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# Note on the Quartic Effective Action of Heterotic String

## — a dictionary of the transformation rules from BdR to others —

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**Abstract**

In this note we try to translate the heterotic supergravity Lagrangian given by Bergshoeff and de Roo [2] to the one which possesses the usual convention, for example, expressed in the Polchinski's book [13] and others [4, 6, 1, 9, 7].

In this note we use the anti-hermitian gauge field, which is convenient when we apply it to the analysis of the characteristic classes.

# 1 Convention

## 1.1 Antisymmetrized symbol

Polchinski [13] and FL [6] define the antisymmetric variables such as

$$T_{[M_1 M_2 \cdots M_p]} \;=\; \frac{1}{p!}\Big( T_{M_1 M_2 \cdots M_p} - T_{M_2 M_1 \cdots M_p} \pm \text{permutations} \Big) \,. \tag{1.1}$$

This definition is also applied in BdR [2, 3]. We can find this result from the following definition in [5]. Chapline and Manton defined the totally antisymmetric gamma matrix $\Gamma^{MNP}$ as $\Gamma^{MNP} = \Gamma^{[M}\Gamma^{N}\Gamma^{P]} = \Gamma^{M}\Gamma^{N}\Gamma^{P}$ if and only if the indices $M, N, P$ are different from one another. This means that the Gamma matrix is defined by

$$\Gamma^{[M}\Gamma^{N}\Gamma^{P]} \;=\; \frac{1}{3!}\Big( \Gamma^{M}\Gamma^{N}\Gamma^{P} \pm \text{permutations} \Big) \,, \tag{1.2}$$

which is consistent with (1.1).

## 1.2 Covariant derivatives

The Lorentz symmetry on the tangent space is important to describe vectors, tensors, and spinors in curved spacetime via vielbeins and inverse vielbeins. Let us now define the Lorentz algebra with respect to the Lorentz generators $\Sigma_{AB}$ such as

$$i[\Sigma_{AB}, \Sigma_{CD}] \;=\; \eta_{AC}\,\Sigma_{BD} + \eta_{BD}\,\Sigma_{AC} - \eta_{AD}\,\Sigma_{BC} - \eta_{BC}\,\Sigma_{AD} \,, \tag{1.3a}$$

$$\{\Gamma_A, \Gamma_B\} \;=\; 2\eta_{AB} \;=\; 2 \cdot \mathbf{1}\text{diag.}(- + + \cdots +) \,, \tag{1.3b}$$

where $\Gamma_A$ is the Dirac gamma matrix which we will discuss in detail. The Lorentz generators acting on scalars, vectors (tensors) and spinors are represented as follows:

$$\left\{ \begin{array}{lll} \Sigma_{AB} \;=\; 0 & & \text{scalar} \\ (\Sigma_{CD})^{A}{}_{B} \;=\; i\big(\delta^A_C\,\eta_{BD} - \delta^A_D\,\eta_{BC}\big) & & \text{vector} \\ \Sigma_{AB} \;=\; \dfrac{i}{2}\Gamma_{AB} & & \text{spinor} \end{array} \right. \tag{1.3c}$$

We introduce vielbeins $e_M{}^A$ and their inverses $E_A{}^M$ from the curved spacetime metric $g_{MN}$ and the tangent space metric $\eta_{AB}$:

$$g_{MN} \;=\; \eta_{AB}\, e_M{}^A\, e_N{}^B \,, \qquad \eta_{AB} \;=\; g_{MN}\, E_A{}^M\, E_B{}^N \,,$$
$$e_M{}^A\, E_A{}^N \;=\; \delta^N_M \,, \qquad E_A{}^M\, e_M{}^B \;=\; \delta^B_A \,.$$

Here we define the covariant derivative and curvature tensor with respect to the affine connection including torsions [11, 12]:

$$D_M(\Gamma,\omega) e_N{}^A \;\equiv\; 0 \;=\; \partial_M e_N{}^A + \omega_M{}^A{}_B\, e_N{}^B - \Gamma^P{}_{NM}\, e_P{}^A \,, \tag{1.4a}$$

$$D_M(\Gamma,\omega) E_A{}^N \;\equiv\; 0 \;=\; \partial_M E_A{}^N - E_B{}^N\, \omega_M{}^B{}_A + \Gamma^N{}_{PM}\, E_A{}^P \,, \tag{1.4b}$$

$$\Gamma^P_{0MN} \;=\; \frac{1}{2} g^{PQ}\big(\partial_M g_{QN} + \partial_N g_{MQ} - \partial_Q g_{MN}\big) \,, \qquad \Gamma^P{}_{[NM]} \;=\; T^P{}_{NM} \,, \tag{1.4c}$$

$$[D_M(\Gamma), D_N(\Gamma)] A_Q \ = \ -R^P{}_{QMN}(\Gamma) A_P + 2T^P{}_{MN}\, D_P(\Gamma) A_Q \ , \tag{1.4d}$$

$$R^P{}_{QMN}(\Gamma) \ = \ \partial_M \Gamma^P{}_{QN} - \partial_N \Gamma^P{}_{QM} + \Gamma^P{}_{RM}\Gamma^R{}_{QN} - \Gamma^R{}_{RN}\Gamma^R{}_{QM} \ . \tag{1.4e}$$

Here the covariant derivative with the local Lorentz transformation is described as

$$D_M(\omega)\phi^i \ = \ \Big\{ \delta^i_j \partial_M - \frac{i}{2} \omega_M{}^{AB} \cdot (\Sigma_{AB})^i{}_j \Big\} \phi^j \ . \tag{1.5a}$$

Then we also define the curvature tensor in terms of the spin connection with(out) torsion:

$$[D_M(\omega), D_N(\omega)]\phi \ = \ -\frac{i}{2} R^{AB}{}_{MN}(\omega)\, \Sigma_{AB}\phi \ , \tag{1.5b}$$

$$R^{AB}{}_{MN}(\omega) \ = \ \partial_M \omega_N{}^{AB} - \partial_N \omega_M{}^{AB} + \omega_M{}^A{}_C\, \omega_N{}^{CB} - \omega_N{}^A{}_C\, \omega_M{}^{CB} \ . \tag{1.5c}$$

The relation between the two curvature tensors are as follows:

$$R^R{}_{PMN}(\Gamma) \ = \ \eta_{BC}\, E_A{}^R\, e_P{}^C\, R^{AB}{}_{MN}(\omega) \ ,$$

$$R^R{}_M(\Gamma) \ = \ g^{PN} R^R{}_{PMN}(\Gamma) \ = \ e_M{}^B\, E_A{}^R\, R^A{}_B(\omega) \ , \qquad R^A{}_B(\omega) \ = \ R^{AC}{}_{BC}(\omega) \ ,$$

$$R(\Gamma) \ = \ R^M{}_M(\Gamma) \ = \ R^A{}_A(\omega) \ = \ R(\omega) \ .$$

## 1.3 Differential forms

We define differential forms on $D$-dimensional Riemannian manifolds ($g_D = \det g_{mn}$). For realistic discussions, we will define them in the curved spacetime with signature $(t,s) = (1, D-1)$ or $(0, D)$, i.e., we will introduce a parameter $t = 0, 1$ in the following definition which shows whether the spacetime is Lorentzian ($t = 1$) or Euclidean ($t = 0$).

$$\omega_p \ \equiv \ \frac{1}{p!} \omega_{M_1 \cdots M_p}\, \mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_p} \ , \tag{1.6a}$$

$$(\text{vol.}) \ \equiv \ \sqrt{|g_D|}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^D \ . \tag{1.6b}$$

It is also necessary to introduce a dual form of the $p$-form via so-called the Hodge dual:

$$*\omega_p \ = \ \frac{\sqrt{|g_D|}}{p!(D-p)!}\, \omega_{M_1 \cdots M_p}\, \varepsilon^{M_1 \cdots M_p}{}_{N_{p+1} \cdots N_D}\, \mathrm{d}x^{N_{p+1}} \wedge \cdots \wedge \mathrm{d}x^{N_D} \ , \tag{1.7a}$$

$$(*1) \ = \ \frac{\sqrt{|g_D|}}{D!}\, \varepsilon_{M_1 \cdots M_D}\, \mathrm{d}x^{M_1} \wedge \cdots \mathrm{d}x^{M_D} \ = \ \sqrt{|g_D|}\, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^D \ = \ (\text{vol.}) \ , \tag{1.7b}$$

where the index "1" does not always implies the first spatial direction; i.e., $\mathrm{d}x^1$ also often implies $\mathrm{d}t$ in the negative signature. Notice that $\varepsilon_{M_1 \cdots M_D}$ and $\varepsilon^{M_1 \cdots M_D}$ are called the invariant tensors whose property is given by

$$\varepsilon^{M_1 \cdots M_n}{}_{M_{n+1} \cdots M_D} \ = \ g^{M_1 N_1} \cdots g^{M_n N_n}\, \varepsilon_{N_1 \ldots N_n M_{n+1} \cdots M_D} \ , \tag{1.8a}$$

$$\varepsilon^{M_1 M_2 \cdots M_D} \ = \ g^{M_1 N_1} \cdots g^{M_D N_D}\, \varepsilon_{N_1 N_2 \cdots N_D} \ = \ g_D^{-1} \varepsilon_{N_1 N_2 \cdots N_D} \ , \tag{1.8b}$$

$$\varepsilon_{12 \cdots D} \ \equiv \ 1 \ , \qquad \varepsilon^{12 \cdots D} \ = \ \frac{1}{g_D} \ , \tag{1.8c}$$

$$T_{M_1 \cdots M_D} \varepsilon^{M_1 \cdots M_D} \ = \ T_{M_1 \cdots M_D} g^{M_1 N_1} \cdots g^{M_D N_D} \varepsilon_{N_1 \cdots N_D} \ = \ T^{N_1 \cdots N_D}\, \varepsilon_{N_1 \cdots N_D} \ . \tag{1.8d}$$

The final line is from the definition that $\varepsilon_{M_1 \ldots M_D}$ is a **tensor**. Using the Hodge star operator and the invariant tensor, we can discuss more properties:

$$* * \omega_p \ = \ (-1)^{p(D-p)+t} \, \omega_p \ , \tag{1.9a}$$

$$\mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_D} \ = \ g_D \, \varepsilon^{M_1 \cdots M_D} \, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^D \ , \tag{1.9b}$$

$$\mathrm{d}^D x \ \equiv \ \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^D \ = \ \frac{1}{D!} \varepsilon_{M_1 \cdots M_D} \, \mathrm{d}x^{M_1} \wedge \cdots \wedge \mathrm{d}x^{M_D} \ , \tag{1.9c}$$

$$g_D \, \varepsilon^{M_1 \cdots M_p}{}_{N_{p+1} \cdots N_D} \cdot \varepsilon_{M_1 \cdots M_p}{}^{L_{p+1} \cdots L_D} \ = \ p!(D-p)! \cdot \delta^{L_{p+1}}_{[N_{p+1}} \cdots \delta^{L_D}_{N_D]} \ . \tag{1.9d}$$

We also introduce an inveriant tensor $\mathcal{E}_{A_1 \cdots A_D}$ in the local Lorentz (or the frame coordinate) system. Introducing the vielbein one-form $e^A = e_M{}^A \, \mathrm{d}x^M$, we write down in such a way as

$$\mathcal{E}^{A_1 A_2 \ldots A_D} \ = \ \eta^{A_1 B_1} \eta^{A_2 B_2} \cdots \eta^{A_D B_D} \, \mathcal{E}_{B_1 B_2 \cdots B_D} \ = \ \eta_D^{-1} \mathcal{E}_{B_1 B_2 \cdots B_D} \ , \tag{1.10a}$$

$$T^{M_1 \cdots M_D} \, \varepsilon_{M_1 \cdots M_D} \ = \ T^{M_1 \cdots M_D} \, e_{M_1}{}^{A_1} \cdots e_{M_D}{}^{A_D} \, \mathcal{E}_{A_1 \cdots A_D} \ = \ T^{A_1 \cdots A_D} \, \mathcal{E}_{A_1 \cdots A_D} \ , \tag{1.10b}$$

$$T_{M_1 \cdots M_D} \, \varepsilon^{M_1 \cdots M_D} \ = \ T_{M_1 \cdots M_D} \, E_{A_1}{}^{M_1} \cdots E_{A_D}{}^{M_D} \, \mathcal{E}^{A_1 \cdots A_D} \ = \ T_{A_1 \cdots A_D} \, \mathcal{E}^{A_1 \cdots A_D} \ , \tag{1.10c}$$

$$\mathcal{E}_{12 \cdots D} \ = \ 1 \ , \qquad \mathcal{E}^{12 \cdots D} \ = \ \frac{1}{\eta_D} \ = \ (-1)^t \ , \tag{1.10d}$$

where $\eta_D \equiv \det \eta_{AB} = (-1)^t$, with the number of minus sign in the signature. Furthermore, we also define the followings:

$$(\mathrm{vol.}) \ = \ \sqrt{|\eta_D|} \, e^1 \wedge \cdots \wedge e^D \ = \ e^1 \wedge \cdots \wedge e^D \ , \tag{1.11a}$$

$$e^{A_1} \wedge \cdots \wedge e^{A_D} \ = \ \eta_D \, \mathcal{E}^{A_1 \cdots A_D} \, e^1 \wedge \cdots \wedge e^D \ , \tag{1.11b}$$

$$e^1 \wedge \cdots \wedge e^D \ = \ \frac{1}{D!} \, \mathcal{E}_{A_1 A_2 \cdots A_D} \, e^{A_1} \wedge \cdots \wedge e^{A_D} \ , \tag{1.11c}$$

$$\eta_D \, \mathcal{E}^{A_1 \cdots A_p}{}_{B_{p+1} \cdots B_D} \cdot \mathcal{E}_{A_1 \cdots A_p}{}^{C_{p+1} \cdots C_D} \ = \ p!(D-p)! \cdot \delta^{C_{p+1}}_{[B_{p+1}} \cdots \delta^{C_D}_{B_D]} \ . \tag{1.11d}$$

## 1.4 Yang-Mills gauge fields: hermitian variables

The covariant derivatives with respect to the Yang-Mills transformation on the field $\phi^i$ in the fundamental representation, and on the field $\varphi^a$ in the adjoint representation, are also defined as

$$\mathscr{D}_M(A)\phi^i \ = \ \partial_M \phi^i - i(A_M)^i{}_j \, \phi^j \qquad \text{fundamental representation ,} \tag{1.12a}$$

$$\left. \begin{aligned} \mathscr{D}_M(A)\chi \ &= \ \partial_M \chi - i[A_M, \chi] \\ \{\mathscr{D}_M(A)\chi\}^a \ &= \ \partial_M \chi^a + f^a{}_{bc} A^b_M \, \chi^c \end{aligned} \right\} \qquad \text{adjoint representation .} \tag{1.12b}$$

The field strength (i.e., the curvature) is defined as

$$[\mathscr{D}_M(A), \mathscr{D}_M(A)]\phi \ = \ -iF_{MN} \, \phi \ , \qquad F_{MN} \ = \ \partial_M A_N - \partial_N A_M - i[A_M, A_N] \ , \tag{1.12c}$$

where the gauge fields $A_M$ and the field strength $F_{MN}$ are described in terms of the gauge symmetry generators $T_a$ such as

$$A_M \ \equiv \ A^a_M T^a \ , \qquad \text{and} \ \ F_{MN} \ = \ F^a_{MN} T^a \ , \tag{1.13a}$$

where

$$\mathrm{tr}(T^a T^b) \ = \ \delta^{ab} \ , \qquad [T^a, T^b] \ = \ i f^{ab}{}_c \, T^c \ , \qquad (T^a)^\dagger \ = \ T^a \ , \qquad (T_a)_b{}^c \ = \ i f_{ba}{}^c \ = \ [\mathrm{ad}(T_a)]_b{}^c \ , \tag{1.13b}$$

$$[T_a,[T_b,T_c]]+[T_b,[T_c,T_a]]+[T_c,[T_a,T_b]] \;=\; 0 \;=\; f_{bcd}f_{ade}+f_{cad}f_{bde}+f_{abd}f_{cde}\;, \tag{1.13c}$$

$$F^a_{MN} \;=\; \partial_M A^a_N - \partial_N A^a_M + f^a{}_{bc}A^b_M A^c_N\;. \tag{1.13d}$$

Due to the above commutation relation we set $T^a$ to be hermitian and the structure constant $f^a{}_{bc}$ to be real.

Comment that the trace symbol "tr" in the above definition is in the fundamental (vector) representation. The exchanging rule between the trace tr in the $SO(n)$ vector and the trace Tr in the $SO(n)$ adjoint representations is given by

$$\mathrm{Tr}(T^2) \;=\; (n-2)\,\mathrm{tr}(T^2)\;, \tag{1.14a}$$

$$\mathrm{Tr}(T^4) \;=\; (n-8)\,\mathrm{tr}(T^4)+3\,\mathrm{tr}(T^2)\,\mathrm{tr}(T^2)\;, \tag{1.14b}$$

$$\mathrm{Tr}(T^6) \;=\; (n-32)\,\mathrm{tr}(T^6)+15\,\mathrm{tr}(T^2)\,\mathrm{tr}(T^4)\;, \tag{1.14c}$$

where $T$ is any linear combination of generators, but this implies the same relations for symmetrized products of different generators.

## 1.5 Yang-Mills gauge fields: anti-hermitian variables

Here we discuss another definition of the Yang-Mills fields in terms of the "anti-hermitian" generators $\widetilde{T}^a$. The algebra is defined as

$$\mathrm{tr}(\widetilde{T}^a\widetilde{T}^b) \;=\; -\delta^{ab}\;, \qquad [\widetilde{T}^a,\widetilde{T}^b] \;=\; f^{ab}{}_c\widetilde{T}^c\;, \qquad (\widetilde{T}^a)^\dagger \;=\; -\widetilde{T}^a\;, \qquad (\widetilde{T}_a)_b{}^c \;=\; f_{ba}{}^c \;=\; [\mathrm{ad}(\widetilde{T}_a)]_b{}^c\;, \tag{1.15a}$$

$$[\widetilde{T}_a,[\widetilde{T}_b,\widetilde{T}_c]]+[\widetilde{T}_b,[\widetilde{T}_c,\widetilde{T}_a]]+[\widetilde{T}_c,[\widetilde{T}_a,\widetilde{T}_b]] \;=\; 0 \;=\; -if_{bcd}f_{ade}-if_{cad}f_{bde}-if_{abd}f_{cde}\;. \tag{1.15b}$$

Note that the structure constant $f^a{}_{bc}$ to be real (and to be same as the one in the previous subsection). The relation between $\widetilde{T}^a$ and the generators $T^a$ is

$$T^a \;=\; i\widetilde{T}^a\;. \tag{1.16}$$

By using this anti-hermitian generators $\widetilde{T}^a$ we re-define the gauge fields

$$\widetilde{A}_M \;\equiv\; A^a_M\widetilde{T}^a \quad \text{with } i\widetilde{A}_M \;=\; A_M\;, \qquad \widetilde{F}_{MN} \;\equiv\; F^a_{MN}\widetilde{T}^a \quad \text{with } i\widetilde{F}_{MN} \;=\; F_{MN}\;. \tag{1.17a}$$

Here we also described the relations between $\widetilde{A}$ and $A$, which is the hermitian gauge fields defined in the previous subsection. Then the covariant derivatives with respect to the Yang-Mills transformation on the field $\phi^i$ in the fundamental representation, and on the field $\widetilde{\varphi}=\varphi^a\widetilde{T}^a$ in the adjoint representation, are also defined as

$$\mathscr{D}_M(\widetilde{A})\phi^i \;=\; \partial_M\phi^i + (\widetilde{A}_M)^i{}_j\,\phi^j \qquad \text{fundamental representation}\;, \tag{1.18a}$$

$$\left.\begin{aligned} \mathscr{D}_M(\widetilde{A})\widetilde{\chi} \;&=\; \partial_M\widetilde{\chi}+[\widetilde{A}_M,\widetilde{\chi}] \\ \{\mathscr{D}_M(\widetilde{A})\chi\}^a \;&=\; \partial_M\chi^a + f^a{}_{bc}A^b_M\,\chi^c \end{aligned}\right\} \qquad \text{adjoint representation}\;. \tag{1.18b}$$

The field strength (i.e., the curvature) is defined as

$$[\mathscr{D}_M(\widetilde{A}),\mathscr{D}_M(\widetilde{A})]\phi \;=\; \widetilde{F}_{MN}\,\phi\;, \qquad \widetilde{F}_{MN} \;=\; \partial_M\widetilde{A}_N - \partial_N\widetilde{A}_M + [\widetilde{A}_M,\widetilde{A}_N]\;, \tag{1.18c}$$

where we should notice that the "component fields" $A^a_M$ and $F^a_{MN}$ are the real fields and they corresponds to the ones in the previous subsection, i.e.,

$$F^a_{MN} \;=\; \partial_M A^a_N - \partial_N A^a_N + f^a{}_{bc}\,A^b_M A^c_N\;. \tag{1.18d}$$

## 1.6 Chern-Simons forms

Here let us introduce two kinds of the Chern-Simons 3-forms, i.e., the Lorentz-Chern-Simons 3-form $\omega_3^{\rm L}$ and the Yang-Mills-Chern-Simons 3-form $\omega_3^{\rm Y}$:

$$\omega_3^{\rm L} = \frac{1}{3!}\omega^{\rm L}_{MNP}\, {\rm d}x^M \wedge {\rm d}x^N \wedge {\rm d}x^P \equiv \Big( \omega^A{}_B \wedge {\rm d}\omega^B{}_A + \frac{2}{3}\omega^A{}_B \wedge \omega^B{}_C \wedge \omega^C{}_A \Big) , \tag{1.19a}$$

$$\omega_3^{\rm Y} = \frac{1}{3!}\omega^{\rm Y}_{MNP}\, {\rm d}x^M \wedge {\rm d}x^N \wedge {\rm d}x^P \equiv {\rm tr}\Big( A \wedge {\rm d}A - \frac{2i}{3} A \wedge A \wedge A \Big) , \tag{1.19b}$$

$$\frac{1}{3!}\omega^{\rm L}_{MNP} = \Big( \omega_{[M}{}^{AB} \partial_N \omega_{P]}{}^{BA} + \frac{2}{3}\omega_{[M}{}^{AB}\omega_N{}^{BC}\omega_{P]}{}^{CA} \Big) , \tag{1.19c}$$

$$\frac{1}{3!}\omega^{\rm Y}_{MNP} = {\rm tr}\Big( A_{[M}\partial_N A_{P]} - \frac{2i}{3} A_{[M}A_N A_{P]} \Big) , \tag{1.19d}$$

where $\omega^A{}_B = \omega_M{}^A{}_B {\rm d}x^M$ and $A = A_M^a T^a {\rm d}x^M$ are ths spin connection and the gauge fields which satisfy the followings

$$R^A{}_B = {\rm d}\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B , \qquad F = {\rm d}A - iA \wedge A . \tag{1.20}$$

Of course the Yang-Mills Chern-Simons 3-form $\widetilde{\omega}_3^{\rm Y}$ with respect to the anti-hermitian generators $\widetilde{T}^a$ can be defined as

$$\widetilde{\omega}_3^{\rm Y} \equiv {\rm tr}\Big( \widetilde{A} \wedge {\rm d}\widetilde{A} + \frac{2}{3}\widetilde{A} \wedge \widetilde{A} \wedge \widetilde{A} \Big) = -\omega_3^{\rm Y} \qquad \text{with} \ \ \widetilde{F} = {\rm d}\widetilde{A} + \widetilde{A} \wedge \widetilde{A} . \tag{1.21}$$

The exterior derivatives of these three-forms are given by

$${\rm d}\omega_3^{\rm L} = R^A{}_B \wedge R^B{}_A = {\rm tr}(R \wedge R) , \tag{1.22a}$$

$${\rm d}\omega_3^{\rm Y} = {\rm tr}(F \wedge F) , \qquad {\rm d}\widetilde{\omega}_3^{\rm Y} = {\rm tr}(\widetilde{F} \wedge \widetilde{F}) = -{\rm tr}(F \wedge F) = -{\rm d}\omega_3^{\rm Y} . \tag{1.22b}$$

## 1.7 $SO(9,1)$ Majorana-Weyl spinor

We define the Dirac conjugate

$$\overline{\psi} = i\psi^\dagger \Gamma^{\hat{0}} , \tag{1.23}$$

where $\Gamma^{\hat{0}}$ lives in the tangent space. Furthermore, we assign the Majorana condition such as

$$\overline{\psi} \equiv \psi^{\rm T} C , \tag{1.24}$$

where $C$ is called the charge conjugate matrix whose generic properties are

$$C = -C^{-1} = -C^{\rm T} , \tag{1.25a}$$

$$C(\Gamma^A)C^{-1} = -(\Gamma^A)^{\rm T} , \qquad C(\Gamma^{A_1 \cdots A_n})C^{-1} = (-)^{[\frac{n+1}{2}]}(\Gamma^{A_1 \cdots A_n})^{\rm T} , \tag{1.25b}$$

where $[\frac{n+1}{2}] = \{1,1,2,2,3,3,\cdots\}$ is the Gauss bracket. The hermitian conjugations of gamma matrices are defined by

$$(\Gamma^A)^\dagger = \Gamma_A = -\Gamma^{\hat{0}}\Gamma^A(\Gamma^{\hat{0}})^{-1} .$$

Among the Dirac gamma matrices there exists a useful identity such as

$$\Gamma^{A_1 A_2 \cdots A_p} \Gamma_{B_1 B_2 \cdots B_q} = \sum_{k=0}^{\min(p,q)} (-1)^{\frac{1}{2}k(2p-k-1)} \frac{p!q!}{(p-k)!(q-k)!k!} \delta^{[A_1}_{[B_1} \cdots \delta^{A_k}_{B_k} \Gamma^{A_{k+1} \cdots A_p]}{}_{B_{k+1} \cdots B_q]} \,. \qquad (1.26)$$

# 2 The descriptions in the BdR framework

From this section we will compare many dynamical variables defined in many articles. Here we describe the dilaton, the spin connection and its curvature, the gauge field and its field strength, and the NS two-form field and its field strength in terms of $\underline{\phi}$, $\{\underline{\omega}, \underline{\Omega}_{\pm}\}$, $\{\underline{R}(\underline{\omega}), \underline{R}(\underline{\Omega}_{\pm})\}$, $\underline{A}$, $\underline{F}$, $\underline{B}$ and $\underline{H}$, respectively, even though the descriptions of the Lagrangian, supersymmetry variation rules become quite dirty.

Notice that we have introduced some corrections of the misleading conventions described in BdR [2]:

$$\underline{\omega}^{\rm Y}_{MNP} \equiv \beta\sqrt{2}\,{\rm tr}\Big(\underline{A}_{[M}\partial_N \underline{A}_{P]} - \frac{2}{3}\underline{A}_{[M}\underline{A}_N\underline{A}_{P]}\Big) \, , \tag{2.1a}$$

$$\underline{R}^{AB}{}_{MN}(\underline{\omega}) \equiv 2\partial_{[M}\underline{\omega}_{N]}{}^{AB} - 2\,\underline{\omega}_{[M}{}^{AC}\,\underline{\omega}_{N]C}{}^B \, . \tag{2.1b}$$

We can easily find that $\underline{A}_M$ is anti-hermitian and the sign of the spin connection $\underline{\omega}$ and $\underline{A}_M$ are different[1] from the ones defined in (1.5) and in (1.17) in the previous section, i.e.,

$$\underline{\omega} \ = \ -\omega \, , \qquad \underline{A}_M \ = \ -\widetilde{A}_M \qquad \text{with} \ \ \underline{F} \ = \ {\rm d}\underline{A} - \underline{A} \wedge \underline{A} \, . \tag{2.2}$$

Before we compare the Lagrangian in [2] and the one in [13], we see the various differences of definitions in each convention:

| Bergshoeff-de Roo [2] with eqs.(2.1) | usual convention |
|---|---|
| $A_{[M_1} B_{M_2} \cdots K_{M_p]} = \frac{1}{p!}\Big(A_{M_1} B_{M_2} \cdots K_{M_p} \pm \text{permutations}\Big)$ | |
| $\underline{C}_p = \underline{C}_{M_1 M_2 \cdots M_p}\, {\rm d}x^{M_1} \wedge {\rm d}x^{M_2} \wedge \cdots \wedge {\rm d}x^{M_p}$ | $C_p = \frac{1}{p!} C_{M_1 M_2 \cdots M_p}\, {\rm d}x^{M_1} \wedge {\rm d}x^{M_2} \wedge \cdots \wedge {\rm d}x^{M_p}$ |
| $\underline{A}_M = \underline{A}^a_M \widetilde{T}^a, \quad (\widetilde{T}^a)^{\dagger} = -\widetilde{T}^a$ | $\widetilde{A}_M = \widetilde{A}^a_M \widetilde{T}^a, \quad (\widetilde{T}^a)^{\dagger} = -\widetilde{T}^a$ |
| — (no explicit descriptions) $[\underline{A}_M = -\widetilde{A}_M]$ | $\widetilde{F}_{MN} = \partial_M \widetilde{A}_N - \partial_N \widetilde{A}_M + (\widetilde{A}_M \widetilde{A}_N - \widetilde{A}_N \widetilde{A}_M)$ |
| — (no explicit descriptions) $[\underline{F} = {\rm d}\underline{A} - \underline{A} \wedge \underline{A}]$ | $\widetilde{F} = {\rm d}\widetilde{A} + \widetilde{A} \wedge \widetilde{A}$ |
| $\underline{R}^{AB}{}_{MN}(\underline{\omega}) = 2\partial_{[M}\underline{\omega}_{N]}{}^{AB} - 2\underline{\omega}_{[M}{}^{AC}\underline{\omega}_{N]C}{}^B$ | $R^{AB}{}_{MN}(\omega) = 2\partial_{[M}\omega_{N]}{}^{AB} + 2\omega_{[M}{}^{AC}\omega_{N]C}{}^B$ |
| — (no explicit descriptions) $[\underline{\omega}^{AB} = -\omega^{AB}]$ | $R^A{}_B(\omega) = {\rm d}\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B$ |
| $\underline{H}_{MNP} = \partial_{[M}\underline{B}_{NP]} - \beta\sqrt{2}\,\underline{\omega}^{\rm Y}_{MNP}$ | $\widetilde{H}_3 = {\rm d}\widetilde{B}_2 - \frac{\kappa_{10}^2}{g_{10}^2}\,\widetilde{\omega}^{\rm Y}_3$ |
| $\underline{\omega}^{\rm Y}_{MNP} = {\rm tr}\Big(\underline{A}_{[M}\partial_N\underline{A}_{P]} - \frac{2}{3}\underline{A}_{[M}\underline{A}_N\underline{A}_{P]}\Big)$ | $\widetilde{\omega}^{\rm Y}_3 = {\rm tr}\Big(\widetilde{A} \wedge {\rm d}\widetilde{A} + \frac{2}{3}\widetilde{A} \wedge \widetilde{A} \wedge \widetilde{A}\Big)$ |
| — (no explicit descriptions) | ${\rm d}\widetilde{H} = \frac{\kappa_{10}^2}{g_{10}^2}\Big[{\rm tr}\{R(\omega) \wedge R(\omega)\} - {\rm tr}(\widetilde{F} \wedge \widetilde{F})\Big]$ |

Note that the trace symbols "tr" in the above table are in the fundamental (vector) representation.

[1] In the same reason, we will interpret the gaugino field as anti-hermitian, i.e., we define $\underline{\chi} = \underline{\chi}^a \widetilde{T}^a$ in terms of the anti-hermitian gauge symmetry generator $\widetilde{T}^a$.

Furthermore, to obtain a positive energy as the eigenvalue of the Hamiltonian, we should treat every kinetic term in the Lagrangian. Notice that, as we will see in the next subsection (and originally in [2]), the kinetic term of the gauge field $\underline{A}_M$ is given in terms of the anti-hermitian gauge field strength in the way as

$$\sqrt{-g}\underline{\phi}^3 \beta \Big[ - \frac{1}{4} \mathrm{tr}(\underline{F}_{MN} \underline{F}^{MN}) \Big] \, . \tag{2.3}$$

I guess[2] that the parameter $\beta$ should be negative to obtain positive energy eigenvalues of the system, because the gauge field $\underline{A}_M$ and its field strength $\underline{F}_{MN}$ are anti-hermitian. Via the Bianchi identity, we will find that the parameter $\alpha$ should be also negative, and this guess might be correct:

$$\alpha, \beta \ < \ 0 \, . \tag{2.4}$$

(Bergshoeff and de Roo made a comment that they constructed $\mathscr{L}(\underline{R}^2)$ in the same analogy as the construction of $\mathscr{L}(\underline{F}^2)$.)

[2]On 2006 8/23. Before that, I thought that the definition of the kinetic term should be modified in the way as $\mathrm{tr}(|\underline{F}_{MN}\underline{F}^{MN}|)$ to obtain positive energy eigenvalues.

### 2.1 Lagrangian

BdR [2] introduced the Lagrangian of ten-dimensional heterotic supergravity (see eq.(B.6) and eq.(B.7) in [2]):

$$\begin{aligned}
\mathscr{L}_{\text{BdR}}(\underline{R}) \;=\; \sqrt{-g}\,\underline{\phi}^{-3}\Big[ & -\frac{1}{2}\mathcal{R}(\underline{\omega}) - \frac{3}{4}\underline{H}_{MNP}\underline{H}^{MNP} + \frac{9}{2}\big(\underline{\phi}^{-1}\partial_M\underline{\phi}\big)^2 \\
& -\frac{1}{2}\overline{\psi}_M\Gamma^{MNP}\mathscr{D}_N(\underline{\omega})\psi_P + 2\sqrt{2}\overline{\lambda}\Gamma^{MN}\mathscr{D}_M(\underline{\omega})\psi_N + 4\overline{\lambda}\mathscr{D}\!\!\!/(\underline{\omega})\lambda \\
& +3\sqrt{2}\overline{\psi}_M\Gamma^N\Gamma^M\lambda\cdot\big(\underline{\phi}^{-1}\partial_N\underline{\phi}\big) - \frac{3}{2}\overline{\psi}_M\Gamma^M\psi_N\cdot\big(\underline{\phi}^{-1}\partial^N\underline{\phi}\big) \\
& +\frac{\sqrt{2}}{16}\underline{H}^{PQR}\Big\{\overline{\psi}_M\Gamma^{[M}\Gamma_{PQR}\Gamma^{N]}\psi_N + 4\sqrt{2}\overline{\psi}_M\Gamma^M{}_{PQR}\lambda - 8\overline{\lambda}\Gamma_{PQR}\lambda\Big\} \\
& +\frac{1}{96}\overline{\psi}^M\Gamma^{ABC}\psi_M\Big\{\overline{\lambda}\Gamma_{ABC}\lambda + \frac{\sqrt{2}}{2}\overline{\lambda}\Gamma_{ABC}\Gamma^N\psi_N \\
& \qquad - \frac{1}{4}\overline{\psi}^N\Gamma_{ABC}\psi_N - \frac{1}{8}\overline{\psi}^N\Gamma_N\Gamma_{ABC}\Gamma^P\psi_P\Big\}\Big]\,, \qquad (2.5a)
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_{\text{BdR}}(\underline{F}^2) \;=\; \sqrt{-g}\,\underline{\phi}^{-3}\beta\Big[ & -\frac{1}{4}\text{tr}\big(\underline{F}_{MN}\underline{F}^{MN}\big) - \frac{1}{2}\text{tr}\{\overline{\chi}\mathscr{D}\!\!\!/(\underline{\omega},\underline{A})\chi\} \\
& -\frac{1}{8}\,\text{tr}\{\overline{\chi}\Gamma^M\Gamma^{AB}(\underline{F}_{AB}+\hat{\underline{F}}_{AB})\}\Big(\psi_M + \frac{\sqrt{2}}{3}\Gamma_M\lambda\Big) + \frac{\sqrt{2}}{16}\,\text{tr}\big(\overline{\chi}\Gamma^{ABC}\chi\big)\hat{\underline{H}}_{ABC} \\
& -\frac{\sqrt{2}}{16\times 24}\,\text{tr}(\overline{\chi}\Gamma^{ABC}\chi)\,\overline{\psi}_M\big(4\Gamma_{ABC}\Gamma^M + 3\Gamma^M\Gamma_{ABC}\big)\lambda \\
& +\frac{1}{96}\,\text{tr}(\overline{\chi}\Gamma^{ABC}\chi)\overline{\lambda}\Gamma_{ABC}\lambda - \frac{\beta}{16\times 24}\,\text{tr}(\overline{\chi}\Gamma^{ABC}\chi)\,\text{tr}(\overline{\chi}\Gamma_{ABC}\chi)\Big]\,, \qquad (2.5b)
\end{aligned}$$

$$\begin{aligned}
\mathscr{L}_{\text{BdR}}(\underline{R}^2) \;=\; \sqrt{-g}\,\underline{\phi}^{-3}\alpha\Big[ & -\frac{1}{4}\underline{R}^{ABMN}(\underline{\Omega}_-)\underline{R}_{ABMN}(\underline{\Omega}_-) - \frac{1}{2}\overline{\psi}^{AB}\mathscr{D}\!\!\!/(\underline{\omega}(e,\psi),\underline{\Omega}_-)\psi_{AB} \\
& -\frac{1}{8}\overline{\psi}_{AB}\Gamma^M\Gamma^{NP}\{\underline{R}^{AB}{}_{NP}(\underline{\Omega}_-) + \hat{\underline{R}}^{AB}{}_{NP}(\underline{\Omega}_-)\}\Big(\psi_M + \frac{\sqrt{2}}{3}\Gamma_M\lambda\Big) \\
& +\frac{\sqrt{2}}{16}\overline{\psi}^{AB}\Gamma^{MNP}\psi_{AB}\hat{\underline{H}}_{MNP} \\
& -\frac{\sqrt{2}}{16\times 24}\,\overline{\psi}_{AB}\Gamma^{CDE}\psi_{AB}\cdot\overline{\psi}_M\big(4\Gamma_{CDE}\Gamma^M + 3\Gamma^M\Gamma_{CDE}\big)\lambda \\
& +\frac{1}{96}\overline{\psi}^{AB}\Gamma^{CDE}\psi_{AB}\big(\overline{\lambda}\Gamma_{CDE}\lambda\big) - \frac{\alpha}{16\times 24}\overline{\psi}^{AB}\Gamma^{FGH}\psi_{AB}\big(\overline{\psi}^{CD}\Gamma_{FGH}\psi_{CD}\big)\Big]\,. \qquad (2.5c)
\end{aligned}$$

Both the parameter $\beta$ and $\alpha$ are dimensionful and are proportional to the Regge slope $\alpha'$.

Derivatives $\mathscr{D}_M(\underline{\omega},\underline{A})$ are the covariant derivatives with respect to Lorentz and Yang-Mills gauge transformations. We define the derivative on fundamental fields $\phi^i$ as[3]

$$\mathscr{D}_M(\underline{\omega},\underline{A},\Gamma)\phi^i \;=\; \partial_M\phi^i + \frac{i}{2}\underline{\omega}_M{}^{AB}(\Sigma_{AB})^i{}_j\,\phi^j - (\underline{A}_M)^i{}_j\,\phi^j + \Gamma^i_{jM}\,\phi^j\,. \qquad (2.6)$$

Note that $\Sigma_{AB}$ is a Lorentz generator whose representation is given by (1.3). We derived this expression from the ones in Polchinski (see eqs. (1.5) and (1.12).) via the field re-defitions (3.4). Notice that we always define the affine connection in the way as (1.4).

[3] I assume that the Lorentz and Clifford algebras in BdR [2] correspond to the ones in Polchinski [13], on January 4, 2006.

## 2.2 Local supersymmetry variations

Here we pick up the local supersymmetry variations descibed in Appendix B of BdR [2]. As mentioned in [2], we write $\delta_{\alpha^n}$ ($\delta_{\beta^m}$) for variations of order $\alpha^n$ ($\beta^m$), while $\delta_0$ corresponds to the terms independent of $\alpha$ and $\beta$:

$$\delta_0 e_M{}^A = \frac{1}{2}\overline{\epsilon}\Gamma^A \psi_M \, , \tag{2.7a}$$

$$\delta_0 \psi_M = \Big(\partial_M - \frac{1}{4}\underline{\Omega}_{+M}{}^{AB}\Gamma_{AB}\Big)\epsilon + \frac{\sqrt{2}}{2}\Big\{\epsilon\big(\overline{\psi}_M\lambda\big) - \psi_M\big(\overline{\epsilon}\lambda\big) + \Gamma^A\lambda\big(\overline{\psi}_M\Gamma_A\epsilon\big)\Big\} \, , \tag{2.7b}$$

$$\delta_0 \underline{B}_{MN} = \frac{\sqrt{2}}{2}\overline{\epsilon}\Gamma_{[M}\psi_{N]} \, , \tag{2.7c}$$

$$\delta_0 \lambda = -\frac{3\sqrt{2}}{8}\,\underline{\phi}^{-1}\not{D}\underline{\phi}\,\epsilon + \frac{1}{8}\Gamma^{ABC}\epsilon\Big(\underline{\hat{H}}_{ABC} - \frac{\sqrt{2}}{24}\overline{\lambda}\Gamma_{ABC}\lambda\Big) \, , \tag{2.7d}$$

$$\underline{\phi}^{-1}\delta_0\underline{\phi} = -\frac{\sqrt{2}}{3}\overline{\epsilon}\lambda \, , \tag{2.7e}$$

$$\delta_0 \underline{A}_M = \frac{1}{2}\overline{\epsilon}\Gamma_M\chi \, , \tag{2.7f}$$

$$\delta_0 \chi = -\frac{1}{4}\Gamma^{AB}\epsilon\,\underline{\hat{F}}_{AB} + \frac{\sqrt{2}}{2}\Big\{\epsilon\big(\overline{\chi}\lambda\big) - \chi\big(\overline{\epsilon}\lambda\big) + \Gamma^A\lambda\big(\overline{\chi}\Gamma_A\epsilon\big)\Big\} \, , \tag{2.7g}$$

$$\delta_0 \underline{\omega}_M{}^{AB}(e,\psi) = \frac{1}{4}\overline{\epsilon}\Gamma_M\psi^{AB} + \frac{1}{2}\overline{\epsilon}\Gamma^{[A}\psi_M{}^{B]} + \frac{3\sqrt{2}}{4}\,\overline{\epsilon}\Gamma_C\psi_M\,\underline{\hat{H}}^{ABC} \, . \tag{2.7h}$$

Notice that the spin connection $\underline{\omega}(e,\psi)$ is the solution of $D_{[M}(\underline{\omega})e_{N]}{}^A = 0$, while $\underline{\omega}(e)$ is the solution of $\mathscr{D}_{[M}(\underline{\omega})e_{N]}{}^A = 0$. Note that Bergshoeff and de Roo define various additional variables such as a spin connection modified by the $H$-flux [2][4], supercovariantizations, and so forth:

$$\underline{\Omega}_{\pm M}{}^{AB} \equiv \underline{\omega}_M{}^{AB}(e,\psi) \pm \frac{3\sqrt{2}}{2}\underline{\hat{H}}_M{}^{AB} \, , \tag{2.8a}$$

$$\begin{aligned}\underline{\hat{H}}_{MNP} &\equiv \underline{H}_{MNP} - \frac{\sqrt{2}}{4}\overline{\psi}_{[M}\Gamma_N\psi_{P]} \\ &= \partial_{[M}\underline{B}_{NP]} - \frac{\sqrt{2}}{4}\overline{\psi}_{[M}\Gamma_N\psi_{P]} - \beta\sqrt{2}\,\mathrm{tr}\Big(\underline{A}_{[M}\partial_N\underline{A}_{P]} - \frac{2}{3}\underline{A}_{[M}\underline{A}_N\underline{A}_{P]}\Big) \\ &\quad + \alpha\sqrt{2}\,\Big(\underline{\Omega}_{-[M}{}^{AB}\,\partial_N\underline{\Omega}_{-P]}{}^{BA} - \frac{2}{3}\underline{\Omega}_{-[M}{}^{AB}\,\underline{\Omega}_{-N}{}^{BC}\,\underline{\Omega}_{-P]}{}^{CA}\Big) \, ,\end{aligned} \tag{2.8b}$$

$$\underline{\hat{F}}_{MN} \equiv \underline{F}_{MN} - \overline{\psi}_{[M}\Gamma_{N]}\chi \, , \tag{2.8c}$$

$$\psi_{MN} \equiv \mathscr{D}_M(\underline{\Omega}_+)\psi_N - \mathscr{D}_N(\underline{\Omega}_+)\psi_M - \frac{\sqrt{2}}{2}\Big\{\psi_M\big(\overline{\psi}_N\lambda\big) - \psi_N\big(\overline{\psi}_M\lambda\big) - \Gamma^P\lambda\,\big(\overline{\psi}_M\Gamma_P\psi_N\big)\Big\} \, . \tag{2.8d}$$

The meaning of hat symbol is the "supercovariantization" of the variables, whose supersymmetry variations do not involve derivatives of the infinitesimal supersymmetry parameter. Subject to this definition, we can find the explicit expression of $\underline{\hat{R}}^{AB}{}_{MN}(\underline{\omega})$, which is not given in the Bergshoeff-de Roo's paper [2]:

$$\underline{\hat{R}}^{AB}{}_{MN}(\underline{\omega}) \equiv \underline{R}^{AB}{}_{MN}(\underline{\omega}) - \frac{1}{2}\overline{\psi}_{[M}\Gamma_{N]}\psi^{AB} - \overline{\psi}_{[M}\Gamma^{[A}\psi_{N]}{}^{B]} + \frac{3\sqrt{2}}{2}\,\overline{\psi}_{[M}\Gamma^C\psi_{N]}\,\underline{\hat{H}}^{AB}{}_C \, . \tag{2.9}$$

We also pick up the supercovariant derivatives at hand:

$$\underline{\phi}^{-1}D_M\underline{\phi} = \underline{\phi}^{-1}\partial_M\underline{\phi} + \frac{\sqrt{2}}{3}\overline{\psi}_M\lambda \, , \tag{2.10a}$$

$$D_M(\underline{\omega})\lambda = \mathscr{D}_M(\underline{\omega})\lambda + \frac{3\sqrt{2}}{8}\,\underline{\phi}^{-1}\not{D}\underline{\phi}\,\psi_M - \frac{1}{8}\Gamma^{ABC}\psi_M\Big(\underline{\hat{H}}_{ABC} - \frac{\sqrt{2}}{24}\,\overline{\lambda}\Gamma_{ABC}\lambda\Big) \, , \tag{2.10b}$$

[4] If we simply write down the spin connection as $\underline{\omega}$, this means $\underline{\omega} = \underline{\omega}(e)$.

$$D_M(\underline{\omega},\underline{A})\chi \ = \ \mathscr{D}_M(\underline{\omega},\underline{A})\chi + \frac{1}{4}\Gamma^{AB}\psi_M\,\hat{\underline{F}}_{AB} - \frac{\sqrt{2}}{2}\Big\{\psi_M\big(\overline{\chi}\lambda\big) - \chi\big(\overline{\psi}_M\lambda\big) + \Gamma^A\lambda\big(\overline{\chi}\Gamma_A\psi_M\big)\Big\} \, . \tag{2.10c}$$

It is both useful and instructive to obtain the supersymmetry algebra from (2.7). The commutator of two supersymmetry variations reads

$$\begin{aligned}
\big[\delta(\epsilon_1),\delta(\epsilon_2)\big] \ &= \ \delta_{\rm P}(\xi^M) + \delta_{\rm Q}(-\xi^M\psi_M) + \delta_{\rm L}(-\xi^M\underline{\Omega}_{-M}{}^{AB}) + \delta_{\rm YM}(-\xi^M\underline{A}_M) \\
&\quad + \delta_{\rm M}(-\tfrac{\sqrt{2}}{2}\xi_M - \xi^N\underline{B}_{NM}) + \delta_{\rm Q}(\epsilon_3) + \delta_{\rm L}(\Lambda^{AB}) \, , && (2.11a) \\
\xi^M \ &= \ \frac{1}{2}\overline{\epsilon}_2\Gamma^M\epsilon_1 \, , && (2.11b) \\
\epsilon_3 \ &= \ -\frac{7\sqrt{2}}{16}\overline{\epsilon}_2\Gamma^A\epsilon_1\,\Gamma_A\lambda + \frac{\sqrt{2}}{32\times 120}\,\overline{\epsilon}_2\Gamma^{ABCDE}\epsilon_1\,\Gamma_{ABCDE}\lambda \, , && (2.11c) \\
\Lambda^{AB} \ &= \ \frac{\beta}{192}\,\overline{\epsilon}_2\Gamma^{[A}\Gamma_{CDE}\Gamma^{B]}\epsilon_1\,{\rm tr}\big(\overline{\chi}\Gamma^{CDE}\chi\big) \, . && (2.11d)
\end{aligned}$$

On the right-hand side of (2.11a), we encounter all gauge transformations of the ten-dimensional super Yang-Mills theory: $\delta_{\rm P}$, $\delta_{\rm Q}$, $\delta_{\rm L}$, $\delta_{\rm YM}$, and $\delta_{\rm M}$ correspond respectively to "general coordinate", "supersymmetry", "local Lorentz", "Yang-Mills" and "antisymmetric tensor gauge" transformations.

The supersymmetry variation of order $\beta$ are given as follows:

$$\begin{aligned}
\delta_\beta\psi_M \ &= \ \frac{\beta}{192}\,\Gamma^{ABC}\Gamma_M\epsilon\,{\rm tr}\big(\overline{\chi}\Gamma_{ABC}\chi\big) \, , && (2.12a) \\
\delta_\beta\underline{B}_{MN} \ &= \ -\beta\sqrt{2}\,{\rm tr}\big\{\underline{A}_{[M}\delta_0\underline{A}_{N]}\big\} \, , && (2.12b) \\
\delta_\beta\lambda \ &= \ \frac{\beta\sqrt{2}}{384}\,\Gamma^{ABC}\epsilon\,{\rm tr}\big(\overline{\chi}\Gamma_{ABC}\chi\big) \, , && (2.12c) \\
\delta_\beta\underline{\omega}_M{}^{AB}(e,\psi) \ &= \ \frac{\beta}{192}\,\overline{\epsilon}\Gamma^{[A}\Gamma_{CDE}\Gamma^{B]}\psi_M\,{\rm tr}\big(\overline{\chi}\Gamma^{CDE}\chi\big) \, . && (2.12d)
\end{aligned}$$

Here the supersymmetry variation of order $\alpha$ are also given such as

$$\begin{aligned}
\delta_\alpha\psi_M \ &= \ \frac{\alpha}{192}\Gamma^{CDE}\Gamma_M\epsilon\,\overline{\psi}^{AB}\Gamma_{CDE}\psi_{AB} \, , && (2.13a) \\
\delta_\alpha\underline{B}_{MN} \ &= \ -\alpha\sqrt{2}\,\underline{\Omega}_{-[M}{}^{AB}\,\delta_0\underline{\Omega}_{-N]}{}^{AB} \, , && (2.13b) \\
\delta_\alpha\lambda \ &= \ \frac{\alpha\sqrt{2}}{384}\,\Gamma^{CDE}\Gamma_M\epsilon\,\overline{\psi}^{AB}\Gamma_{CDE}\psi_{AB} \, . && (2.13c)
\end{aligned}$$

The supersymmetry variations of the supercovariant variables are also obtained. First we write down the zero-th order of $\alpha$ and $\beta$. Next the corrections of first order $\beta$ are described. (Unfortunately, there are no descriptions about the corrections of first order $\alpha$.)

$$\begin{aligned}
\delta_0(\underline{\phi}^{-1}D_A\underline{\phi}) \ &= \ -\frac{\sqrt{2}}{3}\,\overline{\epsilon}D_A(\underline{\Omega}_+)\lambda \, , && (2.14a) \\
\delta_0\underline{\Omega}_{-M}{}^{AB} \ &= \ \frac{1}{2}\overline{\epsilon}\Gamma_M\psi^{AB} \, , && (2.14b) \\
\delta_0\hat{\underline{H}}_{ABC} \ &= \ -\frac{\sqrt{2}}{4}\,\overline{\epsilon}\Gamma_{[A}\psi_{BC]} \, , && (2.14c) \\
\delta_0\psi^{AB} \ &= \ -\frac{1}{4}\Gamma^{CD}\epsilon\,\hat{\underline{R}}^{AB}{}_{CD}(\underline{\Omega}_-) + \frac{\sqrt{2}}{2}\Big\{\epsilon\big(\overline{\psi}^{AB}\lambda\big) - \psi^{AB}\big(\overline{\epsilon}\lambda\big) + \Gamma^C\lambda\,\big(\overline{\psi}^{AB}\Gamma_C\epsilon\big)\Big\} \, , && (2.14d) \\
\delta_0\hat{\underline{F}}_{AB} \ &= \ -\overline{\epsilon}\Gamma_{[A}D_{B]}(\underline{\Omega}_+,\underline{A})\chi \, , && (2.14e) \\
\delta_\beta(\underline{\phi}^{-1}D_A\underline{\phi}) \ &= \ -\frac{\beta\sqrt{2}}{3\times 192}\,\overline{\epsilon}\Gamma_A\Gamma^{BCD}\lambda\,{\rm tr}\big(\overline{\chi}\Gamma_{BCD}\chi\big) \, , && (2.14f)
\end{aligned}$$

$$
\begin{aligned}
\delta_\beta \underline{\hat{H}}_{ABC} &= -\frac{\beta\sqrt{2}}{2}\,\overline{\epsilon}\Gamma_{[A}\,\mathrm{tr}\big(\chi \underline{\hat{F}}_{BC]}\big)\ , && (2.14g)\\
\delta_\beta \psi_{AB} &= \beta\bigg[\frac{3}{4}\,\Gamma^{CD}\epsilon\,\mathrm{tr}\big(\underline{\hat{F}}_{[AB}\underline{\hat{F}}_{CD]}\big) + \frac{1}{48}\Gamma^{CDE}\Gamma_{[A}\epsilon\,\mathrm{tr}\big\{\overline{\chi}\Gamma_{CDE}D_{B]}(\underline{\Omega}_+,\underline{A})\chi\big\} \\
&\qquad + \frac{\sqrt{2}}{256}\,\Gamma^{CDE}\Gamma_{[A}\Gamma^{GH}\epsilon\,\underline{\hat{H}}_{B]GH}\,\mathrm{tr}\big(\overline{\chi}\Gamma_{CDE}\chi\big) \\
&\qquad + \frac{\beta}{96\times 96}\,\Gamma^{CDE}\Gamma_{[A}\Gamma^{FGH}\Gamma_{B]}\epsilon\,\mathrm{tr}\big(\overline{\chi}\Gamma_{CDE}\chi\big)\,\mathrm{tr}\big(\overline{\chi}\Gamma_{FGH}\chi\big)\bigg]\ , && (2.14h)\\
\delta_\beta \underline{\hat{F}}_{AB} &= \frac{\beta}{192}\,\overline{\epsilon}\Gamma_{[A}\Gamma^{CDE}\Gamma_{B]}\chi\,\mathrm{tr}\big(\overline{\chi}\Gamma_{CDE}\chi\big)\ . && (2.14i)
\end{aligned}
$$

I make a comment. The explicit expressions of supersymmetry variations are, of course, just approximate expressions. If you consider not only $\alpha, \beta$ corrections but also the higher order fermions corrections, the corrections of supersymmetry vairations are also corrected. The supercovariant variables such as $\underline{\hat{H}}_{MNP}$, $\underline{\hat{F}}_{MN}$ and $\underline{\hat{R}}_{MN}{}^{AB}(\underline{\omega})$ are influenced by the higher order corrections quite sensitively. Thus, you must take care of any calculations when you study the supersymmetry variations and the construction of the Lagrangian of higher order corrections of fermions.

## 3 Mapping from BdR to Polchinski: bosonic parts

### 3.1 Setup

Here let us write down the actions of heterotic supergravity presented by Bergshoeff and de Roo [2]:

$$\mathscr{L}_{\rm BdR} \ = \ \sqrt{-g}\,\underline{\phi}^{-3}\Big\{ -\frac{1}{2}\underline{\mathcal{R}}(\underline{\omega}) + \frac{9}{2}(\underline{\phi}^{-1}\partial_M\underline{\phi})^2 - \frac{3}{4}\underline{H}_{MNP}\underline{H}^{MNP} - \frac{\beta}{4}\,{\rm tr}\big(\underline{F}_{MN}\underline{F}^{MN}\big) + \cdots \Big\} \, . \tag{3.1}$$

Here we only investigate the bosonic part of the heterotic supergravity. Notice that the action in [6] may include an incorrect coefficient in front of the kinetic term of the gauge field ${\rm tr}(|\underline{F}|^2)$, and more. (Frey and Lippert mentioned that they used the Polchinski's convention with anti-hermitian gauge field $\widetilde{A}_M$ in [13].) Thus we improve the bosonic action in [6] to the one in [13].

There are many equivalent conventions about the supergravity Lagrangian; the one is (3.1) and we would like to obtain a new description of it under some field re-definitions. For example, by using (2.2), we rewrite

$$\mathscr{L}_{\rm new} \ = \ \frac{1}{2\kappa_{10}^2}\sqrt{-g}\,{\rm e}^{-2\Phi}\Big\{ \mathcal{R}(\omega) + 4(\partial_M\Phi)^2 - \frac{1}{2}|\widetilde{H}|^2 + \frac{\kappa_{10}^2}{g_{10}^2}\,{\rm tr}(|\widetilde{F}|^2) + \cdots \Big\} \, , \tag{3.2a}$$

$$\widetilde{H}_3 \ = \ {\rm d}\widetilde{B}_2 - \frac{\kappa_{10}^2}{g_{10}^2}\widetilde{\omega}_3^{\rm Y} + \ldots \, , \tag{3.2b}$$

where $\kappa_{10}$ and $g_{10}$ are the gravitational constant and the Yang-Mills coupling constant in ten-dimensional spacetime, respectively. We adopted the following contraction rule [13]:

$$\int {\rm d}^{10}x\sqrt{-g}\,|C_p|^2 \ = \ \int {\rm d}^{10}x\sqrt{-g}\frac{1}{p!}\,g^{M_1N_1}g^{M_2N_2}\cdots g^{M_pN_p}\,C_{M_1\cdots M_p}C_{N_1\cdots N_p} \, . \tag{3.3}$$

The $p!$ cancels the sum over permutations of the indices, so that each independent component appears with coefficient 1. Notice that the symbol "$|(***)|^2$" does **not** mean that we take the square of the absolute value of $(***)$. Thus we assign the **plus sign** in front of the quadratic of $\widetilde{F}_{MN}$. We also make a comment that we never rescale the spacetime metric.

In order to connect (3.1) to (3.2a), we introduce the following re-definitions:

$$\text{spin connection:} \qquad \underline{\omega}_M{}^{AB} \ = \ -\omega_M{}^{AB} \qquad \text{with} \ \ R^A{}_B \ = \ {\rm d}\omega^A{}_B + \omega^A{}_C\wedge\omega^C{}_B \, , \tag{3.4a}$$

$$\text{non-abelian 1-form:} \qquad \underline{A}_M \ = \ -\widetilde{A}_M \qquad \text{with} \ \ \widetilde{F} \ = \ {\rm d}\widetilde{A} + {\rm d}\widetilde{A}\wedge\widetilde{A} \qquad \text{(anti-hermitian)} \, , \tag{3.4b}$$

$$H_3\text{-form}, B_2\text{-form:} \qquad \underline{H}_{MNP} \ = \ \frac{1}{3!}H'_{MNP} \ = \ \frac{h}{3!}\widetilde{H}_{MNP} \, , \qquad \underline{B}_{MN} \ = \ \frac{1}{2!}B'_{MN} \ = \ \frac{h'}{2!}\widetilde{B}_{MN} \, , \tag{3.4c}$$

$$\text{dilaton:} \qquad \phi^{-3} \ = \ f\,{\rm e}^{-2\Phi} \, . \tag{3.4d}$$

where $c$, $f$, $h$ and $h'$ are real constants. Notice that we rescaled the differential $p$-forms in terms of $1/p!$ because Bergshoeff and de Roo [2] used the relation

$$\underline{H}_{MNP} \ = \ \partial_{[M}\underline{B}_{NP]} + \ldots \, , \tag{3.5}$$

which means they defined the $p$-forms $\underline{C}_p$ in such a way as

$$\underline{C}_p \ \equiv \ \underline{C}_{M_1\cdots M_p}\,{\rm d}x^{M_1}\wedge\cdots\wedge{\rm d}x^{M_p} \, . \tag{3.6}$$

This definition is funny and different from the usual definition in eqs.(1.6).

## 3.2 Transformations

**— Dilaton —**

We compare the first term in (3.1) with the first term in (3.2a) via the field re-definitions (3.4):

$$\begin{aligned} \frac{9}{2}\sqrt{-g}\,\phi^{-3}(\phi^{-1}\partial_M\phi)^2 \;&=\; \frac{9}{2}\sqrt{-g}(f\,\mathrm{e}^{-2\Phi})g^{MN}\Big(\frac{2}{3}\partial_M\Phi\Big)\Big(\frac{2}{3}\partial_N\Phi\Big) \;=\; 2f\,\sqrt{-g}\,\mathrm{e}^{-2\Phi}(\partial_M\Phi)^2 \\ &=\; \frac{4}{2\kappa_{10}^2}\,\sqrt{-g}\,\mathrm{e}^{-2\Phi}(\partial_M\Phi)^2 \;. \end{aligned} \tag{3.7a}$$

Then we obtain

$$f \;\equiv\; \frac{1}{\kappa_{10}^2} \;. \tag{3.7b}$$

**— Scalar curvature —**

Next we compare the Einstein-Hilbert parts in (3.1) and (3.2a). The curvature two-forms should be connected by the re-definition of the spin connection (3.4a):

$$\begin{aligned} \underline{R}^{AB}{}_{MN}(\underline{\omega}) \;&=\; \partial_M\underline{\omega}_N{}^{AB} - \partial_N\underline{\omega}_M{}^{AB} - \underline{\omega}_M{}^{AC}\,\underline{\omega}_{NC}{}^B + \underline{\omega}_N{}^{AC}\,\underline{\omega}_{MC}{}^B \\ &=\; -\Big[\partial_M\omega_M{}^{AB} - \partial_N\omega_N{}^{AB} + \omega_M{}^{AC}\,\omega_{NC}{}^B - \omega_N{}^{AC}\,\omega_{MC}{}^B\Big] \;=\; -R^{AB}{}_{MN}(\omega) \;. \end{aligned} \tag{3.8}$$

The Ricci scalar is rescaled by the inverse vielbein:

$$\underline{\mathcal{R}}(\underline{\omega}) \;=\; E_A{}^M\,E_B{}^N\,\underline{R}^{AB}{}_{MN}(\underline{\omega}) \;=\; -\widetilde{E}_A{}^M\,\widetilde{E}_B{}^N\,R^{AB}{}_{MN}(\omega) \;=\; -\mathcal{R}(\omega) \;. \tag{3.9}$$

Then we obtain

$$-\frac{1}{2}\sqrt{-g}\,\phi^{-3}\,\underline{\mathcal{R}}(\underline{\omega}) \;=\; -\frac{1}{2}(\sqrt{-g})(f\mathrm{e}^{-2\Phi})\Big(-\mathcal{R}(\omega)\Big) \;=\; \frac{f}{2}\,\sqrt{-g}\,\mathrm{e}^{-2\Phi}\mathcal{R}(\omega) \;=\; \frac{1}{2\kappa_{10}^2}\sqrt{-g}\,\mathrm{e}^{-2\Phi}\,\mathcal{R}(\omega) \;, \tag{3.10a}$$

$$\therefore \qquad f \;=\; \frac{1}{\kappa_{10}^2} \;. \tag{3.10b}$$

This is consistent with (3.7b).

**— $H_3$-flux kinetic term —**

The square of the $H_3$-flux in BdR is re-defined as

$$\begin{aligned} \underline{H}_{MNP}\underline{H}^{MNP} \;&=\; g^{MQ}\,g^{NR}\,g^{PS}\,\underline{H}_{MNP}\,\underline{H}_{QRS} \;=\; \Big(\frac{h}{3!}\Big)^2 g^{MQ}\,g^{NR}\,g^{PS}\,\widetilde{H}_{MNP}\,\widetilde{H}_{QRS} \\ &=\; \frac{h^2}{3!}|\widetilde{H}|^2 \;. \end{aligned} \tag{3.11}$$

Then, the kinetic term in (3.1) is re-written by

$$\begin{aligned} -\frac{3}{4}\sqrt{-g}\,\phi^{-3}\underline{H}_{MNP}\underline{H}^{MNP} \;&=\; -\frac{3}{4}(\sqrt{-g})(f\mathrm{e}^{-2\Phi})\Big(\frac{h^2}{3!}\Big)|\widetilde{H}|^2 \;=\; -\frac{h^2}{8}f\,\sqrt{-g}\,\mathrm{e}^{-2\Phi}|\widetilde{H}|^2 \\ &\equiv\; -\frac{1}{4\kappa_{10}^2}\sqrt{-g}\,\mathrm{e}^{-2\Phi}\,|\widetilde{H}|^2 \;, \end{aligned} \tag{3.12a}$$

and we obtain

$$h^2 \;\equiv\; \frac{2}{\kappa_{10}^2 f} \;. \tag{3.12b}$$

— **Non-abelian 1-form kinetic term** —

Let us evaluate the field strength of the gauge field:

$$\underline{F}_{MN} \;=\; \mathrm{d}\underline{A} - \underline{A}\wedge\underline{A} \;=\; -\Big[\mathrm{d}\widetilde{A} + \widetilde{A}\wedge\widetilde{A}\Big] \;=\; -\widetilde{F}\,. \tag{3.13}$$

By using this, we can write down the square of $\underline{F}_{MN}$ in BdR:

$$\begin{aligned}\mathrm{tr}\big(\underline{F}_{MN}\underline{F}^{MN}\big) \;&=\; g^{MP}\,g^{NQ}\,\mathrm{tr}\big(\underline{F}_{MN}\,\underline{F}_{PQ}\big) \;=\; g^{MP}\,g^{NQ}\,\mathrm{tr}\big(\widetilde{F}_{MN}\widetilde{F}_{PQ}\big) \\ &=\; 2|\widetilde{F}|^2\,. \end{aligned}\tag{3.14}$$

Then we can re-write the kinetic term of the gauge field in BdR:

$$-\frac{\beta}{4}\sqrt{-g}\,\phi^{-3}\,\mathrm{tr}\big(\underline{F}_{MN}\underline{F}^{MN}\big) \;=\; -\frac{\beta f}{2}\,\sqrt{-g}\,\mathrm{e}^{-2\Phi}\,\mathrm{tr}(|\widetilde{F}|^2) \;=\; +\frac{1}{2g_{10}^2}\,\sqrt{-g}\,\mathrm{e}^{-2\Phi}\,\mathrm{tr}(|\widetilde{F}|^2)\,, \tag{3.15a}$$

$$\therefore \qquad \beta \;\equiv\; -\frac{1}{g_{10}^2 f}\,. \tag{3.15b}$$

— ***$H_3$*-flux with Chern-Simons 3-form** —

Again we study the $H_3$-fluxes. This fluxes are expanded by the exterior derivative of $B_2$-fields and the Chern-Simons terms. From the expansions we can restrict unknown constants. Let us again write down $H_3$-fluxes:

$$H'_{MNP} \;=\; 3\partial_{[M}B'_{NP]} - 3!\sqrt{2}\beta\,\mathrm{tr}\Big(\underline{A}_{[M}\partial_N\underline{A}_{P]} - \frac{2}{3}\underline{A}_{[M}\underline{A}_N\underline{A}_{P]}\Big)\,, \tag{3.16a}$$

$$H'_{MNP} \;=\; h\,\widetilde{H}_{MNP}\,, \qquad B'_{MN} \;=\; h'\,\widetilde{B}_{MN}\,. \tag{3.16b}$$

On the other hand, the $H$-flux with the Chern-Simons 3-form can be also described as (3.2b) (see, for example, [13]):

$$\widetilde{H}_{MNP} \;=\; 3\partial_{[M}\widetilde{B}_{NP]} - 3!\frac{\kappa_{10}^2}{g_{10}^2}\,\mathrm{tr}\Big(\widetilde{A}_{[M}\partial_N\widetilde{A}_{P]} + \frac{2}{3}\widetilde{A}_{[M}\widetilde{A}_N\widetilde{A}_{P]}\Big)\,. \tag{3.16c}$$

Comparing the $B_2$ terms, we find

$$B'_{MN} \;=\; h\widetilde{B}_{MN} \;\equiv\; h'\widetilde{B}_{MN}\,, \qquad \therefore \qquad h \;\equiv\; h'\,. \tag{3.17}$$

From the second terms in (3.16c) and in (3.16a), we obtain the following non-trivial relation:

$$-3!\sqrt{2}\,\beta\,\mathrm{tr}\big(\underline{A}_{[M}\partial_N\underline{A}_{P]}\big) \;=\; -3!\sqrt{2}\Big(-\frac{1}{g_{10}^2 f}\Big)\,\mathrm{tr}\big(\widetilde{A}_{[M}\partial_N\widetilde{A}_{P]}\big) \;\equiv\; -3!h\cdot\frac{\kappa_{10}^2}{g_{10}^2}\,\mathrm{tr}\big(\widetilde{A}_{[M}\partial_N\widetilde{A}_{P]}\big)\,, \tag{3.18a}$$

$$\therefore \qquad h \;=\; -\frac{\sqrt{2}}{\kappa_{10}^2 f}\,, \tag{3.18b}$$

where we used the relations (3.4b) and (3.15b). In order to be consistent with (3.12b) we should assign $\kappa_{10}^2 f = 1$, which is really consistent with (3.7b). The third terms tell us the other constraint with respect to the constants:

$$\begin{aligned}-3!\sqrt{2}\cdot\beta\cdot\frac{-2}{3}\mathrm{tr}\big(\underline{A}_{[M}\underline{A}_N\underline{A}_{P]}\big) \;&=\; 3!\sqrt{2}\Big(-\frac{1}{g_{10}^2 f}\Big)\cdot\frac{2}{3}\,\mathrm{tr}\big(\widetilde{A}_{[M}\widetilde{A}_N\widetilde{A}_{P]}\big) \\ &=\; 3!h\cdot\frac{\kappa_{10}^2}{g_{10}^2}\cdot\frac{2}{3}\,\mathrm{tr}\big(\widetilde{A}_{[M}\widetilde{A}_N\widetilde{A}_{P]}\big)\,, \end{aligned}\tag{3.19a}$$

$$\therefore \qquad h \;=\; -\frac{\sqrt{2}}{\kappa_{10}^2 f}\,. \tag{3.19b}$$

As we expected, this result corresponds to the previous result (3.18b).

### 3.3 Result

Here we summarize the relations among constants:

$$f \;=\; \frac{1}{\kappa_{10}^2} \,, \qquad h \;=\; h' \;=\; -\frac{\sqrt{2}}{\kappa_{10}^2 f} \;=\; -\sqrt{2} \,, \qquad \beta \;=\; -\frac{1}{g_{10}^2 f} \;=\; -\frac{\kappa_{10}^2}{g_{10}^2} \,. \tag{3.20a}$$

The rescaled Lagrangian (3.2a) can be expressed explicitly:

$$\mathscr{L}_{\text{new}} \;=\; \frac{1}{2\kappa_{10}^2}\sqrt{-g}\,\mathrm{e}^{-2\Phi}\Big\{\mathcal{R}(\omega) + 4(\partial_M \Phi)^2 - \frac{1}{2}|\widetilde{H}|^2 + \frac{\kappa_{10}^2}{g_{10}^2}\,\mathrm{tr}(|\widetilde{F}|^2) + \cdots\Big\} \,. \tag{3.21}$$

In the same analogy as $\beta$, we might fix[5] another constant $\alpha$ to

$$\alpha \;=\; \beta \;=\; -\frac{\kappa_{10}^2}{g_{10}^2} \,. \tag{3.22}$$

### 3.4 Transformation rules

Now we summarize the field re-definition rules:

$$\phi^{-3} \;\equiv\; \frac{1}{\kappa_{10}^2}\exp(-2\Phi) \,, \qquad \underline{\omega}_M{}^{AB} \;\equiv\; -\omega_M{}^{AB} \,, \tag{3.23a}$$

$$\underline{A}_M \;\equiv\; -\widetilde{A}_M \,, \qquad \underline{B}_{MN} \;\equiv\; -\frac{1}{\sqrt{2}}\widetilde{B}_{MN} \,, \qquad \underline{H}_{MNP} \;\equiv\; -\frac{1}{3\sqrt{2}}\widetilde{H}_{MNP} \,, \tag{3.23b}$$

$$\underline{\chi} \;=\; -\widetilde{\chi} \,, \qquad \underline{\lambda} \;\equiv\; \sqrt{2}\lambda \tag{3.23c}$$

$$\psi_M, \epsilon \text{ keep the same variables} \,. \tag{3.23d}$$

where $\underline{\chi}$ and $\underline{\lambda}$ denote the gaugino and the dilatino in BdR [2], while $\widetilde{\chi}$ and $\lambda$ mean the ones in [13, 1]. The reason why we flip the sign of the gaugino is that we want to keep the same signs in front of terms in equations as far as possible. The gaugino $\underline{\chi}$, (fortunately,) appears in equations with pairs of the gauge fields $\underline{A}_M$ or $\underline{F}_{MN}$, which are flipped the sign. We consider that both of the frameworks (BdR and Polchinski) have the anti-hermitian generators of gauge groups. Of course we can perform another field re-definition on the fermions such as $\chi = \widetilde{\chi}/\sqrt{\beta}$, which gives us the "canonical" kinetic terms $\mathscr{L} = -\frac{1}{2}\sqrt{-g}\mathrm{e}^{-2\Phi}\,\mathrm{tr}(\overline{\chi}\mathscr{D}_M(\omega,\widetilde{A})\chi) + \cdots$.

In the next section we will write down the Lagrangian and the supersymmetry variations explicitly. For simplicity, we will omit the symbol "tilde" of the $H$-flux, but we keep this symbol of the gauge field and the gaugino. (Because $A_M$ means the hermitian gauge field, while $\widetilde{A}_M$ is anti-hermitian in the definition.)

[5] Actually, we guess this assignment via the stringy modified Bianchi identity.

# 4 The descriptions in the Polchinski framework

In this section we will explicitly express the Lagrangian and the supersymmetry variation in the Polchinski's framework [13] with anti-hermitian gauge field $\widetilde{A}_M$. On the Bergshoeff-de Roo's framework, please look at section 2.

## 4.1 Lagrangian

Let us write down the Lagrangian of ten-dimensional heterotic supergravity via the field re-defitions (3.23):

$$\begin{aligned}\mathscr{L}(R) \ &= \ \frac{1}{2\kappa_{10}^2}\sqrt{-g}\,\mathrm{e}^{-2\Phi}\Big[\mathcal{R}(\omega) - \frac{1}{12}H_{MNP}H^{MNP} + 4(\partial_M\Phi)^2 \\ &\qquad - \overline{\psi}_M\Gamma^{MNP}\mathscr{D}_N(\omega)\psi_P + 8\,\overline{\lambda}\Gamma^{MN}\mathscr{D}_M(\omega)\psi_N + 16\,\overline{\lambda}\not{\mathscr{D}}(\omega)\lambda \\ &\qquad + 8\,\overline{\psi}_M\Gamma^N\Gamma^M\lambda\,(\partial_N\Phi) - 2\overline{\psi}_M\Gamma^M\psi_N\,(\partial^N\Phi) \\ &\qquad - \frac{1}{24}\,H^{PQR}\Big\{\overline{\psi}_M\Gamma^{[M}\Gamma_{PQR}\Gamma^{N]}\psi_N + 8\,\overline{\psi}_M\Gamma^M{}_{PQR}\lambda - 16\,\overline{\lambda}\Gamma_{PQR}\lambda\Big\} \\ &\qquad + \frac{1}{48}\,\overline{\psi}^M\Gamma^{ABC}\psi_M\Big\{2\overline{\lambda}\Gamma_{ABC}\lambda + \overline{\lambda}\Gamma_{ABC}\Gamma^N\psi_N \\ &\qquad\qquad - \frac{1}{4}\,\overline{\psi}^N\Gamma_{ABC}\psi_N - \frac{1}{8}\,\overline{\psi}^N\Gamma_N\Gamma_{ABC}\Gamma^P\psi_P\Big\}\Big]\,, \end{aligned} \tag{4.1a}$$

$$\begin{aligned}\mathscr{L}(\widetilde{F}^2) \ &= \ -\frac{1}{2\kappa_{10}^2}\frac{\kappa_{10}^2}{2g_{10}^2}\sqrt{-g}\,\mathrm{e}^{-2\Phi}\Big[-\mathrm{tr}\big(\widetilde{F}_{MN}\widetilde{F}^{MN}\big) - 2\,\mathrm{tr}\big\{\overline{\widetilde{\chi}}\not{\mathscr{D}}(\omega,\widetilde{A})\widetilde{\chi}\big\} - \frac{1}{12}\,\mathrm{tr}\big(\overline{\widetilde{\chi}}\Gamma^{ABC}\widetilde{\chi}\big)\hat{H}_{ABC} \\ &\qquad - \frac{1}{2}\,\mathrm{tr}\big\{\overline{\widetilde{\chi}}\Gamma^M\Gamma^{AB}(\widetilde{F}_{AB} + \hat{\widetilde{F}}_{AB})\big\}\Big(\psi_M + \frac{2}{3}\Gamma_M\lambda\Big) \\ &\qquad - \frac{1}{48}\,\mathrm{tr}(\overline{\widetilde{\chi}}\Gamma^{ABC}\widetilde{\chi})\,\overline{\psi}_M\big(4\Gamma_{ABC}\Gamma^M + 3\Gamma^M\Gamma_{ABC}\big)\lambda \\ &\qquad + \frac{1}{12}\,\mathrm{tr}(\overline{\widetilde{\chi}}\Gamma^{ABC}\widetilde{\chi})\overline{\lambda}\Gamma_{ABC}\lambda - \frac{\beta}{96}\,\mathrm{tr}(\overline{\widetilde{\chi}}\Gamma^{ABC}\widetilde{\chi})\,\mathrm{tr}(\overline{\widetilde{\chi}}\Gamma_{ABC}\widetilde{\chi})\Big]\,, \end{aligned} \tag{4.1b}$$

$$\begin{aligned}\mathscr{L}(R^2) \ &= \ -\frac{1}{2\kappa_{10}^2}\frac{\kappa_{10}^2}{2g_{10}^2}\sqrt{-g}\,\mathrm{e}^{-2\Phi}\Big[-R_{ABMN}(\Omega_-)R^{ABMN}(\Omega_-) \\ &\qquad - 2\overline{\psi}^{AB}\not{\mathscr{D}}(\omega(e,\psi),\Omega_-)\psi_{AB} - \frac{1}{12}\overline{\psi}^{AB}\Gamma^{MNP}\psi_{AB}\hat{H}_{MNP} \\ &\qquad + \frac{1}{2}\overline{\psi}_{AB}\Gamma^M\Gamma^{NP}\big\{R^{AB}{}_{NP}(\Omega_-) + \hat{R}^{AB}{}_{NP}(\Omega_-)\big\}\Big(\psi_M + \frac{2}{3}\Gamma_M\lambda\Big) \\ &\qquad - \frac{1}{48}\,\overline{\psi}_{AB}\Gamma^{CDE}\psi_{AB}\cdot\overline{\psi}_M\big(4\Gamma_{CDE}\Gamma^M + 3\Gamma^M\Gamma_{CDE}\big)\lambda \\ &\qquad + \frac{1}{12}\overline{\psi}^{AB}\Gamma^{CDE}\psi_{AB}\big(\overline{\lambda}\Gamma_{CDE}\lambda\big) - \frac{\alpha}{96}\overline{\psi}^{AB}\Gamma^{FGH}\psi_{AB}\big(\overline{\psi}^{CD}\Gamma_{FGH}\psi_{CD}\big)\Big]\,. \end{aligned} \tag{4.1c}$$

Derivatives $\mathscr{D}_M(\omega,\widetilde{A})$ are the covariant derivatives with respect to Lorentz and Yang-Mills gauge transformations. We define the derivative on fundamental fields $\phi^i$ as

$$\mathscr{D}_M(\omega,\widetilde{A},\Gamma)\phi^i \ = \ \partial_M\phi^i - \frac{i}{2}\omega_M{}^{AB}(\Sigma_{AB})^i{}_j\,\phi^j + (\widetilde{A}_M)^i{}_j\,\phi^j + \Gamma^i_{jM}\,\phi^j\,. \tag{4.2}$$

Note that $\Sigma_{AB}$ is a Lorentz generator whose representation is given by (1.3). We derived this expression from the ones in Polchinski (see eqs. (1.5) and (1.12).) via the field re-defitions (3.4).

Notice that we can always re-define the gravitino and dilatino via "mixing" with each other such as

$$\begin{pmatrix} \psi'_M \\ \Gamma_M \lambda' \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix} \begin{pmatrix} \psi_M \\ \Gamma_M \lambda \end{pmatrix} , \qquad b, c \in \mathbb{R} , \qquad c \neq 0 , \tag{4.3}$$

because both $\psi_M$ and $\Gamma_M \lambda$ are same chiralities and belong to the gravity multiplet. In the above Lagrangian we do not obtain the gravitino supersymmetry variation including the gradient of the dilaton, dilatino condensation terms and so forth. For instance, the effects of dilatino condensation has been studied [10] under the field re-definitions of gravitino and dilatino to obtain a useful calculation (commented by Nikolaos Prezas.)

## 4.2 Local supersymmetry variations

Here we pick up the local supersymmetry variations descibed in Appendix B of BdR [2]. As mentioned in [2], we write $\delta_{\alpha^n}$ ($\delta_{\beta^m}$) for variations of order $\alpha^n$ ($\beta^m$), while $\delta_0$ corresponds to the terms independent of $\alpha$ and $\beta$:

$$\begin{aligned}
\delta_0 e_M{}^A &= \frac{1}{2} \overline{\epsilon} \Gamma^A \psi_M , & (4.4a) \\
\delta_0 \psi_M &= \Big( \partial_M + \frac{1}{4} \Omega_{+M}{}^{AB} \Gamma_{AB} \Big) \epsilon + \Big\{ \epsilon \big( \overline{\psi}_M \lambda \big) - \psi_M \big( \overline{\epsilon} \lambda \big) + \Gamma^A \lambda \big( \overline{\psi}_M \Gamma_A \epsilon \big) \Big\} , & (4.4b) \\
\delta_0 B_{MN} &= - \overline{\epsilon} \Gamma_{[M} \psi_{N]} , & (4.4c) \\
\delta_0 \lambda &= - \frac{1}{4} \not{D} \Phi \, \epsilon + \frac{1}{48} \Gamma^{ABC} \epsilon \Big( - \hat{H}_{ABC} - \frac{1}{2} \overline{\lambda} \Gamma_{ABC} \lambda \Big) , & (4.4d) \\
\delta_0 \Phi &= - \overline{\epsilon} \lambda , & (4.4e) \\
\delta_0 \widetilde{A}_M &= \frac{1}{2} \overline{\epsilon} \Gamma_M \widetilde{\chi} , & (4.4f) \\
\delta_0 \widetilde{\chi} &= - \frac{1}{4} \Gamma^{AB} \epsilon \, \hat{\widetilde{F}}_{AB} + \Big\{ \epsilon \big( \overline{\widetilde{\chi}} \lambda \big) - \widetilde{\chi} \big( \overline{\epsilon} \lambda \big) + \Gamma^A \lambda \big( \overline{\widetilde{\chi}} \Gamma_A \epsilon \big) \Big\} , & (4.4g) \\
\delta_0 \omega_M{}^{AB}(e, \psi) &= - \frac{1}{4} \overline{\epsilon} \Gamma_M \psi^{AB} - \frac{1}{2} \overline{\epsilon} \Gamma^{[A} \psi_M{}^{B]} + \frac{1}{4} \, \overline{\epsilon} \Gamma_C \psi_M \, \hat{H}^{ABC} . & (4.4h)
\end{aligned}$$

Notice that the spin connection $\omega(e, \psi)$ is the solution of $D_{[M}(\omega) e_{N]}{}^A = 0$, while $\omega(e)$ is the solution of $\mathscr{D}_{[M}(\omega) e_{N]}{}^A = 0$. Note that Bergshoeff and de Roo define various additional variables such as a spin connection modified by the $H$-flux [2][6], supercovariantizations, and so forth:

$$\begin{aligned}
\Omega_{\pm M}{}^{AB} &\equiv \omega_M{}^{AB}(e, \psi) \pm \frac{1}{2} \hat{H}_M{}^{AB} , & (4.5a) \\
\hat{H}_{MNP} &\equiv H_{MNP} + \frac{3}{2} \overline{\psi}_{[M} \Gamma_N \psi_{P]} \\
&= 3 \partial_{[M} B_{NP]} + \frac{3}{2} \overline{\psi}_{[M} \Gamma_N \psi_{P]} - 6 \frac{\kappa_{10}^2}{g_{10}^2} \, \mathrm{tr} \Big( \widetilde{A}_{[M} \partial_N \widetilde{A}_{P]} + \frac{2}{3} \widetilde{A}_{[M} \widetilde{A}_N \widetilde{A}_{P]} \Big) \\
&\quad + 6 \frac{\kappa_{10}^2}{g_{10}^2} \Big( \Omega_{-[M}{}^{AB} \, \partial_N \Omega_{-P]}{}^{BA} + \frac{2}{3} \Omega_{-[M}{}^{AB} \, \Omega_{-N}{}^{BC} \, \Omega_{-P]}{}^{CA} \Big) , & (4.5b) \\
\mathrm{d} H &= \frac{\kappa_{10}^2}{g_{10}^2} \Big[ \mathrm{tr} \big\{ R(\Omega_-) \wedge R(\Omega_-) \big\} - \mathrm{tr} \big( \widetilde{F} \wedge \widetilde{F} \big) \Big] , & (4.5c) \\
\hat{\widetilde{F}}_{MN} &\equiv \widetilde{F}_{MN} - \overline{\psi}_{[M} \Gamma_{N]} \widetilde{\chi} , & (4.5d) \\
\psi_{MN} &\equiv \mathscr{D}_M(\Omega_+) \psi_N - \mathscr{D}_N(\Omega_+) \psi_M - \Big\{ \psi_M \big( \overline{\psi}_N \lambda \big) - \psi_N \big( \overline{\psi}_M \lambda \big) - \Gamma^P \lambda \, \big( \overline{\psi}_M \Gamma_P \psi_N \big) \Big\} , & (4.5e) \\
\hat{R}^{AB}{}_{MN}(\omega) &\equiv R^{AB}{}_{MN}(\omega) + \frac{1}{2} \overline{\psi}_{[M} \Gamma_{N]} \psi^{AB} + \overline{\psi}_{[M} \Gamma^{[A} \psi_{N]}{}^{B]} + \frac{1}{2} \, \overline{\psi}_{[M} \Gamma^C \psi_{N]} \, \hat{H}^{AB}{}_C . & (4.5f)
\end{aligned}$$

[6] If we simply write down the spin connection as $\omega$, this means $\omega = \omega(e)$.

We also pick up the supercovariant derivatives at hand:

$$D_M\Phi = \partial_M\Phi + \overline{\psi}_M\lambda \,, \tag{4.6a}$$

$$D_M(\omega)\lambda = \mathscr{D}_M(\omega)\lambda + \frac{1}{4}\,\not{D}\Phi\,\psi_M - \frac{1}{48}\Gamma^{ABC}\psi_M\Big(-\hat{H}_{ABC} - \frac{1}{2}\,\overline{\lambda}\Gamma_{ABC}\lambda\Big) \,, \tag{4.6b}$$

$$D_M(\omega,\widetilde{A})\widetilde{\chi} = \mathscr{D}_M(\omega,\widetilde{A})\widetilde{\chi} + \frac{1}{4}\Gamma^{AB}\psi_M\,\hat{\widetilde{F}}_{AB} - \Big\{\psi_M\big(\overline{\widetilde{\chi}}\lambda\big) - \widetilde{\chi}\big(\overline{\psi}_M\lambda\big) + \Gamma^A\lambda\big(\overline{\widetilde{\chi}}\Gamma_A\psi_M\big)\Big\} \,. \tag{4.6c}$$

It is both useful and instructive to obtain the supersymmetry algebra from (4.4). The commutator of two supersymmetry variations reads

$$\big[\delta(\epsilon_1),\delta(\epsilon_2)\big] = \delta_{\rm P}(\xi^M) + \delta_{\rm Q}(-\xi^M\psi_M) + \delta_{\rm L}(\xi^M\Omega_{-M}{}^{AB}) + \delta_{\rm YM}(\xi^M\widetilde{A}_M) + \delta_{\rm M}(-\tfrac{\sqrt{2}}{2}\xi_M + \tfrac{1}{\sqrt{2}}\xi^N B_{NM}) + \delta_{\rm Q}(\epsilon_3) + \delta_{\rm L}(\Lambda^{AB}) \,, \tag{4.7a}$$

$$\xi^M = \frac{1}{2}\overline{\epsilon}_2\Gamma^M\epsilon_1 \,, \tag{4.7b}$$

$$\epsilon_3 = -\frac{7}{8}(\overline{\epsilon}_2\Gamma^A\epsilon_1)\,\Gamma_A\lambda + \frac{1}{16\times 120}\,(\overline{\epsilon}_2\Gamma^{ABCDE}\epsilon_1)\,\Gamma_{ABCDE}\lambda \,, \tag{4.7c}$$

$$\Lambda^{AB} = \frac{\beta}{192}\,\overline{\epsilon}_2\Gamma^{[A}\Gamma_{CDE}\Gamma^{B]}\epsilon_1\,{\rm tr}\big(\overline{\widetilde{\chi}}\Gamma^{CDE}\widetilde{\chi}\big) \,. \tag{4.7d}$$

On the right-hand side of (4.7a), we encounter all gauge transformations of the ten-dimensional super Yang-Mills theory: $\delta_{\rm P}$, $\delta_{\rm Q}$, $\delta_{\rm L}$, $\delta_{\rm YM}$, and $\delta_{\rm M}$ correspond respectively to "general coordinate", "supersymmetry", "local Lorentz", "Yang-Mills" and "antisymmetric tensor gauge" transformations.

The supersymmetry variation of order $\beta$ are given as follows:

$$\delta_\beta\psi_M = \frac{\beta}{192}\,\Gamma^{ABC}\Gamma_M\epsilon\,{\rm tr}\big(\overline{\widetilde{\chi}}\Gamma_{ABC}\widetilde{\chi}\big) \,, \tag{4.8a}$$

$$\delta_\beta B_{MN} = 2\beta\,{\rm tr}\big\{\widetilde{A}_{[M}\delta_0\widetilde{A}_{N]}\big\} \,, \tag{4.8b}$$

$$\delta_\beta\lambda = \frac{\beta}{384}\,\Gamma^{ABC}\epsilon\,{\rm tr}\big(\overline{\widetilde{\chi}}\Gamma_{ABC}\widetilde{\chi}\big) \,, \tag{4.8c}$$

$$\delta_\beta\omega_M{}^{AB}(e,\psi) = -\frac{\beta}{192}\,\overline{\epsilon}\Gamma^{[A}\Gamma_{CDE}\Gamma^{B]}\psi_M\,{\rm tr}\big(\overline{\widetilde{\chi}}\Gamma^{CDE}\widetilde{\chi}\big) \,. \tag{4.8d}$$

Here the supersymmetry variation of order $\alpha$ are also given such as

$$\delta_\alpha\psi_M = \frac{\alpha}{192}\Gamma^{CDE}\Gamma_M\epsilon\,\overline{\psi}^{AB}\Gamma_{CDE}\psi_{AB} \,, \tag{4.9a}$$

$$\delta_\alpha B_{MN} = 2\alpha\,\Omega_{-[M}{}^{AB}\,\delta_0\Omega_{-N]}{}^{AB} \,, \tag{4.9b}$$

$$\delta_\alpha\lambda = \frac{\alpha}{384}\,\Gamma^{CDE}\Gamma_M\epsilon\,\overline{\psi}^{AB}\Gamma_{CDE}\psi_{AB} \,. \tag{4.9c}$$

The supersymmetry variations of the supercovariant variables are also obtained. First we write down the zero-th order of $\alpha$ and $\beta$. Next the corrections of first order $\beta$ are described. (Unfortunately, there are no descriptions about the corrections of first order $\alpha$.)

$$\delta_0(D_A\Phi) = -\frac{\sqrt{2}}{2}\,\overline{\epsilon}D_A(\Omega_+)\lambda \,, \tag{4.10a}$$

$$\delta_0\Omega_{-M}{}^{AB} = -\frac{1}{2}\overline{\epsilon}\Gamma_M\psi^{AB} \,, \tag{4.10b}$$

$$\delta_0\hat{H}_{ABC} = \frac{3}{2}\,\overline{\epsilon}\Gamma_{[A}\psi_{BC]} \,, \tag{4.10c}$$

$$\delta_0\psi^{AB} = \frac{1}{4}\Gamma^{CD}\epsilon\,\hat{R}^{AB}{}_{CD}(\Omega_-) + \Big\{\epsilon\big(\overline{\psi}^{AB}\lambda\big) - \psi^{AB}\big(\overline{\epsilon}\lambda\big) + \Gamma^C\lambda\,\big(\overline{\psi}^{AB}\Gamma_C\epsilon\big)\Big\}\,, \tag{4.10d}$$

$$\delta_0\hat{\widetilde{F}}_{AB} = -\overline{\epsilon}\Gamma_{[A}D_{B]}(\Omega_+,\widetilde{A})\widetilde{\chi}\,, \tag{4.10e}$$

$$\delta_\beta(D_A\Phi) = -\frac{\beta}{192}\,\overline{\epsilon}\Gamma_A\Gamma^{BCD}\lambda\,\mathrm{tr}\big(\overline{\widetilde{\chi}}\Gamma_{BCD}\widetilde{\chi}\big)\,, \tag{4.10f}$$

$$\delta_\beta\hat{H}_{ABC} = \frac{3\beta}{2}\,\overline{\epsilon}\Gamma_{[A}\,\mathrm{tr}\big(\widetilde{\chi}\hat{\widetilde{F}}_{BC]}\big)\,, \tag{4.10g}$$

$$\begin{aligned}\delta_\beta\psi_{AB} = \beta\Bigg[&\frac{3}{4}\,\Gamma^{CD}\epsilon\,\mathrm{tr}\big(\hat{\widetilde{F}}_{[AB}\hat{\widetilde{F}}_{CD]}\big) + \frac{1}{48}\Gamma^{CDE}\Gamma_{[A}\epsilon\,\mathrm{tr}\big\{\overline{\widetilde{\chi}}\Gamma_{CDE}D_{B]}(\Omega_+,\widetilde{A})\widetilde{\chi}\big\} \\ &- \frac{1}{3\times 256}\,\Gamma^{CDE}\Gamma_{[A}\Gamma^{GH}\epsilon\,\hat{H}_{B]GH}\,\mathrm{tr}\big(\overline{\widetilde{\chi}}\Gamma_{CDE}\widetilde{\chi}\big) \\ &+ \frac{\beta}{96\times 96}\,\Gamma^{CDE}\Gamma_{[A}\Gamma^{FGH}\Gamma_{B]}\epsilon\,\mathrm{tr}\big(\overline{\widetilde{\chi}}\Gamma_{CDE}\widetilde{\chi}\big)\,\mathrm{tr}\big(\overline{\widetilde{\chi}}\Gamma_{FGH}\widetilde{\chi}\big)\Bigg]\,,\end{aligned} \tag{4.10h}$$

$$\delta_\beta\hat{\widetilde{F}}_{AB} = \frac{\beta}{192}\,\overline{\epsilon}\Gamma_{[A}\Gamma^{CDE}\Gamma_{B]}\widetilde{\chi}\,\mathrm{tr}\big(\overline{\widetilde{\chi}}\Gamma_{CDE}\widetilde{\chi}\big)\,. \tag{4.10i}$$

Finally we describe an identity among generalized curvature tensors:

$$R_{ABCD}(\Omega_-) = R_{CDAB}(\Omega_+) - \frac{1}{2}(\mathrm{d}H)_{CDAB}\,. \tag{4.11}$$

I have a few comments. To avoid a confusion, we mention about the difference of the modified spin connections between the Polchinski and the BdR:

$$\omega_M{}^{AB} \pm \frac{1}{2}\hat{H}_M{}^{AB}\Big|_{\mathrm{Polchinski}} \equiv -\omega_M{}^{AB} \mp \frac{3\sqrt{2}}{2}\,\hat{H}_M{}^{AB}\Big|_{\mathrm{BdR}}\,, \tag{4.12a}$$

$$\therefore\quad \big(\Omega_{\pm M}{}^{AB}\big)_{\mathrm{Polchinski}} \equiv -\big(\Omega_{\pm M}{}^{AB}\big)_{\mathrm{BdR}}\,. \tag{4.12b}$$

Due to this re-definition, we should take care of the re-definitions of the covariant derivatives and the curvature tensors:

$$\mathscr{D}_M(\Omega_\pm)\Big|_{\mathrm{Polchinski}} = \partial_M - \frac{i}{2}\Omega_{\pm M}{}^{AB}\Sigma_{AB}\Big|_{\mathrm{Polchinski}} \equiv \partial_M + \frac{i}{2}\Omega_{\pm M}{}^{AB}\Sigma_{AB}\Big|_{\mathrm{BdR}} = \mathscr{D}_M(\Omega_\pm)\Big|_{\mathrm{BdR}}\,, \tag{4.13a}$$

$$R^{AB}{}_{MN}(\Omega_\pm)\Big|_{\mathrm{Polchinski}} \equiv -R^{AB}{}_{MN}(\Omega_\pm)\Big|_{\mathrm{BdR}}\,. \tag{4.13b}$$

The explicit expressions of supersymmetry variations are, of course, just approximate expressions. If you consider not only $\alpha$, $\beta$ corrections but also the higher order fermions corrections, the corrections of supersymmetry vairations are also corrected. The supercovariant variables such as $\hat{H}_{MNP}$, $\hat{\widetilde{F}}_{MN}$ and $\hat{R}^{AB}{}_{MN}(\omega)$ are influenced by the higher order corrections quite sensitively. Thus, you must take care of any calculations when you study the supersymmetry variations and the construction of the Lagrangian of higher order corrections of fermions.

Polchinski [13], Becker et al [1] and Kim-Yi [7] adapted

$$\frac{\kappa_{10}^2}{g_{10}^2} \equiv \frac{\alpha'}{4}\,. \tag{4.14}$$

# 5 The descriptions in the KY framework

In this section we will again re-write down the Lagrangian and the supersymmetry variations explicitly. We also keep this symbol of the gauge field and the gaugino. (Because $A_M$ means the hermitian gauge field, while $\widetilde{A}_M$ is anti-hermitian in the definition.)

## 5.1 Transformation rules

Now we summarize the field re-definition rules from BdR to KY [9]:

$$\phi^{-3} \equiv \frac{1}{\kappa_{10}^2} \exp(-2\Phi) , \qquad \underline{\omega}_M{}^{AB} \equiv -\omega_M{}^{AB} , \tag{5.1a}$$

$$\underline{A}_M \equiv -\widetilde{A}_M , \qquad \underline{B}_{MN} \equiv +\sqrt{2} B_{MN} , \qquad \underline{H}_{MNP} \equiv +\frac{\sqrt{2}}{3} H_{MNP} , \tag{5.1b}$$

$$\underline{\chi} = -\widetilde{\chi} , \qquad \underline{\lambda} \equiv \sqrt{2} \lambda , \tag{5.1c}$$

$$\psi_M , \epsilon \text{ keep the same variables} , \tag{5.1d}$$

$$\alpha = \beta = -\frac{\kappa_{10}^2}{g_{10}^2} , \tag{5.1e}$$

where $\underline{\chi}$ and $\underline{\lambda}$ denote the gaugino and the dilatino in BdR [2], while $\widetilde{\chi}$ and $\lambda$ mean the ones in [4, 9]. The reason why we filpped the sign of the gaugino is that we want to keep the same signs in equations. The gaugino always appears with pairing with the gauge field or the gauge field strength, which are flipped via the mapping from the BdR framework to the KY's one. To obtain $\underline{H}_{MNP} = +\frac{\sqrt{2}}{3} H_{MNP}$ with $h = +2\sqrt{2}$, we change the definition of the Lagrangian (3.2a) and the Bianchi identity (3.2b) to

$$\mathscr{L} = \frac{1}{2\kappa_{10}^2} \sqrt{-g} \mathrm{e}^{-2\Phi} \Big\{ \mathcal{R}(\omega) + 4(\partial_M \Phi)^2 - \frac{1}{3} H_{MNP} H^{MNP} + \frac{\kappa_{10}^2}{g_{10}^2} \mathrm{tr}\big( |\widetilde{F}|^2 \big) + \ldots \Big\} , \tag{5.2a}$$

$$H_3 = \mathrm{d} B_2 + \frac{\kappa_{10}^2}{2 g_{10}^2} \widetilde{\omega}_3^{\mathrm{Y}} + \ldots , \tag{5.2b}$$

and we also change (3.16c) to

$$H_{MNP} = 3 \partial_{[M} B_{NP]} + 3! \frac{\kappa_{10}^2}{2 g_{10}^2} \mathrm{tr} \Big( \widetilde{A}_{[M} \partial_N \widetilde{A}_{P]} + \frac{2}{3} \widetilde{A}_{[M} \widetilde{A}_N \widetilde{A}_{P]} \Big) . \tag{5.3}$$

Cardoso et al [4] and Kimura-Yi [9] adapted

$$\kappa_{10}^2 \equiv 2 , \qquad \frac{\kappa_{10}^2}{2 g_{10}^2} \equiv \alpha' . \tag{5.4}$$

### 5.2 Lagrangian

Let us again re-write the Lagrangian of ten-dimensional heterotic supergravity via the field re-defitions (5.1):

$$\begin{aligned}\mathscr{L}(R) \;=\; & \frac{1}{2\kappa_{10}^2}\sqrt{-g}\,\mathrm{e}^{-2\Phi}\bigg[\mathcal{R}(\omega) - \frac{1}{3}H_{MNP}H^{MNP} + 4(\partial_M\Phi)^2 \\ & - \overline{\psi}_M\Gamma^{MNP}\mathscr{D}_N(\omega)\psi_P + 8\,\overline{\lambda}\Gamma^{MN}\mathscr{D}_M(\omega)\psi_N + 16\,\overline{\lambda}\mathscr{D}\!\!\!/(\omega)\lambda \\ & + 8\,\overline{\psi}_M\Gamma^N\Gamma^M\lambda\,(\partial_N\Phi) - 2\overline{\psi}_M\Gamma^M\psi_N\,(\partial^N\Phi) \\ & + \frac{1}{12}\,H^{PQR}\Big\{\overline{\psi}_M\Gamma^{[M}\Gamma_{PQR}\Gamma^{N]}\psi_N + 8\,\overline{\psi}_M\Gamma^M{}_{PQR}\lambda - 16\,\overline{\lambda}\Gamma_{PQR}\lambda\Big\} \\ & + \frac{1}{48}\,\overline{\psi}^M\Gamma^{ABC}\psi_M\Big\{2\overline{\lambda}\Gamma_{ABC}\lambda + \overline{\lambda}\Gamma_{ABC}\Gamma^N\psi_N \\ & \qquad - \frac{1}{4}\,\overline{\psi}^N\Gamma_{ABC}\psi_N - \frac{1}{8}\,\overline{\psi}^N\Gamma_N\Gamma_{ABC}\Gamma^P\psi_P\Big\}\bigg]\,, \end{aligned} \tag{5.5a}$$

$$\begin{aligned}\mathscr{L}(\widetilde{F}^2) \;=\; & -\frac{1}{2\kappa_{10}^2}\frac{\kappa_{10}^2}{2g_{10}^2}\sqrt{-g}\,\mathrm{e}^{-2\Phi}\bigg[-\mathrm{tr}\big(\widetilde{F}_{MN}\widetilde{F}^{MN}\big) - 2\,\mathrm{tr}\big\{\overline{\widetilde{\chi}}\mathscr{D}\!\!\!/(\omega,\widetilde{A})\widetilde{\chi}\big\} + \frac{1}{6}\,\mathrm{tr}\big(\overline{\widetilde{\chi}}\Gamma^{ABC}\widetilde{\chi}\big)\hat{H}_{ABC} \\ & - \frac{1}{2}\,\mathrm{tr}\big\{\overline{\widetilde{\chi}}\Gamma^M\Gamma^{AB}(\widetilde{F}_{AB} + \hat{\widetilde{F}}_{AB})\big\}\Big(\psi_M + \frac{2}{3}\Gamma_M\lambda\Big) \\ & - \frac{1}{48}\,\mathrm{tr}(\overline{\widetilde{\chi}}\Gamma^{ABC}\widetilde{\chi})\,\overline{\psi}_M\big(4\Gamma_{ABC}\Gamma^M + 3\Gamma^M\Gamma_{ABC}\big)\lambda \\ & + \frac{1}{12}\,\mathrm{tr}(\overline{\widetilde{\chi}}\Gamma^{ABC}\widetilde{\chi})\overline{\lambda}\Gamma_{ABC}\lambda - \frac{\beta}{96}\,\mathrm{tr}(\overline{\widetilde{\chi}}\Gamma^{ABC}\widetilde{\chi})\,\mathrm{tr}(\overline{\widetilde{\chi}}\Gamma_{ABC}\widetilde{\chi})\bigg]\,, \end{aligned} \tag{5.5b}$$

$$\begin{aligned}\mathscr{L}(R^2) \;=\; & -\frac{1}{2\kappa_{10}^2}\frac{\kappa_{10}^2}{2g_{10}^2}\sqrt{-g}\,\mathrm{e}^{-2\Phi}\bigg[-R_{ABMN}(\Omega_+)R^{ABMN}(\Omega_+) \\ & - 2\overline{\psi}^{AB}\mathscr{D}\!\!\!/(\omega(e,\psi),\Omega_+)\psi_{AB} + \frac{1}{6}\overline{\psi}^{AB}\Gamma^{MNP}\psi_{AB}\hat{H}_{MNP} \\ & + \frac{1}{2}\overline{\psi}_{AB}\Gamma^M\Gamma^{NP}\big\{R^{AB}{}_{NP}(\Omega_+) + \hat{R}^{AB}{}_{NP}(\Omega_+)\big\}\Big(\psi_M + \frac{2}{3}\Gamma_M\lambda\Big) \\ & - \frac{1}{48}\,\overline{\psi}_{AB}\Gamma^{CDE}\psi_{AB}\cdot\overline{\psi}_M\big(4\Gamma_{CDE}\Gamma^M + 3\Gamma^M\Gamma_{CDE}\big)\lambda \\ & + \frac{1}{12}\overline{\psi}^{AB}\Gamma^{CDE}\psi_{AB}\big(\overline{\lambda}\Gamma_{CDE}\lambda\big) - \frac{\alpha}{96}\overline{\psi}^{AB}\Gamma^{FGH}\psi_{AB}\big(\overline{\psi}^{CD}\Gamma_{FGH}\psi_{CD}\big)\bigg]\,. \end{aligned} \tag{5.5c}$$

Derivatives $\mathscr{D}_M(\omega,\widetilde{A})$ are the covariant derivatives with respect to Lorentz and Yang-Mills gauge transformations. We define the derivative on fundamental fields $\phi^i$ as

$$\mathscr{D}_M(\omega,\widetilde{A},\Gamma)\phi^i \;=\; \partial_M\phi^i - \frac{i}{2}\omega_M{}^{AB}(\Sigma_{AB})^i{}_j\,\phi^j + (\widetilde{A}_M)^i{}_j\,\phi^j + \Gamma^i_{jM}\,\phi^j\,. \tag{5.6}$$

### 5.3 Local supersymmetry variations

Here we pick up the local supersymmetry variations descibed in Appendix B of BdR [2]. As mentioned in [2], we write $\delta_{\alpha^n}$ ($\delta_{\beta^m}$) for variations of order $\alpha^n$ ($\beta^m$), while $\delta_0$ corresponds to the terms independent of $\alpha$ and $\beta$:

$$\delta_0 e_M{}^A \;=\; \frac{1}{2}\overline{\epsilon}\Gamma^A\psi_M\,, \tag{5.7a}$$

$$\delta_0\psi_M \;=\; \Big(\partial_M + \frac{1}{4}\Omega_{-M}{}^{AB}\Gamma_{AB}\Big)\epsilon + \Big\{\epsilon\big(\overline{\psi}_M\lambda\big) - \psi_M\big(\overline{\epsilon}\lambda\big) + \Gamma^A\lambda\big(\overline{\psi}_M\Gamma_A\epsilon\big)\Big\}\,, \tag{5.7b}$$

$$
\begin{aligned}
\delta_0 B_{MN} &= \bar{\epsilon}\Gamma_{[M}\psi_{N]} \ , & (5.7c)\\
\delta_0 \lambda &= -\frac{1}{4}\,\not{D}\Phi\,\epsilon + \frac{1}{24}\Gamma^{ABC}\epsilon\Big(\hat{H}_{ABC} - \frac{1}{4}\bar{\lambda}\Gamma_{ABC}\lambda\Big) \ , & (5.7d)\\
\delta_0 \Phi &= -\bar{\epsilon}\lambda \ , & (5.7e)\\
\delta_0 \widetilde{A}_M &= \frac{1}{2}\bar{\epsilon}\Gamma_M\widetilde{\chi} \ , & (5.7f)\\
\delta_0 \widetilde{\chi} &= -\frac{1}{4}\Gamma^{AB}\epsilon\,\hat{\widetilde{F}}_{AB} + \Big\{\epsilon\big(\bar{\widetilde{\chi}}\lambda\big) - \widetilde{\chi}\big(\bar{\epsilon}\lambda\big) + \Gamma^A\lambda\big(\bar{\widetilde{\chi}}\Gamma_A\epsilon\big)\Big\} \ , & (5.7g)\\
\delta_0 \omega_M{}^{AB}(e,\psi) &= -\frac{1}{4}\bar{\epsilon}\Gamma_M\psi^{AB} - \frac{1}{2}\bar{\epsilon}\Gamma^{[A}\psi_M{}^{B]} - \frac{1}{2}\,\bar{\epsilon}\Gamma_C\psi_M\,\hat{H}^{ABC} \ . & (5.7h)
\end{aligned}
$$

Notice that the spin connection $\omega(e,\psi)$ is the solution of $D_{[M}(\omega)e_{N]}{}^A = 0$, while $\omega(e)$ is the solution of $\mathscr{D}_{[M}(\omega)e_{N]}{}^A = 0$. Note that Bergshoeff and de Roo define various additional variables such as a spin connection modified by the $H$-flux [2][7], supercovariantizations, and so forth:

$$
\begin{aligned}
\Omega_{\pm M}{}^{AB} &\equiv \omega_M{}^{AB}(e,\psi) \pm \hat{H}_M{}^{AB} \ , & (5.8a)\\
\hat{H}_{MNP} &\equiv H_{MNP} - \frac{3}{4}\bar{\psi}_{[M}\Gamma_N\psi_{P]}\\
&= 3\partial_{[M}B_{NP]} - \frac{3}{4}\bar{\psi}_{[M}\Gamma_N\psi_{P]} + 6\frac{\kappa_{10}^2}{2g_{10}^2}\,\mathrm{tr}\Big(\widetilde{A}_{[M}\partial_N\widetilde{A}_{P]} + \frac{2}{3}\widetilde{A}_{[M}\widetilde{A}_N\widetilde{A}_{P]}\Big)\\
&\quad - 6\frac{\kappa_{10}^2}{2g_{10}^2}\,\Big(\Omega_{+[M}{}^{AB}\,\partial_N\Omega_{+P]}{}^{BA} + \frac{2}{3}\Omega_{+[M}{}^{AB}\,\Omega_{+N}{}^{BC}\,\Omega_{+P]}{}^{CA}\Big) \ , & (5.8b)\\
\mathrm{d}H &= -\frac{\kappa_{10}^2}{2g_{10}^2}\Big[\mathrm{tr}\big\{R(\Omega_+)\wedge R(\Omega_+)\big\} - \mathrm{tr}\big(\widetilde{F}\wedge\widetilde{F}\big)\Big] \ , & (5.8c)\\
\hat{\widetilde{F}}_{MN} &\equiv \widetilde{F}_{MN} - \bar{\psi}_{[M}\Gamma_{N]}\widetilde{\chi} \ , & (5.8d)\\
\psi_{MN} &\equiv \mathscr{D}_M(\Omega_-)\psi_N - \mathscr{D}_N(\Omega_-)\psi_M - \Big\{\psi_M\big(\bar{\psi}_N\lambda\big) - \psi_N\big(\bar{\psi}_M\lambda\big) - \Gamma^P\lambda\,\big(\bar{\psi}_M\Gamma_P\psi_N\big)\Big\} \ , & (5.8e)\\
\hat{R}^{AB}{}_{MN}(\omega) &\equiv R^{AB}{}_{MN}(\omega) + \frac{1}{2}\bar{\psi}_{[M}\Gamma_{N]}\psi^{AB} + \bar{\psi}_{[M}\Gamma^{[A}\psi_{N]}{}^{B]} - \bar{\psi}_{[M}\Gamma^C\psi_{N]}\,\hat{H}^{AB}{}_C \ . & (5.8f)
\end{aligned}
$$

We also pick up the supercovariant derivatives at hand:

$$
\begin{aligned}
D_M\Phi &= \partial_M\Phi + \bar{\psi}_M\lambda \ , & (5.9a)\\
D_M(\omega)\lambda &= \mathscr{D}_M(\omega)\lambda + \frac{1}{4}\,\not{D}\Phi\,\psi_M - \frac{1}{24}\Gamma^{ABC}\psi_M\Big(\hat{H}_{ABC} - \frac{1}{4}\,\bar{\lambda}\Gamma_{ABC}\lambda\Big) \ , & (5.9b)\\
D_M(\omega,\widetilde{A})\widetilde{\chi} &= \mathscr{D}_M(\omega,\widetilde{A})\widetilde{\chi} + \frac{1}{4}\Gamma^{AB}\psi_M\,\hat{\widetilde{F}}_{AB} - \Big\{\psi_M\big(\bar{\widetilde{\chi}}\lambda\big) - \widetilde{\chi}\big(\bar{\psi}_M\lambda\big) + \Gamma^A\lambda\big(\bar{\widetilde{\chi}}\Gamma_A\psi_M\big)\Big\} \ . & (5.9c)
\end{aligned}
$$

It is both useful and instructive to obtain the supersymmetry algebra from (5.7). The commutator of two supersymmetry variations reads

$$
\begin{aligned}
\big[\delta(\epsilon_1),\delta(\epsilon_2)\big] &= \delta_{\mathrm{P}}(\xi^M) + \delta_{\mathrm{Q}}(-\xi^M\psi_M) + \delta_{\mathrm{L}}(\xi^M\Omega_{+M}{}^{AB}) + \delta_{\mathrm{YM}}(\xi^M\widetilde{A}_M)\\
&\quad + \delta_{\mathrm{M}}(-\tfrac{\sqrt{2}}{2}\xi_M - \sqrt{2}\xi^N B_{NM}) + \delta_{\mathrm{Q}}(\epsilon_3) + \delta_{\mathrm{L}}(\Lambda^{AB}) \ , & (5.10a)\\
\xi^M &= \frac{1}{2}\bar{\epsilon}_2\Gamma^M\epsilon_1 \ , & (5.10b)\\
\epsilon_3 &= -\frac{7}{8}(\bar{\epsilon}_2\Gamma^A\epsilon_1)\,\Gamma_A\lambda + \frac{1}{16\times 120}\,(\bar{\epsilon}_2\Gamma^{ABCDE}\epsilon_1)\,\Gamma_{ABCDE}\lambda \ , & (5.10c)\\
\Lambda^{AB} &= \frac{\beta}{192}\,\bar{\epsilon}_2\Gamma^{[A}\Gamma_{CDE}\Gamma^{B]}\epsilon_1\,\mathrm{tr}\big(\bar{\widetilde{\chi}}\Gamma^{CDE}\widetilde{\chi}\big) \ . & (5.10d)
\end{aligned}
$$

[7] If we simply write down the spin connection as $\omega$, this means $\omega = \omega(e)$.

On the right-hand side of (5.10a), we encounter all gauge transformations of the ten-dimensional super Yang-Mills theory: $\delta_{\rm P}$, $\delta_{\rm Q}$, $\delta_{\rm L}$, $\delta_{\rm YM}$, and $\delta_{\rm M}$ correspond respectively to "general coordinate", "supersymmetry", "local Lorentz", "Yang-Mills" and "antisymmetric tensor gauge" transformations.

The supersymmetry variation of order $\beta$ are given as follows:

$$\delta_{\beta}\psi_M = \frac{\beta}{192}\,\Gamma^{ABC}\Gamma_M\epsilon\,{\rm tr}\big(\overline{\widetilde{\chi}}\Gamma_{ABC}\widetilde{\chi}\big)\ , \tag{5.11a}$$

$$\delta_{\beta}B_{MN} = -\beta\,{\rm tr}\big\{\widetilde{A}_{[M}\delta_0\widetilde{A}_{N]}\big\}\ , \tag{5.11b}$$

$$\delta_{\beta}\lambda = \frac{\beta}{384}\,\Gamma^{ABC}\epsilon\,{\rm tr}\big(\overline{\widetilde{\chi}}\Gamma_{ABC}\widetilde{\chi}\big)\ , \tag{5.11c}$$

$$\delta_{\beta}\omega_M{}^{AB}(e,\psi) = -\frac{\beta}{192}\,\overline{\epsilon}\Gamma^{[A}\Gamma_{CDE}\Gamma^{B]}\psi_M\,{\rm tr}\big(\overline{\widetilde{\chi}}\Gamma^{CDE}\widetilde{\chi}\big)\ . \tag{5.11d}$$

Here the supersymmetry variation of order $\alpha$ are also given such as

$$\delta_{\alpha}\psi_M = \frac{\alpha}{192}\Gamma^{CDE}\Gamma_M\epsilon\,\overline{\psi}^{AB}\Gamma_{CDE}\psi_{AB}\ , \tag{5.12a}$$

$$\delta_{\alpha}B_{MN} = -\alpha\,\Omega_{+[M}{}^{AB}\,\delta_0\Omega_{+N]}{}^{AB}\ , \tag{5.12b}$$

$$\delta_{\alpha}\lambda = \frac{\alpha}{384}\Gamma^{CDE}\Gamma_M\epsilon\,\overline{\psi}^{AB}\Gamma_{CDE}\psi_{AB}\ . \tag{5.12c}$$

The supersymmetry variations of the supercovariant variables are also obtained. First we write down the zero-th order of $\alpha$ and $\beta$. Next the corrections of first order $\beta$ are described. (Unfortunately, there are no descriptions about the corrections of first order $\alpha$.)

$$\delta_0(D_A\Phi) = -\frac{\sqrt{2}}{2}\,\overline{\epsilon}D_A(\Omega_-)\lambda\ , \tag{5.13a}$$

$$\delta_0\Omega_{+M}{}^{AB} = -\frac{1}{2}\overline{\epsilon}\Gamma_M\psi^{AB}\ , \tag{5.13b}$$

$$\delta_0\hat{H}_{ABC} = -\frac{3}{4}\,\overline{\epsilon}\Gamma_{[A}\psi_{BC]}\ , \tag{5.13c}$$

$$\delta_0\psi^{AB} = \frac{1}{4}\Gamma^{CD}\epsilon\,\hat{R}^{AB}{}_{CD}(\Omega_+) + \Big\{\epsilon\big(\overline{\psi}^{AB}\lambda\big) - \psi^{AB}\big(\overline{\epsilon}\lambda\big) + \Gamma^C\lambda\,\big(\overline{\psi}^{AB}\Gamma_C\epsilon\big)\Big\}\ , \tag{5.13d}$$

$$\delta_0\hat{\widetilde{F}}_{AB} = -\overline{\epsilon}\Gamma_{[A}D_{B]}(\Omega_-,\widetilde{A})\widetilde{\chi}\ , \tag{5.13e}$$

$$\delta_{\beta}(D_A\Phi) = -\frac{\beta}{192}\,\overline{\epsilon}\Gamma_A\Gamma^{BCD}\lambda\,{\rm tr}\big(\overline{\widetilde{\chi}}\Gamma_{BCD}\widetilde{\chi}\big)\ , \tag{5.13f}$$

$$\delta_{\beta}\hat{H}_{ABC} = -\frac{3\beta}{4}\,\overline{\epsilon}\Gamma_{[A}\,{\rm tr}\big(\widetilde{\chi}\hat{\widetilde{F}}_{BC]}\big)\ , \tag{5.13g}$$

$$\begin{aligned}\delta_{\beta}\psi_{AB} = \beta\Bigg[&\frac{3}{4}\Gamma^{CD}\epsilon\,{\rm tr}\big(\hat{\widetilde{F}}_{[AB}\hat{\widetilde{F}}_{CD]}\big) + \frac{1}{48}\Gamma^{CDE}\Gamma_{[A}\epsilon\,{\rm tr}\big\{\overline{\widetilde{\chi}}\Gamma_{CDE}D_{B]}(\Omega_-,\widetilde{A})\widetilde{\chi}\big\} \\ &- \frac{1}{3\times 128}\,\Gamma^{CDE}\Gamma_{[A}\Gamma^{GH}\epsilon\,\hat{H}_{B]GH}\,{\rm tr}\big(\overline{\widetilde{\chi}}\Gamma_{CDE}\widetilde{\chi}\big) \\ &+ \frac{\beta}{96\times 96}\,\Gamma^{CDE}\Gamma_{[A}\Gamma^{FGH}\Gamma_{B]}\epsilon\,{\rm tr}\big(\overline{\widetilde{\chi}}\Gamma_{CDE}\widetilde{\chi}\big)\,{\rm tr}\big(\overline{\widetilde{\chi}}\Gamma_{FGH}\widetilde{\chi}\big)\Bigg]\ ,\end{aligned} \tag{5.13h}$$

$$\delta_{\beta}\hat{\widetilde{F}}_{AB} = \frac{\beta}{192}\,\overline{\epsilon}\Gamma_{[A}\Gamma^{CDE}\Gamma_{B]}\widetilde{\chi}\,{\rm tr}\big(\overline{\widetilde{\chi}}\Gamma_{CDE}\widetilde{\chi}\big)\ . \tag{5.13i}$$

Finally we describe an identity among generalized curvature tensors:

$$R_{ABCD}(\Omega_+) = R_{CDAB}(\Omega_-) + ({\rm d}H)_{CDAB}\ . \tag{5.14}$$

I have a few comments. To avoid a confusion, we mention about the difference of the modified spin connections between the KY [4, 9] and the BdR [2]:

$$\omega_M{}^{AB} \pm \hat{H}_M{}^{AB}\Big|_{\rm KY} \equiv -\omega_M{}^{AB} \pm \frac{3\sqrt{2}}{2}\,\hat{H}_M{}^{AB}\Big|_{\rm BdR} , \tag{5.15a}$$

$$\therefore \qquad \big(\Omega_{\pm M}{}^{AB}\big)_{\rm KY} \equiv -\big(\Omega_{\mp M}{}^{AB}\big)_{\rm BdR} . \tag{5.15b}$$

Due to this re-definition, we should take care of the re-definitions of the covariant derivatives and the curvature tensors:

$$\mathscr{D}_M(\Omega_\pm)\Big|_{\rm KY} = \partial_M - \frac{i}{2}\Omega_{\pm M}{}^{AB}\Sigma_{AB}\Big|_{\rm KY} \equiv \partial_M + \frac{i}{2}\Omega_{\mp M}{}^{AB}\Sigma_{AB}\Big|_{\rm BdR} = \mathscr{D}_M(\Omega_\mp)\Big|_{\rm BdR} , \tag{5.16a}$$

$$R^{AB}{}_{MN}(\Omega_\pm)\Big|_{\rm KY} \equiv -R^{AB}{}_{MN}(\Omega_\mp)\Big|_{\rm BdR} . \tag{5.16b}$$

## 5.4 Serious Error

I found a couple of errors in our previous paper [9]. These are the signs in front of $\text{tr}(\widetilde{F} \wedge \widetilde{F})$ and $\text{tr}(R \wedge R)$ in the Bianchi identity, and the signs in front of $\text{tr}(\widetilde{F}_{MN}\widetilde{F}^{MN})$ and $R_{MNAB}R^{MNAB}$ in the Lagrangian. The reason why I mistook these signs are the followings:

- I naively (and stupidly) assumed that $\alpha$ and $\beta$ were positive constants.
- I guessed the gauge field in [2] was hermitian, because the kinetic term of the gauge field is given by $-\beta\sqrt{-g}\phi^3\text{tr}(F_{MN}F^{MN})$, and because in order to obtain positive energy eigenvalues of the system we should choose the minus sign in front of the kinetic term with respect to the "hermitian" field.
- To keep the hermitian property of the gauge field, I changed the definition of the Yang-Mills-Chern-Simons 3-form in [2]. So far, I have already found a lot of typos in [2]. (For instance, the definition of the curvature tensor is wrong (2.1b). Furthermore, the relative factor in the Yang-Mills-Chern-Simons 3-form is also wrong (2.1a).) So I estimated that they forgot inserting $i$ in front of $A^3$ in the Chern-Simons.
- I also misunderstood that Cardoso et al [4] introduced the wrong definition of the trace. But Now I really understand they chose exactly correct signs in their Lagrangian, even though the definition of the trace looks quite strange.

So we have to compute carefully heterotic action in terms of this note in future works.

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Originally Polchinski showed us the heterotic Larangian with respect to the anti-hermitian gauge field (see chapter 12 in [13]). A few months later after Polchinski published [13], Motl suggested to change any anti-hermtian gauge field to the hermitian fields and Polchinski agreed with his comments. So, in the table of errata in his webpage [14], Polchinski gives us a transformation rules to anti-hermitian to hermitian. In this note, however, I use his original convention, which is now adapted by many people who study flux compactification. Furthermore, the anti-hermitian gauge field is convenient when we analyze the topological characteristic classes (see Nakahara [12]).

[14] See the following URL: http://www.itp.ucsb.edu/~joep/bigbook.html