma_5}{2} \ = \ \overline{(\chi_{\bullet})}\,, \qquad \text{i.e.,} \quad \overline{\chi}^{\bullet} \ \neq \ \overline{(\chi^{\bullet})}\,. \tag{2.23b}$$

3 Complete Lagrangian and variations

3.1 Lagrangian

The complete Lagrangian including quartic fermionic interactions is

$$S \;=\; \int \mathrm{d}^4x\sqrt{-g}\,\Big\{\big(\mathscr{L}^{\mathsf{inv}}_{\mathsf{kin}}+\mathscr{L}_{\mathsf{Pauli}}\big)+\big(\mathscr{L}^{\mathsf{inv}}_{\mathsf{4f}}+\mathscr{L}^{\mathsf{non\text{-}inv}}_{\mathsf{4f}}\big)+\big(\mathscr{L}_{\mathsf{mass}}-V(z,\overline{z},q)\big)\Big\}\,. \tag{3.1a}$$

Each part is explicitly exhibited as follows:

$$\begin{aligned}\mathscr{L}^{\mathsf{inv}}_{\mathsf{kin}} \;=\;& -\frac{1}{2}R+g_{i\overline{j}}\,\nabla_\mu z^i\nabla^\mu\overline{z}^{\overline{j}}+h_{uv}\,\nabla_\mu q^u\nabla^\mu q^v+\mathrm{i}\Big(\overline{\mathcal{N}}_{\Lambda\Sigma}\,\mathcal{F}^{\Lambda-}_{\mu\nu}\,\mathcal{F}^{\Sigma-|\mu\nu}-\mathcal{N}_{\Lambda\Sigma}\,\mathcal{F}^{\Lambda+}_{\mu\nu}\,\mathcal{F}^{\Sigma+|\mu\nu}\Big)\\ &-\frac{\mathrm{i}}{2}g_{i\overline{j}}\Big(\overline{\lambda}^{iA}\gamma^\mu\nabla_\mu\lambda^{\overline{j}}_A+\overline{\lambda}^{\overline{j}}_A\gamma^\mu\nabla_\mu\lambda^{iA}\Big)-\mathrm{i}\Big(\overline{\zeta}^\alpha\gamma^\mu\nabla_\mu\zeta_\alpha+\overline{\zeta}_\alpha\gamma^\mu\nabla_\mu\zeta^\alpha\Big)+\frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\Big(\overline{\psi}^A_\mu\gamma_\sigma\rho_{A\nu\lambda}-\overline{\psi}_{A\mu}\gamma_\sigma\rho^A_{\nu\lambda}\Big)\\ &+\Big\{\Big[-g_{i\overline{j}}\nabla_\mu\overline{z}^{\overline{j}}\Big(\overline{\psi}^\mu_A\lambda^{iA}-\overline{\lambda}^{iA}\gamma^{\mu\nu}\psi_{A\mu}\Big)-2\mathcal{U}^{\alpha A}_u\nabla_\mu q^u\Big(\overline{\psi}^\mu_A\zeta_\alpha-\overline{\zeta}_\alpha\gamma^{\mu\nu}\psi_{A\mu}\Big)\Big]+(\mathsf{h.c.})\Big\}\,,\end{aligned} \tag{3.1b}$$

$$\begin{aligned}\mathscr{L}_{\mathsf{Pauli}} \;=\;& \mathcal{F}^{\Lambda-}_{\mu\nu}(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\Big[4L^\Sigma(\overline{\psi}^{A\mu}\psi^{B\nu})\epsilon_{AB}-4\mathrm{i}\overline{f}^{\Sigma}_{\overline{i}}(\overline{\lambda}^{\overline{i}}_A\gamma^\nu\psi^\mu_B)\epsilon^{AB}+\frac{1}{2}\nabla_i f^\Sigma_j(\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB})\epsilon_{AB}-L^\Sigma(\overline{\zeta}_\alpha\gamma^{\mu\nu}\zeta_\beta)\mathbb{C}^{\alpha\beta}\Big]\\ &+(\mathsf{h.c.})\,,\end{aligned} \tag{3.1c}$$

$$\begin{aligned}\mathscr{L}^{\mathsf{inv}}_{\mathsf{4f}} \;=\;& \frac{\mathrm{i}}{2}\frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\Big\{g_{i\overline{j}}(\overline{\lambda}^{iA}\gamma_\sigma\lambda^{\overline{j}}_B)-2\delta^A_B(\overline{\zeta}^\alpha\gamma_\sigma\zeta_\alpha)\Big\}(\overline{\psi}_{A\mu}\gamma_\lambda\psi^B_\nu)+\Big\{\epsilon_{AB}\mathbb{C}_{\alpha\beta}(\overline{\psi}^A_\mu\zeta^\alpha)(\overline{\psi}^{B\mu}\zeta^\beta)+(\mathsf{h.c.})\Big\}\\ &+2g_{i\overline{j}}(\overline{\lambda}^{iA}\gamma_\mu\psi^B_\nu)(\overline{\lambda}^{\overline{j}}_A\gamma^\mu\psi^\nu_B)+g_{i\overline{j}}(\overline{\psi}^A_\mu\lambda^{\overline{j}}_A)(\overline{\psi}^\mu_B\lambda^{iB})+2(\overline{\psi}^A_\mu\zeta^\alpha)(\overline{\psi}^\mu_A\zeta_\alpha)-2(\overline{\psi}^A_\mu\psi^B_\nu)(\overline{\psi}^\mu_A\psi^\nu_B)\\ &-\frac{1}{6}\Big\{C_{ijk}(\overline{\lambda}^{iA}\gamma^\mu\psi^B_\mu)(\overline{\lambda}^{jC}\lambda^{kD})\epsilon_{AC}\epsilon_{BD}+(\mathsf{h.c.})\Big\}\\ &+\frac{1}{4}\Big(R_{i\overline{j}\ell\overline{k}}+g_{i\overline{k}}\,g_{\ell\overline{j}}-\frac{3}{2}g_{i\overline{j}}\,g_{\ell\overline{k}}\Big)(\overline{\lambda}^{iA}\lambda^{\ell B})(\overline{\lambda}^{\overline{j}}_A\lambda^{\overline{k}}_B)-\Big\{\frac{\mathrm{i}}{12}\nabla_m C_{jk\ell}(\overline{\lambda}^{jA}\lambda^{mB})(\overline{\lambda}^{kC}\lambda^{\ell D})\epsilon_{AC}\epsilon_{BD}+(\mathsf{h.c.})\Big\}\\ &+\frac{1}{4}g_{i\overline{j}}(\overline{\zeta}^\alpha\gamma_\mu\zeta_\alpha)(\overline{\lambda}^{iA}\gamma^\mu\lambda^{\overline{j}}_A)+\frac{1}{2}\mathbb{R}^\alpha{}_{\beta|ts}\,\mathcal{U}^t_{\gamma A}\,\mathcal{U}^s_{\delta B}\,\mathbb{C}^{\delta\eta}(\overline{\zeta}_\alpha\zeta_\eta)(\overline{\zeta}^\beta\zeta^\gamma)\,,\end{aligned} \tag{3.1d}$$

$$\begin{aligned}\mathscr{L}^{\mathsf{non\text{-}inv}}_{\mathsf{4f}} \;=\;& (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\Big\{L^\Lambda L^\Sigma(\overline{\psi}^A_\mu\psi^B_\nu)^{(-)}\epsilon_{AB}\Big[2(\overline{\psi}^{C\mu}\psi^{D\nu})^{(-)}\epsilon_{CD}-(\overline{\zeta}_\alpha\gamma^{\mu\nu}\zeta_\beta)\mathbb{C}^{\alpha\beta}\Big]\\ &\qquad-\mathrm{i}L^\Lambda\,\overline{f}^{\Sigma}_{\overline{i}}(\overline{\lambda}^{\overline{i}}_A\gamma^\nu\psi^\mu_B)^{(-)}\Big[8(\overline{\psi}^A_\mu\psi^B_\nu)^{(-)}-(\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta)\epsilon^{AB}\mathbb{C}^{\alpha\beta}\Big]\\ &\qquad+\frac{\mathrm{i}}{2}L^\Lambda\,\overline{f}^{\Sigma}_{\overline{\ell}}\,g^{k\overline{\ell}}\,C_{ijk}(\overline{\psi}^A_\mu\psi^B_\nu)^{(-)}(\overline{\lambda}^{iC}\gamma^{\mu\nu}\lambda^{jD})\epsilon_{AB}\epsilon_{CD}\\ &\qquad-\overline{f}^{\Lambda}_{\overline{i}}\,\overline{f}^{\Sigma}_{\overline{j}}(\overline{\lambda}^{\overline{i}}_A\gamma^\nu\psi^\mu_B)^{(-)}\Big[2(\overline{\lambda}^{\overline{j}}_C\gamma_\nu\psi_{D\mu})^{(-)}\epsilon^{AB}\epsilon^{CD}-g^{k\overline{j}}\,C_{k\ell m}(\overline{\lambda}^{\ell A}\gamma_{\mu\nu}\lambda^{mB})\Big]\\ &\qquad-\frac{1}{32}C_{ijk}\,C_{\ell mn}\,g^{k\overline{r}}\,g^{n\overline{s}}\,\overline{f}^{\Lambda}_{\overline{r}}\,\overline{f}^{\Sigma}_{\overline{s}}(\overline{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB})(\overline{\lambda}^{mC}\gamma^{\mu\nu}\lambda^{\ell B})\epsilon_{AB}\epsilon_{CD}\\ &\qquad-\frac{1}{8}L^\Lambda(\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta)\mathbb{C}^{\alpha\beta}\Big[\nabla_i f^\Sigma_j(\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB})\epsilon_{AB}-L^\Sigma(\overline{\zeta}_\gamma\gamma^{\mu\nu}\zeta_\delta)\mathbb{C}^{\gamma\delta}\Big]\Big\}\\ &+(\mathsf{h.c.})\,,\end{aligned} \tag{3.1e}$$

$$\begin{aligned}\mathscr{L}_{\mathsf{mass}} \;=\;& \mathrm{g}\Big\{2S_{AB}(\overline{\psi}^A_\mu\gamma^{\mu\nu}\psi^B_\nu)+\mathrm{i}g_{i\overline{j}}W^{iAB}(\overline{\lambda}^{\overline{j}}_A\gamma_\mu\psi^\mu_B)+2\mathrm{i}N^A_\alpha(\overline{\zeta}^\alpha\gamma_\mu\psi^\mu_A)\\ &\quad+\mathcal{M}^{\alpha\beta}(\overline{\zeta}_\alpha\zeta_\beta)+\mathcal{M}^\alpha{}_{iB}(\overline{\zeta}_\alpha\lambda^{iB})+\mathcal{M}_{iA|\ell B}(\overline{\lambda}^{iA}\lambda^{\ell B})\Big\}\\ &+(\mathsf{h.c.})\,,\end{aligned} \tag{3.1f}$$

$$V(z,\overline{z},q) \;=\; \mathrm{g}^2\Big\{\Big(g_{i\overline{j}}\,k^i_\Lambda k^{\overline{j}}_\Sigma+4h_{uv}\,k^u_\Lambda k^v_\Sigma\Big)\overline{L}^\Lambda L^\Sigma+\Big(g^{i\overline{j}}\,f^\Lambda_i\overline{f}^\Sigma_{\overline{j}}-3\overline{L}^\Lambda L^\Sigma\Big)\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma\Big\}\,, \tag{3.1g}$$

where $(\dots)^{(-)}$ denotes the "self-dual" part of the tensors or of the fermion bilinears. Notice that we use $\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}$ rather than $\varepsilon^{\mu\nu\rho\sigma}$ in front of $(\overline{\psi}^A_\mu\gamma_\sigma\rho_{A\nu\lambda}-\overline{\psi}_{A\mu}\gamma_\sigma\rho^A_{\nu\lambda})$ in $\mathscr{L}^{\mathsf{inv}}_{\mathsf{kin}}$ such that the kinetic term behaves as a Lagrangian "density" following from (1.1e). Similar terms also appear in $\mathscr{L}_{4\text{f}}^{\text{inv}}$. The relation between $\varepsilon^{\mu\nu\rho\sigma}$ and $\epsilon^{\mu\nu\rho\sigma}$ is defined in (2.10). The contraction rules are similar to the one in the Chern-Simons term.

We should notice that the contraction of indices of the special Kähler geometry in the fourth line of $\mathscr{L}_{4\text{f}}^{\text{non-inv}}$ in [1] should be corrected to this note.[9]

3.2 Local supersymmetry variations

Local supersymmetry variations of the bosonic fields:

$$\begin{aligned}
\delta V_\mu^a &= -\mathrm{i}(\overline{\psi}_{A\mu}\gamma^a\epsilon^A) - \mathrm{i}(\overline{\psi}_\mu^A\gamma^a\epsilon_A)\,, &(3.2\text{a})\\
\delta A_\mu^\Lambda &= 2\,\overline{L}^\Lambda(\overline{\psi}_{A\mu}\epsilon_B)\epsilon^{AB} + 2\,L^\Lambda(\overline{\psi}_\mu^A\epsilon^B)\epsilon_{AB} + \Big\{\mathrm{i}f_i^\Lambda(\overline{\lambda}^{iA}\gamma_\mu\epsilon^B)\epsilon_{AB} + \mathrm{i}\overline{f}_{\overline{i}}^\Lambda(\overline{\lambda}_A^{\overline{i}}\gamma_\mu\epsilon_B)\epsilon^{AB}\Big\}\,, &(3.2\text{b})\\
\delta z^i &= \overline{\lambda}^{iA}\epsilon_A\,, &(3.2\text{c})\\
\delta \overline{z}^{\overline{i}} &= \overline{\lambda}_A^{\overline{i}}\epsilon^A\,, &(3.2\text{d})\\
\delta q^u &= \mathcal{U}_{\alpha A}^u\Big\{(\overline{\zeta}^\alpha\epsilon^A) + \mathbb{C}^{\alpha\beta}\epsilon^{AB}(\overline{\zeta}_\beta\epsilon_B)\Big\}\,. &(3.2\text{e})
\end{aligned}$$

Local supersymmetry variations of the fermionic fields:[10]

$$\begin{aligned}
\delta\psi_{A\mu} &= \mathcal{D}_\mu\epsilon_A - \frac{1}{4}\Big\{\partial_i\mathcal{K}(\overline{\lambda}^{iB}\epsilon_B) - \partial_{\overline{i}}\mathcal{K}(\overline{\lambda}_B^{\overline{i}}\epsilon^B)\Big\}\psi_{A\mu} - \omega_{vA}{}^B\,\mathcal{U}_{\alpha C}^v\Big\{\epsilon^{CD}\mathbb{C}^{\alpha\beta}(\overline{\zeta}_\beta\epsilon_D) + (\overline{\zeta}^\alpha\epsilon^C)\Big\}\psi_{B\mu}\\
&\quad + \Big\{(A^\nu)_A{}^B\eta_{\mu\nu} + (A'^\nu)_A{}^B\gamma_{\mu\nu}\Big\}\epsilon_B + \Big\{\mathrm{i}g\,S_{AB}\eta_{\mu\nu} + \epsilon_{AB}(T_{\mu\nu}^- + U_{\mu\nu}^+)\Big\}\gamma^\nu\epsilon^B\,, &(3.3\text{a})\\
\delta\lambda^{iA} &= \frac{1}{4}\Big\{\partial_j\mathcal{K}(\overline{\lambda}^{jB}\epsilon_B) - \partial_{\overline{j}}\mathcal{K}(\overline{\lambda}_B^{\overline{j}}\epsilon^B)\Big\}\lambda^{iA} - \omega_v{}^A{}_B\,\mathcal{U}_{\alpha C}^v\Big\{\epsilon^{CD}\mathbb{C}^{\alpha\beta}(\overline{\zeta}_\beta\epsilon_D) + (\overline{\zeta}^\alpha\epsilon^C)\Big\}\lambda^{iB}\\
&\quad - \Gamma_{jk}^i(\overline{\lambda}^{kB}\epsilon_B)\lambda^{jA} + \mathrm{i}\Big\{\nabla_\mu z^i - (\overline{\lambda}^{iA}\psi_{A\mu})\Big\}\gamma^\mu\epsilon^A + G_{\mu\nu}^{i-}\gamma^{\mu\nu}\epsilon_B\,\epsilon^{AB} + D^{iAB}\epsilon_B\,, &(3.3\text{b})\\
\delta\zeta_\alpha &= -\Delta_{v\alpha}{}^\beta\,\mathcal{U}_{\gamma A}^v\Big\{\epsilon^{AB}\mathbb{C}^{\gamma\delta}(\overline{\zeta}_\delta\epsilon_B) + (\overline{\zeta}^\gamma\epsilon^A)\Big\}\zeta_\beta + \frac{1}{4}\Big\{\partial_i\mathcal{K}(\overline{\lambda}^{iB}\epsilon_B) - \partial_{\overline{i}}\mathcal{K}(\overline{\lambda}_B^{\overline{i}}\epsilon^B)\Big\}\zeta_\alpha\\
&\quad + \mathrm{i}\Big\{\mathcal{U}_u^{\beta B}\nabla_\mu q^u - \epsilon^{BC}\mathbb{C}^{\beta\gamma}(\overline{\zeta}_\gamma\psi_{C\mu}) - (\overline{\zeta}^\beta\psi_\mu^B)\Big\}\gamma^\mu\epsilon^A\,\epsilon_{AB}\mathbb{C}_{\alpha\beta} + g\,N_\alpha^A\epsilon_A\,. &(3.3\text{c})
\end{aligned}$$

Supergravity values of the auxiliary fields:

$$\begin{aligned}
(A^\mu)_A{}^B &= -\frac{\mathrm{i}}{4}g_{\overline{k}\ell}\Big\{(\overline{\lambda}_A^{\overline{k}}\gamma^\mu\lambda^{\ell B}) - \delta_A^B(\overline{\lambda}_C^{\overline{k}}\gamma^\mu\lambda^{\ell C})\Big\}\,, &(3.4\text{a})\\
(A'^\mu)_A{}^B &= \frac{\mathrm{i}}{4}g_{\overline{k}\ell}\Big\{(\overline{\lambda}_A^{\overline{k}}\gamma^\mu\lambda^{\ell B}) - \frac{1}{2}\delta_A^B(\overline{\lambda}_C^{\overline{k}}\gamma^\mu\lambda^{\ell C})\Big\} - \frac{\mathrm{i}}{4}\delta_A^B(\overline{\zeta}_\alpha\gamma^\mu\zeta^\alpha)\,, &(3.4\text{b})\\
T_{\mu\nu}^- &= 2\mathrm{i}(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}L^\Sigma\Big\{\widehat{F}_{\mu\nu}^{\Lambda-} + \frac{1}{8}\nabla_i f_j^\Lambda(\overline{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB})\epsilon_{AB} - \frac{1}{4}\mathbb{C}^{\alpha\beta}(\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta)L^\Lambda\Big\}\,, &(3.4\text{c})\\
T_{\mu\nu}^+ &= 2\mathrm{i}(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\overline{L}^\Sigma\Big\{\widehat{F}_{\mu\nu}^{\Lambda+} + \frac{1}{8}\nabla_{\overline{i}}\overline{f}_{\overline{j}}^\Lambda(\overline{\lambda}_A^{\overline{i}}\gamma_{\mu\nu}\lambda_B^{\overline{j}})\epsilon^{AB} - \frac{1}{4}\mathbb{C}_{\alpha\beta}(\overline{\zeta}^\alpha\gamma_{\mu\nu}\zeta^\beta)\overline{L}^\Lambda\Big\}\,, &(3.4\text{d})\\
U_{\mu\nu}^- &= -\frac{\mathrm{i}}{4}\mathbb{C}^{\alpha\beta}(\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta)\,, &(3.4\text{e})\\
U_{\mu\nu}^+ &= -\frac{\mathrm{i}}{4}\mathbb{C}_{\alpha\beta}(\overline{\zeta}^\alpha\gamma_{\mu\nu}\zeta^\beta)\,, &(3.4\text{f})\\
G_{\mu\nu}^{i-} &= -g^{i\overline{j}}\overline{f}_{\overline{j}}^\Gamma(\mathrm{Im}\mathcal{N})_{\Gamma\Lambda}\Big\{\widehat{F}_{\mu\nu}^{\Lambda-} + \frac{1}{8}\nabla_k f_\ell^\Lambda(\overline{\lambda}^{kA}\gamma_{\mu\nu}\lambda^{\ell B})\epsilon_{AB} - \frac{1}{4}\mathbb{C}^{\alpha\beta}(\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta)L^\Lambda\Big\}\,, &(3.4\text{g})\\
G_{\mu\nu}^{\overline{i}+} &= -g^{\overline{i}j}f_j^\Gamma(\mathrm{Im}\mathcal{N})_{\Gamma\Lambda}\Big\{\widehat{F}_{\mu\nu}^{\Lambda+} + \frac{1}{8}\nabla_{\overline{k}}\overline{f}_{\overline{\ell}}^\Lambda(\overline{\lambda}_A^{\overline{k}}\gamma_{\mu\nu}\lambda_B^{\overline{\ell}})\epsilon^{AB} - \frac{1}{4}\mathbb{C}_{\alpha\beta}(\overline{\zeta}^\alpha\gamma_{\mu\nu}\zeta^\beta)\overline{L}^\Lambda\Big\}\,, &(3.4\text{h})\\
D^{iAB} &= \frac{\mathrm{i}}{2}g^{i\overline{j}}C_{\overline{j}\overline{k}\overline{\ell}}(\overline{\lambda}_C^{\overline{k}}\lambda_D^{\overline{\ell}})\epsilon^{AC}\epsilon^{BD} + W^{iAB}\,, &(3.4\text{i})\\
\widehat{F}_{\mu\nu}^\Lambda &= \mathcal{F}_{\mu\nu}^\Lambda + L^\Lambda(\overline{\psi}_\mu^A\psi_\nu^B)\epsilon_{AB} + \overline{L}^\Lambda(\overline{\psi}_{A\mu}\psi_{B\nu})\epsilon^{AB} - \mathrm{i}f_i^\Lambda(\overline{\lambda}^{iA}\gamma_{[\nu}\psi_{\mu]}^B)\epsilon_{AB} - \mathrm{i}\overline{f}_{\overline{i}}^\Lambda(\overline{\lambda}_A^{\overline{i}}\gamma_{[\mu}\psi_{B\nu]})\epsilon^{AB}\,. &(3.4\text{j})
\end{aligned}$$

Fermionic mass matrices in the local supersymmetry variarions:

$$S_{AB} = \frac{\mathrm{i}}{2}(\sigma_x)_A{}^C\,\epsilon_{BC}\,\mathcal{P}_\Lambda^x L^\Lambda\,, \qquad (3.5\text{a})$$

[9] I have already got a comment from D'Auria on Jan 12, 2011.

[10] The symbol "K" (the Kähler two-form in [1]) in $\delta\psi_{A\mu}$ etc. has been corrected to "$\mathcal{K}$" which implies the Kähler potential.

$$
\begin{aligned}
W^{iAB} &= \epsilon^{AB}\, k^i_\Lambda\, \overline{L}^\Lambda + \mathrm{i}(\sigma_x)_C{}^B\, \epsilon^{CA}\, \mathcal{P}^x_\Lambda\, g^{i\bar{j}}\, \overline{f}^\Lambda_{\bar{j}} \,, &(3.5b)\\
N^A_\alpha &= 2\mathcal{U}^A_{\alpha u}\, k^u_\Lambda\, \overline{L}^\Lambda \,, &(3.5c)\\
\mathcal{M}^{\alpha\beta} &= -\mathcal{U}^{\alpha A}_u\, \mathcal{U}^{\beta B}_v\, \epsilon_{AB}\, \nabla^{[u}\, k^{v]}_\Lambda\, L^\Lambda \,, &(3.5d)\\
\mathcal{M}^\alpha{}_{iB} &= -4\mathcal{U}^\alpha_{Bu}\, k^u_\Lambda\, f^\Lambda_i \,, &(3.5e)\\
\mathcal{M}_{iA|\ell B} &= \frac{1}{3}\Big\{ \epsilon_{AB}\, g_{i\bar{j}}\, k^{\bar{j}}_\Lambda\, f^\Lambda_\ell + \mathrm{i}(\sigma_x \epsilon^{-1})_{AB}\, \mathcal{P}^x_\Lambda\, \nabla_\ell f^\Lambda_i \Big\} \,. &(3.5f)
\end{aligned}
$$

3.3 Covariant derivatives

The covariant derivatives in the vector multiplets under the gauging:[11]

$$
\begin{aligned}
\nabla z^i &= \mathrm{d}z^i + g\, A^\Lambda k^i_\Lambda(z) \,, &(3.6a)\\
\nabla \overline{z}^{\bar{i}} &= \mathrm{d}\overline{z}^{\bar{i}} + g\, A^\Lambda k^{\bar{i}}_\Lambda(\overline{z}) \,, &(3.6b)\\
\nabla \lambda^{iA} &\equiv \mathrm{d}\lambda^{iA} - \frac{1}{4}\gamma_{ab}\, \omega^{ab}\, \lambda^{iA} - \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\, \lambda^{iA} + \widehat{\Gamma}^i{}_j\, \lambda^{jA} + \widehat{\omega}^A{}_B\, \lambda^{iB} \,, &(3.6c)\\
\nabla \lambda^{\bar{i}}_A &\equiv \mathrm{d}\lambda^{\bar{i}}_A - \frac{1}{4}\gamma_{ab}\, \omega^{ab}\, \lambda^{\bar{i}}_A + \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\, \lambda^{\bar{i}}_A + \widehat{\Gamma}^{\bar{i}}{}_{\bar{j}}\, \lambda^{\bar{j}}_A + \widehat{\omega}_A{}^B\, \lambda^{\bar{i}}_B \,, &(3.6d)\\
F^\Lambda &\equiv \mathrm{d}A^\Lambda + \frac{1}{2} g\, f^\Lambda{}_{\Sigma\Gamma}\, A^\Sigma \wedge A^\Gamma + \overline{L}^\Lambda(\overline{\psi}_A \wedge \psi_B)\epsilon^{AB} + L^\Lambda(\overline{\psi}^A \wedge \psi^B)\epsilon_{AB} \,. &(3.6e)
\end{aligned}
$$

The covariant derivatives in the hypermultiplets:

$$
\begin{aligned}
\mathcal{U}^{A\alpha} &= \mathcal{U}^{A\alpha}_u\, \nabla q^u \;\equiv\; \mathcal{U}^{A\alpha}_u \Big(\mathrm{d}q^u + g\, A^\Lambda k^u_\Lambda(q) \Big) \,, &(3.7a)\\
\nabla \zeta_\alpha &= \mathrm{d}\zeta_\alpha - \frac{1}{4}\gamma_{ab}\, \omega^{ab}\, \zeta_\alpha - \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\, \zeta_\alpha + \widehat{\Delta}_\alpha{}^\beta\, \zeta_\beta \,, &(3.7b)\\
\nabla \zeta^\alpha &= \mathrm{d}\zeta^\alpha - \frac{1}{4}\gamma_{ab}\, \omega^{ab}\, \zeta^\alpha + \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\, \zeta^\alpha + \widehat{\Delta}^\alpha{}_\beta\, \zeta^\beta \,, &(3.7c)\\
\widehat{\Delta}_\alpha{}^\beta &\equiv \widehat{\Delta}^{\gamma\beta}\, \mathbb{C}_{\gamma\alpha} \,, \qquad \widehat{\Delta}^\alpha{}_\beta \;\equiv\; \mathbb{C}_{\beta\gamma}\, \widehat{\Delta}^{\alpha\gamma} \,. &(3.7d)
\end{aligned}
$$

The covariant derivative $\mathcal{D}_\mu \epsilon_A$ in the local supersymmetry variation $\delta\psi_{A\mu}$ is not clearly defined in [1]. It probably contains not only the spin connection ω^{ab} but also the (gauged) connections $\widehat{\mathcal{Q}}$ and $\widehat{\omega}_A{}^B$. A hint appears in (3.67) in [17] and (9.46) in [19]. The explicit expression is given in (3.2.3) in [18]:[12]

$$
\mathcal{D}_\mu \epsilon_A \;\equiv\; \partial_\mu \epsilon_A - \frac{1}{4}\gamma_{ab}\, \omega^{ab}\, \epsilon_A - \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\, \epsilon_A + \widehat{\omega}_A{}^B\, \epsilon_B \,. \tag{3.8}
$$

Notice that the minus sign in front of the gauged connection $\widehat{\mathcal{Q}}$ should be determined by the chirality of ϵ_A because of the following four facts:

(1) The chirality of ϵ_A corresponds to the ones of $\psi_{A\mu}$, λ^{iA} and ζ_α: (1.5).
(2) The chirality is related to the $U(1)_R$ symmetry combined with the Kähler transformation.
(3) This symmetry is carried by the connection of the $U(1)$-bundle: (1.15).
(4) Their covariant derivatives carry the connection of the $U(1)$-bundle $-\frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}$: (3.6) and (3.7).

Let us exhibit the gauged connections in the above covariant derivatives:

$$
\begin{aligned}
\widehat{\Gamma}^i{}_j &= \Gamma^i{}_j + g\, A^\Lambda \partial_j k^i_\Lambda \,, & \Gamma^i{}_j &= \Gamma^i_{kj}\, \mathrm{d}z^k \,, &(3.9a)\\
\widehat{\mathcal{Q}} &= \mathcal{Q} + g\, A^\Lambda \mathcal{P}^0_\Lambda \,, & \mathcal{Q} &= \mathcal{Q}_i\, \mathrm{d}z^i \,, &(3.9b)\\
\widehat{\omega}^x &= \omega^x + g\, A^\Lambda \mathcal{P}^x_\Lambda \,, & \omega^x &= \omega^x_u\, \mathrm{d}q^u \,, &(3.9c)\\
\widehat{\Delta}^{\alpha\beta} &= \Delta^{\alpha\beta} + g\, A^\Lambda\, \partial_u k^v_\Lambda\, \mathcal{U}^{u|\alpha A}\, \mathcal{U}^\beta_{v|A} \,, & \Delta^{\alpha\beta} &= \Delta^{\alpha\beta}_u \mathrm{d}q^u \,. &(3.9d)
\end{aligned}
$$

[11] The symbols $\wedge$ in the fifth term in $\nabla\lambda^{iA}$ and in $\nabla\lambda^{\bar{i}}_A$ described in [1] are not necessary.
[12] I would like to thank Ulrich Theis for a helpful comment (2011 1/13).

3.4 Gauged curvatures

The supercovariant derivatives and the gauged curvatures in the gravitational sector:[13]

$$
\begin{aligned}
T^a &\equiv \mathcal{D}V^a - \mathrm{i}\overline{\psi}_A \wedge \gamma^a \psi^A = \Big(\mathrm{d}V^a - \omega^a{}_b \wedge V^b\Big) - \mathrm{i}\overline{\psi}_A \wedge \gamma^a \psi^A \,, && \text{(3.10a)}\\
\rho_A &\equiv \mathrm{d}\psi_A - \frac{1}{4}\gamma_{ab}\,\omega^{ab} \wedge \psi_A + \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}} \wedge \psi_A + \widehat{\omega}_A{}^B \wedge \psi_B \equiv \nabla\psi_A \,, && \text{(3.10b)}\\
\rho^A &\equiv \mathrm{d}\psi^A - \frac{1}{4}\gamma_{ab}\,\omega^{ab} \wedge \psi^A - \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}} \wedge \psi^A + \widehat{\omega}^A{}_B \wedge \psi^B \equiv \nabla\psi^A \,, && \text{(3.10c)}\\
R^a{}_b &\equiv \mathrm{d}\omega^a{}_b - \omega^a{}_c \wedge \omega^c{}_b \,, && \text{(3.10d)}\\
\omega_A{}^B &= \frac{\mathrm{i}}{2}\omega^x (\sigma_x)_A{}^B \,, \qquad \omega^A{}_B = \epsilon^{AC}\,\epsilon_{DB}\,\omega_C{}^D \,. && \text{(3.10e)}
\end{aligned}
$$

The gauged curvatures in the vector multiplets and hypermultiplets:

$$
\begin{aligned}
\widehat{R}^i{}_j &= R^i{}_{j\bar{\ell}k}\,\nabla\overline{z}^{\bar{\ell}} \wedge \nabla z^k + \mathrm{g}\,F^\Lambda \partial_j k^i_\Lambda \,, && \text{(3.11a)}\\
\widehat{K} &= K_{i\bar{j}}\,\nabla z^i \wedge \nabla\overline{z}^{\bar{j}} + \mathrm{g}\,F^\Lambda \mathcal{P}^0_\Lambda \,, && \text{(3.11b)}\\
\widehat{\Omega}^x &= \Omega^x_{uv}\,\nabla q^u \wedge \nabla q^v + \mathrm{g}\,F^\Lambda \mathcal{P}^x_\Lambda \,, && \text{(3.11c)}\\
\widehat{\mathbb{R}}^{\alpha\beta} &= \mathbb{R}^{\alpha\beta}_{uv}\,\nabla q^u \wedge \nabla q^v + \mathrm{g}\,A^\Lambda\,\partial_u k^v_\Lambda\,\mathcal{U}^{u|\alpha A}\,\mathcal{U}^\beta_{v|A} \,, && \text{(3.11d)}
\end{aligned}
$$

where the originral curvatures are defined as follows:

$$
\begin{aligned}
R^i{}_{j\bar{\ell}k} &= \partial_{\bar{\ell}}\Gamma^i{}_{jk} \,, & \Gamma^i_{jk} &= g^{i\bar{\ell}}(\partial_k g_{j\bar{\ell}}) \,, && \text{(3.12a)}\\
K &= K_{i\bar{j}}\,\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^{\bar{j}} = \frac{\mathrm{i}}{2\pi}\mathcal{K}_{i\bar{j}}\,\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^{\bar{j}} \,, & \mathcal{K}_{i\bar{j}} &= g_{i\bar{j}} = \partial_i\partial_{\bar{j}}\mathcal{K} \,, && \text{(3.12b)}\\
\Omega^x &= \mathrm{d}\omega^x + \frac{1}{2}\epsilon^{xyz}\,\omega^y \wedge \omega^z = \Omega^x_{uv}\,\mathrm{d}q^u \wedge \mathrm{d}q^v \,, &&&& \text{(3.12c)}\\
\mathbb{R}^{\alpha\beta} &= \mathrm{d}\Delta^{\alpha\beta} + \Delta^{\alpha\gamma} \wedge \Delta^{\delta\beta}\,\mathbb{C}_{\gamma\delta} = \mathbb{R}^{\alpha\beta}_{uv}\,\mathrm{d}q^u \wedge \mathrm{d}q^v \,. &&&& \text{(3.12d)}
\end{aligned}
$$

3.5 Comments: towards a symplectic covariant formulation

The Lagrangians $\mathscr{L}_{\mathsf{Pauli}}$ (3.1c) and $\mathscr{L}_{\mathsf{4f}}^{\mathsf{non\text{-}inv}}$ are not invariant under the symplectic rotations because they only involve L^Λ (and its Kähler covariant derivative $f_i^\Lambda = \nabla_i L^\Lambda$), i.e., the half degrees of the symplectic vector $V = (L^\Lambda, M_\Sigma)$. In order to construct the symplectic invariant Lagrangian, i.e., the electric-magnetic covariant Lagrangian, one has to introduce additional terms depending on M_Σ (and its Kähler covariant derivative $h_{\Sigma i} = \nabla_i M_\Sigma$).

Actually, the literature [9] discussed an extension of an $\mathcal{N} = 2$ gauged supergravity coupled to tensor multiplets in order to include these variables. This is an well-established extension of the standard (electrically) gauged supergravity [1] because [9] introduced the "magnetic" part $\omega^x_I m^{I\Lambda} M_\Lambda (\equiv \mathcal{P}^{x\Lambda} M_\Lambda)$ coupled to the tensor fields in eqs.(2.13)-(2.15) there. In particular cases, the electric Killing prepotential is given by the electric charges as $\mathcal{P}^x_\Lambda = \omega^x_I e^I_\Lambda$, while the newly introduced magnetic Killing prepotentials are (implicitly) given by $\mathcal{P}^{x\Lambda} = \omega^x_I m^{I\Lambda}$. Indeed these magnetic part of the Killing prepotentials explicitly appear in papers on (non)geometric flux compactifications (see (4.1) in [20], (A.2)-(A.5) in [21], etc.). This is one of the important step to complete the symplectically covariant formalism of $\mathcal{N} = 2$ gauged supergravity.

[13] The explicit form of $\mathcal{D}V^a$ can be seen by comparing between (A.1) in [1] and (B.1) in [9].

4 Fixing the conversion rule

4.1 (Anti-)self-dual tensor

The sign of the operator $-\frac{\mathrm{i}}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}$ which acts on an (anti-)self-dual tensor (1.3) is flipped to $+\frac{\mathrm{i}}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}$:

$$(\mathcal{T}^{(\pm)}_{\mu\nu})^{\mathsf{new}} \ \equiv\ \frac{1}{2}\Big(\mathcal{T}_{\mu\nu} \pm \frac{\mathrm{i}}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}\mathcal{T}^{\rho\sigma}\Big) \ =\ \mathcal{T}^{(\pm)}_{\mu\nu} \quad \text{with} \quad +\frac{\mathrm{i}}{2}\sqrt{-g}\,\varepsilon_{\mu\nu}{}^{\rho\sigma}(\mathcal{T}^{(\pm)}_{\rho\sigma})^{\mathsf{new}} \ =\ \pm(\mathcal{T}^{(\pm)}_{\mu\nu})^{\mathsf{new}}\,. \tag{4.1}$$

Then we change the name of $\mathcal{T}^{(-)}_{\mu\nu}$ from a "self-dual" tensor in [1] to an "anti-self-dual" tensor in later sections. We also refer to $\mathcal{T}^{(+)}_{\mu\nu}$ as a self-dual tensor from an "anti-self-dual" tensor in [1]. This indicates that we simply change the names but mathematical expressions of $\mathcal{T}^{(\pm)}_{\mu\nu}$ are unchanged.

4.2 Gamma matrix and signature of the spacetime metric

A simple way to flip the signature via the Clifford algebra (1.4a) is changing such as

$$\gamma_a \ \equiv\ -\mathrm{i}(\gamma_a)^{\mathsf{new}} \qquad \Big(\gamma_a \ \neq\ +\mathrm{i}(\gamma_a)^{\mathsf{new}}\Big) \tag{4.2}$$

in (1.4) if we consider the Dirac conjugate compared to [17] (or compared to my convention in appendix A.3). Then the signature is natually flipped to

$$2\eta_{ab} \ =\ \{\gamma_a, \gamma_b\} \ =\ \{-\mathrm{i}(\gamma_a)^{\mathsf{new}}, -\mathrm{i}(\gamma_b)^{\mathsf{new}}\} \ =\ -\{(\gamma_a)^{\mathsf{new}}, (\gamma_b)^{\mathsf{new}}\} \ \equiv\ -2(\eta_{ab})^{\mathsf{new}}\,, \tag{4.3a}$$

$$(\eta_{ab})^{\mathsf{new}} \ \equiv\ -\eta_{ab} \ =\ (-,+,+,+)\,, \tag{4.3b}$$

$$\gamma^a \ =\ \eta^{ab}\gamma_b \ =\ (-\eta^{ab})^{\mathsf{new}}\big\{-\mathrm{i}(\gamma_b)^{\mathsf{new}}\big\} \ =\ \mathrm{i}(\gamma^a)^{\mathsf{new}}\,. \tag{4.3c}$$

If we change the signature, $\overline{\chi}$ should be attached "i" and γ_0 should be changed to γ^0. But if we follow the above re-definition (4.2), it is automatically done:

$$\begin{aligned}\overline{\chi} \ &=\ \chi^{\dagger}\gamma_0 \ =\ \chi^{\dagger}\{-\mathrm{i}(\gamma_0)^{\mathsf{new}}\} \ =\ \chi^{\dagger}\{-\mathrm{i}(\gamma^0)^{\mathsf{new}}(\eta_{00})^{\mathsf{new}}\} \ =\ \mathrm{i}\chi^{\dagger}(\gamma^0)^{\mathsf{new}} \\ &=\ \overline{\chi}^{\mathsf{new}}\,.\end{aligned} \tag{4.4}$$

This is nothing but the Dirac conjugate defined as in [17] and as in appendix A.3. This implies that we do directly use the descriptions of all fermionic terms from the original ones in [1] without changing any factors.

Caused by this sign-flipping, the signs of terms carrying odd number of spacetime metric in the Lagrangian, the variations, and all other equations are also flipped, whilst the sign of terms containing even number of the metrics is kept invariant. Notice that we have to be careful if the gamma matrices such as $\gamma^{\mu} = g^{\mu\nu}\gamma_{\nu}$ are contracted; i.e., we have to avoid inconsistently redundant multi-flippings:

$$\det(g_{\mu\nu}) \ =\ (-1)^4\det(g_{\mu\nu})^{\mathsf{new}} \ =\ (\det g_{\mu\nu})^{\mathsf{new}}\,, \tag{4.5a}$$

$$\mathcal{A}_{\mu}\mathcal{B}^{\mu} \ =\ g^{\mu\nu}\mathcal{A}_{\mu}\mathcal{B}_{\nu} \ =\ (-g^{\mu\nu})^{\mathsf{new}}\mathcal{A}_{\mu}\mathcal{B}_{\nu} \ =\ -(\mathcal{A}_{\mu}\mathcal{B}^{\mu})^{\mathsf{new}}\,, \tag{4.5b}$$

$$\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu} \ =\ g^{\mu\rho}g^{\nu\sigma}\,\mathcal{T}_{\mu\nu}\mathcal{T}_{\rho\sigma} \ =\ (-g^{\mu\rho})^{\mathsf{new}}(-g^{\nu\sigma})^{\mathsf{new}}\,\mathcal{T}_{\mu\nu}\mathcal{T}_{\rho\sigma} \ =\ (\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu})^{\mathsf{new}}\,, \tag{4.5c}$$

$$\mathrm{i}\gamma_{\mu}\mathcal{A}^{\mu} \ =\ \mathrm{i}g^{\mu\rho}\gamma_{\mu}\mathcal{A}_{\rho} \ =\ \mathrm{i}(-g^{\mu\rho})^{\mathsf{new}}\{-\mathrm{i}(\gamma_{\mu})^{\mathsf{new}}\}\mathcal{A}_{\rho} \ =\ -(\gamma_{\mu}\mathcal{A}^{\mu})^{\mathsf{new}}\,, \tag{4.5d}$$

$$\mathrm{i}\gamma^{\mu}\mathcal{A}_{\mu} \ =\ \mathrm{i}g^{\mu\nu}\gamma_{\nu}\mathcal{A}_{\mu} \ =\ \mathrm{i}(-g^{\mu\nu})^{\mathsf{new}}\{-\mathrm{i}(\gamma_{\nu})^{\mathsf{new}}\}\mathcal{A}_{\mu} \ =\ -(\gamma^{\mu}\mathcal{A}_{\mu})^{\mathsf{new}} \ =\ -(\gamma_{\mu}\mathcal{A}^{\mu})^{\mathsf{new}}\,, \tag{4.5e}$$

$$\gamma_{\mu}\gamma^{\mu} \ =\ g^{\mu\nu}\gamma_{\mu}\gamma_{\nu} \ =\ (-g^{\mu\nu})^{\mathsf{new}}\{-\mathrm{i}(\gamma_{\mu})^{\mathsf{new}}\}\{-\mathrm{i}(\gamma_{\nu})^{\mathsf{new}}\} \ =\ (\gamma_{\mu}\gamma^{\mu})^{\mathsf{new}}\,, \tag{4.5f}$$

$$\mathcal{T}_{\mu\nu}\gamma^{\mu\nu} \ =\ (-g^{\mu\rho})^{\mathsf{new}}(-g^{\nu\sigma})^{\mathsf{new}}\mathcal{T}_{\mu\nu}(-\mathrm{i})^2(\gamma_{\rho\sigma})^{\mathsf{new}} \ =\ -(\mathcal{T}_{\mu\nu}\gamma^{\mu\nu})^{\mathsf{new}}\,, \tag{4.5g}$$

$$\mathcal{T}^{\mu\nu}\gamma_{\mu\nu} \ =\ (-g^{\mu\rho})^{\mathsf{new}}(-g^{\nu\sigma})^{\mathsf{new}}\mathcal{T}_{\rho\sigma}(-\mathrm{i})^2(\gamma_{\mu\nu})^{\mathsf{new}} \ =\ -(\mathcal{T}^{\mu\nu}\gamma_{\mu\nu})^{\mathsf{new}} \ =\ -(\mathcal{T}_{\mu\nu}\gamma^{\mu\nu})^{\mathsf{new}}\,. \tag{4.5h}$$

The keypoint is that we have to rewrite the contractions of the indices in terms of the spacetime metric, which is also flipped. In particular, we have to be careful for $\gamma_{\mu}\mathcal{A}^{\mu} = \gamma^{\mu}\mathcal{A}_{\mu} = g^{\mu\nu}\gamma_{\mu}\mathcal{A}_{\nu}$.

4.3 Natural four-form

The orientation of $\mathrm{d}^4x = -\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ in a spacetime in [1] is different from my experiences. Here let us flip the overall sign to $(\mathrm{d}^4x)^{\mathsf{new}} = +\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ as in appendix A.4. Because of this sign flipping and the property of $\varepsilon_{\mu\nu\rho\sigma}$ (A.13), the symbol $\epsilon^{\mu\nu\rho\sigma}$ in (1.1e) might be or might not be re-defined. Let us check it:

$$(\mathrm{d}^4x)^{\mathsf{new}} = \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = +\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \,, \tag{4.6a}$$

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \equiv \alpha_3\,\varepsilon^{\mu\nu\rho\sigma}(\mathrm{d}^4x)^{\mathsf{new}} \equiv -\frac{(\epsilon^{\mu\nu\rho\sigma})^{\mathsf{new}}}{(\sqrt{-g})^{\mathsf{new}}}\Big\{\sqrt{-g}\,\mathrm{d}^4x\Big\}^{\mathsf{new}} . \tag{4.6b}$$

Multiplying $\varepsilon_{\mu\nu\rho\sigma}$, we compute the left-hand-side and the middle-side in the second line respectively:

$$\text{(LHS)} = 4!\,(\mathrm{d}^4x)^{\mathsf{new}} \,, \qquad \text{(MS)} = \frac{4!\alpha_3}{g^{\mathsf{new}}}\,(\mathrm{d}^4x)^{\mathsf{new}} \qquad \to \qquad \alpha_3 = g^{\mathsf{new}} . \tag{4.6c}$$

Noticing $g^{\mathsf{new}} = \det(g_{\mu\nu})^{\mathsf{new}} = g$, we find that the $(\epsilon^{\mu\nu\rho\sigma})^{\mathsf{new}}$ is nothing but the original $\epsilon^{\mu\nu\rho\sigma}$:

$$(\epsilon^{\mu\nu\rho\sigma})^{\mathsf{new}} = -g\,\varepsilon^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} \,, \qquad (\epsilon^{0123})^{\mathsf{new}} = \epsilon^{0123} = -g\,\varepsilon^{0123} = -1 . \tag{4.7a}$$

In the same analogy of the epsilon tensor, we also define $(\epsilon_{\mu\nu\rho\sigma})^{\mathsf{new}} = \epsilon_{\mu\nu\rho\sigma}$:

$$(\epsilon_{\mu\nu\rho\sigma})^{\mathsf{new}} \equiv (g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta}\,\epsilon^{\mu\nu\rho\sigma})^{\mathsf{new}} = g_{\mu\alpha}g_{\nu\beta}g_{\rho\gamma}g_{\sigma\delta}\,\epsilon^{\mu\nu\rho\sigma} = g(-g)\varepsilon^{\mu\nu\rho\sigma} = (-g)\varepsilon_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma} \,, \tag{4.7b}$$

$$(\epsilon_{0123})^{\mathsf{new}} = \epsilon_{0123} = -g . \tag{4.7c}$$

Applying (4.6) to the wedge products, we obtain the followings:

$$\begin{aligned} F \wedge F &= \mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = -\Big\{\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\Big(\sqrt{-g}\,\mathrm{d}^4x\Big)\Big\}^{\mathsf{new}} \\ &= -\sqrt{-g}\,\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\Big\{\sqrt{-g}\,(\mathrm{d}^4x)^{\mathsf{new}}\Big\} \\ &= -2\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}^{\mu\nu}\Big\{\sqrt{-g}\,(\mathrm{d}^4x)^{\mathsf{new}}\Big\} , \end{aligned} \tag{4.8a}$$

$$\begin{aligned} F \wedge *F &= \mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}_{\rho\sigma}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = \frac{g\sqrt{-g}}{2!}\mathcal{F}_{\mu\nu}\mathcal{F}^{\lambda\gamma}\,\varepsilon_{\rho\sigma\lambda\gamma}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \\ &= \frac{g\sqrt{-g}}{2!}\mathcal{F}_{\mu\nu}\mathcal{F}^{\lambda\gamma}\,\varepsilon_{\rho\sigma\lambda\gamma}\varepsilon^{\mu\nu\rho\sigma}\,(\mathrm{d}^4x)^{\mathsf{new}} = \frac{g\sqrt{-g}}{2!}\mathcal{F}_{\mu\nu}\mathcal{F}^{\lambda\gamma}\Big\{\frac{2!}{g}\big(\delta^\mu_\lambda\delta^\nu_\gamma - \delta^\nu_\lambda\delta^\mu_\gamma\big)\Big\}(\mathrm{d}^4x)^{\mathsf{new}} \\ &= 2\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\Big\{\sqrt{-g}\,(\mathrm{d}^4x)^{\mathsf{new}}\Big\} . \end{aligned} \tag{4.8b}$$

It would be useful to introduce a new symbol $\not\epsilon_{\mu\nu\rho\sigma}$, which is related to $\varepsilon_{\mu\nu\rho\sigma}$ and $\epsilon_{\mu\nu\rho\sigma}$ as follows:

$$\not\epsilon_{\mu\nu\rho\sigma} \equiv \sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma} = \frac{\epsilon_{\mu\nu\rho\sigma}}{\sqrt{-g}} \,, \qquad \not\epsilon_{0123} = \sqrt{-g} \,, \tag{4.9a}$$

$$\not\epsilon^{\mu\nu\rho\sigma} \equiv \sqrt{-g}\,\varepsilon^{\mu\nu\rho\sigma} = \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}} \,, \qquad \not\epsilon^{0123} = -\frac{1}{\sqrt{-g}} \,, \tag{4.9b}$$

$$\not\epsilon_{\mu\nu\rho\sigma}\not\epsilon^{\mu\nu\rho\sigma} = -4! \qquad \not\epsilon_{\mu\nu\rho\sigma}\not\epsilon^{\alpha\nu\rho\sigma} = -3!\,\delta^\alpha_\mu \,, \qquad \not\epsilon_{\mu\nu\rho\sigma}\not\epsilon^{\alpha\beta\rho\sigma} = -2!\Big(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\beta_\mu\delta^\alpha_\nu\Big) . \tag{4.9c}$$

4.4 Affine connection, spin connection, and curvature

Following the redefinition (4.3), let us consider the spin connection one-form $\omega^a{}_b$ in [1] under the following sign flipping

$$\omega^a{}_b \equiv -(\omega^a{}_b)^{\mathsf{new}} \qquad \to \qquad \omega^{ab} = \eta^{bc}\,\omega^a{}_c = (-\eta^{bc})^{\mathsf{new}}(-\omega^a{}_c)^{\mathsf{new}} = +(\omega^{ab})^{\mathsf{new}} . \tag{4.10}$$

Combining (4.2) and (4.10), we can see the followings:

$$-\omega^a{}_b \wedge V^b = -(-\omega^a{}_b)^{\mathsf{new}} \wedge V^b = +(\omega^a{}_b)^{\mathsf{new}} \wedge V^b \,, \tag{4.11a}$$

$$-\frac{1}{4}\gamma_{ab}\,\omega^{ab} = -\frac{1}{4}\big\{(-\mathrm{i})^2(\gamma_{ab})^{\mathsf{new}}\big\}(+\omega^{ab})^{\mathsf{new}} = +\frac{1}{4}(\gamma_{ab})^{\mathsf{new}}(\omega^{ab})^{\mathsf{new}} . \tag{4.11b}$$

Refered to [16, 17] and other works of literature and using the re-definitions (4.3) and (4.10), we re-define the curvature two-form $R^a{}_b = \mathrm{d}\omega^a{}_b - \omega^a{}_c \wedge \omega^c{}_b$ in (3.10d) in the following way:

$$\begin{aligned} R^a{}_b &= \mathrm{d}\omega^a{}_b - \omega^a{}_c \wedge \omega^c{}_b \;=\; \mathrm{d}(-\omega^a{}_b)^{\text{new}} - (-\omega^a{}_c)^{\text{new}} \wedge (-\omega^c{}_b)^{\text{new}} \;=\; -\Big(\mathrm{d}(\omega^a{}_b)^{\text{new}} + (\omega^a{}_c)^{\text{new}} \wedge (\omega^c{}_b)^{\text{new}}\Big) \\ &\equiv -(R^a{}_b)^{\text{new}} \,, \qquad (4.12\text{a}) \\ R^{ab} &= \mathrm{d}\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} \;=\; \mathrm{d}(\omega^{ab})^{\text{new}} - (-\omega^a{}_c)^{\text{new}} \wedge (\omega^{cb})^{\text{new}} \;=\; \Big(\mathrm{d}(\omega^{ab})^{\text{new}} + (\omega^a{}_c)^{\text{new}} \wedge (\omega^{cb})^{\text{new}}\Big) \\ &= +(R^{ab})^{\text{new}} \;=\; (\eta^{bc})^{\text{new}} (R^a{}_c)^{\text{new}} \,. \qquad (4.12\text{b}) \end{aligned}$$

We impose that the scalar curvature of the spin connection $(R(\omega))^{\text{new}}$ must correspond to the scalar curvature of the affine connection $(R(\Gamma))^{\text{new}}$ in the converted system [16]:

$$\begin{aligned} \big(R(\omega)\big)^{\text{new}} &= (R^{ab}{}_{\mu\nu})^{\text{new}} V_a^\mu V_b^\nu \;=\; \delta_\mu^\rho (g^{\nu\sigma})^{\text{new}} (R^{ab}{}_{\rho\sigma})^{\text{new}} V_a^\mu V_{b\nu} \;\equiv\; \delta_\mu^\rho (g^{\nu\sigma})^{\text{new}} (R^\mu{}_{\nu\rho\sigma})^{\text{new}} \\ &= \big(R(\Gamma)\big)^{\text{new}} \,, \qquad (4.13\text{a}) \\ \big(R^\mu{}_{\nu\rho\sigma}(\Gamma)\big)^{\text{new}} &\equiv \partial_\rho \Gamma^\mu{}_{\sigma\nu} - \partial_\sigma \Gamma^\mu{}_{\rho\nu} + \Gamma^\mu{}_{\rho\lambda}\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\mu{}_{\sigma\lambda}\Gamma^\lambda{}_{\rho\nu} \;=\; -R^\mu{}_{\nu\rho\sigma}(\Gamma) \,. \qquad (4.13\text{b}) \end{aligned}$$

where the curvature of the affine connection is redefined from (1.2) as [16, 17]. Recall that the sign of the scalar curvature $R(\omega)$ under the flipping (4.10) is not changed. In the same analogy, the scalar curvature $R(\Gamma)$ is also invariant under the flipping (4.13):

$$\big(R(\Gamma)\big)^{\text{new}} \;=\; \delta_\mu^\rho (g^{\nu\sigma})^{\text{new}} (R^\mu{}_{\nu\rho\sigma})^{\text{new}} \;=\; \delta_\mu^\rho (-g^{\nu\sigma})(-R^\mu{}_{\nu\rho\sigma}) \,. \qquad (4.14)$$

In order that the kinetic energy of the gravitational fields (i.e., the fluctuation fields from the background metric) should be non-negative, the minus sign of the Einstein-Hilbert action in [1] (or the one in (3.1b)) has to be defined by a different contraction on the scalar curvature as defined in [23]:

$$-R([1]) \;\equiv\; -g^{\rho\sigma} R^\mu{}_{\rho\sigma\mu}(\Gamma) \;=\; -(-g^{\rho\sigma})^{\text{new}}(-R^\mu{}_{\rho\sigma\mu})^{\text{new}} \;=\; -(-g^{\rho\sigma})^{\text{new}}(+R^\mu{}_{\rho\mu\sigma})^{\text{new}} \;=\; \big(R(\Gamma)\big)^{\text{new}} \,. \qquad (4.15\text{a})$$

To keep the correspondence of the two scalar curvature $R(\omega) = R(\Gamma)$ even in [1], we have to fix the definitin of the curvature of the spin connection $R^{ab}{}_{\mu\nu}(\omega)$ via the definition of the curvature of the affine connection as in the left-hand-side of (4.15a):

$$R(\omega) \;\equiv\; R(\Gamma) \;=\; g^{\rho\mu} R^\nu{}_{\rho\mu\nu}(\Gamma) \;=\; g^{\rho\mu} R^{ab}{}_{\mu\nu}(\omega) V_a^\nu V_{b\rho} \;=\; R^{ab}{}_{\mu\nu}(\omega) V_a^\nu V_b^\mu \;=\; -\big(R(\omega)\big)^{\text{new}} \,. \qquad (4.15\text{b})$$

Indeed, the works of literature [23, 26][14] adopted the Einstein-Hilbert action $\mathscr{L}_{\text{EH}} = -\frac{1}{2}R$ with the contraction (the left-hand-side of (4.15a)), whilst the convention $\mathscr{L}_{\text{EH}} = +\frac{1}{2}R$ with (4.14) is [16, 17, 19, 24]. Note that all these works of literature [16, 17, 19, 23, 24] adopt the mostly plus signature (4.3). An typical discussion can be seen in section 4.5.3 of [24].

The main point to fix the sign of the Einstein-Hilbert action is that the kinetic term of the fluctuation fields $g_{\mu\nu} = \eta^{\text{flat}}_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2)$ behaves in a similar way as the one of the scalar fields as

$$\mathscr{L}_{\text{fluc}} \;\sim\; -\partial_\mu h_{\nu\lambda} \partial^\mu h^{\nu\lambda} \qquad \longleftrightarrow \qquad \mathscr{L}_{\text{scalar}} \;=\; -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi \;=\; +\frac{1}{2}(\partial_0 \phi)^2 + \ldots \,. \qquad (4.16)$$

This form implies that the sign of the signature is also sensitive to the sign of the Einstein-Hilbert action. Eventually we adopt the following re-definition with using the set of redefinitions (4.3), (4.10) and (4.13) via (4.15a):

$$\mathscr{L}_{\text{EH}} \;=\; -\frac{1}{2}R([1]) \;=\; +\frac{1}{2}R^{\text{new}} \,. \qquad (4.17)$$

We summarize all variations in appendix B.

[14] Bernard de Wit also introduced the Ricci tensor in [26] different from the ones in [16, 17]. Strictly speaking, however, curvature in [26] corresponds to (1.2) in this note, whilst the definition of the Ricci tensor is given as $R_{\mu\nu} = R^\rho{}_{\mu\rho\sigma}$ as in [16]. Eventually the scalar curvature of [26] is equal to the one of [23]. See the Remark in appendix B.

4.5 Structure on gauge kinetic term

Here we rewrite the gauge kinetic term discussed in (2.15):

$$\mathscr{L}_{\mathsf{kin}}^{\mathsf{gauge}} \ = \ \mathrm{i}\Big(\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^{\Lambda-}\mathcal{F}^{\Sigma-|\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^{\Lambda+}\mathcal{F}^{\Sigma+|\mu\nu}\Big) \ = \ (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^{\Lambda}\mathcal{F}^{\Sigma\mu\nu} + (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^{\Lambda}\widetilde{\mathcal{F}}^{\Sigma\mu\nu} \,. \tag{4.18}$$

Since the definition of $\mathcal{F}_{\mu\nu}^{\pm}$ is given in (4.1) and the above form is given by contraction of two spacetime metrics without any gamma matrices, the above expressions are unchanged under the new conventions:

$$\mathscr{L}_{\mathsf{kin}}^{\mathsf{gauge}} \ = \ \mathrm{i}\Big(\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^{\Lambda-}\mathcal{F}^{\Sigma-|\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^{\Lambda+}\mathcal{F}^{\Sigma+|\mu\nu}\Big)^{\mathsf{new}} \ = \ \Big[(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^{\Lambda}\mathcal{F}^{\Sigma\mu\nu} + (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}_{\mu\nu}^{\Lambda}\widetilde{\mathcal{F}}^{\Sigma\mu\nu}\Big]^{\mathsf{new}} . \tag{4.19}$$

The (generalized) magnetic dual of the gauge field strength (2.16) and (2.17) is also redefined:

$$\pm\frac{\mathrm{i}}{2}\frac{\delta\mathscr{L}}{\delta\mathcal{F}^{\Lambda\pm|\mu\nu}} \ = \ \mathcal{G}_{\Lambda\mu\nu}^{\pm} \ = \ (\mathcal{G}_{\Lambda\mu\nu}^{\pm})^{\mathsf{new}} \ = \ \pm\frac{\mathrm{i}}{2}\Big(\frac{\delta\mathscr{L}}{\delta\mathcal{F}^{\Lambda\pm|\mu\nu}}\Big)^{\mathsf{new}} , \tag{4.20a}$$

$$\frac{1}{2}\frac{\delta\mathscr{L}}{\delta\widetilde{\mathcal{F}}^{\Lambda\mu\nu}} \ = \ \mathcal{G}_{\Lambda\mu\nu} \ = \ (\mathcal{G}_{\Lambda\mu\nu})^{\mathsf{new}} \ = \ \Big(\frac{1}{2}\frac{\delta\mathscr{L}}{\delta\widetilde{\mathcal{F}}^{\Lambda\mu\nu}}\Big)^{\mathsf{new}} , \tag{4.20b}$$

$$\frac{1}{2}\frac{\delta\mathscr{L}}{\delta\mathcal{F}^{\Lambda\mu\nu}} \ = \ \widetilde{\mathcal{G}}_{\Lambda\mu\nu} \ = \ (\widetilde{\mathcal{G}}_{\Lambda\mu\nu})^{\mathsf{new}} \ = \ \Big(\frac{1}{2}\frac{\delta\mathscr{L}}{\delta\mathcal{F}^{\Lambda\mu\nu}}\Big)^{\mathsf{new}} . \tag{4.20c}$$

4.6 Summary

Here let us summarize the conversion rules:

$$\gamma_a \ = \ -\mathrm{i}(\gamma_a)^{\mathsf{new}} \qquad \eta^{ab} \ = \ -(\eta^{ab})^{\mathsf{new}} \qquad \gamma^a \ = \ \mathrm{i}(\gamma^a)^{\mathsf{new}} , \tag{4.21a}$$

$$\overline{\chi} \ = \ \overline{\chi}^{\mathsf{new}} , \tag{4.21b}$$

$$\mathrm{d}^4 x \ = \ -(\mathrm{d}^4 x)^{\mathsf{new}} , \qquad \epsilon^{\mu\nu\rho\sigma} \ = \ +(\epsilon^{\mu\nu\rho\sigma})^{\mathsf{new}} \ = \ -g\,\varepsilon^{\mu\nu\rho\sigma} , \tag{4.21c}$$

$$\omega^a{}_b \ = \ -(\omega^a{}_b)^{\mathsf{new}} , \qquad \omega^{ab} \ = \ +(\omega^{ab})^{\mathsf{new}} , \tag{4.21d}$$

$$R^a{}_b(\omega) \ = \ -\big(R^a{}_b(\omega)\big)^{\mathsf{new}} , \qquad R^{ab}(\omega) \ = \ +\big(R^{ab}(\omega)\big)^{\mathsf{new}} , \qquad R^{\mu}{}_{\nu\rho\sigma}(\Gamma) \ = \ -\big(R^{\mu}{}_{\nu\rho\sigma}(\Gamma)\big)^{\mathsf{new}} , \tag{4.21e}$$

$$R_{\mu\nu}(\Gamma) \ = \ R^{\rho}{}_{\mu\nu\rho}(\Gamma) \ = \ -R^{\rho}{}_{\mu\rho\nu}(\Gamma) \ = \ +\big(R^{\rho}{}_{\mu\rho\nu}(\Gamma)\big)^{\mathsf{new}} \ = \ +\big(R_{\mu\nu}(\Gamma)\big)^{\mathsf{new}} , \tag{4.21f}$$

$$R(\Gamma) \ = \ g^{\mu\nu}R_{\mu\nu}(\Gamma) \ = \ (-g^{\mu\nu})^{\mathsf{new}}\big(R_{\mu\nu}(\Gamma)\big)^{\mathsf{new}} \ = \ -\big(R(\Gamma)\big)^{\mathsf{new}} \ = \ -\big(R(\omega)\big)^{\mathsf{new}} \ = \ +R(\omega) \,. \tag{4.21g}$$

[Comment] I wanted to redefine the normalization of the two-forms in the quaternionic geometry in order to use the definition of the differential forms in appendix A.4. However, once I perform it, I have to further redefine many equations in many lieteratures which have already been established, except for [19, 17] (see appendix C.2). In the present stage I am embarassed about redefining the metric h_{uv} with or without changing the parameter λ, which should appear in interaction terms derived from the supersymmetry variations. This is caused by the lack of my knowledge of the quaternionic geometry. Therefore (屈辱的だが) I do not change the normalization of the two-forms K^x, Ω^x, $\mathbb{R}^{\alpha\beta}$ and the normalization of the kinetic term of the real scalar fields q^u. **I have to rewrite the action in terms of the "canonical" definitions as in [19, 17] with checking supersymmetry invariance in this note, including the gravitational coupling constant! (February 12, 2011)**

4.7 Additional redefinition: normalization of two-forms and more

We change the normalizations of the two-forms (2.1) in the following ways:

$$\mathcal{F}_{\mu\nu}^{\Lambda} \ \equiv \ \frac{1}{2!}F_{\mu\nu}^{\Lambda} , \qquad \widetilde{\mathcal{F}}_{\mu\nu}^{\Lambda} \ \equiv \ \frac{1}{2!}\widetilde{F}_{\mu\nu}^{\Lambda} \ = \ \frac{\sqrt{-g}}{2!\cdot 2!}\,\varepsilon_{\mu\nu\rho\sigma}F^{\Lambda\rho\sigma} , \qquad \widehat{F}_{\mu\nu}^{\Lambda} \ \equiv \ \frac{1}{2!}(\widehat{F}_{\mu\nu}^{\Lambda})^{\mathsf{new}} , \tag{4.22a}$$

$$\mathcal{G}_{\Lambda\mu\nu} \ \equiv \ \frac{1}{2!}G_{\Lambda\mu\nu} , \qquad \widetilde{\mathcal{G}}_{\Lambda\mu\nu} \ \equiv \ \frac{1}{2!}\widetilde{G}_{\Lambda\mu\nu} , \qquad \mathcal{G}_{\Lambda\mu\nu}^{\pm} \ \equiv \ \frac{1}{2!}G_{\Lambda\mu\nu}^{\pm} , \tag{4.22b}$$

$$T_{\mu\nu}^{\pm} \ \equiv \ \frac{1}{2!}(T_{\mu\nu}^{\pm})^{\mathsf{new}'} \ \equiv \ \frac{\mathrm{i}}{2!}(T_{\mu\nu}^{\pm})^{\mathsf{new}} , \qquad U_{\mu\nu}^{\pm} \ \equiv \ \frac{1}{2!}(U_{\mu\nu}^{\pm})^{\mathsf{new}'} \ \equiv \ \frac{\mathrm{i}}{2!}(U_{\mu\nu}^{\pm})^{\mathsf{new}} , \tag{4.22c}$$

$$G_{\mu\nu}^{i-} \ \equiv \ \frac{1}{2!}(G_{\mu\nu}^{i-})^{\mathsf{new}'} \ \equiv \ -\frac{1}{2!}(G_{\mu\nu}^{i-})^{\mathsf{new}} , \qquad G_{\mu\nu}^{\bar{i}+} \ \equiv \ \frac{1}{2!}(G_{\mu\nu}^{\bar{i}+})^{\mathsf{new}'} \ \equiv \ -\frac{1}{2!}(G_{\mu\nu}^{\bar{i}+})^{\mathsf{new}} . \tag{4.22d}$$

Note that we attached the extra factors "i" in front of $(T^{\pm}_{\mu\nu})^{\text{new}}$ and $(U^{\pm}_{\mu\nu})^{\text{new}}$ for later convenience. We also attached the additional minus signs in $(G^{i-}_{\mu\nu})^{\text{new}}$ and $(G^{\bar{i}+}_{\mu\nu})^{\text{new}}$.

Due to the above redefinition of two-forms, the kinetic term of the gauge fields (4.19) is rewritten as

$$\begin{aligned} \mathscr{L}^{\text{gauge}}_{\text{kin}} \ &= \ \mathrm{i}\Big(\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{\Lambda-}_{\mu\nu}\mathcal{F}^{\Sigma-|\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{\Lambda+}_{\mu\nu}\mathcal{F}^{\Sigma+|\mu\nu}\Big) \ = \ (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}\mathcal{F}^{\Sigma\mu\nu} + (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\mathcal{F}^{\Lambda}_{\mu\nu}\widetilde{\mathcal{F}}^{\Sigma\mu\nu} \\ &= \ \frac{\mathrm{i}}{4}\Big(\overline{\mathcal{N}}_{\Lambda\Sigma}F^{\Lambda-}_{\mu\nu}F^{\Sigma-|\mu\nu} - \mathcal{N}_{\Lambda\Sigma}F^{\Lambda+}_{\mu\nu}F^{\Sigma+|\mu\nu}\Big) \ = \ \frac{1}{4}(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} + \frac{1}{4}(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}\widetilde{F}^{\Sigma\mu\nu}\,. \end{aligned} \tag{4.23}$$

The definition of the magnetic dual of the gauge field strength (4.20) is modified as follows:

$$G^{\pm}_{\Lambda\mu\nu} \ \equiv \ \pm 2\mathrm{i}\frac{\delta\mathscr{L}}{\delta F^{\Lambda\pm|\mu\nu}}\,, \qquad G_{\Lambda\mu\nu} \ = \ 2\frac{\delta\mathscr{L}}{\delta\widetilde{F}^{\Lambda\mu\nu}}\,, \qquad \widetilde{G}_{\Lambda\mu\nu} \ = \ 2\frac{\delta\mathscr{L}}{\delta F^{\Lambda\mu\nu}}\,. \tag{4.24}$$

In addition, we further flip the sign of the auxiliary field $(A'^{\mu})_A{}^B$ as

$$(A'^{\mu})_A{}^B \ \equiv \ -\big((A'^{\mu})_A{}^B\big)^{\text{new}}\,. \tag{4.25}$$

In [1] the vierbein one-form is represented as $V^a = V^a_\mu\,\mathrm{d}x^\mu$ with $g_{\mu\nu} = \eta_{ab}V^a_\mu V^b_\nu$. It will be changed to $e^a = e^a_\mu\,\mathrm{d}x^\mu$ with $g_{\mu\nu} = \eta_{ab}\,e^a_\mu\,e^b_\nu$ to avoid confusions with the holomorphic sections, the scalar potential and so forth.

5 Conversion

Following section 4, we convert all the expressions in section 3 to the new descriptions.

5.1 Lagrangian

Caused by the changing of the orientation (4.6), we redefine the sign of the action as $S = -S^{\text{new}}$. Each part of the new Lagrangian is explicitly exhibited as follows (without writing the symbol "new" in any descriptions):

$$\begin{aligned}
\mathscr{L}_{\text{kin}}^{\text{inv}} \;=\;& \frac{1}{2}R - g_{i\bar{j}}\,\nabla_\mu z^i \nabla^\mu \bar{z}^{\bar{j}} - h_{uv}\,\nabla_\mu q^u \nabla^\mu q^v + \frac{1}{4}(\text{Im}\mathcal{N})_{\Lambda\Sigma}\,F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} + \frac{1}{4}(\text{Re}\mathcal{N})_{\Lambda\Sigma}\,F^{\Lambda}_{\mu\nu}\widetilde{F}^{\Sigma\mu\nu} \\
& + \frac{1}{2}g_{i\bar{j}}\Big(\overline{\lambda}^{iA}\gamma^\mu\nabla_\mu\lambda^{\bar{j}}_A + \overline{\lambda}^{\bar{j}}_A\gamma^\mu\nabla_\mu\lambda^{iA}\Big) + \Big(\overline{\zeta}^\alpha\gamma^\mu\nabla_\mu\zeta_\alpha + \overline{\zeta}_\alpha\gamma^\mu\nabla_\mu\zeta^\alpha\Big) - \mathrm{i}\frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\Big(\overline{\psi}^A_\mu\gamma_\sigma\rho_{A\nu\lambda} - \overline{\psi}_{A\mu}\gamma_\sigma\rho^A_{\nu\lambda}\Big) \\
& + \Big\{\Big[g_{i\bar{j}}\nabla_\mu\bar{z}^{\bar{j}}\Big(\overline{\psi}^\mu_A\lambda^{iA} - \overline{\lambda}^{iA}\gamma^{\mu\nu}\psi_{A\mu}\Big) + 2\mathcal{U}^{\alpha A}_u\nabla_\mu q^u\Big(\overline{\psi}^\mu_A\zeta_\alpha - \overline{\zeta}_\alpha\gamma^{\mu\nu}\psi_{A\mu}\Big)\Big] + (\text{h.c.})\Big\}\,, \qquad (5.1\text{a})
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_{\text{Pauli}} \;=\;& \frac{1}{2}F^{\Lambda-}_{\mu\nu}(\text{Im}\mathcal{N})_{\Lambda\Sigma}\Big[4L^\Sigma(\overline{\psi}^{A\mu}\psi^{B\nu})\epsilon_{AB} - 4\overline{f}^\Sigma_{\bar{i}}(\overline{\lambda}^{\bar{i}}_A\gamma^\nu\psi^\mu_B)\epsilon^{AB} - \frac{1}{2}\nabla_i f^\Sigma_j(\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB})\epsilon_{AB} + L^\Sigma(\overline{\zeta}_\alpha\gamma^{\mu\nu}\zeta_\beta)\mathbb{C}^{\alpha\beta}\Big] \\
& + (\text{h.c.})\,, \qquad (5.1\text{b})
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_{4\text{f}}^{\text{inv}} \;=\;& -\frac{\mathrm{i}}{2}\frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\Big\{g_{i\bar{j}}(\overline{\lambda}^{iA}\gamma_\sigma\lambda^{\bar{j}}_B) - 2\delta^A_B(\overline{\zeta}^\alpha\gamma_\sigma\zeta_\alpha)\Big\}(\overline{\psi}_{A\mu}\gamma_\lambda\psi^B_\nu) - \Big\{\epsilon_{AB}\mathbb{C}_{\alpha\beta}(\overline{\psi}^A_\mu\zeta^\alpha)(\overline{\psi}^{B\mu}\zeta^\beta) + (\text{h.c.})\Big\} \\
& + 2g_{i\bar{j}}(\overline{\lambda}^{iA}\gamma_\mu\psi^B_\nu)(\overline{\lambda}^{\bar{j}}_A\gamma^\mu\psi^\nu_B) - g_{i\bar{j}}(\overline{\psi}^A_\mu\lambda^{\bar{j}}_A)(\overline{\psi}^\mu_B\lambda^{iB}) - 2(\overline{\psi}^A_\mu\zeta^\alpha)(\overline{\psi}^\mu_A\zeta_\alpha) - 2(\overline{\psi}^A_\mu\psi^B_\nu)(\overline{\psi}^\mu_A\psi^\nu_B) \\
& - \frac{\mathrm{i}}{6}\Big\{C_{ijk}(\overline{\lambda}^{iA}\gamma^\mu\psi^B_\mu)(\overline{\lambda}^{jC}\lambda^{kD})\epsilon_{AC}\epsilon_{BD} - (\text{h.c.})\Big\} \\
& + \frac{1}{4}\Big(R_{i\bar{j}\ell\bar{k}} + g_{i\bar{k}}\,g_{\ell\bar{j}} - \frac{3}{2}g_{i\bar{j}}\,g_{\ell\bar{k}}\Big)(\overline{\lambda}^{iA}\lambda^{\ell B})(\overline{\lambda}^{\bar{j}}_A\lambda^{\bar{k}}_B) - \Big\{\frac{\mathrm{i}}{12}\nabla_m C_{jk\ell}(\overline{\lambda}^{jA}\lambda^{mB})(\overline{\lambda}^{kC}\lambda^{\ell D})\epsilon_{AC}\epsilon_{BD} + (\text{h.c.})\Big\} \\
& + \frac{1}{4}g_{i\bar{j}}(\overline{\zeta}^\alpha\gamma_\mu\zeta_\alpha)(\overline{\lambda}^{iA}\gamma^\mu\lambda^{\bar{j}}_A) + \frac{1}{2}\mathbb{R}^\alpha{}_{\beta|ts}\,\mathcal{U}^t_{\gamma A}\,\mathcal{U}^s_{\delta B}\,\mathbb{C}^{\delta\eta}(\overline{\zeta}_\alpha\zeta_\eta)(\overline{\zeta}^\beta\zeta^\gamma)\,, \qquad (5.1\text{c})
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_{4\text{f}}^{\text{non-inv}} \;=\;& (\text{Im}\mathcal{N})_{\Lambda\Sigma}\Big\{L^\Lambda L^\Sigma(\overline{\psi}^A_\mu\psi^B_\nu)^{(-)}\epsilon_{AB}\Big[2(\overline{\psi}^{C\mu}\psi^{D\nu})^{(-)}\epsilon_{CD} + (\overline{\zeta}_\alpha\gamma^{\mu\nu}\zeta_\beta)\mathbb{C}^{\alpha\beta}\Big] \\
& \qquad - L^\Lambda\,\overline{f}^\Sigma_{\bar{i}}(\overline{\lambda}^{\bar{i}}_A\gamma^\nu\psi^\mu_B)^{(-)}\Big[8(\overline{\psi}^A_\mu\psi^B_\nu)^{(-)} + (\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta)\epsilon^{AB}\mathbb{C}^{\alpha\beta}\Big] \\
& \qquad - \frac{\mathrm{i}}{2}L^\Lambda\,\overline{f}^\Sigma_{\bar{\ell}}\,g^{k\bar{\ell}}\,C_{ijk}(\overline{\psi}^A_\mu\psi^B_\nu)^{(-)}(\overline{\lambda}^{iC}\gamma^{\mu\nu}\lambda^{jD})\epsilon_{AB}\epsilon_{CD} \\
& \qquad + \overline{f}^\Lambda_{\bar{i}}\,\overline{f}^\Sigma_{\bar{j}}(\overline{\lambda}^{\bar{i}}_A\gamma^\nu\psi^\mu_B)^{(-)}\Big[2(\overline{\lambda}^{\bar{j}}_C\gamma_\nu\psi_{D\mu})^{(-)}\epsilon^{AB}\epsilon^{CD} + \mathrm{i}g^{k\bar{j}}\,C_{k\ell m}(\overline{\lambda}^{\ell A}\gamma_{\mu\nu}\lambda^{mB})\Big] \\
& \qquad - \frac{1}{32}C_{ijk}\,C_{\ell mn}\,g^{k\bar{r}}\,g^{n\bar{s}}\,\overline{f}^\Lambda_{\bar{r}}\,\overline{f}^\Sigma_{\bar{s}}(\overline{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB})(\overline{\lambda}^{mC}\gamma^{\mu\nu}\lambda^{\ell B})\epsilon_{AB}\epsilon_{CD} \\
& \qquad - \frac{1}{8}L^\Lambda(\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta)\mathbb{C}^{\alpha\beta}\Big[\nabla_i f^\Sigma_j(\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB})\epsilon_{AB} - L^\Sigma(\overline{\zeta}_\gamma\gamma^{\mu\nu}\zeta_\delta)\mathbb{C}^{\gamma\delta}\Big]\Big\} \\
& + (\text{h.c.})\,, \qquad (5.1\text{d})
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_{\text{mass}} \;=\;& \mathrm{g}\Big\{-2S_{AB}(\overline{\psi}^A_\mu\gamma^{\mu\nu}\psi^B_\nu) - g_{i\bar{j}}W^{iAB}(\overline{\lambda}^{\bar{j}}_A\gamma_\mu\psi^\mu_B) - 2N^A_\alpha(\overline{\zeta}^\alpha\gamma_\mu\psi^\mu_A) \\
& \quad + \mathcal{M}^{\alpha\beta}(\overline{\zeta}_\alpha\zeta_\beta) + \mathcal{M}^\alpha{}_{iB}(\overline{\zeta}_\alpha\lambda^{iB}) + \mathcal{M}_{iA|\ell B}(\overline{\lambda}^{iA}\lambda^{\ell B})\Big\} \\
& + (\text{h.c.})\,, \qquad (5.1\text{e})
\end{aligned}$$

$$V(z,\bar{z},q) \;=\; \mathrm{g}^2\Big\{\Big(g_{i\bar{j}}\,k^i_\Lambda k^{\bar{j}}_\Sigma + 4h_{uv}\,k^u_\Lambda k^v_\Sigma\Big)\overline{L}^\Lambda L^\Sigma + \Big(g^{i\bar{j}}\,f^\Lambda_i\overline{f}^\Sigma_{\bar{j}} - 3\overline{L}^\Lambda L^\Sigma\Big)\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma\Big\}\,, \qquad (5.1\text{f})$$

where $(\ldots)^{(-)}$ denotes the "anti-self-dual" part of the tensors or of the fermion bilinears.

5.2 Local supersymmetry variations

Local supersymmetry variations of the bosonic fields:

$$\delta e^a_\mu = (\overline{\psi}_{A\mu}\gamma^a\epsilon^A) + (\overline{\psi}^A_\mu\gamma^a\epsilon_A)\,, \tag{5.2a}$$

$$\delta A^\Lambda_\mu = 2\,\overline{L}^\Lambda(\overline{\psi}_{A\mu}\epsilon_B)\epsilon^{AB} + 2\,L^\Lambda(\overline{\psi}^A_\mu\epsilon^B)\epsilon_{AB} + \Big\{f^\Lambda_i(\overline{\lambda}^{iA}\gamma_\mu\epsilon^B)\epsilon_{AB} + \overline{f}^\Lambda_{\overline{i}}(\overline{\lambda}^{\overline{i}}_A\gamma_\mu\epsilon_B)\epsilon^{AB}\Big\}\,, \tag{5.2b}$$

$$\delta z^i = \overline{\lambda}^{iA}\epsilon_A\,, \tag{5.2c}$$

$$\delta \overline{z}^{\overline{i}} = \overline{\lambda}^{\overline{i}}_A\epsilon^A\,, \tag{5.2d}$$

$$\delta q^u = \mathcal{U}^u_{\alpha A}\Big\{(\overline{\zeta}^\alpha\epsilon^A) + \mathbb{C}^{\alpha\beta}\epsilon^{AB}(\overline{\zeta}_\beta\epsilon_B)\Big\}\,. \tag{5.2e}$$

Local supersymmetry variations of the fermionic fields:

$$\begin{aligned}\delta\psi_{A\mu} &= \mathcal{D}_\mu\epsilon_A - \frac{1}{4}\Big\{\partial_i\mathcal{K}(\overline{\lambda}^{iB}\epsilon_B) - \partial_{\overline{i}}\mathcal{K}(\overline{\lambda}^{\overline{i}}_B\epsilon^B)\Big\}\psi_{A\mu} - \omega_{vA}{}^B\,\mathcal{U}^v_{\alpha C}\Big\{\epsilon^{CD}\mathbb{C}^{\alpha\beta}(\overline{\zeta}_\beta\epsilon_D) + (\overline{\zeta}^\alpha\epsilon^C)\Big\}\psi_{B\mu}\\ &\quad + \Big\{(A^\nu)_A{}^B\eta_{\mu\nu} + (A'^\nu)_A{}^B\gamma_{\mu\nu}\Big\}\epsilon_B - \Big\{\mathrm{g}\,S_{AB}\eta_{\mu\nu} + \frac{1}{2}\epsilon_{AB}(T^-_{\mu\nu} + U^+_{\mu\nu})\Big\}\gamma^\nu\epsilon^B\,,\end{aligned} \tag{5.3a}$$

$$\begin{aligned}\delta\lambda^{iA} &= \frac{1}{4}\Big\{\partial_j\mathcal{K}(\overline{\lambda}^{jB}\epsilon_B) - \partial_{\overline{j}}\mathcal{K}(\overline{\lambda}^{\overline{j}}_B\epsilon^B)\Big\}\lambda^{iA} - \omega_v{}^A{}_B\,\mathcal{U}^v_{\alpha C}\Big\{\epsilon^{CD}\mathbb{C}^{\alpha\beta}(\overline{\zeta}_\beta\epsilon_D) + (\overline{\zeta}^\alpha\epsilon^C)\Big\}\lambda^{iB}\\ &\quad - \Gamma^i_{jk}(\overline{\lambda}^{kB}\epsilon_B)\lambda^{jA} - \Big\{\nabla_\mu z^i - (\overline{\lambda}^{iA}\psi_{A\mu})\Big\}\gamma^\mu\epsilon^A + \frac{1}{2}G^{i-}_{\mu\nu}\gamma^{\mu\nu}\epsilon_B\,\epsilon^{AB} + D^{iAB}\epsilon_B\,,\end{aligned} \tag{5.3b}$$

$$\begin{aligned}\delta\zeta_\alpha &= -\Delta_{v\alpha}{}^\beta\,\mathcal{U}^v_{\gamma A}\Big\{\epsilon^{AB}\mathbb{C}^{\gamma\delta}(\overline{\zeta}_\delta\epsilon_B) + (\overline{\zeta}^\gamma\epsilon^A)\Big\}\zeta_\beta + \frac{1}{4}\Big\{\partial_i\mathcal{K}(\overline{\lambda}^{iB}\epsilon_B) - \partial_{\overline{i}}\mathcal{K}(\overline{\lambda}^{\overline{i}}_B\epsilon^B)\Big\}\zeta_\alpha\\ &\quad - \Big\{\mathcal{U}^{\beta B}_u\nabla_\mu q^u - \epsilon^{BC}\mathbb{C}^{\beta\gamma}(\overline{\zeta}_\gamma\psi_{C\mu}) - (\overline{\zeta}^\beta\psi^B_\mu)\Big\}\gamma^\mu\epsilon^A\,\epsilon_{AB}\mathbb{C}_{\alpha\beta} + \mathrm{g}\,N^A_\alpha\epsilon_A\,.\end{aligned} \tag{5.3c}$$

Supergravity values of the auxiliary fields:

$$(A^\mu)_A{}^B = \frac{1}{4}g_{\overline{k}\ell}\Big\{(\overline{\lambda}^{\overline{k}}_A\gamma^\mu\lambda^{\ell B}) - \delta^B_A(\overline{\lambda}^{\overline{k}}_C\gamma^\mu\lambda^{\ell C})\Big\}\,, \tag{5.4a}$$

$$(A'^\mu)_A{}^B = \frac{1}{4}g_{\overline{k}\ell}\Big\{(\overline{\lambda}^{\overline{k}}_A\gamma^\mu\lambda^{\ell B}) - \frac{1}{2}\delta^B_A(\overline{\lambda}^{\overline{k}}_C\gamma^\mu\lambda^{\ell C})\Big\} - \frac{1}{4}\delta^B_A(\overline{\zeta}_\alpha\gamma^\mu\zeta^\alpha)\,, \tag{5.4b}$$

$$T^-_{\mu\nu} = 2(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}L^\Sigma\Big\{\widehat{F}^{\Lambda-}_{\mu\nu} - \frac{1}{4}\nabla_i f^\Lambda_j(\overline{\lambda}^{iA}\gamma_{\mu\nu}\lambda^{jB})\epsilon_{AB} + \frac{1}{2}\mathbb{C}^{\alpha\beta}(\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta)L^\Lambda\Big\}\,, \tag{5.4c}$$

$$T^+_{\mu\nu} = 2(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\overline{L}^\Sigma\Big\{\widehat{F}^{\Lambda+}_{\mu\nu} - \frac{1}{4}\nabla_{\overline{i}}\overline{f}^\Lambda_{\overline{j}}(\overline{\lambda}^{\overline{i}}_A\gamma_{\mu\nu}\lambda^{\overline{j}}_B)\epsilon^{AB} + \frac{1}{2}\mathbb{C}_{\alpha\beta}(\overline{\zeta}^\alpha\gamma_{\mu\nu}\zeta^\beta)\overline{L}^\Lambda\Big\}\,, \tag{5.4d}$$

$$U^-_{\mu\nu} = \frac{1}{2}\mathbb{C}^{\alpha\beta}(\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta)\,, \tag{5.4e}$$

$$U^+_{\mu\nu} = \frac{1}{2}\mathbb{C}_{\alpha\beta}(\overline{\zeta}^\alpha\gamma_{\mu\nu}\zeta^\beta)\,, \tag{5.4f}$$

$$G^{i-}_{\mu\nu} = g^{i\overline{j}}\overline{f}^\Gamma_{\overline{j}}(\mathrm{Im}\mathcal{N})_{\Gamma\Lambda}\Big\{\widehat{F}^{\Lambda-}_{\mu\nu} - \frac{1}{4}\nabla_k f^\Lambda_\ell(\overline{\lambda}^{kA}\gamma_{\mu\nu}\lambda^{\ell B})\epsilon_{AB} + \frac{1}{2}\mathbb{C}^{\alpha\beta}(\overline{\zeta}_\alpha\gamma_{\mu\nu}\zeta_\beta)L^\Lambda\Big\}\,, \tag{5.4g}$$

$$G^{\overline{i}+}_{\mu\nu} = g^{\overline{i}j}f^\Gamma_j(\mathrm{Im}\mathcal{N})_{\Gamma\Lambda}\Big\{\widehat{F}^{\Lambda+}_{\mu\nu} - \frac{1}{4}\nabla_{\overline{k}}\overline{f}^\Lambda_{\overline{\ell}}(\overline{\lambda}^{\overline{k}}_A\gamma_{\mu\nu}\lambda^{\overline{\ell}}_B)\epsilon^{AB} + \frac{1}{2}\mathbb{C}_{\alpha\beta}(\overline{\zeta}^\alpha\gamma_{\mu\nu}\zeta^\beta)\overline{L}^\Lambda\Big\}\,, \tag{5.4h}$$

$$D^{iAB} = \frac{\mathrm{i}}{2}g^{i\overline{j}}C_{\overline{j}\overline{k}\overline{\ell}}(\overline{\lambda}^{\overline{k}}_C\lambda^{\overline{\ell}}_D)\epsilon^{AC}\epsilon^{BD} + W^{iAB}\,, \tag{5.4i}$$

$$\widehat{F}^\Lambda_{\mu\nu} = F^\Lambda_{\mu\nu} + 2L^\Lambda(\overline{\psi}^A_\mu\psi^B_\nu)\epsilon_{AB} + 2\overline{L}^\Lambda(\overline{\psi}_{A\mu}\psi_{B\nu})\epsilon^{AB} - 2f^\Lambda_i(\overline{\lambda}^{iA}\gamma_{[\nu}\psi^B_{\mu]})\epsilon_{AB} - 2\overline{f}^\Lambda_{\overline{i}}(\overline{\lambda}^{\overline{i}}_A\gamma_{[\mu}\psi_{B\nu]})\epsilon^{AB}\,. \tag{5.4j}$$

Fermionic mass matrices in the local supersymmetry variarions:

$$S_{AB} = \frac{\mathrm{i}}{2}(\sigma_x)_A{}^C\,\epsilon_{BC}\,\mathcal{P}^x_\Lambda L^\Lambda\,, \tag{5.5a}$$

$$W^{iAB} = \epsilon^{AB}\,k^i_\Lambda\,\overline{L}^\Lambda + \mathrm{i}(\sigma_x)_C{}^B\,\epsilon^{CA}\,\mathcal{P}^x_\Lambda\,g^{i\overline{j}}\,\overline{f}^\Lambda_{\overline{j}}\,, \tag{5.5b}$$

$$N^A_\alpha = 2\mathcal{U}^A_{\alpha u}\,k^u_\Lambda\,\overline{L}^\Lambda\,, \tag{5.5c}$$

$$\mathcal{M}^{\alpha\beta} = -\mathcal{U}^{\alpha A}_u\,\mathcal{U}^{\beta B}_v\,\epsilon_{AB}\,\nabla^{[u}\,k^{v]}_\Lambda\,L^\Lambda\,, \tag{5.5d}$$

$$\mathcal{M}^\alpha{}_{iB} = -4\,\mathcal{U}^\alpha_{Bu}\,k^u_\Lambda\,f^\Lambda_i\,, \tag{5.5e}$$

$$\mathcal{M}_{iA|\ell B} = \frac{1}{3}\Big\{\epsilon_{AB}\,g_{i\overline{j}}\,k^{\overline{j}}_\Lambda\,f^\Lambda_\ell + \mathrm{i}(\sigma_x\epsilon^{-1})_{AB}\,\mathcal{P}^x_\Lambda\,\nabla_\ell f^\Lambda_i\Big\}\,. \tag{5.5f}$$

5.3 Covariant derivatives

The covariant derivatives in the vector multiplets under the gauging:

$$\begin{aligned}
\nabla z^i &= \mathrm{d}z^i + \mathrm{g}\, A^\Lambda k^i_\Lambda(z)\,, && (5.6a)\\
\nabla \overline{z}^{\overline{i}} &= \mathrm{d}\overline{z}^{\overline{i}} + \mathrm{g}\, A^\Lambda k^{\overline{i}}_\Lambda(\overline{z})\,, && (5.6b)\\
\nabla \lambda^{iA} &\equiv \mathrm{d}\lambda^{iA} + \frac{1}{4}\gamma_{ab}\,\omega^{ab}\,\lambda^{iA} - \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\,\lambda^{iA} + \widehat{\Gamma}^i{}_j\,\lambda^{jA} + \widehat{\omega}^A{}_B\,\lambda^{iB}\,, && (5.6c)\\
\nabla \lambda^{\overline{i}}_A &\equiv \mathrm{d}\lambda^{\overline{i}}_A + \frac{1}{4}\gamma_{ab}\,\omega^{ab}\,\lambda^{\overline{i}}_A + \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\,\lambda^{\overline{i}}_A + \widehat{\Gamma}^{\overline{i}}{}_{\overline{j}}\,\lambda^{\overline{j}}_A + \widehat{\omega}_A{}^B\,\lambda^{\overline{i}}_B\,, && (5.6d)\\
F^\Lambda &\equiv \mathrm{d}A^\Lambda + \frac{1}{2}\mathrm{g}\, f^\Lambda{}_{\Sigma\Gamma}\,A^\Sigma\wedge A^\Gamma + \overline{L}^\Lambda(\overline{\psi}_A\wedge\psi_B)\epsilon^{AB} + L^\Lambda(\overline{\psi}^A\wedge\psi^B)\epsilon_{AB}\,. && (5.6e)
\end{aligned}$$

The covariant derivatives in the hypermultiplets:

$$\begin{aligned}
\mathcal{U}^{A\alpha} &= \mathcal{U}^{A\alpha}_u\,\nabla q^u \;\equiv\; \mathcal{U}^{A\alpha}_u\Big(\mathrm{d}q^u + \mathrm{g}\,A^\Lambda k^u_\Lambda(q)\Big)\,, && (5.7a)\\
\nabla\zeta_\alpha &= \mathrm{d}\zeta_\alpha + \frac{1}{4}\gamma_{ab}\,\omega^{ab}\,\zeta_\alpha - \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\,\zeta_\alpha + \widehat{\Delta}_\alpha{}^\beta\,\zeta_\beta\,, && (5.7b)\\
\nabla\zeta^\alpha &= \mathrm{d}\zeta^\alpha + \frac{1}{4}\gamma_{ab}\,\omega^{ab}\,\zeta^\alpha + \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\,\zeta^\alpha + \widehat{\Delta}^\alpha{}_\beta\,\zeta^\beta\,, && (5.7c)\\
\widehat{\Delta}_\alpha{}^\beta &\equiv \widehat{\Delta}^{\gamma\beta}\,\mathbb{C}_{\gamma\alpha}\,, \qquad \widehat{\Delta}^\alpha{}_\beta \;\equiv\; \mathbb{C}_{\beta\gamma}\,\widehat{\Delta}^{\alpha\gamma}\,. && (5.7d)
\end{aligned}$$

The covariant derivative $\mathcal{D}_\mu\epsilon_A$ in the local supersymmetry variation $\delta\psi_{A\mu}$ is

$$\mathcal{D}_\mu\epsilon_A \;\equiv\; \partial_\mu\epsilon_A + \frac{1}{4}\gamma_{ab}\,\omega^{ab}\,\epsilon_A - \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\,\epsilon_A + \widehat{\omega}_A{}^B\,\epsilon_B\,. \qquad (5.8)$$

Let us exhibit the gauged connections in the above covariant derivatives:

$$\begin{aligned}
\widehat{\Gamma}^i{}_j &= \Gamma^i{}_j + \mathrm{g}\,A^\Lambda\partial_j k^i_\Lambda\,, & \Gamma^i{}_j &= \Gamma^i_{kj}\,\mathrm{d}z^k\,, && (5.9a)\\
\widehat{\mathcal{Q}} &= \mathcal{Q} + \mathrm{g}\,A^\Lambda\mathcal{P}^0_\Lambda\,, & \mathcal{Q} &= \mathcal{Q}_i\,\mathrm{d}z^i\,, && (5.9b)\\
\widehat{\omega}^x &= \omega^x + \mathrm{g}\,A^\Lambda\mathcal{P}^x_\Lambda\,, & \omega^x &= \omega^x_u\,\mathrm{d}q^u\,, && (5.9c)\\
\widehat{\Delta}^{\alpha\beta} &= \Delta^{\alpha\beta} + \mathrm{g}\,A^\Lambda\,\partial_u k^v_\Lambda\,\mathcal{U}^{u|\alpha A}\,\mathcal{U}^\beta_{v|A}\,, & \Delta^{\alpha\beta} &= \Delta^{\alpha\beta}_u\mathrm{d}q^u\,. && (5.9d)
\end{aligned}$$

5.4 Gauged curvatures

The supercovariant derivatives and the gauged curvatures in the gravitational sector:

$$\begin{aligned}
T^a &\equiv \mathcal{D}e^a + \overline{\psi}_A\wedge\gamma^a\psi^A \;=\; \Big(\mathrm{d}e^a + \omega^a{}_b\wedge e^b\Big) + \overline{\psi}_A\wedge\gamma^a\psi^A\,, && (5.10a)\\
\rho_A &\equiv \mathrm{d}\psi_A + \frac{1}{4}\gamma_{ab}\,\omega^{ab}\wedge\psi_A + \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\wedge\psi_A + \widehat{\omega}_A{}^B\wedge\psi_B \;\equiv\; \nabla\psi_A\,, && (5.10b)\\
\rho^A &\equiv \mathrm{d}\psi^A + \frac{1}{4}\gamma_{ab}\,\omega^{ab}\wedge\psi^A - \frac{\mathrm{i}}{2}\widehat{\mathcal{Q}}\wedge\psi^A + \widehat{\omega}^A{}_B\wedge\psi^B \;\equiv\; \nabla\psi^A\,, && (5.10c)\\
R^a{}_b &\equiv \mathrm{d}\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b\,. && (5.10d)
\end{aligned}$$

The gauged curvatures in the vector multiplets and hypermultiplets:

$$\begin{aligned}
\widehat{R}^i{}_j &= R^i{}_{j\overline{\ell}k}\,\nabla\overline{z}^{\overline{\ell}}\wedge\nabla z^k + \mathrm{g}\,F^\Lambda\partial_j k^i_\Lambda\,, && (5.11a)\\
\widehat{K} &= K_{i\overline{j}}\,\nabla z^i\wedge\nabla\overline{z}^{\overline{j}} + \mathrm{g}\,F^\Lambda\mathcal{P}^0_\Lambda\,, && (5.11b)\\
\widehat{\Omega}^x &= \Omega^x_{uv}\,\nabla q^u\wedge\nabla q^v + \mathrm{g}\,F^\Lambda\mathcal{P}^x_\Lambda\,, && (5.11c)\\
\widehat{\mathbb{R}}^{\alpha\beta} &= \mathbb{R}^{\alpha\beta}_{uv}\,\nabla q^u\wedge\nabla q^v + \mathrm{g}\,A^\Lambda\,\partial_u k^v_\Lambda\,\mathcal{U}^{u|\alpha A}\,\mathcal{U}^\beta_{v|A}\,, && (5.11d)
\end{aligned}$$

where the originral curvatures are defined as follows:

$$R^i{}_{j\bar{\ell}k} \;=\; \partial_{\bar{\ell}}\Gamma^i_{jk}\,, \qquad \Gamma^i_{jk} \;=\; g^{i\bar{\ell}}(\partial_k g_{j\bar{\ell}})\,, \tag{5.12a}$$

$$K \;=\; K_{i\bar{j}}\,\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}} \;=\; \frac{\mathrm{i}}{2\pi}\mathcal{K}_{i\bar{j}}\,\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}}\,, \qquad \mathcal{K}_{i\bar{j}} \;=\; g_{i\bar{j}} \;=\; \partial_i\partial_{\bar{j}}\mathcal{K}\,, \tag{5.12b}$$

$$\Omega^x \;=\; \mathrm{d}\omega^x + \frac{1}{2}\epsilon^{xyz}\,\omega^y \wedge \omega^z \;=\; \Omega^x_{uv}\,\mathrm{d}q^u \wedge \mathrm{d}q^v\,, \tag{5.12c}$$

$$\mathbb{R}^{\alpha\beta} \;=\; \mathrm{d}\Delta^{\alpha\beta} + \Delta^{\alpha\gamma} \wedge \Delta^{\delta\beta}\,\mathbb{C}_{\gamma\delta} \;=\; \mathbb{R}^{\alpha\beta}_{uv}\,\mathrm{d}q^u \wedge \mathrm{d}q^v\,. \tag{5.12d}$$

6 Reduced Lagrangian without four-fermi terms

In this section we reduce the complete system demonstrated in section 5 to the one without more than three fermionic contributions. The higher order terms in fermionic fields only contribute to a specific configuration where (expectation values of) fermionic condensations modify the equations of motion.

6.1 Lagrangian

Each part of the new Lagrangian up to four-fermions is explicitly exhibited as follows:

$$\begin{aligned}\mathscr{L}_{\text{kin}}^{\text{inv}} &= \frac{1}{2}R - g_{i\overline{j}}\,\nabla_\mu z^i \nabla^\mu \overline{z}^{\overline{j}} - h_{uv}\,\nabla_\mu q^u \nabla^\mu q^v + \frac{1}{4}(\text{Im}\mathcal{N})_{\Lambda\Sigma}\,F_{\mu\nu}^{\Lambda}F^{\Sigma\mu\nu} + \frac{1}{4}(\text{Re}\mathcal{N})_{\Lambda\Sigma}\,F_{\mu\nu}^{\Lambda}\widetilde{F}^{\Sigma\mu\nu} \\ &\quad + \frac{1}{2}g_{i\overline{j}}\Big(\overline{\lambda}^{iA}\gamma^\mu\nabla_\mu\lambda_A^{\overline{j}} + \overline{\lambda}_A^{\overline{j}}\gamma^\mu\nabla_\mu\lambda^{iA}\Big) + \Big(\overline{\zeta}^\alpha\gamma^\mu\nabla_\mu\zeta_\alpha + \overline{\zeta}_\alpha\gamma^\mu\nabla_\mu\zeta^\alpha\Big) - \mathrm{i}\frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\Big(\overline{\psi}_\mu^A\gamma_\sigma\rho_{A\nu\lambda} - \overline{\psi}_{A\mu}\gamma_\sigma\rho_{\nu\lambda}^A\Big) \\ &\quad + \Big\{\Big[g_{i\overline{j}}\nabla_\mu\overline{z}^{\overline{j}}\Big(\overline{\psi}_A^\mu\lambda^{iA} - \overline{\lambda}^{iA}\gamma^{\mu\nu}\psi_{A\mu}\Big) + 2\mathcal{U}_u^{\alpha A}\nabla_\mu q^u\Big(\overline{\psi}_A^\mu\zeta_\alpha - \overline{\zeta}_\alpha\gamma^{\mu\nu}\psi_{A\mu}\Big)\Big] + (\text{h.c.})\Big\}\,, \end{aligned} \tag{6.1a}$$

$$\begin{aligned}\mathscr{L}_{\text{Pauli}} &= \frac{1}{2}F_{\mu\nu}^{\Lambda-}(\text{Im}\mathcal{N})_{\Lambda\Sigma}\Big[4L^\Sigma(\overline{\psi}^{A\mu}\psi^{B\nu})\epsilon_{AB} - 4\overline{f}_{\overline{i}}^{\Sigma}(\overline{\lambda}_A^{\overline{i}}\gamma^\nu\psi_B^\mu)\epsilon^{AB} - \frac{1}{2}\nabla_i f_j^\Sigma(\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB})\epsilon_{AB} + L^\Sigma(\overline{\zeta}_\alpha\gamma^{\mu\nu}\zeta_\beta)\mathbb{C}^{\alpha\beta}\Big] \\ &\quad + (\text{h.c.})\,, \end{aligned} \tag{6.1b}$$

$$\mathscr{L}_{\text{4f}}^{\text{inv}} = \mathcal{O}(\text{fermion}^4)\,, \qquad \mathscr{L}_{\text{4f}}^{\text{non-inv}} = \mathcal{O}(\text{fermion}^4)\,, \tag{6.1c}$$

$$\begin{aligned}\mathscr{L}_{\text{mass}} &= \mathrm{g}\Big\{ - 2S_{AB}(\overline{\psi}_\mu^A\gamma^{\mu\nu}\psi_\nu^B) - g_{i\overline{j}}W^{iAB}(\overline{\lambda}_A^{\overline{j}}\gamma_\mu\psi_B^\mu) - 2N_\alpha^A(\overline{\zeta}^\alpha\gamma_\mu\psi_A^\mu) \\ &\qquad + \mathcal{M}^{\alpha\beta}(\overline{\zeta}_\alpha\zeta_\beta) + \mathcal{M}^\alpha{}_{iB}(\overline{\zeta}_\alpha\lambda^{iB}) + \mathcal{M}_{iA|\ell B}(\overline{\lambda}^{iA}\lambda^{\ell B})\Big\} \\ &\quad + (\text{h.c.})\,, \end{aligned} \tag{6.1d}$$

$$V(z,\overline{z},q) = \mathrm{g}^2\Big\{\Big(g_{i\overline{j}}\,k_\Lambda^i k_\Sigma^{\overline{j}} + 4h_{uv}\,k_\Lambda^u k_\Sigma^v\Big)\overline{L}^\Lambda L^\Sigma + \Big(g^{i\overline{j}}\,f_i^\Lambda\overline{f}_{\overline{j}}^\Sigma - 3\overline{L}^\Lambda L^\Sigma\Big)\mathcal{P}_\Lambda^x\mathcal{P}_\Sigma^x\Big\}\,, \tag{6.1e}$$

6.2 Local supersymmetry variations

The local supersymmetry variations of the bosonic fields are trivial if we consider a Lorentz-invariant configuration constraind by the vanishing expectation values of the fermionic fields $\langle\text{fermion}\rangle = 0$:

$$\delta e_\mu^a = (\overline{\psi}_{A\mu}\gamma^a\epsilon^A) + (\overline{\psi}_\mu^A\gamma^a\epsilon_A)\,, \tag{6.2a}$$

$$\delta A_\mu^\Lambda = 2\,\overline{L}^\Lambda(\overline{\psi}_{A\mu}\epsilon_B)\epsilon^{AB} + 2\,L^\Lambda(\overline{\psi}_\mu^A\epsilon^B)\epsilon_{AB} + \Big\{f_i^\Lambda(\overline{\lambda}^{iA}\gamma_\mu\epsilon^B)\epsilon_{AB} + \overline{f}_{\overline{i}}^\Lambda(\overline{\lambda}_A^{\overline{i}}\gamma_\mu\epsilon_B)\epsilon^{AB}\Big\}\,, \tag{6.2b}$$

$$\delta z^i = \overline{\lambda}^{iA}\epsilon_A\,, \tag{6.2c}$$

$$\delta\overline{z}^{\overline{i}} = \overline{\lambda}_A^{\overline{i}}\epsilon^A\,, \tag{6.2d}$$

$$\delta q^u = \mathcal{U}_{\alpha A}^u\Big\{(\overline{\zeta}^\alpha\epsilon^A) + \mathbb{C}^{\alpha\beta}\epsilon^{AB}(\overline{\zeta}_\beta\epsilon_B)\Big\}\,. \tag{6.2e}$$

The local supersymmetry variations of the fermionic fields up to three-fermions provide non-trivial (algebraic) equations for the bosonic fields even under the configuration $\langle\text{fermion}\rangle = 0$:

$$\delta\psi_{A\mu} = \mathcal{D}_\mu\epsilon_A - \Big\{\mathrm{g}\,S_{AB}\eta_{\mu\nu} + \frac{1}{2}\epsilon_{AB}T_{\mu\nu}^-\Big\}\gamma^\nu\epsilon^B + \mathcal{O}(\text{fermion}^3)\,, \tag{6.3a}$$

$$\delta\lambda^{iA} = -\nabla_\mu z^i\gamma^\mu\epsilon^A + \frac{1}{2}G_{\mu\nu}^{i-}\gamma^{\mu\nu}\epsilon_B\,\epsilon^{AB} + W^{iAB}\epsilon_B + \mathcal{O}(\text{fermion}^3)\,, \tag{6.3b}$$

$$\delta\zeta_\alpha = -\mathcal{U}_u^{\beta B}\nabla_\mu q^u\gamma^\mu\epsilon^A\,\epsilon_{AB}\mathbb{C}_{\alpha\beta} + \mathrm{g}\,N_\alpha^A\epsilon_A + \mathcal{O}(\text{fermion}^3)\,. \tag{6.3c}$$

Supergravity values of the auxiliary fields:

$$
\begin{aligned}
T^-_{\mu\nu} &= 2(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} L^\Sigma F^{\Lambda -}_{\mu\nu} + \mathcal{O}(\text{fermion}^2)\,, && \text{(6.4a)} \\
T^+_{\mu\nu} &= 2(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} \overline{L}^\Sigma F^{\Lambda +}_{\mu\nu} + \mathcal{O}(\text{fermion}^2)\,, && \text{(6.4b)} \\
G^{i-}_{\mu\nu} &= g^{i\overline{j}} \overline{f}{}^\Gamma_{\overline{j}} (\mathrm{Im}\mathcal{N})_{\Gamma\Lambda} F^{\Lambda -}_{\mu\nu} + \mathcal{O}(\text{fermion}^2)\,, && \text{(6.4c)} \\
G^{\overline{i}+}_{\mu\nu} &= g^{\overline{i}j} f^\Gamma_j (\mathrm{Im}\mathcal{N})_{\Gamma\Lambda} F^{\Lambda +}_{\mu\nu} + \mathcal{O}(\text{fermion}^2)\,, && \text{(6.4d)} \\
D^{iAB} &= W^{iAB} + \mathcal{O}(\text{fermion}^2)\,. && \text{(6.4e)} \\
\widehat{F}^\Lambda_{\mu\nu} &= F^\Lambda_{\mu\nu} + \mathcal{O}(\text{fermion}^2)\,, && \text{(6.4f)}
\end{aligned}
$$

where the following objects only contribute to more than third orders of the fermions in the variations (6.3):

$$
\begin{aligned}
(A^\mu)_A{}^B &= \mathcal{O}(\text{fermion}^2)\,, \qquad (A'^\mu)_A{}^B = \mathcal{O}(\text{fermion}^2)\,, && \text{(6.5a)} \\
U^-_{\mu\nu} &= \mathcal{O}(\text{fermion}^2)\,, \qquad\quad U^+_{\mu\nu} = \mathcal{O}(\text{fermion}^2)\,. && \text{(6.5b)}
\end{aligned}
$$

Note that the fermionic mass matrices in the local supersymmetry variarions are not reduced from (5.5). The covariant derivatives (5.6), (5.7) and (5.8), the connections (5.9), and the (gauged) curvatures (5.10), (5.11) and (5.12) are also not reduced.

6.3 Equations of motion

Let us write down the equations of motion for all the dynamical fields in the reduced gauged Lagrangian (6.1). Notice that the configuration with $\langle\text{fermion}\rangle = 0$ makes all the equations of motion for the ferimonic fields trivial: The fermion bilinear terms in $\mathscr{L}^{\text{inv}}_{\text{kin}}$ (6.1a), the Pauli part $\mathscr{L}_{\text{Pauli}}$ (6.1b) and the mass terms $\mathscr{L}_{\text{mass}}$ (6.1d) do not contribute to the equations of motion because they bring fermionic fields taken the vanishing expectation values. This implies that the Lagrangian parts which affect the equations of motion are only $\mathscr{L}^{\text{inv}}_{\text{kin}} - V$. Under this condition, we focus only on the equations of motion for the bosonic fields.

6.3.1 Variation of the metric $g_{\mu\nu}$

The term $\sqrt{-g}(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} F^\Lambda_{\mu\nu} \widetilde{F}^{\Sigma\mu\nu}$ in $\mathscr{L}^{\text{inv}}_{\text{kin}}$ (6.1a) is independent of $g_{\mu\nu}$. The scalar potential $V(z, \overline{z}, q)$ (6.1e) does not depend on the spacetime metric $g_{\mu\nu}$. Let us evaluate the variation of the metric:

$$
0 = \frac{\delta S}{\delta g^{\mu\nu}}\bigg|_{\langle\text{fermion}\rangle=0} = \int \mathrm{d}^4 x \frac{\delta}{\delta g^{\mu\nu}} \Big[\sqrt{-g} \Big(\mathscr{L}^{\text{inv}}_{\text{kin}} - V \Big) \Big]\,, \tag{6.6a}
$$

$$
\begin{aligned}
&\delta\Big\{ \sqrt{-g} \Big(\frac{1}{2} R - V \Big) \Big\} \\
&\quad = \frac{1}{2} \sqrt{-g}\, \delta g^{\mu\nu} \Big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} V \Big) + \frac{1}{2} \partial_\rho \big(\sqrt{-g}\, g^{\mu\nu} \delta\Gamma^\rho{}_{\nu\mu} \big) - \frac{1}{2} \partial_\nu \big(\sqrt{-g}\, g^{\mu\nu} \delta\Gamma^\rho{}_{\rho\mu} \big)\,,
\end{aligned} \tag{6.6b}
$$

$$
\begin{aligned}
&\delta\Big\{ \sqrt{-g} \Big(- g_{i\overline{j}} \nabla_\rho z^i \nabla^\rho \overline{z}^{\overline{j}} - h_{uv} \nabla_\rho q^u \nabla^\rho q^v + \frac{1}{4} (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} F^\Lambda_{\rho\sigma} F^{\Sigma\rho\sigma} + \frac{1}{4} (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} F^\Lambda_{\rho\sigma} \widetilde{F}^{\Sigma\rho\sigma} \Big) \Big\} \\
&\quad = -\frac{1}{2} \sqrt{-g}\, \delta g^{\mu\nu}\, g_{\mu\nu} \Big(- g_{i\overline{j}} \nabla_\rho z^i \nabla^\rho \overline{z}^{\overline{j}} - h_{uv} \nabla_\rho q^u \nabla^\rho q^v + \frac{1}{4} (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} F^\Lambda_{\rho\sigma} F^{\Sigma\rho\sigma} \Big) \\
&\qquad + \sqrt{-g}\, \delta g^{\mu\nu} \Big(- g_{i\overline{j}} \nabla_{(\mu} z^i \nabla_{\nu)} \overline{z}^{\overline{j}} - h_{uv} \nabla_{(\mu} q^u \nabla_{\nu)} q^v + \frac{1}{2} (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} F^\Lambda_{(\mu|\rho|} F^\Sigma_{\nu)\sigma} g^{\rho\sigma} \Big)\,.
\end{aligned} \tag{6.6c}
$$

Summarizing the above, we perform the variation of the action as follows:

$$
\begin{aligned}
\delta S &= \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g}\, \delta g^{\mu\nu} \bigg[\Big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} V \Big) - \Big(2 g_{i\overline{j}} \nabla_{(\mu} z^i \nabla_{\nu)} \overline{z}^{\overline{j}} + 2 h_{uv} \nabla_{(\mu} q^u \nabla_{\nu)} q^v - (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} F^\Lambda_{(\mu|\rho|} F^\Sigma_{\nu)\sigma}\, g^{\rho\sigma} \Big) \\
&\qquad\qquad - g_{\mu\nu} \Big(- g_{i\overline{j}} \nabla_\rho z^i \nabla^\rho \overline{z}^{\overline{j}} - h_{uv} \nabla_\rho q^u \nabla^\rho q^v + \frac{1}{4} (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} F^\Lambda_{\rho\sigma} F^{\Sigma\rho\sigma} \Big) \bigg] \\
&\quad + \frac{1}{2} \int \mathrm{d}^4 x\, \partial_\mu \Big[\big(\sqrt{-g}\, g^{\rho\sigma}\, \delta\Gamma^\mu{}_{\sigma\rho} \big) - \big(\sqrt{-g}\, g^{\nu\mu}\, \delta\Gamma^\rho{}_{\rho\mu} \big) \Big]\,.
\end{aligned} \tag{6.7}
$$

Then we obtain the equation of motion in the bulk:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, R \;=\; T_{\mu\nu} - g_{\mu\nu} V \,, \tag{6.8a}$$

$$\begin{aligned} T_{\mu\nu} \;&=\; \frac{1}{4} (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} \Big(- 4 F_{\mu\rho}^{\Lambda} F_{\nu\sigma}^{\Sigma}\, g^{\rho\sigma} + g_{\mu\nu} F_{\rho\sigma}^{\Lambda} F^{\Sigma\rho\sigma} \Big) \\ &\quad + g_{i\overline{j}} \Big(2 \nabla_{(\mu} z^i \nabla_{\nu)} \overline{z}^{\overline{j}} - g_{\mu\nu} \nabla_{\rho} z^i \nabla^{\rho} \overline{z}^{\overline{j}} \Big) + h_{uv} \Big(2 \nabla_{\mu} q^u \nabla_{\nu} q^v - g_{\mu\nu} \nabla_{\rho} q^u \nabla^{\rho} q^v \Big) \,. \end{aligned} \tag{6.8b}$$

Note that the parentheses of the indices imply the symmetrization defined as $A_{(\mu} B_{\nu)} = \frac{1}{2}(A_{\mu} B_{\nu} + A_{\nu} B_{\mu})$. The parentheses in the kinetic terms of the hyperscalars q^u and of the gauge fields A_{μ}^{Λ} are trivially removed due to the symmetry. Only the kinetic term of the complex scalar fields are non-trivial.

We should notice the total derivative terms in the second line of (6.7) if we cannot ignore the contribution of the boundary of the spacetime which we consider. For instance, if the spacetime is AdS which has the boundary at infinity, the contribution of the total derivative terms should be quite sensitive [22].

6.3.2 Variation of the gauge fields A_{μ}^{Λ}

Here let us compute the equations of motion for the gauge fields A_{μ}^{Λ} whose constituents are the gauge fields A_{μ}^{i} in the vector multiplets and the graviphoton A_{μ}^{0} in the gravitational multiplet. The scalar potential does not contain any gauge fields. Then the variation is

$$0 \;=\; \frac{\delta S}{\delta A_{\mu}^{\Lambda}} \bigg|_{\langle \text{fermion} \rangle = 0} \;=\; \int \mathrm{d}^4 x \frac{\delta}{\delta A_{\mu}^{\Lambda}} \Big[\sqrt{-g}\, \mathscr{L}_{\text{kin}}^{\text{inv}} \Big] \,, \tag{6.9a}$$

$$\begin{aligned} \delta \big\{ \sqrt{-g}\, \mathscr{L}_{\text{kin}}^{\text{inv}} \big\} \;&=\; \sqrt{-g} \Big[- g_{i\overline{j}} \Big((\delta \nabla_{\mu} z^i) \nabla^{\mu} \overline{z}^{\overline{j}} + \nabla_{\mu} z^i (\delta \nabla^{\mu} \overline{z}^{\overline{j}}) \Big) - 2 h_{uv} (\delta \nabla_{\mu} q^u) \nabla^{\mu} q^v \\ &\qquad + \frac{1}{2} (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} (\delta F_{\mu\nu}^{\Lambda}) F^{\Sigma\mu\nu} + \frac{1}{2} (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} (\delta F_{\mu\nu}^{\Lambda}) \widetilde{F}^{\Sigma\mu\nu} \Big] \,, \end{aligned} \tag{6.9b}$$

$$\begin{aligned} (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} F_{\mu\nu}^{\Lambda} (\delta \widetilde{F}^{\Sigma\mu\nu}) \;&=\; (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} F^{\Sigma\mu\nu} (\delta \widetilde{F}_{\mu\nu}^{\Lambda}) \;=\; (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} \Big[\frac{\sqrt{-g}}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\Sigma\mu\nu} (\delta F^{\Lambda\rho\sigma}) \Big] \\ &=\; (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} \widetilde{F}_{\rho\sigma}^{\Sigma} (\delta F^{\Lambda\rho\sigma}) \;=\; (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} (\delta F_{\mu\nu}^{\Lambda}) \widetilde{F}^{\Sigma\mu\nu} \,. \end{aligned} \tag{6.9c}$$

Each variation in the above equation is evaluated:

$$\delta \nabla_{\mu} z^i \;=\; \mathrm{g}\, \delta A_{\mu}^{\Lambda}\, k_{\Lambda}^{i} \,, \qquad \delta \nabla_{\mu} \overline{z}^{\overline{j}} \;=\; \mathrm{g}\, \delta A_{\mu}^{\Lambda}\, k_{\Lambda}^{\overline{j}} \,, \qquad \delta \nabla_{\mu} q^u \;=\; \mathrm{g}\, \delta A_{\mu}^{\Lambda}\, k_{\Lambda}^{u} \,, \tag{6.10a}$$

$$\begin{aligned} F^{\Sigma\mu\nu} \delta F_{\mu\nu}^{\Lambda} \;&=\; F^{\Sigma\mu\nu} \Big(\partial_{\mu} \delta A_{\nu}^{\Lambda} - \partial_{\nu} \delta A_{\mu}^{\Lambda} + \mathrm{g}\, f^{\Lambda}{}_{\Gamma\Delta}\, \delta A_{\mu}^{\Gamma} A_{\nu}^{\Delta} + \mathrm{g}\, f^{\Lambda}{}_{\Gamma\Delta}\, A_{\mu}^{\Gamma}\, \delta A_{\nu}^{\Delta} \Big) \\ &=\; 2 F^{\Sigma\mu\nu} \Big(- \partial_{\nu} \delta A_{\mu}^{\Lambda} + \mathrm{g}\, f^{\Lambda}{}_{\Gamma\Delta}\, \delta A_{\mu}^{\Gamma} A_{\nu}^{\Delta} \Big) \,. \end{aligned} \tag{6.10b}$$

The derivative terms of the variation can be rewritten as

$$\sqrt{-g} \Big[\frac{1}{2} (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} F^{\Sigma\mu\nu} \big(- 2 \partial_{\nu} \delta A_{\mu}^{\Lambda} \big) \Big] \;=\; \delta A_{\mu}^{\Lambda} \partial_{\nu} \Big[\sqrt{-g} (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} F^{\Sigma\mu\nu} \Big] - \partial_{\nu} \Big[\sqrt{-g} (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} \delta A_{\mu}^{\Lambda} F^{\Sigma\mu\nu} \Big] \,, \tag{6.11a}$$

$$\sqrt{-g} \Big[\frac{1}{2} (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} \widetilde{F}^{\Sigma\mu\nu} \big(- 2 \partial_{\nu} \delta A_{\mu}^{\Lambda} \big) \Big] \;=\; \delta A_{\mu}^{\Lambda} \partial_{\nu} \Big[\sqrt{-g} (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} \widetilde{F}^{\Sigma\mu\nu} \Big] - \partial_{\nu} \Big[\sqrt{-g} (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} \delta A_{\mu}^{\Lambda} \widetilde{F}^{\Sigma\mu\nu} \Big] \,. \tag{6.11b}$$

Indeed the second terms in the right-hand-sides are total derivative part of the Lagrangian. Summarizing the above, we describe the variation of the action in terms of the gauge fields in the following form:

$$\begin{aligned} \delta S \;=\; & \int \mathrm{d}^4 x \sqrt{-g}\, \delta A_{\mu}^{\Lambda} \bigg[\frac{1}{\sqrt{-g}} \partial_{\nu} \Big[\sqrt{-g} (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} F^{\Sigma\mu\nu} \Big] + \frac{1}{\sqrt{-g}} \partial_{\nu} \Big[\sqrt{-g} (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} \widetilde{F}^{\Sigma\mu\nu} \Big] \\ &\qquad + \mathrm{g} \Big(f^{\Gamma}{}_{\Lambda\Delta} \Big[(\mathrm{Im}\mathcal{N})_{\Gamma\Sigma} F^{\Sigma\mu\nu} + (\mathrm{Re}\mathcal{N})_{\Gamma\Sigma} \widetilde{F}^{\Sigma\mu\nu} \Big] A_{\nu}^{\Delta} - g_{i\overline{j}} \Big[k_{\Lambda}^{i}\, \nabla^{\mu} \overline{z}^{\overline{j}} + k_{\Lambda}^{\overline{j}}\, \nabla^{\mu} z^i \Big] - 2 h_{uv}\, k_{\Lambda}^{u}\, \nabla^{\mu} q^v \Big) \bigg] \\ & - \int \mathrm{d}^4 x\, \partial_{\nu} \bigg[\Big(\sqrt{-g} (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} \delta A_{\mu}^{\Lambda} F^{\Sigma\mu\nu} \Big) + \Big(\sqrt{-g} (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} \delta A_{\mu}^{\Lambda} \widetilde{F}^{\Sigma\mu\nu} \Big) \bigg] \,. \end{aligned} \tag{6.12}$$

Then the equation of motion for the gauge fields in the bulk is given as follows:

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}}\partial_\nu\Big[\sqrt{-g}(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}F^{\Sigma\mu\nu}\Big] + \frac{1}{\sqrt{-g}}\partial_\nu\Big[\sqrt{-g}(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\widetilde{F}^{\Sigma\mu\nu}\Big] \\ &\quad + \mathrm{g}\Big(f^\Gamma{}_{\Lambda\Delta}\Big[(\mathrm{Im}\mathcal{N})_{\Gamma\Sigma}F^{\Sigma\mu\nu} + (\mathrm{Re}\mathcal{N})_{\Gamma\Sigma}\widetilde{F}^{\Sigma\mu\nu}\Big]A_\nu^\Delta - g_{i\overline{j}}\Big[k_\Lambda^i\,\nabla^\mu\overline{z}^{\overline{j}} + k_\Lambda^{\overline{j}}\,\nabla^\mu z^i\Big] - 2h_{uv}\,k_\Lambda^u\,\nabla^\mu q^v\Big)\,. \end{aligned} \tag{6.13}$$

We can simplify this form in terms of the magnetic dual of the gauge field strength discussed in (4.24). In the ignorance of higher orders in fermionic fields given by (6.1), the on-shell magnetic dual of the gauge field strength (i.e., under the condition $\langle\text{fermion}\rangle = 0$) is explicitly described as

$$G^+_{\Lambda\mu\nu} = \mathcal{N}_{\Lambda\Sigma}F^{\Lambda+}_{\mu\nu}\,, \qquad G^-_{\Lambda\mu\nu} = \overline{\mathcal{N}}_{\Lambda\Sigma}F^{\Lambda-}_{\mu\nu}\,, \tag{6.14a}$$

$$G_{\Lambda\mu\nu} = (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}F^\Sigma_{\mu\nu} - (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}\widetilde{F}^\Sigma_{\mu\nu}\,, \qquad \widetilde{G}_{\Lambda\mu\nu} = (\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}F^\Sigma_{\mu\nu} + (\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\widetilde{F}^\Sigma_{\mu\nu}\,. \tag{6.14b}$$

Then (6.13) is given in terms of the magnetic dual $\widetilde{G}_{\Lambda\mu\nu}$ in such a way as

$$0 = \frac{1}{\sqrt{-g}}\partial_\nu\Big[\sqrt{-g}\,\widetilde{G}^{\mu\nu}_\Lambda\Big] + \mathrm{g}\Big(f^\Gamma{}_{\Lambda\Delta}\widetilde{G}^{\mu\nu}_\Gamma A^\Delta_\nu - g_{i\overline{j}}\Big[k^i_\Lambda\,\nabla^\mu\overline{z}^{\overline{j}} + k^{\overline{j}}_\Lambda\,\nabla^\mu z^i\Big] - 2h_{uv}\,k^u_\Lambda\,\nabla^\mu q^v\Big)\,. \tag{6.15}$$

We should notice the total derivative terms in (6.12) if we cannot ignore the contribution of the boundary of the spacetime which we consider. For instance, if the spacetime is AdS which has the boundary at infinity, the contribution of the total derivative terms should be quite sensitive.

6.3.3 Variation of the complex scalar fields z^i

Notice that the period matrix $\mathcal{N}_{\Lambda\Sigma}$ in $\mathscr{L}^{\mathsf{inv}}_{\mathsf{kin}}$ (6.1a) and the scalar potential $V(z,\overline{z},q)$ (6.1e) are functions of the complex scalars z^i and $\overline{z}^{\overline{j}}$. Then the variation is given as follows:

$$0 = \frac{\delta S}{\delta z^i}\bigg|_{\langle\text{fermion}\rangle=0} = \int \mathrm{d}^4x\,\frac{\delta}{\delta z^i}\Big[\sqrt{-g}\Big(\mathscr{L}^{\mathsf{inv}}_{\mathsf{kin}} - V\Big)\Big]\,, \tag{6.16a}$$

$$\delta\{\sqrt{-g}\,\mathscr{L}^{\mathsf{inv}}_{\mathsf{kin}}\} = \sqrt{-g}\Big[-(\delta g_{k\overline{\ell}})\nabla_\mu z^k\nabla^\mu\overline{z}^{\overline{\ell}} - g_{k\overline{\ell}}(\delta\nabla_\mu z^k)\nabla^\mu\overline{z}^{\overline{\ell}} + \frac{1}{4}\delta(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} + \frac{1}{4}\delta(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}F^\Lambda_{\mu\nu}\widetilde{F}^{\Sigma\mu\nu}\Big]\,, \tag{6.16b}$$

$$\begin{aligned} -\delta\{\sqrt{-g}\,V\} &= -\sqrt{-g}\,\frac{\partial V}{\partial z^i}\,\delta z^i \\ &= -\mathrm{g}^2\sqrt{-g}\Big\{\frac{\partial}{\partial z^i}\Big(g_{k\overline{\ell}}\,k^k_\Lambda k^{\overline{\ell}}_\Sigma\,\overline{L}^\Lambda L^\Sigma\Big) + \frac{\partial}{\partial z^i}\Big[\big(g^{k\overline{\ell}}f^\Lambda_k\overline{f}^{\Sigma}_{\overline{\ell}} - 3\overline{L}^\Lambda L^\Sigma\big)\mathcal{P}^x_\Lambda\mathcal{P}^x_\Sigma\Big]\Big\}\delta z^i\,. \end{aligned} \tag{6.16c}$$

More detail expressions are

$$\delta g_{k\overline{\ell}} = (\partial_i g_{k\overline{\ell}})\delta z^i\,, \qquad \delta\nabla_\mu z^k = \partial_\mu(\delta z^k) + \mathrm{g}A^\Lambda_\mu(\partial_i k^k_\Lambda)\delta z^i\,, \tag{6.16d}$$

$$\delta(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma} = \frac{(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}}{\partial z^i}\delta z^i\,, \qquad \delta(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma} = \frac{(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}}{\partial z^i}\delta z^i\,. \tag{6.16e}$$

The derivative of the variation in (6.16b) is rewritten as

$$-\sqrt{-g}\partial_\mu(\delta z^k)\Big[g_{k\overline{\ell}}\nabla^\mu\overline{z}^{\overline{\ell}}\Big] = \delta z^k\partial_\mu\Big[\sqrt{-g}\,g_{k\overline{\ell}}\nabla^\mu\overline{z}^{\overline{\ell}}\Big] - \partial_\mu\Big(\sqrt{-g}\,\delta z^k\,g_{k\overline{\ell}}\nabla^\mu\overline{z}^{\overline{\ell}}\Big)\,. \tag{6.17}$$

The variation of the action by the scalar fields z^i is given as

$$\begin{aligned} \delta S &= \int \mathrm{d}^4x\,\sqrt{-g}\,\delta z^i\bigg[\frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}\,g_{i\overline{\ell}}\nabla^\mu\overline{z}^{\overline{\ell}}\Big) - \partial_i g_{k\overline{\ell}}\,\nabla_\mu z^k\nabla^\mu\overline{z}^{\overline{\ell}} - \mathrm{g}\,A^\Lambda_\mu(\partial_i k^k_\Lambda)\,g_{k\overline{\ell}}\,\nabla^\mu\overline{z}^{\overline{\ell}} \\ &\qquad + \frac{1}{4}F^\Lambda_{\mu\nu}\Big(\partial_i(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}F^{\Sigma\mu\nu} + \partial_i(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\widetilde{F}^{\Sigma\mu\nu}\Big)\bigg] \\ &\quad - \int \mathrm{d}^4x\,\partial_\mu\Big(\sqrt{-g}\,\delta z^i\,g_{i\overline{\ell}}\nabla^\mu\overline{z}^{\overline{\ell}}\Big)\,. \end{aligned} \tag{6.18}$$

Then the equation of motion for the complex scalar fields in the bult is given as

$$
\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}\,g_{i\overline{\ell}}\nabla^\mu\overline{z}^{\overline{\ell}}\Big) - \partial_i g_{k\overline{\ell}}\nabla_\mu z^k\nabla^\mu\overline{z}^{\overline{\ell}} - \mathrm{g}\,A_\mu^\Lambda(\partial_i k_\Lambda^k)\,g_{k\overline{\ell}}\nabla^\mu\overline{z}^{\overline{\ell}} \\
&\quad + \frac{1}{4}F_{\mu\nu}^\Lambda\Big[\partial_i(\mathrm{Im}\mathcal{N})_{\Lambda\Sigma}F^{\Sigma\mu\nu} + \partial_i(\mathrm{Re}\mathcal{N})_{\Lambda\Sigma}\widetilde{F}^{\Sigma\mu\nu}\Big] - \frac{\partial V}{\partial z^i}\,, && (6.19a) \\
\frac{\partial V}{\partial z^i} &= \mathrm{g}^2\bigg[\frac{\partial}{\partial z^i}\Big(g_{k\overline{\ell}}\,k_\Lambda^k k_\Sigma^{\overline{\ell}}\,\overline{L}^\Lambda L^\Sigma\Big) + \frac{\partial}{\partial z^i}\Big[\big(g^{k\overline{\ell}}f_k^\Lambda\overline{f}_{\overline{\ell}}^\Sigma - 3\overline{L}^\Lambda L^\Sigma\big)\mathcal{P}_\Lambda^x\mathcal{P}_\Sigma^x\Big]\bigg]\,. && (6.19b)
\end{aligned}
$$

We should notice the total derivative terms in (6.18) if we cannot ignore the contribution of the boundary of the spacetime which we consider. For instance, if the spacetime is AdS which has the boundary at infinity, the contribution of the total derivative terms should be quite sensitive.

6.3.4 Variation of the real scalar fields q^u

Notice that the scalar potential $V(z,\overline{z},q)$ is a function of the real scalar fields q^u. The variation is given as follows:

$$
\begin{aligned}
0 &= \frac{\delta S}{\delta q^u}\bigg|_{\langle\text{fermion}\rangle=0} = \int d^4x\,\frac{\delta}{\delta q^u}\Big[\sqrt{-g}\Big(\mathscr{L}_{\text{kin}}^{\text{inv}} - V\Big)\Big]\,, && (6.20a) \\
\delta\big\{\sqrt{-g}\,\mathscr{L}_{\text{kin}}^{\text{inv}}\big\} &= \sqrt{-g}\Big[-(\delta h_{st})\nabla_\mu q^s\nabla^\mu q^t - 2h_{st}(\delta\nabla_\mu q^s)\nabla^\mu q^t\Big]\,, && (6.20b) \\
-\delta\big\{\sqrt{-g}\,V\big\} &= -\sqrt{-g}\,\frac{\partial V}{\partial q^u}\,\delta q^u \\
&= -\mathrm{g}^2\sqrt{-g}\Big\{\frac{\partial}{\partial q^u}\Big(4h_{st}k_\Lambda^s k_\Sigma^t\Big)\overline{L}^\Lambda L^\Sigma + \big(g^{i\overline{j}}f_i^\Lambda\overline{f}_{\overline{j}}^\Sigma - 3\overline{L}^\Lambda L^\Sigma\big)\frac{\partial}{\partial q^u}\Big(\mathcal{P}_\Lambda^x\mathcal{P}_\Sigma^x\Big)\Big\}\delta q^u\,. && (6.20c)
\end{aligned}
$$

Here let me exhibit two variations in (6.20b):

$$
\delta h_{st} = \frac{\partial h_{st}}{\partial q^u}\delta q^u\,, \qquad \delta\nabla_\mu q^s = \partial_\mu q^s + \mathrm{g}\,A_\mu^\Lambda\frac{\partial k_\Lambda^s}{\partial q^u}\,\delta q^u\,. \tag{6.20d}
$$

The derivative of the variation in (6.20b) is rewritten as

$$
\sqrt{-g}\,\partial_\mu(\delta q^s)\Big[-2h_{st}\nabla^\mu q^t\Big] = \delta q^s\partial_\mu\Big[2\sqrt{-g}\,h_{st}\nabla^\mu q^t\Big] - \partial_\mu\Big(2\sqrt{-g}\,\delta q^s\,h_{st}\nabla^\mu q^t\Big)\,. \tag{6.21}
$$

The variation of the action by the real scalar fields q^u is given as

$$
\begin{aligned}
\delta S &= \int d^4x\,\sqrt{-g}\,\delta q^u\bigg[-\frac{\partial h_{st}}{\partial q^u}\nabla_\mu q^s\nabla^\mu q^t + \frac{1}{\sqrt{-g}}\partial_\mu\Big(2\sqrt{-g}\,h_{ut}\nabla^\mu q^t\Big) - 2\mathrm{g}\,A_\mu^\Lambda\frac{\partial k_\Lambda^s}{\partial q^u}h_{st}\nabla^\mu q^t - \frac{\partial V}{\partial q^u}\bigg] \\
&\quad - \int d^4x\,\partial_\mu\Big(2\sqrt{-g}\,\delta q^u\,h_{uv}\nabla^\mu q^v\Big)\,. && (6.22)
\end{aligned}
$$

Then the equation of motion for the real scalar fields in the bulk is given as

$$
\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}\,h_{ut}\nabla^\mu q^t\Big) - \frac{1}{2}\frac{\partial h_{st}}{\partial q^u}\nabla_\mu q^s\nabla^\mu q^t - \mathrm{g}\,A_\mu^\Lambda\frac{\partial k_\Lambda^s}{\partial q^u}h_{st}\nabla^\mu q^t - \frac{1}{2}\frac{\partial V}{\partial q^u}\,, && (6.23a) \\
\frac{\partial V}{\partial q^u} &= \mathrm{g}^2\Big\{\frac{\partial}{\partial q^u}\Big(4h_{st}k_\Lambda^s k_\Sigma^t\Big)\overline{L}^\Lambda L^\Sigma + \big(g^{i\overline{j}}f_i^\Lambda\overline{f}_{\overline{j}}^\Sigma - 3\overline{L}^\Lambda L^\Sigma\big)\frac{\partial}{\partial q^u}\Big(\mathcal{P}_\Lambda^x\mathcal{P}_\Sigma^x\Big)\Big\}\,. && (6.23b)
\end{aligned}
$$

We should notice the total derivative terms in (6.22) if we cannot ignore the contribution of the boundary of the spacetime which we consider. For instance, if the spacetime is AdS which has the boundary at infinity, the contribution of the total derivative terms should be quite sensitive.

6.4 Noether currents and conserved charges

Appendix

A Conventions in TK-NOTEs

A.1 Contraction rule on antisymmetric tensors

We introduce the following simplified form:

$$|F_p|^2 \equiv \frac{1}{p!} g^{\mu_1\nu_1} \cdots g^{\mu_p\nu_p} F_{\mu_1\cdots\mu_p} F_{\nu_1\cdots\nu_p} , \tag{A.1}$$

where $F_{\mu_1\cdots\mu_p}$ is a totally antisymmetric tensor, i.e., the component of a p-form. The coefficient $1/p!$ is adopted to normalize an each term appearing in the explicit expansion of $|F_p|^2$ to unity.

A.2 Antisymmetrized symbol

The totally anti-symmetrized symbol is defined in terms of the square bracket:

$$T_{[\mu_1\mu_2\cdots\mu_p]} = \frac{1}{p!}\Big(T_{\mu_1\mu_2\cdots\mu_p} - T_{\mu_2\mu_1\cdots\mu_p} \pm \text{permutations} \Big) . \tag{A.2}$$

As an example, an anti-symmetrization of three gamma matrices is defined by

$$\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]} = \frac{1}{3!}\Big(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho} + \gamma^{\nu}\gamma^{\rho}\gamma^{\rho} + \gamma^{\rho}\gamma^{\mu}\gamma^{\nu} - \gamma^{\mu}\gamma^{\rho}\gamma^{\nu} - \gamma^{\nu}\gamma^{\rho}\gamma^{\rho} - \gamma^{\rho}\gamma^{\nu}\gamma^{\mu} \Big) . \tag{A.3}$$

A.3 Clifford algebra, gamma matrix and spinor

Clifford algebra defined in the local Lorentz frame:

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab} , \qquad \eta_{ab} = \text{diag.}(-,+,+,\ldots,+) . \tag{A.4}$$

Here we define the chirality operator $\widehat{\Gamma}$ in $d = 2k+2$ dimensional spacetime with Lorentz signature:

$$\widehat{\Gamma} \equiv \mathrm{i}^{-k} \gamma^0 \gamma^1 \cdots \gamma^{d-1} , \tag{A.5}$$

in particular, $\gamma^5 \equiv -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ in four-dimensional Minkowski spacetime. Note that all superscripts are the local Lorentz coordinate indices, since a spinor can be defined in the local Lorentz frame (or the tangent space of the geometry), in which the Dirac gamma matrix is also defined. In the Lorentzian spacetime, almost all matrices are hermitian except for γ^0, which is anti-hermitian (see the definition (A.9)).

Let us define the Dirac conjugate

$$\overline{\psi} \equiv \mathrm{i}\psi^{\dagger}\gamma^0 , \tag{A.6}$$

where γ^0 lives in the orthogonal frame space in D-dimensional space. Furthermore, we assign the Majorana condition such as

$$\psi \equiv C\overline{\psi}^{\mathrm{T}} \leftrightarrow \overline{\psi} \equiv \psi^{\mathrm{T}} C , \tag{A.7}$$

where C is called the charge conjugate matrix whose generic properties are

$$C = -C^{-1} = -C^{\mathrm{T}} , \tag{A.8a}$$

$$C(\gamma^a)C^{-1} = -(\gamma^a)^{\mathrm{T}} , \qquad C(\gamma^{a_1\cdots a_n})C^{-1} = (-)^{[\frac{n+1}{2}]}(\gamma^{a_1\cdots a_n})^{\mathrm{T}} , \tag{A.8b}$$

where $[\frac{n+1}{2}] = \{1,1,2,2,3,3,\cdots\}$ is the Gauss bracket. The hermitian conjugates of gamma matrices are defined by

$$(\gamma^a)^{\dagger} = \gamma_a = -\gamma^0\gamma^a(\gamma^0)^{-1} . \tag{A.9}$$

Among the Dirac gamma matrices there exists a useful identity such as

$$\gamma^{a_1a_2\cdots a_p}\gamma_{b_1b_2\cdots b_q} = \sum_{k=0}^{\min(p,q)} (-1)^{\frac{1}{2}k(2p-k-1)} \frac{p!q!}{(p-k)!(q-k)!k!} \delta^{[a_1}_{[b_1} \cdots \delta^{a_k}_{b_k} \gamma^{a_{k+1}\cdots a_p]}{}_{b_{k+1}\cdots b_q]} . \tag{A.10}$$

A.4 Differential forms

We define differential forms on D-dimensional geometry ($g_D = \det g_{\mu\nu}$). For concrete calculations, let us define them in the curved spacetime with the Lorentzian signature. First, a differential p-form and the volume form are defined as

$$\omega_p \equiv \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}\,, \tag{A.11a}$$

$$(\text{vol.}) \equiv \sqrt{-g_D}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1}\,. \tag{A.11b}$$

It is also necessary to introduce a dual form of the p-form via so-called the Hodge dual:

$$*\omega_p = \frac{\sqrt{-g_D}}{p!(D-p)!}\,\omega_{\mu_1\cdots\mu_p}\,\varepsilon^{\mu_1\cdots\mu_p}{}_{\nu_{p+1}\cdots\nu_D}\,\mathrm{d}x^{\nu_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{\nu_D}\,, \tag{A.12a}$$

$$(*1) = \frac{\sqrt{-g_D}}{D!}\,\varepsilon_{\mu_1\cdots\mu_D}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_D} = \sqrt{-g_D}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} = (\text{vol.})\,, \tag{A.12b}$$

where $\varepsilon_{\mu_1\cdots\mu_D}$ and $\varepsilon^{\mu_1\cdots\mu_D}$ are called the invariant tensors whose property is given by

$$\varepsilon^{\mu_1\cdots\mu_n}{}_{\mu_{n+1}\cdots\mu_D} = g^{\mu_1\nu_1}\cdots g^{\mu_n\nu_n}\,\varepsilon_{\nu_1\ldots\nu_n\mu_{n+1}\cdots\mu_D}\,, \tag{A.13a}$$

$$\varepsilon^{\mu_1\mu_2\cdots\mu_D} = g^{\mu_1\nu_1}\cdots g^{\mu_D\nu_D}\,\varepsilon_{\nu_1\nu_2\cdots\nu_D} = \frac{1}{g_D}\varepsilon_{\mu_1\mu_2\cdots\mu_D}\,, \tag{A.13b}$$

$$\varepsilon_{01\cdots(D-1)} \equiv 1\,, \qquad \varepsilon^{01\cdots(D-1)} = \frac{1}{g_D} < 0\,, \tag{A.13c}$$

$$T_{\mu_1\cdots\mu_D}\varepsilon^{\mu_1\cdots\mu_D} = T_{\mu_1\cdots\mu_D}\,g^{\mu_1\nu_1}\cdots g^{\mu_D\nu_D}\,\varepsilon_{\nu_1\cdots\nu_D} = T^{\nu_1\cdots\nu_D}\,\varepsilon_{\nu_1\cdots\nu_D}\,. \tag{A.13d}$$

The final line is from the definition that $\varepsilon_{\mu_1\ldots\mu_D}$ is a **tensor**. Using the Hodge star operator and the invariant tensor, we can discuss more properties:[15]

$$**\,\omega_p = (-1)^{p(D-p)+1}\,\omega_p\,, \tag{A.14a}$$

$$\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_D} = g_D\,\varepsilon^{\mu_1\cdots\mu_D}\,\mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1}\,, \tag{A.14b}$$

$$\mathrm{d}^Dx \equiv \mathrm{d}x^0\wedge\cdots\wedge\mathrm{d}x^{D-1} = \frac{1}{D!}\,\varepsilon_{\mu_1\cdots\mu_D}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_D}\,, \tag{A.14c}$$

$$g_D\,\varepsilon^{\mu_1\cdots\mu_p}{}_{\nu_{p+1}\cdots\nu_D}\cdot\varepsilon_{\mu_1\cdots\mu_p}{}^{\rho_{p+1}\cdots\rho_D} = p!(D-p)!\cdot\delta^{\rho_{p+1}}_{[\nu_{p+1}}\cdots\delta^{\rho_D}_{\nu_D]}\,. \tag{A.14d}$$

We also introduce an inveriant tensor $\mathcal{E}_{a_1\cdots a_D}$ in the local Lorentz (or the frame coordinate) system. Introducing the vielbein one-form $e^a = e^a_\mu\,\mathrm{d}x^\mu$, we write down in such a way as

$$\mathcal{E}^{a_1a_2\ldots a_D} = \eta^{a_1b_1}\eta^{a_2b_2}\cdots\eta^{a_Db_D}\,\mathcal{E}_{b_1b_2\cdots b_D} = \frac{1}{\eta_D}\mathcal{E}_{a_1a_2\cdots a_D} = -\mathcal{E}_{a_1a_2\cdots a_D}\,, \tag{A.15a}$$

$$T^{\mu_1\cdots\mu_D}\,\varepsilon_{\mu_1\cdots\mu_D} = T^{\mu_1\cdots\mu_D}\,e^{a_1}_{\mu_1}\cdots e^{a_D}_{\mu_D}\,\mathcal{E}_{a_1\cdots a_D} = T^{a_1\cdots a_D}\,\mathcal{E}_{a_1\cdots a_D}\,, \tag{A.15b}$$

$$T_{\mu_1\cdots\mu_D}\,\varepsilon^{\mu_1\cdots\mu_D} = T_{\mu_1\cdots\mu_D}\,e^{\mu_1}_{a_1}\cdots e^{\mu_D}_{a_D}\,\mathcal{E}^{a_1\cdots a_D} = T_{a_1\cdots a_D}\,\mathcal{E}^{a_1\cdots a_D}\,, \tag{A.15c}$$

$$\mathcal{E}_{01\cdots(D-1)} \equiv 1\,, \qquad \mathcal{E}^{01\cdots(D-1)} = \frac{1}{\eta_D} = -1\,, \tag{A.15d}$$

where $\eta_D \equiv \det\eta_{ab} = -1$, with the number of minus sign in the signature. Furthermore, using $\sqrt{-\eta_D} = 1$, we also define the followings:

$$(\text{vol.}) = e^0\wedge e^1\wedge\cdots\wedge e^{D-1}\,, \tag{A.16a}$$

$$e^{a_1}\wedge\cdots\wedge e^{a_D} = -\mathcal{E}^{a_1\cdots a_D}\,e^0\wedge e^1\wedge\cdots\wedge e^{D-1}\,, \tag{A.16b}$$

$$e^0\wedge\cdots\wedge e^{D-1} = \frac{1}{D!}\,\mathcal{E}_{a_1a_2\cdots a_D}\,e^{a_1}\wedge\cdots\wedge e^{a_D}\,, \tag{A.16c}$$

$$\mathcal{E}^{a_1\cdots a_p}{}_{b_{p+1}\cdots b_D}\cdot\mathcal{E}_{a_1\cdots a_p}{}^{c_{p+1}\cdots c_D} = -p!(D-p)!\cdot\delta^{c_{p+1}}_{[b_{p+1}}\cdots\delta^{c_D}_{b_D]}\,. \tag{A.16d}$$

[15] Different from the definition here, the orientation of d^4x in [1] is opposite to (vol.), i.e., $\mathrm{d}^4x = -(\text{vol.})$ while the normalization of the epsilon tensor is $\varepsilon_{0123} = +1$ as usual. Other definitions on differential form in [1] are same as mine.

B Einstein-Hilbert action

B.1 Sign of the action

Here let us list up all variations of the signs of the Einstein-Hilbert action discussed in section 4.4:

Einstein-Hilbert	Ricci tensor	signature	Riemann curvature	references
$\mathscr{L}_{\mathsf{EH}} = +\frac{1}{2}R$	$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$	$(-,+,+,+)$	$R^{\mu}{}_{\nu\rho\sigma} = +2(\partial_{[\rho}\Gamma^{\mu}{}_{\sigma]\nu} + \Gamma^{\mu}{}_{[\rho\vert\lambda\vert}\Gamma^{\lambda}{}_{\sigma]\nu})$	[2, 4, 6, 10, 16, 17, 19, 24]
		$(+,-,-,-)$	$R^{\mu}{}_{\nu\rho\sigma} = -2(\partial_{[\rho}\Gamma^{\mu}{}_{\sigma]\nu} + \Gamma^{\mu}{}_{[\rho\vert\lambda\vert}\Gamma^{\lambda}{}_{\sigma]\nu})$	[3, 5, 7]
	$R_{\mu\nu} = R^{\rho}{}_{\mu\nu\rho}$	$(-,+,+,+)$	$R^{\mu}{}_{\nu\rho\sigma} = -2(\partial_{[\rho}\Gamma^{\mu}{}_{\sigma]\nu} + \Gamma^{\mu}{}_{[\rho\vert\lambda\vert}\Gamma^{\lambda}{}_{\sigma]\nu})$	
		$(+,-,-,-)$	$R^{\mu}{}_{\nu\rho\sigma} = +2(\partial_{[\rho}\Gamma^{\mu}{}_{\sigma]\nu} + \Gamma^{\mu}{}_{[\rho\vert\lambda\vert}\Gamma^{\lambda}{}_{\sigma]\nu})$	
$\mathscr{L}_{\mathsf{EH}} = -\frac{1}{2}R$	$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$	$(-,+,+,+)$	$R^{\mu}{}_{\nu\rho\sigma} = -2(\partial_{[\rho}\Gamma^{\mu}{}_{\sigma]\nu} + \Gamma^{\mu}{}_{[\rho\vert\lambda\vert}\Gamma^{\lambda}{}_{\sigma]\nu})$	[25, 26, 27, 28]
		$(+,-,-,-)$	$R^{\mu}{}_{\nu\rho\sigma} = +2(\partial_{[\rho}\Gamma^{\mu}{}_{\sigma]\nu} + \Gamma^{\mu}{}_{[\rho\vert\lambda\vert}\Gamma^{\lambda}{}_{\sigma]\nu})$	[29]
	$R_{\mu\nu} = R^{\rho}{}_{\mu\nu\rho}$	$(-,+,+,+)$	$R^{\mu}{}_{\nu\rho\sigma} = +2(\partial_{[\rho}\Gamma^{\mu}{}_{\sigma]\nu} + \Gamma^{\mu}{}_{[\rho\vert\lambda\vert}\Gamma^{\lambda}{}_{\sigma]\nu})$	[23]
		$(+,-,-,-)$	$R^{\mu}{}_{\nu\rho\sigma} = -2(\partial_{[\rho}\Gamma^{\mu}{}_{\sigma]\nu} + \Gamma^{\mu}{}_{[\rho\vert\lambda\vert}\Gamma^{\lambda}{}_{\sigma]\nu})$	[1, 9]

where we used the anti-symmetrized symbol (A.2):

$$2\Big(\partial_{[\rho}\Gamma^{\mu}{}_{\sigma]\nu} + \Gamma^{\mu}{}_{[\rho|\lambda|}\Gamma^{\lambda}{}_{\sigma]\nu}\Big) \ = \ \partial_{\rho}\Gamma^{\mu}{}_{\sigma\nu} - \partial_{\sigma}\Gamma^{\mu}{}_{\rho\nu} + \Gamma^{\mu}{}_{\rho\lambda}\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\mu}{}_{\sigma\lambda}\Gamma^{\lambda}{}_{\rho\nu}\,. \tag{B.1}$$

[Remark]:
I was confused the convention of the Bernard de Wit and his company [25, 26, 27, 28], because the kinetic terms of scalar fields indicates the mostly plus signature, while the Ricci tensor is defined as $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$ as usual. Notice that they definitely use the mostly plus signature. However, the curvature is exactly same as (1.2) in this note. Notice the positions of indices! B. de Wit defines his curvature tensor $R_{\mu\nu\rho}{}^{\sigma}(= -R_{\mu\nu}{}^{\sigma}{}_{\rho}) \equiv \partial_{\mu}\Gamma^{\sigma}{}_{\nu\rho} - \partial_{\nu}\Gamma^{\sigma}{}_{\mu\rho} + \Gamma^{\sigma}{}_{\mu\tau}\Gamma^{\tau}{}_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\tau}\Gamma^{\tau}{}_{\mu\rho}$ (see (2.25) in [26]), which corresponds to $-R^{\sigma}{}_{\rho\mu\nu}$ in (1.2) in this note. We can see a justification of this convention in appendix A of [3].

We should also notice that Vandoren and his company corrected the sign of the Einstein-Hilbert action from the JHEP version of [3] to the ones in [5, 7]. We can see the corrected version as arXiv:0909.1743v4, and they explicitly mention this comment in the footnote 2 of [5].

B.2 Formulae

$$\begin{aligned}
g \ &\equiv \ \det g_{\mu\nu}\,, && \text{(B.2a)}\\
\delta g^{\mu\nu} \ &= \ -g^{\mu\rho}g^{\nu\sigma}\,\delta g_{\rho\sigma}\,, && \text{(B.2b)}\\
\delta\sqrt{-g} \ &= \ \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu} \ = \ -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}\,, && \text{(B.2c)}\\
\delta R_{\mu\nu} \ &= \ \nabla_{\rho}\delta\Gamma^{\rho}{}_{\nu\mu} - \nabla_{\nu}\delta\Gamma^{\rho}{}_{\rho\mu}\,, && \text{(B.2d)}\\
\delta(\sqrt{-g}\,R) \ &= \ \sqrt{-g}\,R_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2}\sqrt{-g}\,Rg_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\big(\nabla_{\rho}\delta\Gamma^{\rho}{}_{\nu\mu} - \nabla_{\nu}\delta\Gamma^{\rho}{}_{\rho\mu}\big)\\
\ &= \ \sqrt{-g}\,R_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2}\sqrt{-g}\,Rg_{\mu\nu}\delta g^{\mu\nu} + \partial_{\rho}\big(\sqrt{-g}\,g^{\mu\nu}\delta\Gamma^{\rho}{}_{\nu\mu}\big) - \partial_{\nu}\big(\sqrt{-g}\,g^{\mu\nu}\delta\Gamma^{\rho}{}_{\rho\mu}\big)\,. && \text{(B.2e)}
\end{aligned}$$

C Minor differences of convention among literature

C.1 Convention of $Sp(2m)$ curvatures

There is a comment on a difference between the convention in [1] and the one in [9]. The $Sp(2m)$ metric $\mathbb{C}_{\alpha\beta}$ in [1] (used in this note) is expressed as $\mathbb{R}_{\alpha\beta}$ in the JHEP version of [9], while the curvature of the $Sp(2m)$ bundle is changed from $\mathbb{R}^{\alpha\beta}$ in [1] to $\mathcal{R}^{\alpha\beta}$ in [9]. These changes are bit confusing and we summarize as follows:

	[1] and arXiv version of [9]	JHEP version of [9]
$Sp(2m)$ metric	$\mathbb{C}_{\alpha\beta}$	$\mathbb{R}_{\alpha\beta}$
$Sp(2m)$ curvature	$\mathbb{R}^{\alpha\beta}$	$\mathcal{R}^{\alpha\beta}$

C.2 Structure on quaternionic geometry

In [1, 21], the hyper-Kähler forms, the curvatures of the $SU(2)$-bundle and of the $Sp(2m)$-bundle are expanded as

$$K^x = K^x_{uv}\, \mathrm{d}q^u \wedge \mathrm{d}q^v \,, \qquad \Omega^x = \Omega^x_{uv}\, \mathrm{d}q^u \wedge \mathrm{d}q^v \,, \qquad \mathbb{R}^{\alpha\beta} = \mathbb{R}^{\alpha\beta}_{uv}\, \mathrm{d}q^u \wedge \mathrm{d}q^v \,. \tag{C.1}$$

Following the convention in appendix A.4, let us redefine the components of them as follows [19]:

$$K^x = K^x_{uv}\, \mathrm{d}q^u \wedge \mathrm{d}q^v \equiv \frac{1}{2!}(K^x_{uv})^{\mathsf{new}} \mathrm{d}q^u \wedge \mathrm{d}q^v \,, \tag{C.2a}$$

$$\Omega^x = \Omega^x_{uv}\, \mathrm{d}q^u \wedge \mathrm{d}q^v \equiv \frac{1}{2!}(\Omega^x_{uv})^{\mathsf{new}} \mathrm{d}q^u \wedge \mathrm{d}q^v \,, \tag{C.2b}$$

$$\mathbb{R}^{\alpha\beta} = \mathbb{R}^{\alpha\beta}_{uv}\, \mathrm{d}q^u \wedge \mathrm{d}q^v \equiv \frac{1}{2!}(\mathbb{R}^{\alpha\beta}_{uv})^{\mathsf{new}} \mathrm{d}q^u \wedge \mathrm{d}q^v \,. \tag{C.2c}$$

However, the three complex structures J^x, J^y and J^z and the metrics ϵ_{AB} and $\mathbb{C}_{\alpha\beta}$ are kept unchanged because they are purely mathematical objects, while the objects in (C.2) are described by dynamical fields which explicitly appear in the Lagarngian and various variation rules in section 3. We also do not redefine the spin connections ω^x and $\Delta^{\alpha\beta}$. We have to analyze how the quaternionic metric h_{uv}, the vielbein $\mathcal{U}^{\alpha A}_u$ and the constant λ would be changed under the above redefinitions.

Since the complex structures are invariant by definition, we can see the redefinition of the metric easily:

$$(J^x)^u{}_v = h^{uw} K^x_{wv} = (h^{uw})^{\mathsf{new}} (K^x_{wv})^{\mathsf{new}} = 2(h^{uw})^{\mathsf{new}} K^x_{wv} \,, \tag{C.3a}$$

$$\therefore \quad h^{uv} \equiv 2(h^{uv})^{\mathsf{new}} \,, \quad h_{uv} \equiv \frac{1}{2}(h_{uv})^{\mathsf{new}} \quad \to \quad \mathcal{U}^{\alpha A}_u \equiv \frac{1}{\sqrt{2}}(\mathcal{U}^{\alpha A}_u)^{\mathsf{new}} \,, \quad \mathcal{U}^u_{\alpha A} \equiv \sqrt{2}(\mathcal{U}^u_{\alpha A})^{\mathsf{new}} \,, \tag{C.3b}$$

$$\mathrm{d}s^2 = h_{uv}\, \mathrm{d}q^u \otimes \mathrm{d}q^v = \frac{1}{2}(h_{uv})^{\mathsf{new}} \mathrm{d}q^u \otimes \mathrm{d}q^v \equiv \frac{1}{2}(\mathrm{d}s^2)^{\mathsf{new}} \,. \tag{C.3c}$$

Let us redifine (1.27) to see the redefinition of $\lambda = -1$ of [1, 21]:

$$h^{st} \Omega^x_{us} \Omega^y_{tv} = -\lambda^2 \delta^{xy} h_{uv} + \lambda \epsilon^{xyz} \Omega^z_{uv} \,, \tag{C.4a}$$

$$\to \quad \Big(\frac{2}{2^2}\Big)(h^{st})^{\mathsf{new}} (\Omega^x_{us})^{\mathsf{new}} (\Omega^y_{tv})^{\mathsf{new}} = -\frac{1}{2}(\lambda^2)^{\mathsf{new}} \delta^{xy} (h_{uv})^{\mathsf{new}} + \frac{1}{2}(\lambda)^{\mathsf{new}} \epsilon^{xyz} (\Omega^z_{uv})^{\mathsf{new}} \,, \tag{C.4b}$$

$$\therefore \quad \lambda^{\mathsf{new}} = \lambda \,. \tag{C.4c}$$

Substituting the above results and the redefition of the spacetime metric into (1.26), we obtain the renewed canonical kinetic term of the scalar fields [19, 17]:

$$\begin{aligned} \mathscr{L}^{\mathsf{hyper}}_{\mathsf{kin}} &= -\lambda\, g^{\mu\nu} h_{uv}\, \partial_\mu q^u \partial_\nu q^v = -\lambda^{\mathsf{new}} (-g^{\mu\nu})^{\mathsf{new}} \Big(\frac{1}{2} h_{uv}\Big)^{\mathsf{new}} \partial_\mu q^u \partial_\nu q^v \\ &= -\frac{1}{2}\big(h_{uv}\, \partial_\mu q^u \partial^\mu q^v\big)^{\mathsf{new}} \,. \end{aligned} \tag{C.5}$$

Now we see that $\sqrt{2}$ appears in many terms caused by the rescaling of the metric h_{uv}. It would be helpful to redefine the fermion masses to reduce redundant factors (February 12, 2011).

References

[1] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and T. Magri, "*$\mathcal{N}=2$ supergravity and $\mathcal{N}=2$ super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map*," [J. Geom. Phys. **23** (1997) 111], [arXiv:hep-th/9605032].

[2] J.P. Gauntlett, S. Kim, O. Varela and D. Waldram, "*Consistent supersymmetric Kaluza–Klein truncations with massive modes*," [JHEP **0904** (2009) 102], [arXiv:0901.0676 [hep-th]].

[3] K. Hristov, H. Looyestijn and S. Vandoren, "*Maximally supersymmetric solutions of $D=4$ $\mathcal{N}=2$ gauged supergravity*," [JHEP **0911** (2009) 115], [arXiv:0909.1743 [hep-th]].

[4] S.L. Cacciatori and D. Klemm, "*Supersymmetric AdS_4 black holes and attractors*," [JHEP **1001** (2010) 085], [arXiv:0911.4926 [hep-th]].

[5] K. Hristov, H. Looyestijn and S. Vandoren, "*BPS black holes in $\mathcal{N}=2$ $D=4$ gauged supergravities*," [JHEP **1008** (2010) 103], [arXiv:1005.3650 [hep-th]].

[6] A. Donos, J.P. Gauntlett, N. Kim and O. Varela, "*Wrapped M5-branes, consistent truncations and AdS/CMT*," [JHEP **1012** (2010) 003], [arXiv:1009.3805 [hep-th]].

[7] K. Hristov and S. Vandoren, "*Static supersymmetric black holes in AdS_4 with spherical symmetry*," [arXiv:1012.4314 [hep-th]].

[8] R. D'Auria, S. Ferrara and M. Trigiante, "*On the supergravity formulation of mirror symmetry in generalized Calabi-Yau manifolds*," [Nucl. Phys. B **780** (2007) 28], [arXiv:hep-th/0701247].

[9] R. D'Auria, L. Sommovigo and S. Vaulà, "*$\mathcal{N}=2$ supergravity Lagrangian coupled to tensor multiplets with electric and magnetic fluxes*," [JHEP **0411** (2004) 028], [arXiv:hep-th/0409097].

[10] D. Cassani, "*Reducing democratic type II supergravity on $SU(3)\times SU(3)$ structures*," [JHEP **0806** (2008) 027], [arXiv:0804.0595 [hep-th]].

[11] L.J. Romans, "*Supersymmetric, cold and lukewarm black holes in cosmological Einstein-Maxwell theory*," [Nucl. Phys. B **383** (1992) 395], [arXiv:hep-th/9203018].

[12] M.M. Caldarelli and D. Klemm, "*Supersymmetry of anti-de Sitter black holes*," [Nucl. Phys. B **545** (1999) 434], [arXiv:hep-th/9808097].

[13] W.A. Sabra, "*Anti-de Sitter BPS black holes in $\mathcal{N}=2$ gauged supergravity*," [Phys. Lett. B **458** (1999) 36], [arXiv:hep-th/9903143].

[14] A.H. Chamseddine and W.A. Sabra, "*Magnetic and dyonic black holes in $D=4$ gauged supergravity*," [Phys. Lett. B **485** (2000) 301], [arXiv:hep-th/0003213].

[15] G. Dall'Agata and A. Gnecchi, "*Flow equations and attractors for black holes in $\mathcal{N}=2$ $U(1)$ gauged supergravity*," [arXiv:1012.3756 [hep-th]].

[16] M. Nakahara, "*Geometry, topology and physics*," *Boca Raton, USA: Taylor & Francis (2003) 573 p.*

[17] 今村 洋介, "超重力理論ノート" [2007 年 9 月版].

[18] S. Bellucci, S. Ferrara and A. Marrani, "*Supersymmetric mechanics. Vol. 2: The attractor mechanism and space time singularities*," Lect. Notes Phys. **701** (2006) 1.

[19] A. Van Proeyen, "*$\mathcal{N}=2$ supergravity in $d=4,5,6$ and its matter couplings*," the extended version of Lectures given during the semester "Supergravity, superstrings and M-theory" at Institut Henri Poincaré, Paris, November 2000. [http://itf.fys.kuleuven.be/~toine/LectParis.pdf].

[20] M. Graña, J. Louis and D. Waldram, "*$SU(3)\times SU(3)$ compactification and mirror duals of magnetic fluxes*," [JHEP **0704** (2007) 101], [arXiv:hep-th/0612237].

[21] D. Cassani, S. Ferrara, A. Marrani, J.F. Morales and H. Samtleben, "*A special road to AdS vacua*," [JHEP **1002** (2010) 027], [arXiv:0911.2708 [hep-th]].

[22] M.J. Duff, "*TASI lectures on branes, black holes and anti-de Sitter space*," based on talks delivered at the Theoretical Advanced Study Institute, Boulder, Colorado, Jun 1999 and the Banff Summer School, Alberta, Canada, Jun-Jul 1999, [arXiv:hep-th/9912164].

[23] 内山 龍雄, "一般相対性理論," 裳華房.

[24] 太田 信義, "超弦理論・ブレイン・M 理論," シュプリンガー・フェアラーク東京.

[25] B. de Wit and H. Nicolai, "*$\mathcal{N} = 8$ supergravity*," [Nucl. Phys. B **208** (1982) 323].

[26] B. de Wit, "*Introduction to supergravity*," [Lectures] given at 1984 Spring School on Supersymmetry and Supergravity, Trieste, Italy, Apr 4-14, 1984.

[27] B. de Wit, H. Samtleben and M. Trigiante, "*On Lagrangians and gaugings of maximal supergravities*," [Nucl. Phys. B **655** (2003) 93], [arXiv:hep-th/0212239].

[28] B. de Wit, H. Samtleben and M. Trigiante, "*The maximal $D = 4$ supergravities*," [JHEP **0706** (2007) 049] [arXiv:0705.2101 [hep-th]].

[29] L.D. ランダウ, E.M. リフシッツ, "場の古典論," 理論物理学教程, 東京図書; (L.D. Landau and E.M. Lifschitz, "*The classical theory of fields*," Elsevier Science).