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Supergravities in ten and eleven dimensions

— as a dictionary —

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Abstract

In this note we start eleven-dimensional supergravity which explicitly contains fermionic terms. We connect it to the type IIA supergravity action, and perform T-duality to obtain the type IIB supergravity. Further we also study the type I and heterotic supergravity actions.

In this note we impose the self-dual condition on the Ramond-Ramond five-form field strength F_5 by hand in type IIB supergravity.

1 Strategy

We start from the eleven-dimensional supergravity. Along various kinds of duality transformations, we connect five significant supergravity Lagrangian including fermions. We also investigate local supersymmetry transformations in each theory. The reason is that, unfortunately, we only find the bosonic parts of supergravity theories while the local supersymmetry transformation rules are explicitly introduced. This situation is terrible to consider the Killing spinor equations and equations of motion, as well as the Bianchi identity in the presense of fermions. To complement such a difficulty, we will derive the supergravity Lagrangians including fermions and their interactions, explicitly.

It might be quite useful to fix the convention of fields and transformations for our future work on the investigation of duality transformations in terms of Killing spinors and spinorial geometry.

Contents

2 Eleven-dimensional supergravity

Here let us summarize the convantion which appears in the note [4, 7]:

$$S = \int \mathrm{d}^{11}x\, \mathscr{L} \ , \tag{2.1a}$$

$$\begin{aligned} 2\kappa_{11}^2\, \mathscr{L} &= \beta_0 \sqrt{-g_{11}}\, R(e,\omega) - \frac{1}{2}\sqrt{-g_{11}}\, \overline{\Psi}_M \Gamma^{MNP} D_N[\tfrac{1}{2}(\omega + \widehat{\omega})]\Psi_P - \frac{1}{48}\sqrt{-g_{11}}\, F_{MNPQ}F^{MNPQ} \\ &\quad + \frac{1}{2}\beta_1 \sqrt{-g_{11}}\, \overline{\Psi}_M \widetilde{\Gamma}^{MNPQRS}\Psi_N\, (F + \widehat{F})_{PQRS} \\ &\quad + \beta_2\, \hat{\varepsilon}^{MNPQRSUVWXY}\, F_{MNPQ}\, F_{RSUV}\, C_{WXY} \ , \end{aligned} \tag{2.1b}$$

where $\widehat{\omega}_{MAB}$ and $\widehat{F}_{MNPQ}$ are the supercovariantizations of ω_{MAB} and F_{MNPQ}, respectively [5, 4]. The supercovariantization means that its supersymmetry variation does not contain derivatives of the supersymmetry parameter. However, such extension introduce fermion bilinear terms in it, which might be irrelevant in this note. Then we truncate such supercovariantized fields as

$$\widehat{\omega}_{MAB} = \omega_{MAB} \ , \quad \widehat{F}_{MNPQ} = F_{MNPQ} \ , \tag{2.1c}$$

where ω_{MAB} is a torsionless spin connection. We should also notice that $\hat{\varepsilon}^{MNPQRSUWXY}$ is not the invariant tensor but the "antisymmetric symbol", whose normalization is given as $\hat{\varepsilon}^{012\cdots\natural} = 1$. This is related to the tensor $\varepsilon^{MNPQRSUWXY}$ as

$$\hat{\varepsilon}^{MNPQRSUVWXY} = g_{11}\, \varepsilon^{MNPQRSUVWXY} \ . \tag{2.1d}$$

The transformation rule under the local supersymmetry is given as

$$\delta e_M{}^A = \frac{1}{2}\bar{\varepsilon}\Gamma^A \Psi_M \ , \quad \delta C_{MNP} = \alpha_2\, \bar{\varepsilon}\Gamma_{[MN}\Psi_{P]} \ , \tag{2.1e}$$

$$\delta\Psi_M = 2D_M(\omega)\varepsilon + 2T_M{}^{NPQR}\, \varepsilon F_{NPQR} \ . \tag{2.1f}$$

In the above formulation, various objects are defined in the following way:

$$\overline{\Psi}_M = i\Psi_M^\dagger \Gamma^0 \ , \quad D_M(\omega) = \partial_M - \frac{i}{2}\omega_{MAB}\Sigma^{AB} \ , \tag{2.1g}$$

$$\widetilde{\Gamma}^{MNPQRS} = \Gamma^{MNPQRS} + 12 g^{M[P}\Gamma^{QR}g^{S]N} \ , \tag{2.1h}$$

$$T_M{}^{NPQR} = \frac{\alpha_1}{2}\Big(\Gamma_M{}^{NPQR} - 8\delta_M^{[N}\Gamma^{PQR]} \Big) \ . \tag{2.1i}$$

For later convenience, we introduced some unfixed coefficients β_0, β_1 and β_2 in the Lagrangian, and α_1 and α_2 in the supersymmetry transformation rule. These are closely related to each other to preserve symmetry of the action:

$$\beta_2 = -\frac{2\alpha_1\beta_1}{\alpha_2} = \frac{2}{216}\beta_1 = -\frac{1}{144}\alpha_1 \ , \quad \alpha_1\beta_1 = -\frac{1}{96\cdot 288\beta_0} \ . \tag{2.2a}$$

To arrange the usual form of the Einstein-Hilbert action, we fix the constant β_0 to be unity: $\beta_0 = 1$. In this setting we should choose the following explicit solution:

$$\alpha_1 \ = \ \frac{1}{144} \ , \quad \alpha_2 \ = \ -\frac{3}{2} \ , \quad \beta_1 \ = \ -\frac{1}{192} \ , \quad \beta_2 \ = \ -\frac{1}{(144)^2} \ . \tag{2.2b}$$

Let us rewrite the above forms (2.1) to connect other supergravity actions in ten-dimensional spacetime, which appear in the Polchinski's book [15] and in the book written by Becker, Becker and Schwarz [1]. Let us rescale the gravitino

$$\Psi_M \ \to \ 2\Psi_M \ . \tag{2.3}$$

Substituting (2.1c), (2.2b) and (2.3), we obtain the 11-dimensional supergravity action, which we will mainly use, instead of (2.1). First, the Lagrangian itself is

$$\begin{aligned} 2\kappa_{11}^2 \mathscr{L} \ &= \ \sqrt{-g}\, R(e,\omega) - 2\sqrt{-g}\, \overline{\Psi}_M \Gamma^{MNP} D_N(\omega) \Psi_P - \frac{1}{48}\sqrt{-g}\, F_{MNPQ} F^{MNPQ} \\ &\ \ \ \ - \frac{1}{48}\sqrt{-g}\, \overline{\Psi}_M \widetilde{\Gamma}^{MNPQRS} \Psi_N \, F_{PQRS} \\ &\ \ \ \ - \frac{1}{(144)^2} \hat{\varepsilon}^{MNPQRSUVWXY} \, F_{MNPQ}\, F_{RSUV}\, C_{WXY} \ . \end{aligned} \tag{2.4a}$$

The transformation rule under the local supersymmetry is given as

$$\delta e_M{}^A \ = \ \overline{\varepsilon} \Gamma^A \Psi_M \ , \quad \delta C_{MNP} \ = \ -3 \overline{\varepsilon} \Gamma_{[MN} \Psi_{P]} \ , \tag{2.5a}$$

$$\delta \Psi_M \ = \ D_M(\omega) \varepsilon + T_M{}^{NPQR} \, \varepsilon F_{NPQR} \ . \tag{2.5b}$$

In addition, the combination of the gamma matrices $T_M{}^{NPQR}$ is rewritten in the following way:

$$T_M{}^{NPQR} \ = \ \frac{1}{288}\Big(\Gamma_M{}^{NPQR} - 8 \delta_M^{[N} \Gamma^{PQR]} \Big) \ . \tag{2.5c}$$

Here it is worth describing the bosonic part of the action in terms of the differential form to compare the conventions in [15, 1]. Let us first extract the bosonic part of the action:

$$\begin{aligned} S_{\text{boson}} \ &= \ \frac{1}{2\kappa_{11}^2} \int \mathrm{d}^{11}x \, \sqrt{-g} \left(R(e,\omega) - \frac{1}{48} F_{MNPQ} F^{MNPQ} \right) \\ &\ \ \ \ - \frac{1}{2\kappa_{11}^2 (144)^2} \int \mathrm{d}^{11} x \, \hat{\varepsilon}^{MNPQRSUVWXY} \, F_{MNPQ} F_{RSTU} C_{WXY} \ . \end{aligned} \tag{2.6}$$

Notice that the symbol $\hat{\varepsilon}^{MNPQRSUWXY}$ does not depend on the curved space coordinates. It is just a number. Using the convention in appendix A, the last term is rewritten as

$$\begin{aligned} & - \frac{1}{(144)^2} \int \mathrm{d}^{11} x \, \hat{\varepsilon}^{MNPQRSUVWXY} \, F_{MNPQ} F_{RSUV} C_{WXY} \\ & \ \ = \ - \frac{1}{(144)^2} \int \mathrm{d}^{11} x \, g \, \varepsilon^{MNPQRSUVWXY} \, F_{MNPQ} F_{RSUV} C_{WXY} \end{aligned}$$

$$= -\frac{1}{(144)^2} \int \mathrm{d}x^M \wedge \mathrm{d}x^N \wedge \cdots \wedge \mathrm{d}x^Y \, F_{MNPQ} F_{RSUV} C_{WXY}$$

$$= -\frac{4!4!3!}{(144)^2} \int F_4 \wedge F_4 \wedge C_3 \; = \; -\frac{1}{6} \int F_4 \wedge F_4 \wedge C_3 \; . \tag{2.7}$$

Then, the bosonic action is simplified in the following way:

$$S_{\text{boson}} \; = \; \frac{1}{2\kappa_{11}^2} \int \Big((\text{vol.}) \Big\{ R(e, \omega) - \frac{1}{2} |F_4|^2 \Big\} - \frac{1}{6} F_4 \wedge F_4 \wedge C_3 \Big) \; . \tag{2.8}$$

3 Type IIA supergravity in ten dimensions

In this section let us derive the type IIA supergravity from the eleven-dimensional supergravity. The derivation rule has already been investigated very well. Here we follow a convention introduced by [14], because the Lagrangians both in eleven- and in ten-dimensional spacetime are common forms in modern sense, while the fermion transformation rule itself is not explicitly discussed.

3.1 Duality transformation rules

Here let us introduce the duality transformation rules from the eleven-dimensional supergravity to the type IIA supergravity, and vice versa. We only fucus on the bosonic parts. The former rule is given by

$$\hat{g}_{MN} \ = \ \mathrm{e}^{-\frac{2}{3}\Phi} g_{MN} + \mathrm{e}^{\frac{4}{3}\Phi} A_M A_N \ , \quad \hat{C}_{MNP} \ = \ C_{MNP} \ , \tag{3.1a}$$

$$\hat{g}_{M\natural} \ = \ \mathrm{e}^{\frac{4}{3}\Phi} A_M \ , \quad \hat{C}_{MN\natural} \ = \ \frac{2}{3} B_{MN} \ , \quad \hat{g}_{\natural\natural} \ = \ \exp\Big(\frac{4}{3}\Phi\Big) \ . \tag{3.1b}$$

Note that the objects with $\hat{}$ denote the ones in eleven-dimensions, while the others in ten dimensions. In addition, the symbol $\natural$ indicates the eleventh direction in the eleven-dimensional spacetime. In the same way, the latter rule is also given by

$$g_{MN} \ = \ \big(\hat{g}_{\natural\natural}\big)^{\frac{1}{2}} \Big(\hat{g}_{MN} - \frac{\hat{g}_{M\natural}\, \hat{g}_{\natural N}}{\hat{g}_{\natural\natural}}\Big) \ , \quad C_{MNP} \ = \ \hat{C}_{MNP} \ , \tag{3.2a}$$

$$A_M \ = \ \frac{\hat{g}_{M\natural}}{\hat{g}_{\natural\natural}} \ , \quad B_{MN} \ = \ \frac{3}{2} \hat{C}_{MN\natural} \ , \quad \Phi \ = \ \frac{3}{4} \log\big(\hat{g}_{\natural\natural}\big) \ . \tag{3.2b}$$

In this note we omit the derivation of the above rules (see section *5.1* in [14]).

4 Type IIB supergravity in ten dimensions

In this section let us derive the type IIB supergravity from the type IIA supergravity using the convention introduced by [14].

4.1 Duality transformation rules

Here let us introduce the duality transformation rules from the type IIA supergravity to the type IIB supergravity, and vice versa. We only fucus on the bosonic parts. The former rule is given by

$$\hat{g}_{MN} \ = \ g_{MN} - \frac{1}{g_{99}}\Big(g_{M9}g_{9N} + B^{(1)}_{M9}B^{(1)}_{9N}\Big) \, , \quad \hat{g}_{9M} \ = \ \frac{B^{(1)}_{9M}}{g_{99}} \, , \quad \hat{g}_{99} \ = \ \frac{1}{g_{99}} \, , \tag{4.1a}$$

$$\hat{C}_{9MN} \ = \ \frac{2}{3}\Big(B^{(2)}_{MN} + \frac{2B^{(2)}_{9[M}g_{N]9}}{g_{99}}\Big) \, , \tag{4.1b}$$

$$\hat{C}_{MNP} \ = \ \frac{8}{3}C_{9MNP} + \varepsilon^{ij}B^{(i)}_{9[M}B^{(j)}_{NP]} + \frac{\varepsilon^{ij}B^{(i)}_{9[M}B^{(j)}_{|9|N}g_{P]9}}{g_{99}} \, , \tag{4.1c}$$

$$\hat{B}_{MN} \ = \ B^{(1)}_{MN} + \frac{2B^{(1)}_{9[M}g_{N]9}}{g_{99}} \, , \quad \hat{B}_{9M} \ = \ \frac{g_{9M}}{g_{99}} \, , \tag{4.1d}$$

$$\hat{A}_M \ = \ -B^{(2)}_{9M} + CB^{(1)}_{9M} \, , \quad \hat{A}_9 \ = \ C \, , \quad \hat{\Phi} \ = \ \Phi - \frac{1}{2}\log\big(g_{99}\big) \, . \tag{4.1e}$$

Note that the objects with $\hat{}$ denote the ones in type IIA, while the others in type IIB. The two different two-form fields $B^{(i)}_{MN}$ implies $B^{(1)}_{MN} = B^{\rm NSNS}_{MN}$ and $B^{(2)}_{MN} = C^{\rm RR}_{MN}$, respectively. We normalize $\varepsilon^{12} = 1$. In the same way, the latter rule is also given by

$$g_{MN} \ = \ \hat{g}_{MN} - \frac{1}{\hat{g}_{99}}\Big(\hat{g}_{M9}\hat{g}_{9N} + \hat{B}_{M9}\hat{B}_{9N}\Big) \, , \quad g_{9M} \ = \ \frac{\hat{B}_{9M}}{\hat{g}_{99}} \, , \quad g_{99} \ = \ \frac{1}{\hat{g}_{99}} \, , \tag{4.2a}$$

$$C_{9MNP} \ = \ \frac{3}{8}\left(\hat{C}_{MNP} - \hat{A}_{[M}\hat{B}_{NP]} + \frac{\hat{g}_{9[M}\hat{B}_{NP]}\hat{A}_9}{\hat{g}_{99}} - \frac{3}{2}\frac{\hat{g}_{9[M}\hat{C}_{NP]9}}{\hat{g}_{99}}\right) \, , \tag{4.2b}$$

$$B^{(1)}_{MN} \ = \ \hat{B}_{MN} + \frac{2\hat{g}_{9[M}\hat{B}_{N]9}}{\hat{g}_{99}} \, , \quad B^{(1)}_{9M} \ = \ \frac{\hat{g}_{9M}}{\hat{g}_{99}} \, , \tag{4.2c}$$

$$B^{(2)}_{MN} \ = \ \frac{3}{2}\hat{C}_{MN9} - 2\hat{A}_{[M}\hat{B}_{N]9} + \frac{2\hat{g}_{9[M}\hat{B}_{N]9}\hat{A}_9}{\hat{g}_{99}} \, , \quad B^{(2)}_{9M} \ = \ -\hat{A}_M + \frac{\hat{A}_9\hat{g}_{9M}}{\hat{g}_{99}} \, , \tag{4.2d}$$

$$\Phi \ = \ \hat{\Phi} - \frac{1}{2}\log\big(\hat{g}_{99}\big) \, , \quad C \ = \ \hat{A}_9 \, . \tag{4.2e}$$

In this note we omit the derivation of the above rules (see section *5.1* in [14]). Notice that the RR four-form field C_{MNPQ} without containing the nineth-direction is given by the self-duality condition on its field strength:

$$\mathrm{d}C_4 \ = \ F_5 \ = \ *_{10}F_5 \, . \tag{4.3}$$

5 Type I supergravity in ten dimensions

The type I supergravity contains the vielbein $e_M{}^A$, the dilaton Φ, the one-form anti-hermitian gauge field $\widetilde{A}_1$, and the Ramond-Ramond two-form field $B^{(2)}_{MN} = C_{MN}$, as bosonic fields, and the gravitino Ψ_M, the dilatino λ and the gaugino χ. Here we will follow the convention given by Polchinski [15, 14].

The relation between the type IIB and the type I is as follows: The type I supergravity action is obtained from the type IIB supergravity with truncating the fields $B^{(1)}_{MN}$, C and C_{MNPQ}, which are not invariant under the Ω-projection, i.e., the worldsheet parity projection. Further, we newly introduce a one-form gauge field $\widetilde{A}_1$ from the open string modes, and modify the Ramond-Ramond three-form field strength F_3 in the following way:

$$F_3 \ \to \ F'_3 \ = \ \mathrm{d}C_2 - \frac{\kappa_{10}^2}{g_{10}^2}\omega_3^{\mathrm{Y}} \ , \quad \omega_3^{\mathrm{Y}} \ \equiv \ \mathrm{tr}\Big(\widetilde{A} \wedge \mathrm{d}\widetilde{A} + \frac{2}{3}\widetilde{A} \wedge \widetilde{A} \wedge \widetilde{A}\Big) \ , \tag{5.1}$$

where g_{10} and ω_3^{Y} are called the Yang-Mills coupling and the Yang-Mills Chern-Simons three-form in ten-dimensional spacetime, respectively. Notice that the one-form gauge field $\widetilde{A}_1$ couples to anti-hermitian generator of the gauge group[1], in the same way as the one in the heterotic supergravity [8], which will appear in the next section. Notice that the Ramond-Ramond three-form F'_3 should be further modified via the Green-Schwarz anomaly cancellation mechanism.

[1]The one-form gauge field A_1 in [14] is given as a hermitian matrix field. See appendix A.12 for the difference between hermitian and anti-hermitian gauge fields.

6 Heterotic supergravity in ten dimensions

The ten-dimensional heterotic supergravity contains the vielbein $e_M{}^A$, the NS-NS two-form field B_{MN}, the dilaton Φ, the anti-hermitian gauge field $\widetilde{A}_M$, as bosonic fields, and the gravitino Ψ_M, the dilatino λ and the gaugino χ. Here we will follow the convention given by Polchinski [15, 14]. In the next section we will expand the heterotic theory of lowest order to the one of higher-order correction in α' [2].

The S-duality rule between the type I and the heterotic Lagrangian in ten dimensions is quite simple. We follow the discussion to [14]:

$$\hat{g}_{MN} \ = \ \mathrm{e}^{-\Phi} g_{MN} \ , \quad \hat{\Phi} \ = \ -\Phi \ , \tag{6.1a}$$

$$\hat{F}'_3 \ = \ H'_3 \ , \quad \hat{A}_1 \ = \ A_1 \ , \tag{6.1b}$$

where the fields with $\hat{}$ denote the ones in the type I, while the ones without hat in the heterotic theory. Notice that the NS-NS three-form field strength H'_3 is also modified by introducing the Yang-Mills Chern-Simons three-form ω_3^{Y}:

$$H'_3 \ = \ \mathrm{d}B_2 - \frac{\kappa_{10}^2}{g_{10}^2} \omega_3^{\mathrm{Y}} \ , \quad \omega_3^{\mathrm{Y}} \ \equiv \ \mathrm{tr}\Big(\widetilde{A} \wedge \mathrm{d}\widetilde{A} + \frac{2}{3} \widetilde{A} \wedge \widetilde{A} \wedge \widetilde{A} \Big) \ . \tag{6.2}$$

Notice that the NS-NS three-form should be further modified via the Green-Schwarz anomaly cancellation mechanism.

7 Heterotic supergravity in ten dimensions, II

The heterotic supergravity with higher-order corrections in α' has been well investigated [2]. Thus, in this section, we introduce such a corrected Lagrangian in the heterotic theory given by [2, 8].

7.1 Lagrangian with higher-order corrections in α'

Let us write down the Lagrangian of ten-dimensional heterotic supergravity [2], which is an extended verions of the heterotic theory with higher-order corrections in α':

$$
\begin{aligned}
\mathscr{L}_{\text{total}} &= \mathscr{L}(R) + \mathscr{L}(\widetilde{F}^2) + \mathscr{L}(R^2) \,, && (7.1a) \\
2\kappa_{10}^2 \mathscr{L}(R) &= \sqrt{-g}\, \mathrm{e}^{-2\Phi} \bigg[R(\omega) - \frac{1}{12} H_{MNP} H^{MNP} + 4(\nabla_M \Phi)^2 \\
&\quad - \overline{\psi}_M \Gamma^{MNP} D_N(\omega) \psi_P + 8\, \overline{\lambda}\, \Gamma^{MN} D_M(\omega) \psi_N + 16\, \overline{\lambda} \not{D}(\omega) \lambda \\
&\quad + 8\, \overline{\psi}_M \Gamma^N \Gamma^M \lambda \, (\nabla_N \Phi) - 2 \overline{\psi}_M \Gamma^M \psi_N \, (\nabla^N \Phi) \\
&\quad - \frac{1}{24}\, H^{PQR} \Big\{ \overline{\psi}_M \Gamma^{[M} \Gamma_{PQR} \Gamma^{N]} \psi_N + 8\, \overline{\psi}_M \Gamma^M{}_{PQR} \lambda - 16\, \overline{\lambda} \Gamma_{PQR} \lambda \Big\} \\
&\quad + \frac{1}{48}\, \overline{\psi}^M \Gamma^{ABC} \psi_M \Big\{ 2 \overline{\lambda} \Gamma_{ABC} \lambda + \overline{\lambda} \Gamma_{ABC} \Gamma^N \psi_N \\
&\qquad - \frac{1}{4}\, \overline{\psi}^N \Gamma_{ABC} \psi_N - \frac{1}{8}\, \overline{\psi}^N \Gamma_N \Gamma_{ABC} \Gamma^P \psi_P \Big\} \bigg] \,, && (7.1b) \\
2\kappa_{10}^2 \mathscr{L}(\widetilde{F}^2) &= - \frac{\kappa_{10}^2}{2 g_{10}^2} \sqrt{-g}\, \mathrm{e}^{-2\Phi} \bigg[- \mathrm{tr}\big(\widetilde{F}_{MN} \widetilde{F}^{MN} \big) - 2\, \mathrm{tr}\big\{ \overline{\widetilde{\chi}} \not{D}(\omega, \widetilde{A}) \widetilde{\chi} \big\} - \frac{1}{12}\, \mathrm{tr}\big(\overline{\widetilde{\chi}} \Gamma^{ABC} \widetilde{\chi} \big) \hat{H}_{ABC} \\
&\quad - \frac{1}{2}\, \mathrm{tr}\big\{ \overline{\widetilde{\chi}} \Gamma^M \Gamma^{AB} (\widetilde{F}_{AB} + \hat{\widetilde{F}}_{AB}) \big\} \Big(\psi_M + \frac{2}{3} \Gamma_M \lambda \Big) \\
&\quad - \frac{1}{48}\, \mathrm{tr}(\overline{\widetilde{\chi}} \Gamma^{ABC} \widetilde{\chi})\, \overline{\psi}_M \big(4 \Gamma_{ABC} \Gamma^M + 3 \Gamma^M \Gamma_{ABC} \big) \lambda \\
&\quad + \frac{1}{12}\, \mathrm{tr}(\overline{\widetilde{\chi}} \Gamma^{ABC} \widetilde{\chi}) \overline{\lambda} \Gamma_{ABC} \lambda - \frac{\beta}{96}\, \mathrm{tr}(\overline{\widetilde{\chi}} \Gamma^{ABC} \widetilde{\chi})\, \mathrm{tr}(\overline{\widetilde{\chi}} \Gamma_{ABC} \widetilde{\chi}) \bigg] \,, && (7.1c) \\
2\kappa_{10}^2 \mathscr{L}(R^2) &= - \frac{\kappa_{10}^2}{2 g_{10}^2} \sqrt{-g}\, \mathrm{e}^{-2\Phi} \bigg[- R_{ABMN}(\Omega_-) R^{ABMN}(\Omega_-) \\
&\quad - 2 \overline{\psi}^{AB} \not{D}(\omega(e,\psi), \Omega_-) \psi_{AB} - \frac{1}{12} \overline{\psi}^{AB} \Gamma^{MNP} \psi_{AB} \hat{H}_{MNP} \\
&\quad + \frac{1}{2} \overline{\psi}_{AB} \Gamma^M \Gamma^{NP} \big\{ R^{AB}{}_{NP}(\Omega_-) + \hat{R}^{AB}{}_{NP}(\Omega_-) \big\} \Big(\psi_M + \frac{2}{3} \Gamma_M \lambda \Big) \\
&\quad - \frac{1}{48}\, \overline{\psi}_{AB} \Gamma^{CDE} \psi_{AB} \cdot \overline{\psi}_M \big(4 \Gamma_{CDE} \Gamma^M + 3 \Gamma^M \Gamma_{CDE} \big) \lambda \\
&\quad + \frac{1}{12} \overline{\psi}^{AB} \Gamma^{CDE} \psi_{AB} \big(\overline{\lambda} \Gamma_{CDE} \lambda \big) - \frac{\alpha}{96} \overline{\psi}^{AB} \Gamma^{FGH} \psi_{AB} \big(\overline{\psi}^{CD} \Gamma_{FGH} \psi_{CD} \big) \bigg] \,. && (7.1d)
\end{aligned}
$$

The ten-dimensional gravitational constant κ_{10}, the Yang-Mills coupling g_{10} are related to

$$\frac{\kappa_{10}^2}{g_{10}^2} \equiv \frac{\alpha'}{4} . \tag{7.2}$$

Derivatives $D_M(\omega, \widetilde{A})$ are the covariant derivatives with respect to Lorentz and Yang-Mills gauge transformations. We define the derivative on fundamental fields ϕ^i as

$$D_M(\omega, \widetilde{A}, \Gamma)\phi^i = \partial_M \phi^i - \frac{i}{2}\omega_M{}^{AB}(\Sigma_{AB})^i{}_j\, \phi^j + (\widetilde{A}_M)^i{}_j\, \phi^j + \Gamma^i{}_{jM}\, \phi^j . \tag{7.3}$$

Note that Σ_{AB} is a Lorentz generator whose representation is given by (A.4). Notice that we can always re-define the gravitino and dilatino via "mixing" with each other such as

$$\left(\begin{array}{c} \psi'_M \\ \Gamma_M \lambda' \end{array} \right) = \left(\begin{array}{cc} 1 & b \\ 0 & c \end{array} \right) \left(\begin{array}{c} \psi_M \\ \Gamma_M \lambda \end{array} \right) , \quad b, c \in \mathbb{R} , \quad c \neq 0 , \tag{7.4}$$

because both ψ_M and $\Gamma_M \lambda$ are same chiralities and belong to the gravity multiplet. In the above Lagrangian we do not obtain the gravitino supersymmetry variation including the gradient of the dilaton, dilatino condensation terms and so forth.

7.2 Local supersymmetry variations

Following to [2], let us show the local supersymmetry variations We write δ_{α^n} (δ_{β^m}) for variations of order α^n (β^m), while δ_0 corresponds to the terms independent of parameters[2] α and β:

$$\delta_0 e_M{}^A = \frac{1}{2}\overline{\epsilon}\Gamma^A \psi_M , \tag{7.5a}$$

$$\delta_0 \psi_M = \Big(\partial_M + \frac{1}{4}\Omega_{+M}{}^{AB}\Gamma_{AB}\Big)\epsilon + \Big\{ \epsilon\big(\overline{\psi}_M \lambda\big) - \psi_M\big(\overline{\epsilon}\lambda\big) + \Gamma^A \lambda \big(\overline{\psi}_M \Gamma_A \epsilon\big) \Big\} , \tag{7.5b}$$

$$\delta_0 B_{MN} = -\overline{\epsilon}\Gamma_{[M}\psi_{N]} , \tag{7.5c}$$

$$\delta_0 \lambda = -\frac{1}{4}\mathscr{D}\!\!\!/\Phi\, \epsilon + \frac{1}{48}\Gamma^{ABC}\epsilon\Big(- \hat{H}_{ABC} - \frac{1}{2}\overline{\lambda}\Gamma_{ABC}\lambda\Big) , \tag{7.5d}$$

$$\delta_0 \Phi = -\overline{\epsilon}\lambda , \tag{7.5e}$$

$$\delta_0 \widetilde{A}_M = \frac{1}{2}\overline{\epsilon}\Gamma_M \widetilde{\chi} , \tag{7.5f}$$

$$\delta_0 \widetilde{\chi} = -\frac{1}{4}\Gamma^{AB}\epsilon\, \hat{\widetilde{F}}_{AB} + \Big\{ \epsilon\big(\overline{\widetilde{\chi}}\lambda\big) - \widetilde{\chi}\big(\overline{\epsilon}\lambda\big) + \Gamma^A \lambda \big(\overline{\widetilde{\chi}}\Gamma_A \epsilon\big) \Big\} , \tag{7.5g}$$

$$\delta_0 \omega_M{}^{AB}(e,\psi) = -\frac{1}{4}\overline{\epsilon}\Gamma_M \psi^{AB} - \frac{1}{2}\overline{\epsilon}\Gamma^{[A}\psi_M{}^{B]} + \frac{1}{4}\,\overline{\epsilon}\Gamma_C \psi_M\, \hat{H}^{ABC} . \tag{7.5h}$$

Notice that the spin connection $\omega(e,\psi)$ is the solution of $\mathscr{D}_{[M}(\omega)e_{N]}{}^A = 0$, while $\omega(e)$ is the solution of $D_{[M}(\omega)e_{N]}{}^A = 0$. Note that various objects such as a spin connection[3] modified by the H-flux,

[2]Compared to [2] and [2], we assign the expansion parameters α and β to $\alpha = \beta = -\kappa_{10}^2/g_{10}^2$.

[3]If we simply write down the spin connection as ω, this means $\omega = \omega(e)$.

supercovariantizations $\hat{H}$ and $\hat{F}$, and so forth:

$$\Omega_{\pm M}{}^{AB} \equiv \omega_M{}^{AB}(e,\psi) \pm \frac{1}{2}\hat{H}_M{}^{AB} , \tag{7.6a}$$

$$\begin{aligned}\hat{H}_{MNP} &\equiv H_{MNP} + \frac{3}{2}\overline{\psi}_{[M}\Gamma_N\psi_{P]} \\ &= 3\partial_{[M}B_{NP]} + \frac{3}{2}\overline{\psi}_{[M}\Gamma_N\psi_{P]} - 6\frac{\kappa_{10}^2}{g_{10}^2}\,\mathrm{tr}\Big(\widetilde{A}_{[M}\partial_N\widetilde{A}_{P]} + \frac{2}{3}\widetilde{A}_{[M}\widetilde{A}_N\widetilde{A}_{P]}\Big) \\ &\quad + 6\frac{\kappa_{10}^2}{g_{10}^2}\Big(\Omega_{-[M}{}^{AB}\,\partial_N\Omega_{-P]}{}^{BA} + \frac{2}{3}\Omega_{-[M}{}^{AB}\,\Omega_{-N}{}^{BC}\,\Omega_{-P]}{}^{CA}\Big) ,\end{aligned} \tag{7.6b}$$

$$\mathrm{d}H = \frac{\kappa_{10}^2}{g_{10}^2}\Big[\mathrm{tr}\big\{R(\Omega_-)\wedge R(\Omega_-)\big\} - \mathrm{tr}\big(\widetilde{F}\wedge\widetilde{F}\big)\Big] , \tag{7.6c}$$

$$\hat{\widetilde{F}}_{MN} \equiv \widetilde{F}_{MN} - \overline{\psi}_{[M}\Gamma_{N]}\widetilde{\chi} , \tag{7.6d}$$

$$\begin{aligned}\psi_{MN} &\equiv D_M(\Omega_+)\psi_N - D_N(\Omega_+)\psi_M \\ &\quad - \Big\{\psi_M\big(\overline{\psi}_N\lambda\big) - \psi_N\big(\overline{\psi}_M\lambda\big) - \Gamma^P\lambda\,\big(\overline{\psi}_M\Gamma_P\psi_N\big)\Big\} ,\end{aligned} \tag{7.6e}$$

$$\hat{R}^{AB}{}_{MN}(\omega) \equiv R^{AB}{}_{MN}(\omega) + \frac{1}{2}\overline{\psi}_{[M}\Gamma_{N]}\psi^{AB} + \overline{\psi}_{[M}\Gamma^{[A}\psi_{N]}{}^{B]} + \frac{1}{2}\,\overline{\psi}_{[M}\Gamma^C\psi_{N]}\,\hat{H}^{AB}{}_C . \tag{7.6f}$$

We also pick up the supercovariant derivatives $\mathscr{D}_M$ at hand:

$$\mathscr{D}_M\Phi = \nabla_M\Phi + \overline{\psi}_M\lambda , \tag{7.7a}$$

$$\mathscr{D}_M(\omega)\lambda = D_M(\omega)\lambda + \frac{1}{4}\,\mathscr{D}\!\!\!/\Phi\,\psi_M - \frac{1}{48}\Gamma^{ABC}\psi_M\Big(- \hat{H}_{ABC} - \frac{1}{2}\,\overline{\lambda}\Gamma_{ABC}\lambda\Big) , \tag{7.7b}$$

$$\mathscr{D}_M(\omega,\widetilde{A})\widetilde{\chi} = D_M(\omega,\widetilde{A})\widetilde{\chi} + \frac{1}{4}\Gamma^{AB}\psi_M\,\hat{\widetilde{F}}_{AB} - \Big\{\psi_M\big(\overline{\widetilde{\chi}}\lambda\big) - \widetilde{\chi}\big(\overline{\psi}_M\lambda\big) + \Gamma^A\lambda\big(\overline{\widetilde{\chi}}\Gamma_A\psi_M\big)\Big\} . \tag{7.7c}$$

It is both useful and instructive to obtain the supersymmetry algebra from (7.5). The commutator of two supersymmetry variations reads

$$\begin{aligned}\big[\delta(\epsilon_1),\delta(\epsilon_2)\big] &= \delta_{\mathrm{P}}(\xi^M) + \delta_{\mathrm{Q}}(-\xi^M\psi_M) + \delta_{\mathrm{L}}(\xi^M\Omega_{-M}{}^{AB}) + \delta_{\mathrm{YM}}(\xi^M\widetilde{A}_M) \\ &\quad + \delta_{\mathrm{M}}(-\tfrac{\sqrt{2}}{2}\xi_M + \tfrac{1}{\sqrt{2}}\xi^N B_{NM}) + \delta_{\mathrm{Q}}(\epsilon_3) + \delta_{\mathrm{L}}(\Lambda^{AB}) ,\end{aligned} \tag{7.8a}$$

$$\xi^M = \frac{1}{2}\overline{\epsilon}_2\Gamma^M\epsilon_1 , \tag{7.8b}$$

$$\epsilon_3 = -\frac{7}{8}(\overline{\epsilon}_2\Gamma^A\epsilon_1)\,\Gamma_A\lambda + \frac{1}{16\times 120}\,(\overline{\epsilon}_2\Gamma^{ABCDE}\epsilon_1)\,\Gamma_{ABCDE}\lambda , \tag{7.8c}$$

$$\Lambda^{AB} = \frac{\beta}{192}\,\overline{\epsilon}_2\Gamma^{[A}\Gamma_{CDE}\Gamma^{B]}\epsilon_1\,\mathrm{tr}\big(\overline{\widetilde{\chi}}\Gamma^{CDE}\widetilde{\chi}\big) . \tag{7.8d}$$

On the right-hand side of (7.8a), we encounter all gauge transformations of the ten-dimensional super Yang-Mills theory: δ_{P}, δ_{Q}, δ_{L}, δ_{YM}, and δ_{M} correspond respectively to "general coordinate", "supersymmetry", "local Lorentz", "Yang-Mills" and "antisymmetric tensor gauge" transformations.

The supersymmetry variation of order β are given as follows:

$$\delta_\beta \psi_M = \frac{\beta}{192} \Gamma^{ABC} \Gamma_M \epsilon \, \mathrm{tr}\big(\overline{\widetilde{\chi}} \Gamma_{ABC} \widetilde{\chi}\big) , \tag{7.9a}$$

$$\delta_\beta B_{MN} = 2\beta \, \mathrm{tr}\big\{ \widetilde{A}_{[M} \delta_0 \widetilde{A}_{N]} \big\} , \tag{7.9b}$$

$$\delta_\beta \lambda = \frac{\beta}{384} \Gamma^{ABC} \epsilon \, \mathrm{tr}\big(\overline{\widetilde{\chi}} \Gamma_{ABC} \widetilde{\chi}\big) , \tag{7.9c}$$

$$\delta_\beta {\omega_M}^{AB}(e,\psi) = -\frac{\beta}{192} \overline{\epsilon} \Gamma^{[A} \Gamma_{CDE} \Gamma^{B]} \psi_M \, \mathrm{tr}\big(\overline{\widetilde{\chi}} \Gamma^{CDE} \widetilde{\chi}\big) . \tag{7.9d}$$

Here the supersymmetry variation of order α are also given such as

$$\delta_\alpha \psi_M = \frac{\alpha}{192} \Gamma^{CDE} \Gamma_M \epsilon \, \overline{\psi}^{AB} \Gamma_{CDE} \psi_{AB} , \tag{7.10a}$$

$$\delta_\alpha B_{MN} = 2\alpha \, {\Omega_{-[M}}^{AB} \, \delta_0 {\Omega_{-N]}}^{AB} , \tag{7.10b}$$

$$\delta_\alpha \lambda = \frac{\alpha}{384} \Gamma^{CDE} \Gamma_M \epsilon \, \overline{\psi}^{AB} \Gamma_{CDE} \psi_{AB} . \tag{7.10c}$$

The supersymmetry variations of the supercovariant variables are also obtained. First we write down the zero-th order of α and β. Next the corrections of first order β are described. (Unfortunately, there are no descriptions about the corrections of first order α [2].)

$$\delta_0(\mathscr{D}_A \Phi) = -\frac{\sqrt{2}}{2} \overline{\epsilon} \mathscr{D}_A(\Omega_+) \lambda , \tag{7.11a}$$

$$\delta_0 {\Omega_{-M}}^{AB} = -\frac{1}{2} \overline{\epsilon} \Gamma_M \psi^{AB} , \tag{7.11b}$$

$$\delta_0 \hat{H}_{ABC} = \frac{3}{2} \overline{\epsilon} \Gamma_{[A} \psi_{BC]} , \tag{7.11c}$$

$$\delta_0 \psi^{AB} = \frac{1}{4} \Gamma^{CD} \epsilon \, {\hat{R}^{AB}}{}_{CD}(\Omega_-) + \Big\{ \epsilon \big(\overline{\psi}^{AB} \lambda \big) - \psi^{AB} \big(\overline{\epsilon} \lambda \big) + \Gamma^C \lambda \, \big(\overline{\psi}^{AB} \Gamma_C \epsilon \big) \Big\} , \tag{7.11d}$$

$$\delta_0 \hat{\widetilde{F}}_{AB} = -\overline{\epsilon} \Gamma_{[A} \mathscr{D}_{B]}(\Omega_+, \widetilde{A}) \widetilde{\chi} , \tag{7.11e}$$

$$\delta_\beta(\mathscr{D}_A \Phi) = -\frac{\beta}{192} \overline{\epsilon} \Gamma_A \Gamma^{BCD} \lambda \, \mathrm{tr}\big(\overline{\widetilde{\chi}} \Gamma_{BCD} \widetilde{\chi}\big) , \tag{7.11f}$$

$$\delta_\beta \hat{H}_{ABC} = \frac{3\beta}{2} \overline{\epsilon} \Gamma_{[A} \, \mathrm{tr}\big(\widetilde{\chi} \hat{\widetilde{F}}_{BC]} \big) , \tag{7.11g}$$

$$\begin{aligned} \delta_\beta \psi_{AB} = \beta \bigg[& \frac{3}{4} \Gamma^{CD} \epsilon \, \mathrm{tr}\big(\hat{\widetilde{F}}_{[AB} \hat{\widetilde{F}}_{CD]} \big) + \frac{1}{48} \Gamma^{CDE} \Gamma_{[A} \epsilon \, \mathrm{tr}\big\{ \overline{\widetilde{\chi}} \Gamma_{CDE} \mathscr{D}_{B]}(\Omega_+, \widetilde{A}) \widetilde{\chi} \big\} \\ & - \frac{1}{3 \times 256} \Gamma^{CDE} \Gamma_{[A} \Gamma^{GH} \epsilon \, \hat{H}_{B]GH} \, \mathrm{tr}\big(\overline{\widetilde{\chi}} \Gamma_{CDE} \widetilde{\chi} \big) \\ & + \frac{\beta}{96 \times 96} \Gamma^{CDE} \Gamma_{[A} \Gamma^{FGH} \Gamma_{B]} \epsilon \, \mathrm{tr}\big(\overline{\widetilde{\chi}} \Gamma_{CDE} \widetilde{\chi} \big) \, \mathrm{tr}\big(\overline{\widetilde{\chi}} \Gamma_{FGH} \widetilde{\chi} \big) \bigg] , \end{aligned} \tag{7.11h}$$

$$\delta_\beta \hat{\widetilde{F}}_{AB} = \frac{\beta}{192} \overline{\epsilon} \Gamma_{[A} \Gamma^{CDE} \Gamma_{B]} \widetilde{\chi} \, \mathrm{tr}\big(\overline{\widetilde{\chi}} \Gamma_{CDE} \widetilde{\chi} \big) . \tag{7.11i}$$

Finally we describe an identity among generalized curvature tensors:

$$R_{ABCD}(\Omega_-) = R_{CDAB}(\Omega_+) - \frac{1}{2} (\mathrm{d}H)_{CDAB} . \tag{7.12}$$

The explicit expressions of supersymmetry variations are, of course, just approximate expressions. If you consider not only α, β corrections but also the higher order fermions corrections, the corrections of supersymmetry vairations are also corrected. The supercovariant variables such as $\hat{H}_{MNP}$, $\hat{\tilde{F}}_{MN}$ and $\hat{R}^{AB}{}_{MN}(\omega)$ are influenced by the higher order corrections quite sensitively. Thus, you must take care of any calculations when you study the supersymmetry variations and the construction of the Lagrangian of higher order corrections of fermions.

Appendix

A Convention

A.1 Contraction rule on antisymmetric tensors

We introduce the following simplified form:

$$|F_p|^2 \;\equiv\; \frac{1}{p!} g^{M_1 N_1} \cdots g^{M_p N_p} \, F_{M_1 \cdots M_p} F_{N_1 \cdots N_p} \; , \tag{A.1}$$

where $F_{M_1 \cdots M_p}$ is a totally antisymmetric tensor, i.e., the component of a p-form. The coefficient $1/p!$ is adopted to normalize each term appearing in the explicit expansion of $|F_p|^2$ to unity.

A.2 Antisymmetrized symbol

The totally anti-symmetrized symbol is defined in terms of the square bracket:

$$T_{[M_1 M_2 \cdots M_p]} \;=\; \frac{1}{p!} \Big(T_{M_1 M_2 \cdots M_p} - T_{M_2 M_1 \cdots M_p} \pm \text{permutations} \Big) \; . \tag{A.2}$$

This Gamma matrix is defined by

$$\Gamma^{MNP} \;=\; \Gamma^{[M} \Gamma^N \Gamma^{P]} \;=\; \frac{1}{3!} \Big(\Gamma^M \Gamma^N \Gamma^P \pm \text{permutations} \Big) \; . \tag{A.3}$$

A.3 Lorentz algebra

The Lorentz symmetry on the tangent space is important to describe vectors, tensors, and spinors in curved spacetime via vielbeins and inverse vielbeins. Let us now define the Lorentz algebra in Euclidean space with respect to the Lorentz generators Σ_{AB} such as

$$i[\Sigma_{AB}, \Sigma_{CD}] \;=\; \eta_{AC} \, \Sigma_{BD} + \eta_{BD} \, \Sigma_{AC} - \eta_{AD} \, \Sigma_{BC} - \eta_{BC} \, \Sigma_{AD} \; , \tag{A.4a}$$

where eta_{AB} is the metric in the local Lorentz frame whose signature is defined by the signature in the original curved space geometry. To show the signature itself, it is very convenient to use the local Lorentz frame metric η_{AB} in the following way:

$$\eta_{AB} \;=\; \text{diag.}(\underbrace{-,-,\ldots,-}_{t};\underbrace{+,+,\ldots,+}_{s}) \; . \tag{A.4b}$$

Here t and s denote the number of directions with minus (plus) signatures. Mainly we will discuss the cases as $(t,s) = (1, D-1)$ or $(t,s) = (0, D)$. The relation between the metrics in the local Lorentz frame and in the curved spacetime will be given later.

Let us go back to the discussions on the local Lorentz generator. The Lorentz generators acting on scalars, vectors (tensors) and spinors are represented as follows:

$$\begin{cases} \Sigma_{AB} \ = \ 0 & \text{scalar} \\ (\Sigma_{CD})^A{}_B \ = \ i\big(\delta^A_C \, \eta_{BD} - \delta^A_D \, \eta_{BC}\big) & \text{vector} \\ \Sigma_{AB} \ = \ \dfrac{i}{2}\Gamma_{AB} & \text{spinor} \end{cases} \tag{A.4c}$$

where Γ_A is the Dirac gamma matrix which satisfies the Clifford algebra

$$\{\Gamma_A, \Gamma_B\} \ = \ 2\eta_{AB} \ . \tag{A.5}$$

Here we define the chirality operator $\widehat{\Gamma}$ in $d = 2k+2$ dimensional spacetime with Lorentz signature:

$$\widehat{\Gamma} \ \equiv \ i^{-k} \Gamma^0 \Gamma^1 \cdots \Gamma^{d-1} \ , \tag{A.6}$$

where all superscripts are the local Lorentz coordinate indices, since a spinor can be defined in the local Lorentz frame (or the tangent space of the geometry), in which the Dirac gamma matrix is also defined. On the other hand, the chirality operator in $d = 2n$ dimensional space with Euclidean signature is defined as

$$\widehat{\Gamma} \ \equiv \ i^{-n} \Gamma^1 \Gamma^2 \cdots \Gamma^d \ . \tag{A.7}$$

The difference between (A.6) and (A.7) mainly comes from the hermiticity on the gamma matrices: in the Lorentzian spacetime, almost all matrices are hermitian except for Γ^0, which is anti-hermitian (see the definition (A.11)), while all the matrices in the Euclidean space are hermitian.

A.4 Dirac conjugate and charge conjugate on spinors

We define the Dirac conjugate

$$\overline{\psi} \ \equiv \ i \psi^{\dagger} \Gamma^{\hat{0}} \ , \tag{A.8}$$

where $\Gamma^{\hat{0}}$ lives in the tangent space. Furthermore, we assign the Majorana condition such as

$$\overline{\psi} \ \equiv \ \psi^{\mathrm{T}} C \ , \tag{A.9}$$

where C is called the charge conjugate matrix whose generic properties in $D = 2k$ are

$$C^\dagger C \;=\; 1\,, \quad C^{-1} \;=\; C^\dagger \;=\; C^{\rm T} \;=\; (-1)^{k+[k/2]} C\,, \tag{A.10a}$$

$$C(\Gamma^A)C^{-1} \;=\; (-1)^{k+[k/2]}(\Gamma^A)^{\rm T}\,, \quad C(\Gamma^{A_1 \cdots A_n})C^{-1} \;=\; (-)^{[\frac{n+1}{2}]}(\Gamma^{A_1\cdots A_n})^{\rm T}\,, \tag{A.10b}$$

where $[\frac{n+1}{2}] \;=\; \{1,1,2,2,3,3,\cdots\}$ is the Gauss bracket. The hermitian conjugates of gamma matrices are defined by

$$(\Gamma^A)^\dagger \;=\; \Gamma_A \;=\; -\Gamma^{\hat{0}}\Gamma^A(\Gamma^{\hat{0}})^{-1}\,. \tag{A.11}$$

Among the Dirac gamma matrices there exists a useful identity such as

$$\begin{aligned}
&\Gamma^{A_1 A_2 \cdots A_p}\Gamma_{B_1 B_2 \cdots B_q} \\
&\quad = \sum_{k=0}^{\min(p,q)} (-1)^{\frac{1}{2}k(2p-k-1)} \frac{p!q!}{(p-k)!(q-k)!k!} \delta^{[A_1}_{[B_1} \cdots \delta^{A_k}_{B_k} \Gamma^{A_{k+1}\cdots A_p]}{}_{B_{k+1}\cdots B_q]}\,.
\end{aligned} \tag{A.12}$$

A.5 Covariant derivatives and curvature tensors

We introduce vielbeins ${e_M}^A$ and their inverses ${E_A}^M$, which come from the spacetime metric g_{MN} and the metric η_{AB} on orthogonal frame via $g_{MN} = \eta_{AB}\, {e_M}^A\, {e_N}^B$ and $\eta_{AB} = g_{MN}\, {E_A}^M\, {E_B}^N$. By using these geometrical variables, let us define the covariant derivatives $D_M(\omega, \Gamma)$ such as

$$D_M(\Gamma) A_N \;=\; \partial_M A_N - \Gamma^P{}_{NM} A_P\,, \tag{A.13a}$$

$$D_M(\Gamma) A^N \;=\; \partial_M A^N + \Gamma^N{}_{PM} A^P\,, \tag{A.13b}$$

$$D_M(\Gamma) g_{NP} \;\equiv\; 0 \;=\; \partial_M g_{NP} - \Gamma^Q{}_{NM} g_{QP} - \Gamma^Q{}_{PM} g_{NQ}\,, \tag{A.13c}$$

$$D_M(\Gamma) g^{NP} \;\equiv\; 0 \;=\; \partial_M g^{NP} + \Gamma^N{}_{QM} g^{QP} + \Gamma^P{}_{QM} g^{NQ}\,, \tag{A.13d}$$

$$D_M(\omega,\Gamma) {e_N}^A \;\equiv\; 0 \;=\; \partial_M {e_N}^A + {\omega_M}^A{}_B\, {e_N}^B - \Gamma^P{}_{NM}\, {e_P}^A\,, \tag{A.13e}$$

$$D_M(\omega,\Gamma) {E_A}^N \;\equiv\; 0 \;=\; \partial_M {E_A}^N - {E_B}^N\, {\omega_M}^B{}_A + \Gamma^N{}_{PM}\, {E_A}^P\,, \tag{A.13f}$$

$$[D_M(\Gamma), D_N(\Gamma)] A_Q \;=\; -R^P{}_{QMN}(\Gamma) A_P + 2T^P{}_{MN}\, D_P(\Gamma) A_Q\,, \tag{A.13g}$$

$$R^P{}_{QMN}(\Gamma) \;=\; \partial_M \Gamma^P{}_{QN} - \partial_N \Gamma^P{}_{QM} + \Gamma^P{}_{RM}\Gamma^R{}_{QN} - \Gamma^P{}_{RN}\Gamma^R{}_{QM}\,. \tag{A.13h}$$

Note that A_M in the above equations are vector. $\Gamma^P{}_{MN}$ is the affine connection whose two lower indices are not symmetric in general case. The antisymmetric part of the affine connection $\Gamma^P{}_{[MN]}$ is defined as a torsion $T^P{}_{MN}$, while the symmetric part $\Gamma^P{}_{(MN)}$ is given in terms of the Levi-Civita connection $\Gamma^P_{(\rm L)MN}$ and torsion, which we will show from the metricity condition (A.13c). First we prepare the followings:

$$\Gamma^P{}_{MN} \;=\; \Gamma^P{}_{(MN)} + \Gamma^P{}_{[MN]}\,, \quad \Gamma^P{}_{[NM]} \;=\; T^P{}_{NM}\,, \tag{A.14a}$$

$$\Gamma^P_{(\mathrm{L})MN} \ = \ \frac{1}{2} g^{PQ} \big(\partial_M g_{QN} + \partial_N g_{MQ} - \partial_Q g_{MN} \big) \ . \tag{A.14b}$$

Next we investigate the symmetric part $\Gamma^P{}_{(MN)}$. The metricity condition gives

$$\begin{aligned}
0 \ &= \ -D_M(\Gamma) g_{NP} \ = \ -\partial_M g_{NP} + \Gamma^Q{}_{NM} g_{QP} + \Gamma^Q{}_{PM} g_{NQ} \ , && \text{(A.15a)} \\
0 \ &= \ D_N(\Gamma) g_{PM} \ = \ \partial_N g_{PM} - \Gamma^Q{}_{PN} g_{QM} - \Gamma^Q{}_{MN} g_{PQ} \ , && \text{(A.15b)} \\
0 \ &= \ D_P(\Gamma) g_{MN} \ = \ \partial_P g_{MN} - \Gamma^Q{}_{MP} g_{QN} - \Gamma^Q{}_{NP} g_{MQ} \ . && \text{(A.15c)}
\end{aligned}$$

Summing (A.15a), (A.15b) and (A.15c), we obtain

$$\begin{aligned}
0 \ &= \ \Big(\partial_P g_{MN} + \partial_N g_{PM} - \partial_M g_{NP} \Big) - 2 T^Q{}_{MN} g_{PQ} - 2 T^Q{}_{MP} g_{QN} - 2 \Gamma^Q{}_{(PN)} g_{MQ} \ , \\
&\therefore \quad \Gamma^Q{}_{(PN)} \ = \ \Gamma^Q_{(\mathrm{L})PN} - T_P{}^Q{}_N - T_N{}^Q{}_P \ . && \text{(A.16)}
\end{aligned}$$

Then the affine connection is also given in terms of the Levi-Civita connection and the other:

$$\Gamma^P{}_{MN} \ = \ \Gamma^P_{(\mathrm{L})MN} + K^P{}_{MN} \ , \quad K^P{}_{MN} \ \equiv \ T^P{}_{MN} - T_M{}^P{}_N - T_N{}^P{}_M \ . \tag{A.17a}$$

The tensor $K^P{}_{MN}$ is called the contorsion, which has the following property:

$$K_{MNP} \ = \ g_{MQ} K^Q{}_{NP} \ = \ T_{MNP} - T_{NMP} - T_{PMN} \ = \ -K_{NMP} \ . \tag{A.17b}$$

It is worth discussing the Riemann tensor induced from the Levi-Civita connection (A.14b):

$$\begin{aligned}
R^P{}_{QMN}(\Gamma_{(\mathrm{L})}) \ &\equiv \ \partial_M \Gamma^P_{(\mathrm{L})QN} - \partial_N \Gamma^P_{(\mathrm{L})QM} + \Gamma^P_{(\mathrm{L})LM} \Gamma^L_{(\mathrm{L})QN} - \Gamma^P_{(\mathrm{L})LN} \Gamma^L_{(\mathrm{L})QM} \\
&= \ -R^P{}_{QNM}(\Gamma_{(\mathrm{L})} \ . && \text{(A.18)}
\end{aligned}$$

This Riemann tensor has various significant (anti)symmetries under exchanges of indices. Let us carefully analyze them by using $R_{PQMN}(\Gamma_{(\mathrm{L})}) = g_{PR} R^R{}_{QMN}(\Gamma_{(\mathrm{L})})$:

$$\begin{aligned}
R_{PQMN}(\Gamma_{(\mathrm{L})}) \ &= \ g_{PR} \Big(\partial_M \Gamma^R_{(\mathrm{L})QN} + \Gamma^R_{(\mathrm{L})LM} \Gamma^L_{(\mathrm{L})QN} \Big) - \big(M \leftrightarrow N \big) \\
&= \ \frac{1}{2} g_{PR} \partial_M \Big\{ g^{RK} \Big(\partial_Q g_{KN} + \partial_N g_{QK} - \partial_K g_{QN} \Big) \Big\} + g_{PR} \Gamma^R_{(\mathrm{L})LM} \Gamma^L_{(\mathrm{L})QN} - \big(M \leftrightarrow N \big) \\
&= \ \frac{1}{2} \partial_M \Big(\partial_Q g_{PN} + \partial_N g_{QP} - \partial_P g_{QN} \Big) - \frac{1}{2} g^{KL} \partial_M g_{PL} \Big(\partial_Q g_{KN} + \partial_N g_{QK} - \partial_K g_{QN} \Big) \\
&\ \ \ \ + \frac{1}{4} g^{LK} \Big(\partial_L g_{PM} + \partial_M g_{LP} - \partial_P g_{LM} \Big) \Big(\partial_Q g_{KN} + \partial_N g_{QK} - \partial_K g_{QN} \Big) - \big(M \leftrightarrow N \big) \\
&= \ \frac{1}{2} \partial_M \Big(\partial_Q g_{PN} - \partial_P g_{QN} \Big) - \frac{1}{2} g^{KL} \partial_M g_{PL} \Big(\partial_Q g_{KN} + \partial_N g_{QK} - \partial_K g_{QN} \Big) \\
&\ \ \ \ + \frac{1}{4} g^{LK} \Big(\partial_L g_{PM} + \partial_M g_{LP} - \partial_P g_{LM} \Big) \Big(\partial_Q g_{KN} + \partial_N g_{QK} - \partial_K g_{QN} \Big) - \big(M \leftrightarrow N \big) \\
&= \ \frac{1}{2} \partial_M \Big(\partial_Q g_{PN} - \partial_P g_{QN} \Big) - \frac{1}{2} \partial_N \Big(\partial_Q g_{PM} - \partial_P g_{QM} \Big) \\
&\ \ \ \ + \frac{1}{4} g^{KL} \Big(\partial_Q g_{KM} + \partial_M g_{QK} - \partial_K g_{QM} \Big) \Big(\partial_P g_{LN} + \partial_N g_{PL} - \partial_L g_{PN} \Big)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{4}g^{KL}\Big(\partial_Q g_{KN} + \partial_N g_{QK} - \partial_K g_{QN}\Big)\Big(\partial_P g_{LM} + \partial_M g_{PL} - \partial_L g_{PM}\Big) \\
=\ & \frac{1}{2}\partial_M\Big(\partial_Q g_{PN} - \partial_P g_{QN}\Big) - \frac{1}{2}\partial_N\Big(\partial_Q g_{PM} - \partial_P g_{QM}\Big) \\
&+ g_{KL}\Big(\Gamma^K_{(\mathrm{L})QM}\Gamma^L_{(\mathrm{L})PN} - \Gamma^K_{(\mathrm{L})QN}\Gamma^L_{(\mathrm{L})PM}\Big) \ .
\end{aligned} \tag{A.19a}
$$

Thus we can easily find the following remarkable relations:

$$
R_{PQMN}(\Gamma_{(\mathrm{L})}) \ = \ -R_{PQNM}(\Gamma_{(\mathrm{L})}) \ = \ -R_{QPMN}(\Gamma_{(\mathrm{L})}) \ = \ R_{MNPQ}(\Gamma_{(\mathrm{L})}) \ . \tag{A.19b}
$$

Here we also describe the first and second Bianchi identity on this Riemann tensor:

$$
\text{1st:} \quad 0 \ = \ R^M{}_{NPQ}(\Gamma_{(\mathrm{L})}) + R^M{}_{PQN}(\Gamma_{(\mathrm{L})}) + R^M{}_{QNP}(\Gamma_{(\mathrm{L})}) \ , \tag{A.20a}
$$

$$
\text{2nd:} \quad 0 \ = \ \nabla_M R^N{}_{PQR}(\Gamma_{(\mathrm{L})}) + \nabla_Q R^N{}_{PRM}(\Gamma_{(\mathrm{L})}) + \nabla_R R^N{}_{PMQ}(\Gamma_{(\mathrm{L})}) \ . \tag{A.20b}
$$

Nest, let us introduce the covariant derivative induced by the local Lorentz transformation acting on a generic field ϕ^i as

$$
D_M(\omega)\phi^i \ = \ \Big\{\delta^i_j\,\partial_M - \frac{i}{2}\omega_M{}^{AB}\cdot(\Sigma_{AB})^i{}_j\Big\}\phi^j \ , \tag{A.21}
$$

where Σ_{AB} is the Lorentz generator whose explicit form depends on the representation of the field ϕ^i. The curvature tensor associated with this covariant derivative is given in terms of the spin connection

$$
[D_M(\omega), D_N(\omega)]\phi \ = \ -\frac{i}{2}R^{AB}{}_{MN}(\omega)\,\Sigma_{AB}\phi \ , \tag{A.22a}
$$

$$
R^{AB}{}_{MN}(\omega) \ = \ \partial_M\omega_N{}^{AB} - \partial_N\omega_M{}^{AB} + \omega_M{}^A{}_C\,\omega_N{}^{CB} - \omega_N{}^A{}_C\,\omega_M{}^{CB} \ . \tag{A.22b}
$$

These are closely related via the vielbein and its inverse in the following way:

$$
R^R{}_{PMN}(\Gamma) \ = \ \eta_{BC}\,E_A{}^R\,e_P{}^C\,R^{AB}{}_{MN}(\omega) \ , \tag{A.23a}
$$

$$
R^R{}_M(\Gamma) \ = \ g^{PN}R^R{}_{PMN}(\Gamma) \ = \ e_M{}^B\,E_A{}^R\,R^A{}_B(\omega) \ , \quad R^A{}_B(\omega) \ = \ R^{AC}{}_{BC}(\omega) \ , \tag{A.23b}
$$

$$
R(\Gamma) \ = \ R^M{}_M(\Gamma) \ = \ R^A{}_A(\omega) \ = \ R(\omega) \ . \tag{A.23c}
$$

It is useful to give a comment on vielbein. We will meet a vielbein with indices at different positions such as $e^M{}_A$, and so forth. This can be regarded as the inverse $E_A{}^M$:

$$
e^M{}_A \ = \ g^{MN}\delta_{AB}\,e_N{}^B \ , \tag{A.24a}
$$

$$
e_M{}^A\,e^N{}_A \ = \ g^{NP}\delta_{AB}\,e_M{}^A e_P{}^B \ = \ g^{NP}g_{MP} \ = \ \delta^N_M \ = \ e_M{}^A\,E_A{}^N \ , \tag{A.24b}
$$

$$
\therefore \quad e^N{}_A \ = \ E_A{}^N \ . \tag{A.24c}
$$

In the same way, we also prove $E^A{}_M = e_M{}^A$.

A.6 Riemann tensor of Levi-Civita connection

Here we analyze the Riemann tensor of the Levi-Civita connection on a hermitian complex manifold. Notice that the "Levi-Civita" connection $\widetilde{\Gamma}_{(\mathrm{L})}$ indicates the torsionless part of the hermitian connection, whose properties are given in (C.29):

$$\begin{aligned}
\widetilde{R}_{pqmn}(\widetilde{\Gamma}_{(\mathrm{L})}) &= \widetilde{R}_{pqmn}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ -\widetilde{R}_{pqnm}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ -\widetilde{R}_{qpmn}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ \widetilde{R}_{mnpq}(\widetilde{\Gamma}_{(\mathrm{L})}) \\
&= \frac{1}{2}\partial_m \Big[\partial_q \widetilde{g}_{pn} - \partial_p \widetilde{g}_{qn} \Big] - \frac{1}{2}\partial_n \Big[\partial_q \widetilde{g}_{pm} - \partial_p \widetilde{g}_{qm} \Big] + \widetilde{g}_{KL} \Big[\widetilde{\Gamma}^K_{(\mathrm{L})qm} \widetilde{\Gamma}^L_{(\mathrm{L})pn} - \widetilde{\Gamma}^K_{(\mathrm{L})qn} \widetilde{\Gamma}^L_{(\mathrm{L})pm} \Big] \\
&= 0 \, , \qquad \text{(A.25a)} \\
\widetilde{R}_{pqm\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) &= \widetilde{R}_{pqm\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ -\widetilde{R}_{pq\overline{n}m}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ -\widetilde{R}_{qpm\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ \widetilde{R}_{m\overline{n}pq}(\widetilde{\Gamma}_{(\mathrm{L})}) \\
&= \frac{1}{2}\partial_m \Big[\partial_q \widetilde{g}_{p\overline{n}} - \partial_p \widetilde{g}_{q\overline{n}} \Big] - \frac{1}{2}\partial_{\overline{n}} \Big[\partial_q \widetilde{g}_{pm} - \partial_p \widetilde{g}_{qm} \Big] + \widetilde{g}_{KL} \Big[\widetilde{\Gamma}^K_{(\mathrm{L})qm} \widetilde{\Gamma}^L_{(\mathrm{L})p\overline{n}} - \widetilde{\Gamma}^K_{(\mathrm{L})q\overline{n}} \widetilde{\Gamma}^L_{(\mathrm{L})pm} \Big] \\
&= \frac{1}{2}\partial_m \Big[\partial_q \widetilde{g}_{p\overline{n}} - \partial_p \widetilde{g}_{q\overline{n}} \Big] + \widetilde{g}_{k\overline{l}} \Big[\widetilde{\Gamma}^k_{(\mathrm{L})qm} \widetilde{\Gamma}^{\overline{l}}_{(\mathrm{L})p\overline{n}} - \widetilde{\Gamma}^k_{(\mathrm{L})pm} \widetilde{\Gamma}^{\overline{l}}_{(\mathrm{L})q\overline{n}} \Big] \, , \qquad \text{(A.25b)} \\
\widetilde{R}_{pq\overline{m}\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) &= \widetilde{R}_{pq\overline{m}\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ -\widetilde{R}_{pq\overline{n}\overline{m}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ -\widetilde{R}_{qp\overline{m}\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ \widetilde{R}_{\overline{m}\overline{n}pq}(\widetilde{\Gamma}_{(\mathrm{L})}) \\
&= \frac{1}{2}\partial_{\overline{m}} \Big[\partial_q \widetilde{g}_{p\overline{n}} - \partial_p \widetilde{g}_{q\overline{n}} \Big] - \frac{1}{2}\partial_{\overline{n}} \Big[\partial_q \widetilde{g}_{p\overline{m}} - \partial_p \widetilde{g}_{q\overline{m}} \Big] \\
&\quad + \widetilde{g}_{KL} \Big[\widetilde{\Gamma}^K_{(\mathrm{L})q\overline{m}} \widetilde{\Gamma}^L_{(\mathrm{L})p\overline{n}} - \widetilde{\Gamma}^K_{(\mathrm{L})q\overline{n}} \widetilde{\Gamma}^L_{(\mathrm{L})p\overline{m}} \Big] \, , \qquad \text{(A.25c)} \\
\widetilde{R}_{p\overline{q}m\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) &= \widetilde{R}_{p\overline{q}m\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ -\widetilde{R}_{p\overline{q}\overline{n}m}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ -\widetilde{R}_{\overline{q}pm\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ \widetilde{R}_{m\overline{n}p\overline{q}}(\widetilde{\Gamma}_{(\mathrm{L})}) \\
&= \frac{1}{2}\partial_m \partial_{\overline{q}} \widetilde{g}_{p\overline{n}} + \frac{1}{2}\partial_{\overline{n}} \partial_p \widetilde{g}_{\overline{q}m} + \widetilde{g}_{KL} \Big[\widetilde{\Gamma}^K_{(\mathrm{L})\overline{q}m} \widetilde{\Gamma}^L_{(\mathrm{L})p\overline{n}} - \widetilde{\Gamma}^K_{(\mathrm{L})\overline{q}\overline{n}} \widetilde{\Gamma}^L_{(\mathrm{L})pm} \Big] \, , \qquad \text{(A.25d)} \\
\widetilde{R}_{p\overline{q}\overline{m}\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) &= \widetilde{R}_{p\overline{q}\overline{m}\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ -\widetilde{R}_{p\overline{q}\overline{n}\overline{m}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ -\widetilde{R}_{\overline{q}p\overline{m}\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ \widetilde{R}_{\overline{m}\overline{n}p\overline{q}}(\widetilde{\Gamma}_{(\mathrm{L})}) \\
&= \frac{1}{2}\partial_{\overline{m}} \partial_{\overline{q}} \widetilde{g}_{p\overline{n}} - \frac{1}{2}\partial_{\overline{n}} \partial_{\overline{q}} \widetilde{g}_{p\overline{m}} + \widetilde{g}_{\overline{k}l} \Big[\widetilde{\Gamma}^{\overline{k}}_{(\mathrm{L})\overline{q}\overline{m}} \widetilde{\Gamma}^l_{(\mathrm{L})p\overline{n}} - \widetilde{\Gamma}^{\overline{k}}_{(\mathrm{L})\overline{q}\overline{n}} \widetilde{\Gamma}^l_{(\mathrm{L})p\overline{m}} \Big] \, , \qquad \text{(A.25e)} \\
\widetilde{R}_{\overline{p}\overline{q}\overline{m}\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) &= \widetilde{R}_{\overline{p}\overline{q}\overline{m}\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ -\widetilde{R}_{\overline{p}\overline{q}\overline{n}\overline{m}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ -\widetilde{R}_{\overline{q}\overline{p}\overline{m}\overline{n}}(\widetilde{\Gamma}_{(\mathrm{L})}) \ = \ \widetilde{R}_{\overline{m}\overline{n}\overline{p}\overline{q}}(\widetilde{\Gamma}_{(\mathrm{L})}) \\
&= \frac{1}{2}\partial_{\overline{m}} \Big[\partial_{\overline{q}} \widetilde{g}_{\overline{p}\overline{n}} - \partial_{\overline{p}} \widetilde{g}_{\overline{q}\overline{n}} \Big] - \frac{1}{2}\partial_{\overline{n}} \Big[\partial_{\overline{q}} \widetilde{g}_{\overline{p}\overline{m}} - \partial_{\overline{p}} \widetilde{g}_{\overline{q}\overline{m}} \Big] + \widetilde{g}_{KL} \Big[\widetilde{\Gamma}^K_{(\mathrm{L})\overline{q}\overline{m}} \widetilde{\Gamma}^L_{(\mathrm{L})\overline{p}\overline{n}} - \widetilde{\Gamma}^K_{(\mathrm{L})\overline{q}\overline{n}} \widetilde{\Gamma}^L_{(\mathrm{L})\overline{p}\overline{m}} \Big] \\
&= 0 \, . \qquad \text{(A.25f)}
\end{aligned}$$

Here we used the expression (A.19a).

A.7 Contorsion

The curved space metric g_{MN} has a back-reaction from the (con)torsion if it exists on the manifold. However, the metricity condition itself is free from this back-reaction. Suppose g_{MN} and $\hat{g}_{MN}$ be the metrics on the manifolds with and without torsion, respectively. These satisfy the following metricity conditions individually:

$$0 \;=\; \nabla_M g_{NP} \;=\; \partial_M g_{NP} - \Gamma^Q_{(\mathrm{L})NM} g_{QP} - \Gamma^Q_{(\mathrm{L})PM} g_{NQ} \, , \tag{A.26a}$$

$$\begin{aligned} 0 \;&=\; D_M(\Gamma) \hat{g}_{NP} \;=\; \partial_M \hat{g}_{NP} - \Gamma^Q{}_{NM} \hat{g}_{QP} - \Gamma^Q{}_{PM} \hat{g}_{NQ} \\ &=\; \partial_M \hat{g}_{NP} - \Big(\hat{\Gamma}^Q_{(\mathrm{L})NM} + K^Q{}_{NM} \Big) \hat{g}_{QP} - \Big(\hat{\Gamma}^Q_{(\mathrm{L})PM} + K^Q{}_{PM} \Big) \hat{g}_{NQ} \\ &=\; \hat{\nabla}_M \hat{g}_{NP} - K_{PNM} - K_{NPM} \, . \end{aligned} \tag{A.26b}$$

Note that $\hat{\Gamma}^Q_{(\mathrm{L})NM}$ and $\hat{\nabla}_M$ are given in terms of the the metric $\hat{g}_{MN}$. Since the contorsion is antisymmetric $K_{PNM} = -K_{NPM}$, we find that the metricity condition even in the presence of torsion has the same form as the one in the absence of torsion:

$$0 \;=\; \hat{\nabla}_M \hat{g}_{NP} \, . \tag{A.26c}$$

Then we need not worry about the existence of torsion when we use the metricity condition (A.26).

The above discussion is quite important when we decompose the spin connection into the contorsion part and the other. We start from the vielbein postulate $0 = D_M(\omega, \Gamma) e_N{}^A$ and obtain

$$\begin{aligned} \omega_{MAB} \;&=\; -E_B{}^N \partial_M e_{NA} + \Gamma^P{}_{NM} \, e_{PA} E_B{}^N \\ &=\; -E_B{}^N \partial_M e_{NA} + \Big(\Gamma^P_{(\mathrm{L})NM} + K^P{}_{NM} \Big) e_{PA} E_B{}^N \\ &\equiv\; \omega^{(\mathrm{L})}_{MAB} + K_{ABM} \, , \end{aligned} \tag{A.27}$$

where we defined the spin connection $\omega^{(\mathrm{L})}_{MAB}$ given by the Levi-Civita connection $\Gamma_{(\mathrm{L})}$. The second term in the right-hand side is given by the contorsion tensor, which is, by definition (A.17), antisymmetric under the exchange of the former two indices $K_{ABM} = -K_{BAM}$. Now it is useful to check the antisymmetry of the Levi-Civita spin connection:

$$\begin{aligned} \omega^{(\mathrm{L})}_{MBA} \;&=\; -E_A{}^N \partial_M e_{NB} + \Gamma^P_{(\mathrm{L})NM} \, e_{PB} E_A{}^N \;=\; -E_A{}^N \partial_M \Big(g_{NQ} E_B{}^Q \Big) + \Gamma^P_{(\mathrm{L})NM} \, e_{PB} E_A{}^N \\ &=\; -E_A{}^N E_B{}^Q \partial_M g_{NQ} - E_{AQ} \partial_M E_B{}^Q + \Gamma^P_{(\mathrm{L})NM} \, e_{PB} E_A{}^N \\ &=\; -E_A{}^N E_B{}^Q \Big(\Gamma^P_{(\mathrm{L})NM} g_{PQ} + \Gamma^P_{(\mathrm{L})QM} g_{NP} \Big) - E_{AQ} \partial_M E_B{}^Q + \Gamma^P_{(\mathrm{L})NM} \, e_{PB} E_A{}^N \\ &=\; E_B{}^Q \partial_M e_{QA} - \Gamma^P_{(\mathrm{L})QM} \, e_{PA} E_B{}^Q \\ &=\; -\omega^{(\mathrm{L})}_{MAB} \, , \end{aligned} \tag{A.28}$$

where we used (A.24) and the metricity condition (A.26). Substituting (A.17) and (A.28) into (A.27), we confirm that the spin connection with torsion is also antisymmetric:

$$\omega_{MAB} \;=\; -\omega_{MBA} \, . \tag{A.29}$$

A.8 Bismut torsion

Suppose the complex structure $J_M{}^N$ is covariantly constant with respect to the connection $\Gamma_- = \Gamma_{(\mathrm{L})} - H$ with contorsion $K = -H$:

$$0 \;=\; D_M(\Gamma_-) J_N{}^P \;=\; \nabla_M J_N{}^P + H^R{}_{NM} J_R{}^P - H^P{}_{RM} J_N{}^R \; . \tag{A.30}$$

where we assigned $D_M(\Gamma_{(\mathrm{L})}) = \nabla_M$. By using this we express the Nijenhuis tensor such as

$$\mathcal{N}_{MNP} \;\equiv\; J_M{}^Q \nabla_{[Q} J_{N]P} - J_N{}^Q \nabla_{[Q} J_{M]P} \;=\; \Big(H_{MNP} - 3 J_{[M}{}^Q J_N{}^R H_{P]QR} \Big) \; ,$$

which tells us

$$H_{MNP} \;=\; \mathcal{N}_{MNP} + 3 J_{[M}{}^Q J_N{}^R H_{P]QR} \; .$$

We also show an identitiy in terms of (A.30):

$$J_{[M}{}^Q \nabla_{|Q|} J_{NP]} \;=\; -2 J_{[M}{}^Q J_N{}^R H_{P]QR} \; .$$

This is nothing but the Bismut torsion defined by

$$T^{(\mathrm{B})}_{MNP} \;=\; \frac{3}{2} J_M{}^Q J_N{}^R J_P{}^S \nabla_{[Q} J_{RS]} \;=\; -\frac{3}{2} J_{[M}{}^Q \nabla_{|Q|} J_{NP]} \; .$$

Summarizing the above facts, we find

$$H_{MNP} \;=\; \mathcal{N}_{MNP} + T^{(\mathrm{B})}_{MNP} \; . \tag{A.31}$$

which denotes that the contorsion in the affine connection corresponds to the Bismut torsion $T^{(\mathrm{B})}$ if the geometry is complex ($\mathcal{N}_{MNP} = 0$):

$$H_{MNP} \;=\; T^{(\mathrm{B})}_{MNP} \;=\; \frac{3}{2} J_M{}^Q J_N{}^R J_P{}^S \nabla_{[Q} J_{RS]} \;=\; -\frac{3}{2} J_{[M}{}^Q \nabla_{|Q|} J_{NP]} \; . \tag{A.32}$$

Especially, in the heterotic superstring theory, the NS-NS three-form flux H appears in the Bismut connection itself [11]. In this scenario we naturally choose $a = -1$.

Here we show the explicit computation with using $J_M{}^N J_N{}^P = -\delta_M^P$:

$$\begin{aligned}
H_{MNP} \;&=\; \frac{3}{2} J_M{}^Q J_N{}^R J_P{}^S \nabla_{[Q} J_{RS]} \; , \\
\frac{1}{2} J_M{}^Q J_N{}^R J_P{}^S \nabla_Q J_{RS} \;&=\; \frac{1}{2} \nabla_Q \Big(J_M{}^Q J_N{}^R J_P{}^S J_{RS} \Big) - \frac{1}{2} J_{RS} \nabla_Q \Big(J_M{}^Q J_N{}^R J_P{}^S \Big) \\
&=\; \frac{1}{2} \nabla_Q \big(J_M{}^Q J_{NP} \big) \\
&\quad\; - \frac{1}{2} J_{RS} \Big\{ (\nabla_Q J_M{}^Q) J_N{}^R J_P{}^S + (\nabla_Q J_N{}^R) J_M{}^Q J_P{}^S + (\nabla_Q J_P{}^S) J_M{}^Q J_N{}^R \Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}J_M{}^Q(\nabla_Q J_{NP}) - \frac{1}{2}(\nabla_Q J_{NP})J_M{}^Q + \frac{1}{2}J_M{}^Q\nabla_Q J_{PN} \\
&= -\frac{1}{2}J_M{}^Q(\nabla_Q J_{NP}) \, , && \text{(A.33a)} \\
\therefore \quad H_{MNP} &= -\frac{1}{2}J_M{}^Q\nabla_Q J_{NP} - \frac{1}{2}J_N{}^Q\nabla_Q J_{PM} - \frac{1}{2}J_P{}^Q\nabla_Q J_{MN} \\
&= -\frac{3}{2}J_{[M}{}^Q\nabla_{|Q|}J_{NP]} \, . && \text{(A.33b)}
\end{aligned}
$$

Now let us rewrite this in terms of the (complex) differential forms. Due to the above analysis we have already understood that the NS-flux (or Bismut torsion) is the sum of $(2,1)$-form and $(1,2)$-form with respect to the complex structure $J_M{}^N$. Then

$$
\begin{aligned}
H_{MNP} &= \frac{3}{2}J_M{}^Q J_N{}^R J_P{}^S \nabla_{[Q}J_{RS]} \;=\; \frac{3}{2}J_M{}^Q J_N{}^R J_P{}^S \partial_{[Q}J_{RS]} \, , && \text{(A.34a)} \\
H &= \frac{1}{3!}H_{MNP}\, dx^M \wedge dx^N \wedge dx^P \;=\; \frac{1}{4}J_M{}^Q J_N{}^R J_P{}^S \partial_{[Q}J_{RS]}\, dx^M \wedge dx^N \wedge dx^P \\
&= \frac{1}{4}J_m{}^Q J_n{}^R J_{\overline{p}}{}^S \partial_Q J_{RS}\, dz^m \wedge dz^n \wedge d\overline{z}^{\overline{p}} + \frac{1}{4}J_m{}^Q J_{\overline{n}}{}^R J_p{}^S \partial_Q J_{RS}\, dz^m \wedge d\overline{z}^{\overline{n}} \wedge dz^p \\
&\quad + \frac{1}{4}J_{\overline{m}}{}^Q J_n{}^R J_p{}^S \partial_Q J_{RS}\, d\overline{z}^{\overline{m}} \wedge dz^n \wedge dz^p + \frac{1}{4}J_m{}^Q J_{\overline{n}}{}^R J_{\overline{p}}{}^S \partial_Q J_{RS}\, dz^m \wedge d\overline{z}^{\overline{n}} \wedge d\overline{z}^{\overline{p}} \\
&\quad + \frac{1}{4}J_{\overline{m}}{}^Q J_n{}^R J_{\overline{p}}{}^S \partial_Q J_{RS}\, d\overline{z}^{\overline{m}} \wedge dz^n \wedge d\overline{z}^{\overline{p}} + \frac{1}{4}J_{\overline{m}}{}^Q J_{\overline{n}}{}^R J_p{}^S \partial_Q J_{RS}\, d\overline{z}^{\overline{m}} \wedge d\overline{z}^{\overline{n}} \wedge dz^p \\
&= \frac{1}{2}J_m{}^Q J_n{}^R J_{\overline{p}}{}^S \partial_Q J_{RS}\, dz^m \wedge dz^n \wedge d\overline{z}^{\overline{p}} + \frac{1}{2}J_{\overline{m}}{}^Q J_n{}^R J_{\overline{p}}{}^S \partial_Q J_{RS}\, d\overline{z}^{\overline{m}} \wedge dz^n \wedge d\overline{z}^{\overline{p}} \\
&= \frac{1}{2}i^2(-i)\partial_m J_{n\overline{p}}\, dz^m \wedge dz^n \wedge d\overline{z}^{p} + \frac{1}{2}(-i)^2 i\partial_{\overline{m}} J_{n\overline{p}}\, d\overline{z}^{\overline{m}} \wedge dz^n \wedge d\overline{z}^{\overline{p}} \\
&= \frac{i}{2}\big(\partial - \overline{\partial}\big)J \, , && \text{(A.34b)}
\end{aligned}
$$

Notice that the component of the complex structure is given by $J_m{}^n = i\delta_m^n$, $J_{\overline{m}}{}^{\overline{n}} = -i\delta_{\overline{m}}^{\overline{n}}$ and $J_{mn} = J_{\overline{mn}} = 0$.

A.9 Differential forms

We define differential forms on D-dimensional geometry ($g_D = \det g_{mn}$). For realistic discussions, we will define them in the curved spacetime with signature $(t,s) = (1, D-1)$ or $(0, D)$, i.e., we will introduce a parameter $t = 0,1$ in the following definition which shows whether the spacetime is Lorentzian ($t=1$) or Euclidean ($t=0$).

$$
\begin{aligned}
\omega_p &\equiv \frac{1}{p!}\omega_{M_1\cdots M_p}\, dx^{M_1} \wedge \cdots \wedge dx^{M_p} \, , && \text{(A.35a)} \\
(\text{vol.}) &\equiv \sqrt{|g_D|}\, dx^1 \wedge \cdots \wedge dx^D \, . && \text{(A.35b)}
\end{aligned}
$$

It is also necessary to introduce a dual form of the p-form via so-called the Hodge dual:

$$
*\omega_p = \frac{\sqrt{|g_D|}}{p!(D-p)!}\,\omega_{M_1\cdots M_p}\,\varepsilon^{M_1\cdots M_p}{}_{N_{p+1}\cdots N_D}\, dx^{N_{p+1}} \wedge \cdots \wedge dx^{N_D} \, , \qquad \text{(A.36a)}
$$

$$(*1) \;=\; \frac{\sqrt{|g_D|}}{D!}\,\varepsilon_{M_1\cdots M_D}\,\mathrm{d}x^{M_1}\wedge\cdots\mathrm{d}x^{M_D} \;=\; \sqrt{|g_D|}\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^D \;=\; (\text{vol.})\;, \tag{A.36b}$$

where the index "1" does not always implies the first spatial direction; i.e., $\mathrm{d}x^1$ also often implies $\mathrm{d}t$ in the negative signature. Notice that $\varepsilon_{M_1\cdots M_D}$ and $\varepsilon^{M_1\cdots M_D}$ are called the invariant tensors whose property is given by

$$\varepsilon^{M_1\cdots M_n}{}_{M_{n+1}\cdots M_D} \;=\; g^{M_1N_1}\cdots g^{M_nN_n}\,\varepsilon_{N_1\ldots N_nM_{n+1}\cdots M_D}\;, \tag{A.37a}$$

$$\varepsilon^{M_1M_2\cdots M_D} \;=\; g^{M_1N_1}\cdots g^{M_DN_D}\,\varepsilon_{N_1N_2\cdots N_D} \;=\; g_D^{-1}\varepsilon_{N_1N_2\cdots N_D}\;, \tag{A.37b}$$

$$\varepsilon_{12\cdots D} \;\equiv\; 1\;, \quad \varepsilon^{12\cdots D} \;=\; \frac{1}{g_D}\;, \tag{A.37c}$$

$$T_{M_1\cdots M_D}\varepsilon^{M_1\cdots M_D} \;=\; T_{M_1\cdots M_D}g^{M_1N_1}\cdots g^{M_DN_D}\varepsilon_{N_1\cdots N_D} \;=\; T^{N_1\cdots N_D}\,\varepsilon_{N_1\cdots N_D}\;. \tag{A.37d}$$

The final line is from the definition that $\varepsilon_{M_1\ldots M_D}$ is a **tensor**. Using the Hodge star operator and the invariant tensor, we can discuss more properties:

$$**\,\omega_p \;=\; (-1)^{p(D-p)+t}\,\omega_p\;, \tag{A.38a}$$

$$\mathrm{d}x^{M_1}\wedge\cdots\wedge\mathrm{d}x^{M_D} \;=\; g_D\,\varepsilon^{M_1\cdots M_D}\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^D\;, \tag{A.38b}$$

$$\mathrm{d}^Dx \;\equiv\; \mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^D \;=\; \frac{1}{D!}\,\varepsilon_{M_1\cdots M_D}\,\mathrm{d}x^{M_1}\wedge\cdots\wedge\mathrm{d}x^{M_D}\;, \tag{A.38c}$$

$$g_D\,\varepsilon^{M_1\cdots M_p}{}_{N_{p+1}\cdots N_D}\cdot\varepsilon_{M_1\cdots M_p}{}^{L_{p+1}\cdots L_D} \;=\; p!(D-p)!\cdot\delta^{L_{p+1}}_{[N_{p+1}}\cdots\delta^{L_D}_{N_D]}\;. \tag{A.38d}$$

We also introduce an inveriant tensor $\mathcal{E}_{A_1\cdots A_D}$ in the local Lorentz (or the frame coordinate) system. Introducing the vielbein one-form $e^A = e_M{}^A\,\mathrm{d}x^M$, we write down in such a way as

$$\mathcal{E}^{A_1A_2\ldots A_D} \;=\; \eta^{A_1B_1}\eta^{A_2B_2}\cdots\eta^{A_DB_D}\,\mathcal{E}_{B_1B_2\cdots B_D} \;=\; \eta_D^{-1}\mathcal{E}_{B_1B_2\cdots B_D}\;, \tag{A.39a}$$

$$T^{M_1\cdots M_D}\,\varepsilon_{M_1\cdots M_D} \;=\; T^{M_1\cdots M_D}\,e_{M_1}{}^{A_1}\cdots e_{M_D}{}^{A_D}\,\mathcal{E}_{A_1\cdots A_D} \;=\; T^{A_1\cdots A_D}\,\mathcal{E}_{A_1\cdots A_D}\;, \tag{A.39b}$$

$$T_{M_1\cdots M_D}\,\varepsilon^{M_1\cdots M_D} \;=\; T_{M_1\cdots M_D}\,E_{A_1}{}^{M_1}\cdots E_{A_D}{}^{M_D}\,\mathcal{E}^{A_1\cdots A_D} \;=\; T_{A_1\cdots A_D}\,\mathcal{E}^{A_1\cdots A_D}\;, \tag{A.39c}$$

$$\mathcal{E}_{12\cdots D} \;=\; 1\;, \quad \mathcal{E}^{12\cdots D} \;=\; \frac{1}{\eta_D} \;=\; (-1)^t\;, \tag{A.39d}$$

where $\eta_D \equiv \det\eta_{AB} = (-1)^t$, with the number of minus sign in the signature. Furthermore, using $\sqrt{|\eta_D|} = 1$, we also define the followings:

$$(\text{vol.}) \;=\; e^1\wedge\cdots\wedge e^D\;, \tag{A.40a}$$

$$e^{A_1}\wedge\cdots\wedge e^{A_D} \;=\; \eta_D\,\mathcal{E}^{A_1\cdots A_D}\,e^1\wedge\cdots\wedge e^D\;, \tag{A.40b}$$

$$e^1\wedge\cdots\wedge e^D \;=\; \frac{1}{D!}\,\mathcal{E}_{A_1A_2\cdots A_D}\,e^{A_1}\wedge\cdots\wedge e^{A_D}\;, \tag{A.40c}$$

$$\eta_D\,\mathcal{E}^{A_1\cdots A_p}{}_{B_{p+1}\cdots B_D}\cdot\mathcal{E}_{A_1\cdots A_p}{}^{C_{p+1}\cdots C_D} \;=\; p!(D-p)!\cdot\delta^{C_{p+1}}_{[B_{p+1}}\cdots\delta^{C_D}_{B_D]}\;. \tag{A.40d}$$

A.10 Yang-Mills gauge fields: hermitian variables

The covariant derivatives with respect to the Yang-Mills transformation on the field ϕ^i in the fundamental representation, and on the field φ^a in the adjoint representation, are also defined as

$$D_M(A)\phi^i = \partial_M \phi^i - i(A_M)^i{}_j \, \phi^j \qquad \text{fundamental representation ,} \tag{A.41a}$$

$$\left.\begin{aligned} D_M(A)\chi &= \partial_M \chi - i[A_M, \chi] \\ \{D_M(A)\chi\}^a &= \partial_M \chi^a + f^a{}_{bc} A_M^b \, \chi^c \end{aligned}\right\} \qquad \text{adjoint representation .} \tag{A.41b}$$

The field strength (i.e., the curvature) is defined as

$$[D_M(A), D_M(A)]\phi = -iF_{MN}\,\phi \,, \qquad F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N] \,, \tag{A.41c}$$

where the gauge fields A_M and the field strength F_{MN} are described in terms of the gauge symmetry generators T_a such as

$$A_M \equiv A_M^a T^a \,, \quad \text{and} \quad F_{MN} = F_{MN}^a T^a \,, \tag{A.42a}$$

where T^a is a hermitian generator of the group $(T^a)^\dagger = T^a$, which also satisfies

$$\mathrm{tr}(T^a T^b) = \delta^{ab} \,, \quad [T^a, T^b] = if^{ab}{}_c T^c \,, \quad (T_a)_b{}^c = if_{ba}{}^c = [\mathrm{ad}(T_a)]_b{}^c \,, \tag{A.42b}$$

$$[T_a,[T_b,T_c]] + [T_b,[T_c,T_a]] + [T_c,[T_a,T_b]] = 0 = f_{bcd}f_{ade} + f_{cad}f_{bde} + f_{abd}f_{cde} \,, \tag{A.42c}$$

$$F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + f^a{}_{bc} A_M^b A_N^c \,. \tag{A.42d}$$

Due to the above commutation relation we set the structure constant $f^a{}_{bc}$ to be real.

Comment that the trace symbol "tr" in the above definition is in the fundamental (vector) representation. The exchanging rule between the trace tr in the $SO(n)$ vector and the trace Tr in the $SO(n)$ adjoint representations is given by

$$\mathrm{Tr}(T^2) = (n-2)\,\mathrm{tr}(T^2) \,, \tag{A.43a}$$

$$\mathrm{Tr}(T^4) = (n-8)\,\mathrm{tr}(T^4) + 3\,\mathrm{tr}(T^2)\,\mathrm{tr}(T^2) \,, \tag{A.43b}$$

$$\mathrm{Tr}(T^6) = (n-32)\,\mathrm{tr}(T^6) + 15\,\mathrm{tr}(T^2)\,\mathrm{tr}(T^4) \,, \tag{A.43c}$$

where T is any linear combination of generators, but this implies the same relations for symmetrized products of different generators.

A.11 Yang-Mills gauge fields: anti-hermitian variables

Here we discuss another definition of the Yang-Mills fields in terms of the "anti-hermitian" generators $\widetilde{T}^a$. The algebra is defined as

$$(\widetilde{T}^a)^\dagger \ = \ -\widetilde{T}^a \ , \quad \mathrm{tr}(\widetilde{T}^a \widetilde{T}^b) \ = \ -\delta^{ab} \ , \tag{A.44a}$$

$$[\widetilde{T}^a, \widetilde{T}^b] \ = \ f^{ab}{}_c \widetilde{T}^c \ , \quad (\widetilde{T}_a)_b{}^c \ = \ f_{ba}{}^c \ = \ [\mathrm{ad}(\widetilde{T}_a)]_b{}^c \ , \tag{A.44b}$$

$$[\widetilde{T}_a, [\widetilde{T}_b, \widetilde{T}_c]] + [\widetilde{T}_b, [\widetilde{T}_c, \widetilde{T}_a]] + [\widetilde{T}_c, [\widetilde{T}_a, \widetilde{T}_b]] \ = \ 0 \ = \ -i f_{bcd} f_{ade} - i f_{cad} f_{bde} - i f_{abd} f_{cde} \ . \tag{A.44c}$$

Note that the structure constant $f^a{}_{bc}$ to be real (and to be same as the one in the previous subsection). The relation between $\widetilde{T}^a$ and the generators T^a is

$$T^a \ = \ i \widetilde{T}^a \ . \tag{A.45}$$

By using this anti-hermitian generators $\widetilde{T}^a$ we re-define the gauge fields

$$\widetilde{A}_M \ \equiv \ A_M^a \widetilde{T}^a \quad \text{with } i\widetilde{A}_M \ = \ A_M \ , \quad \widetilde{F}_{MN} \ \equiv \ F_{MN}^a \widetilde{T}^a \quad \text{with } i\widetilde{F}_{MN} \ = \ F_{MN} \ . \tag{A.46a}$$

Here we also described the relations between $\widetilde{A}$ and A, which is the hermitian gauge fields defined in the previous subsection. Then the covariant derivatives with respect to the Yang-Mills transformation on the field ϕ^i in the fundamental representation, and on the field $\widetilde{\varphi} = \varphi^a \widetilde{T}^a$ in the adjoint representation, are also defined as

$$D_M(\widetilde{A})\phi^i \ = \ \partial_M \phi^i + (\widetilde{A}_M)^i{}_j \, \phi^j \qquad \text{fundamental representation ,} \tag{A.47a}$$

$$\left.\begin{aligned} D_M(\widetilde{A})\widetilde{\chi} \ &= \ \partial_M \widetilde{\chi} + [\widetilde{A}_M, \widetilde{\chi}] \\ \{D_M(\widetilde{A})\chi\}^a \ &= \ \partial_M \chi^a + f^a{}_{bc} A_M^b \, \chi^c \end{aligned}\right\} \qquad \text{adjoint representation .} \tag{A.47b}$$

The field strength (i.e., the curvature) is defined as

$$[D_M(\widetilde{A}), D_M(\widetilde{A})]\phi \ = \ \widetilde{F}_{MN} \, \phi \ , \quad \widetilde{F}_{MN} \ = \ \partial_M \widetilde{A}_N - \partial_N \widetilde{A}_M + [\widetilde{A}_M, \widetilde{A}_N] \ , \tag{A.47c}$$

where we should notice that the "component fields" A_M^a and F_{MN}^a are the real fields and they corresponds to the ones in the previous subsection, i.e.,

$$F_{MN}^a \ = \ \partial_M A_N^a - \partial_N A_N^a + f^a{}_{bc} \, A_M^b A_N^c \ . \tag{A.47d}$$

A.12 Chern-Simons three-forms

Here let us introduce two kinds of the Chern-Simons three-forms, i.e., the Lorentz-Chern-Simons three-form ω_3^{L} and the Yang-Mills-Chern-Simons three-form ω_3^{Y}:

$$\omega_3^{\mathrm{L}} \ = \ \frac{1}{3!} \omega_{MNP}^{\mathrm{L}} \, \mathrm{d}x^M \wedge \mathrm{d}x^N \wedge \mathrm{d}x^P \ \equiv \ \Big(\omega^A{}_B \wedge \mathrm{d}\omega^B{}_A + \frac{2}{3} \omega^A{}_B \wedge \omega^B{}_C \wedge \omega^C{}_A \Big) \ , \tag{A.48a}$$

$$\omega_3^{\mathrm{Y}} \;=\; \frac{1}{3!}\omega^{\mathrm{Y}}_{MNP}\,\mathrm{d}x^M \wedge \mathrm{d}x^N \wedge \mathrm{d}x^P \;\equiv\; \mathrm{tr}\Big(A \wedge \mathrm{d}A - \frac{2i}{3} A \wedge A \wedge A\Big) \; , \tag{A.48b}$$

$$\frac{1}{3!}\omega^{\mathrm{L}}_{MNP} \;=\; \Big(\omega_{[M}{}^{AB}\partial_N \omega_{P]}{}^{BA} + \frac{2}{3}\omega_{[M}{}^{AB}\omega_N{}^{BC}\omega_{P]}{}^{CA}\Big) \; , \tag{A.48c}$$

$$\frac{1}{3!}\omega^{\mathrm{Y}}_{MNP} \;=\; \mathrm{tr}\Big(A_{[M}\partial_N A_{P]} - \frac{2i}{3} A_{[M}A_N A_{P]}\Big) \; , \tag{A.48d}$$

where $\omega^A{}_B = \omega_M{}^A{}_B \mathrm{d}x^M$ and $A = A^a_M T^a \mathrm{d}x^M$ are ths spin connection and the gauge fields which satisfy the followings

$$R^A{}_B \;=\; \mathrm{d}\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B \; , \quad F \;=\; \mathrm{d}A - iA \wedge A \; . \tag{A.49}$$

Of course the Yang-Mills Chern-Simons 3-form $\widetilde{\omega}_3^{\mathrm{Y}}$ with respect to the anti-hermitian generators $\widetilde{T}^a$ can be defined as

$$\widetilde{\omega}_3^{\mathrm{Y}} \;\equiv\; \mathrm{tr}\Big(\widetilde{A} \wedge \mathrm{d}\widetilde{A} + \frac{2}{3}\widetilde{A} \wedge \widetilde{A} \wedge \widetilde{A}\Big) \;=\; -\omega_3^{\mathrm{Y}} \quad \text{with } \widetilde{F} \;=\; \mathrm{d}\widetilde{A} + \widetilde{A} \wedge \widetilde{A} \; . \tag{A.50}$$

The exterior derivatives of these three-forms are given by

$$\mathrm{d}\omega_3^{\mathrm{L}} \;=\; R^A{}_B \wedge R^B{}_A \;=\; \mathrm{tr}(R \wedge R) \; , \tag{A.51a}$$

$$\mathrm{d}\omega_3^{\mathrm{Y}} \;=\; \mathrm{tr}(F \wedge F) \; , \quad \mathrm{d}\widetilde{\omega}_3^{\mathrm{Y}} \;=\; \mathrm{tr}(\widetilde{F} \wedge \widetilde{F}) \;=\; -\mathrm{tr}(F \wedge F) \;=\; -\mathrm{d}\omega_3^{\mathrm{Y}} \; . \tag{A.51b}$$

B Conventions

Here we show the redefinition rules between the variables in the BdR ($\underline{\varphi}$) and the ones (φ) in this note in heterotic theory:

$$\underline{\phi}^{-3} \equiv \frac{1}{\kappa_{10}^2}\exp(-2\Phi) , \quad \underline{\omega}_M{}^{AB} \equiv -\omega_M{}^{AB} , \tag{B.1a}$$

$$\underline{A}_M \equiv -\widetilde{A}_M , \quad \underline{B}_{MN} \equiv +\sqrt{2}B_{MN} , \quad \underline{H}_{MNP} \equiv +\frac{\sqrt{2}}{3}H_{MNP} , \tag{B.1b}$$

$$\underline{\chi} = \chi^a\underline{T}^a \equiv -\chi^a\widetilde{T}^a = -\widetilde{\chi} , \quad \underline{\lambda} \equiv \sqrt{2}\lambda , \tag{B.1c}$$

$$\psi_M, \epsilon \text{ keep the same variables} , \tag{B.1d}$$

$$\alpha = \beta = -\frac{\kappa_{10}^2}{g_{10}^2} , \tag{B.1e}$$

$$\kappa_{10}^2 = 2 , \quad \frac{\kappa_{10}^2}{2g_{10}^2} = \alpha' . \tag{B.1f}$$

C Geometries: complex, hermitian and Kähler manifolds

C.1 Complex coordinates

In this section we discuss various conditions on differential manifolds. We start from a Riemannian manifold whose coordinates are given in terms of x^M, where the indices M runs 1 to $D = 2d$. First we re-name the coordinates such as $x^{d+m} \equiv y^{m'}$, where $m, m' = 1, \ldots, d$, the we define "(anti-)holomorphic" coordinates in the complex frame such as

$$z^m \equiv \frac{1}{\sqrt{2}}\big(x^m + iy^{m'}\big) , \quad \overline{z}^{\overline{m}} \equiv \frac{1}{\sqrt{2}}\big(x^m - iy^{m'}\big) , \tag{C.1a}$$

$$x^m = \frac{1}{\sqrt{2}}\big(z^m + \overline{z}^{\overline{m}}\big) , \quad y^{m'} = -\frac{i}{\sqrt{2}}\big(z^m - \overline{z}^{\overline{m}}\big) . \tag{C.1b}$$

The derivatives $\partial/\partial z^m$ are given as

$$\frac{\partial}{\partial z^m} = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x^m} - i\frac{\partial}{\partial y^{m'}}\Big) , \quad \frac{\partial}{\partial \overline{z}^{\overline{m}}} = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x^m} + i\frac{\partial}{\partial y^{m'}}\Big) , \tag{C.2a}$$

$$\frac{\partial}{\partial x^m} = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial z^m} + \frac{\partial}{\partial \overline{z}^{\overline{m}}}\Big) , \quad \frac{\partial}{\partial y^{m'}} = \frac{i}{\sqrt{2}}\Big(\frac{\partial}{\partial z^m} - \frac{\partial}{\partial \overline{z}^{\overline{m}}}\Big) . \tag{C.2b}$$

C.2 Metric on complex manifold: hermitian metric

We should also re-define the metric on the geometry. The metric associated with the bosonic operators $x^M = (x^m, y^{m'})$ is given as

$$g_{MN}\, \mathrm{d}x^M \mathrm{d}x^N = g_{mn}\, \mathrm{d}x^m \mathrm{d}x^n + g_{mn'}\, \mathrm{d}x^m \mathrm{d}y^{n'} + g_{m'n}\, \mathrm{d}y^{m'} \mathrm{d}x^n + g_{m'n'}\, \mathrm{d}y^{m'} \mathrm{d}y^{n'} . \tag{C.3}$$

This line element is invariant under the rotation with $SO(D)$ group. We also define the line element given in terms of the complex coordinates z^m and $\overline{z}^{\overline{m}}$:

$$\widetilde{g}{}^{(z,\overline{z})}_{MN} \mathrm{d}Z^M \mathrm{d}Z^N = \widetilde{g}_{m\overline{n}}\, \mathrm{d}z^m \mathrm{d}\overline{z}^{\overline{n}} + \widetilde{g}_{\overline{m}n}\, \mathrm{d}\overline{z}^{\overline{m}} \mathrm{d}z^n . \tag{C.4}$$

This line element is invariant under the rotation with $U(d)$ group, the subgroup of $SO(2d)$. We should notice that this rotation group is the structure group. Now let us find the relation between g_{MN} in (C.3) and $\widetilde{g}_{m\overline{n}}$ in (C.4) via (C.1). We impose

$$\widetilde{g}_{mn} = \widetilde{g}_{\overline{m}\overline{n}} = 0 \tag{C.5}$$

on the metric since we assume that the geometry is a complex manifold:

$$\widetilde{g}_{mn} = 0 = \frac{\partial x^p}{\partial z^m}\frac{\partial x^q}{\partial z^n} g_{pq} + \frac{\partial x^p}{\partial z^m}\frac{\partial y^{q'}}{\partial z^n} g_{pq'} + \frac{\partial y^{p'}}{\partial z^m}\frac{\partial x^q}{\partial z^n} g_{p'q} + \frac{\partial y^{p'}}{\partial z^m}\frac{\partial y^{q'}}{\partial z^n} g_{p'q'}$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\delta^p_m \delta^q_n \, g_{pq} - i\delta^p_m \delta^{q'}_n \, g_{pq'} - i\delta^{p'}_m \delta^q_n \, g_{p'q} - \delta^{p'}_m \delta^{q'}_n \, g_{p'q'}\Big) \;, && \text{(C.6a)}\\
\widetilde{g}_{\overline{m}\overline{n}} \;=\; 0 \;&=\; \frac{\partial x^p}{\partial \overline{z}^{\overline{m}}}\frac{\partial x^q}{\partial \overline{z}^{\overline{n}}} g_{pq} + \frac{\partial x^p}{\partial \overline{z}^{\overline{m}}}\frac{\partial y^{q'}}{\partial \overline{z}^{\overline{n}}} g_{pq'} + \frac{\partial y^{p'}}{\partial \overline{z}^{\overline{m}}}\frac{\partial x^q}{\partial \overline{z}^{\overline{n}}} g_{p'q} + \frac{\partial y^{p'}}{\partial \overline{z}^{\overline{m}}}\frac{\partial y^{q'}}{\partial \overline{z}^{\overline{n}}} g_{p'q'} \\
&= \frac{1}{2}\Big(\delta^p_m \delta^q_n \, g_{pq} + i\delta^p_m \delta^{q'}_n \, g_{pq'} + i\delta^{p'}_m \delta^q_n \, g_{p'q} - \delta^{p'}_m \delta^{q'}_n \, g_{p'q'}\Big) \;. && \text{(C.6b)}
\end{aligned}
$$

This indicates a strong condition which is imposed on the metric

$$
g_{pq} - g_{p'q'} \;=\; 0 \;, \qquad g_{pq'} + g_{p'q} \;=\; 0 \;, \tag{C.7a}
$$

$$
\therefore \quad g_{pq} \;=\; g_{p'q'} \;, \qquad g_{p'q} \;=\; -g_{pq'} \;, \qquad g_{p'p} \;=\; -g_{pp'} \;=\; 0 \;, \tag{C.7b}
$$

where we used the symmetry $g_{MN} = g_{NM}$. This decomposes the original structure group $SO(2d)$ to $U(d)$. Notice that the local Lorentz group is not reduced. Let us further investigate it:

$$
\begin{aligned}
g_{pq} \;&=\; \bigg(\frac{\partial z^m}{\partial x^p}\frac{\partial \overline{z}^{\overline{n}}}{\partial x^q}\widetilde{g}_{m\overline{n}} + \frac{\partial \overline{z}^{\overline{m}}}{\partial x^p}\frac{\partial z^n}{\partial x^q}\widetilde{g}_{\overline{m}n}\bigg) \;=\; \frac{1}{2}\Big(\delta^m_p \delta^{\overline{n}}_q \, \widetilde{g}_{m\overline{n}} + \delta^{\overline{m}}_p \delta^n_q \, \widetilde{g}_{\overline{m}n}\Big) \\
&= \frac{1}{2}\Big(\delta^m_p \delta^{\overline{n}}_q + \delta^m_q \delta^{\overline{n}}_p\Big)\widetilde{g}_{m\overline{n}} \;, && \text{(C.8a)}\\
g_{p'q'} \;&=\; \bigg(\frac{\partial z^m}{\partial y^{p'}}\frac{\partial \overline{z}^{\overline{n}}}{\partial y^{q'}}\widetilde{g}_{m\overline{n}} + \frac{\partial \overline{z}^{\overline{m}}}{\partial y^{p'}}\frac{\partial z^n}{\partial y^{q'}}\widetilde{g}_{\overline{m}n}\bigg) \;=\; \frac{1}{2}\Big(\delta^m_{p'} \delta^{\overline{n}}_{q'} \, \widetilde{g}_{m\overline{n}} + \delta^{\overline{m}}_{p'} \delta^n_{q'} \, \widetilde{g}_{\overline{m}n}\Big) \\
&= \frac{1}{2}\Big(\delta^m_{p'} \delta^{\overline{n}}_{q'} + \delta^m_{q'} \delta^{\overline{n}}_{p'}\Big)\widetilde{g}_{m\overline{m}} \;, && \text{(C.8b)}\\
g_{pq'} \;&=\; \bigg(\frac{\partial z^m}{\partial x^p}\frac{\partial \overline{z}^{\overline{n}}}{\partial y^{q'}}\widetilde{g}_{m\overline{n}} + \frac{\partial \overline{z}^{\overline{m}}}{\partial x^p}\frac{\partial z^n}{\partial y^{q'}}\widetilde{g}_{\overline{m}n}\bigg) \;=\; -\frac{i}{2}\Big(\delta^m_p \delta^{\overline{n}}_{q'} \, \widetilde{g}_{m\overline{n}} - \delta^{\overline{m}}_p \delta^n_{q'} \, \widetilde{g}_{\overline{m}n}\Big) \\
&= -\frac{i}{2}\Big(\delta^m_p \delta^{\overline{n}}_{q'} - \delta^m_{q'} \delta^{\overline{n}}_p\Big)\widetilde{g}_{m\overline{n}} \;, && \text{(C.8c)}\\
g_{p'q} \;&=\; \bigg(\frac{\partial z^m}{\partial y^{p'}}\frac{\partial \overline{z}^{\overline{n}}}{\partial x^q}\widetilde{g}_{m\overline{n}} + \frac{\partial \overline{z}^{\overline{m}}}{\partial y^{p'}}\frac{\partial z^n}{\partial x^q}\widetilde{g}_{\overline{m}n}\bigg) \;=\; \frac{i}{2}\Big(\delta^m_{p'} \delta^{\overline{n}}_q \, \widetilde{g}_{m\overline{n}} - \delta^{\overline{m}}_{p'} \delta^n_q \, \widetilde{g}_{m\overline{n}}\Big) \\
&= \frac{i}{2}\Big(\delta^m_{p'} \delta^{\overline{n}}_q - \delta^m_q \delta^{\overline{n}}_{p'}\Big)\widetilde{g}_{m\overline{n}} \;, && \text{(C.8d)}
\end{aligned}
$$

where we used the symmetry $\widetilde{g}_{m\overline{n}} = \widetilde{g}_{\overline{n}m}$. For later discussion, we define the inverse of g_{pq} as

$$
g^{pq} \;=\; \Big(\delta^p_m \delta^q_{\overline{n}} + \delta^q_m \delta^p_{\overline{n}}\Big)\widetilde{g}^{m\overline{n}} \;. \tag{C.9}
$$

Then, the determinant of the metric $g = \det(g_{MN})$ can be represented in such a way as

$$
g \;\equiv\; \det g_{MN} \;=\; \det \begin{pmatrix} g_{pq} & g_{pq'} \\ g_{p'q} & g_{p'q'} \end{pmatrix} \;=\; \det g_{pq} \cdot \det\big(g_{p'q'} - g_{p'p} g^{pq} g_{qq'}\big) \;. \tag{C.10}
$$

The first determinant is given by using (C.8):

$$
\det g_{pq} \;=\; \det \Big(\tfrac{1}{2}\big(\delta^m_p \delta^{\overline{n}}_q + \delta^m_q \delta^{\overline{n}}_p\big)\widetilde{g}_{m\overline{n}}\Big) \;. \tag{C.11a}
$$

Let us consider the second determinant carefully:

$$\begin{aligned} g_{p'p} g^{pq} g_{qq'} &= \frac{1}{4} \Big(\delta^m_{p'} \delta^{\overline{n}}_q - \delta^m_q \delta^{\overline{n}}_{p'} \Big) \widetilde{g}_{m\overline{n}} \, \Big(\delta^p_r \delta^q_{\overline{s}} + \delta^q_r \delta^p_{\overline{s}} \Big) \widetilde{g}^{r\overline{s}} \, \Big(\delta^k_p \delta^{\overline{l}}_{q'} - \delta^k_{q'} \delta^{\overline{l}}_p \Big) \widetilde{g}_{k\overline{l}} \\ &= \frac{1}{4} \Big(\delta^k_{p'} \delta^{\overline{l}}_{q'} + \delta^k_{q'} \delta^{\overline{l}}_{p'} \Big) \widetilde{g}_{k\overline{l}} \, , \end{aligned} \tag{C.11b}$$

$$\therefore \quad \det \big(g_{p'q'} - g_{p'p} g^{pq} g_{qq'} \big) \ = \ \det \Big(\tfrac{1}{4} \big(\delta^k_{p'} \delta^{\overline{l}}_{q'} + \delta^k_{q'} \delta^{\overline{l}}_{p'} \big) \widetilde{g}_{k\overline{l}} \Big) \, . \tag{C.11c}$$

Summarizing the above, we obtain the following expression in terms of $\widetilde{g} \equiv \det \widetilde{g}_{m\overline{N}}$:

$$\begin{aligned} g &= \det \Big(\tfrac{1}{2} \big(\delta^m_p \delta^{\overline{n}}_q + \delta^m_q \delta^{\overline{n}}_p \big) \widetilde{g}_{m\overline{n}} \Big) \cdot \det \Big(\tfrac{1}{4} \big(\delta^k_{p'} \delta^{\overline{l}}_{q'} + \delta^k_{q'} \delta^{\overline{l}}_{p'} \big) \widetilde{g}_{k\overline{l}} \Big) \\ &= \frac{1}{2^d} \Big\{ \det \Big(\tfrac{1}{2} \big(\delta^m_p \delta^{\overline{n}}_q + \delta^m_q \delta^{\overline{n}}_p \big) \widetilde{g}_{m\overline{n}} \Big) \Big\}^2 \ = \ \frac{1}{2^d} \cdot \widetilde{g}^2 \, . \end{aligned} \tag{C.12}$$

Note that we still use the original expression g in later discussion, if there are no confusions.

C.3 General coordinate transformations of connections

It is worth mentioning the transformation rule of the affine connection under the general coordinate transformation. This transformation can be, for instance, derived from the general coordinate transformation of a tensor $\nabla_M A^N$ from the $(x^M; A^N)$-frame to the $(\widetilde{x}^m; \widetilde{A}^n)$-frame:

$$\begin{aligned} \nabla_M A^N &= \partial_M A^N + \Gamma^N{}_{PM} A^P \ = \ \Big(\frac{\partial \widetilde{x}^m}{\partial x^M} \frac{\partial}{\partial \widetilde{x}^m} \Big) \Big(\frac{\partial x^N}{\partial \widetilde{x}^n} \widetilde{A}^n \Big) + \Gamma^N{}_{PM} \Big(\frac{\partial x^P}{\partial \widetilde{x}^p} \widetilde{A}^p \Big) \\ &\equiv \frac{\partial \widetilde{x}^m}{\partial x^M} \frac{\partial x^N}{\partial \widetilde{x}^n} \Big(\widetilde{\nabla}_m \widetilde{A}^n \Big) \ = \ \frac{\partial \widetilde{x}^m}{\partial x^M} \frac{\partial x^N}{\partial \widetilde{x}^n} \Big(\widetilde{\partial}_m \widetilde{A}^n + \widetilde{\Gamma}^n{}_{pm} \widetilde{A}^p \Big) \, . \end{aligned} \tag{C.13}$$

where A^N is an arbitrary contravariant vector, and the affine connection $\Gamma^N{}_{PM}$ can contain contorsion. Compared to the first line and the second line in the right-hand side, we easily find the relation between $\Gamma^N{}_{PM}$ and $\widetilde{\Gamma}^n{}_{pm}$:

$$\Gamma^N{}_{PM} \frac{\partial x^P}{\partial \widetilde{x}^p} \widetilde{A}^p \ = \ \frac{\partial \widetilde{x}^m}{\partial x^M} \frac{\partial x^N}{\partial \widetilde{x}^n} \widetilde{\Gamma}^n{}_{pm} \widetilde{A}^p - \frac{\partial \widetilde{x}^m}{\partial x^M} \frac{\partial^2 x^N}{\partial \widetilde{x}^m \partial \widetilde{x}^p} \widetilde{A}^p \, , \tag{C.14a}$$

$$\therefore \quad \Gamma^N{}_{PM} \ = \ \frac{\partial x^N}{\partial \widetilde{x}^n} \frac{\partial \widetilde{x}^p}{\partial x^P} \frac{\partial \widetilde{x}^m}{\partial x^M} \widetilde{\Gamma}^n{}_{pm} + \frac{\partial x^N}{\partial \widetilde{x}^p} \frac{\partial^2 \widetilde{x}^p}{\partial x^P \partial x^M} \, . \tag{C.14b}$$

Here we also discuss the transformation rule of the spin connection from the x^M-coordinate frame to the $\widetilde{x}^m$-coordinate frame. To do so, we should also define the vielbein and its inverse:

$$g_{MN} \ = \ \delta_{AB} \, e_M{}^A \, e_N{}^B \, , \qquad e_M{}^A \, E_A{}^N \ = \ \delta^N_M \, , \qquad E_A{}^M \, e_M{}^B \ = \ \delta^B_A \, , \tag{C.15}$$

where δ_{AB} is the orthogonal metric in the local Lorentz frame. Since the vielbein $e_M{}^A$ and its inverse $E_A{}^M$ are vectors under the general coordinate transformation[4], they transform in the fol-

[4]Notice that the local Lorentz coordinates is not transformed under the general coordinate transformations.

lowing way:

$$e_M{}^A \;=\; \frac{\partial \widetilde{x}^m}{\partial x^M} \widetilde{e}_m{}^A \; , \qquad E_B{}^N \;=\; \frac{\partial x^N}{\partial \widetilde{x}^n} \, \widetilde{E}_B{}^n \; . \tag{C.16}$$

Focus on the vielbein postulate given by the following equation and its transformation:

$$\begin{aligned} 0 \;=\; D_M(\omega, \Gamma) e_N{}^A \;&=\; \partial_M e_N{}^A + \omega_M{}^A{}_B \, e_N{}^B - \Gamma^P{}_{NM} \, e_P{}^A \qquad \text{(C.17a)} \\ &=\; \Big(\frac{\partial \widetilde{x}^m}{\partial x^M} \frac{\partial}{\partial \widetilde{x}^m} \Big) \Big(\frac{\partial \widetilde{x}^n}{\partial x^N} \, \widetilde{e}_n{}^A \Big) + \omega_M{}^A{}_B \Big(\frac{\partial \widetilde{x}^n}{\partial x^N} \, \widetilde{e}_n{}^B \Big) \\ &\quad - \Big(\frac{\partial x^P}{\partial \widetilde{x}^p} \frac{\partial \widetilde{x}^n}{\partial x^N} \frac{\partial \widetilde{x}^m}{\partial x^M} \widetilde{\Gamma}^p{}_{nm} + \frac{\partial x^P}{\partial \widetilde{x}^p} \frac{\partial^2 \widetilde{x}^p}{\partial x^N \partial x^M} \Big) \Big(\frac{\partial \widetilde{x}^q}{\partial x^P} \, \widetilde{e}_q{}^A \Big) \; . \qquad \text{(C.17b)} \end{aligned}$$

The equation in the first line is applicable to any Riemannian manifold even in the presence of torsion. Here we used the transformation rule of the affine connection (C.14). This equation behaves as a vector-valued tensor under the general coordinate transformation from the x^M-frame to the $\widetilde{x}^m$-frame:

$$\begin{aligned} D_M(\omega, \Gamma) e_N{}^A \;&=\; \frac{\partial \widetilde{x}^m}{\partial x^M} \frac{\partial \widetilde{x}^n}{\partial x^N} \Big(\widetilde{D}_m(\widetilde{\omega}, \widetilde{\Gamma}) \widetilde{e}_n{}^A \Big) \\ &=\; \frac{\partial \widetilde{x}^m}{\partial x^M} \frac{\partial \widetilde{x}^n}{\partial x^N} \Big(\widetilde{\partial}_m \widetilde{e}_n{}^A + \widetilde{\omega}_m{}^A{}_B \, \widetilde{e}_n{}^B - \widetilde{\Gamma}^p{}_{nm} \, \widetilde{e}_p{}^A \Big) \; . \end{aligned} \tag{C.18}$$

Comparing the above equations (C.17b) and (C.18), we find that the spin connection transforms as a vector (or a tensor) under the general coordinate transformation:

$$\begin{aligned} \frac{\partial \widetilde{x}^m}{\partial x^M} \frac{\partial \widetilde{x}^n}{\partial x^N} \, \widetilde{\omega}_m{}^A{}_B \, \widetilde{e}_n{}^B \;&=\; \frac{\partial \widetilde{x}^n}{\partial x^N} \, \omega_M{}^A{}_B \, \widetilde{e}_n{}^B + \frac{\partial^2 \widetilde{x}^n}{\partial x^M \partial x^N} \, \widetilde{e}_n{}^A - \frac{\partial^2 \widetilde{x}^n}{\partial x^M \partial x^N} \, \widetilde{e}_n{}^A \\ &=\; \frac{\partial \widetilde{x}^n}{\partial x^N} \, \omega_M{}^A{}_B \, \widetilde{e}_n{}^B \; , \qquad \text{(C.19a)} \\ \therefore \quad \omega_M{}^A{}_B \;&=\; \frac{\partial \widetilde{x}^m}{\partial x^M} \, \widetilde{\omega}_m{}^A{}_B \; . \qquad \text{(C.19b)} \end{aligned}$$

Notice, however, the spin connection does not behave as a vector under the $SO(2d)$ local Lorentz transformation in the same way as the gauge transformation of the non-abelian gauge field.

To impose the condition that the manifold is hermitian (C.22) on the spin connection, we should find the relation between the affine connection and the spin connection via the vielbein postulate:

$$\Gamma^P{}_{NM} \;=\; E_A{}^P \Big(\partial_M e_N{}^A + \omega_M{}^A{}_B \, e_N{}^B \Big) \; . \tag{C.20}$$

We also note that the spin connection can be described in terms of the vielbein and the affine connection:

$$\omega_M{}^A{}_B \;=\; -E_B{}^N \partial_M e_N{}^A + \Gamma^P{}_{NM} e_P{}^A E_B{}^N \; . \tag{C.21}$$

Note that these forms are, of course, applicable in any coordinate frame.

C.4 Further analysis on hermitian manifold

Let us further discuss the complex manifold. Actually, the condition (C.5) is nothing but the definition of the hermitian metric. Furthermore, the complex manifold whose metric is given by the hermitian manifold is the hermitian manifold by Yano [17]. However, Nakahara [13] discusses a different definition of the hermitian manifold. We follow the Yano's definition.

We want to use the formulation of topological invariants on a complex $SU(3)$-structure manifold in the presence of non-trivial torsion. In heterotic string compactification scenario, the compactified six-dimensional manifold is an $SU(3)$-structure manifold, if we impose an low energy effective theory in four-dimensional spacetime is an $\mathcal{N}=1$ supersymmetric theory. In the $SU(3)$-structure manifold affine connections with non-trivial values are of pure type $\Gamma^m{}_{np}$ and of mixed type $\Gamma^m{}_{\overline{n}p}$, and their complex conjugates. This condition is nothing but the condition on a hermitian manifold (for the definitions of the hermitian manifold, see [17], not [13]; for the discussions of the $SU(3)$-structure manifold, see [16]). Let us impose that the holomorphic covariant derivative along the holomorphic tangent vector should keep the holomorphicity:

$$\widetilde{\nabla}_m \frac{\partial}{\partial z^n} \ = \ \widetilde{\Gamma}^p{}_{nm} \frac{\partial}{\partial z^p} \, , \quad \widetilde{\nabla}_m \frac{\partial}{\partial \overline{z}^{\overline{n}}} \ = \ \widetilde{\Gamma}^{\overline{p}}{}_{\overline{n}m} \frac{\partial}{\partial \overline{z}^{\overline{p}}} \, , \tag{C.22a}$$

$$\widetilde{\nabla}_{\overline{m}} \frac{\partial}{\partial \overline{z}^{\overline{n}}} \ = \ \widetilde{\Gamma}^{\overline{p}}{}_{\overline{n}\overline{m}} \frac{\partial}{\partial \overline{z}^{\overline{p}}} \, , \quad \widetilde{\nabla}_{\overline{m}} \frac{\partial}{\partial z^n} \ = \ \widetilde{\Gamma}^p{}_{n\overline{m}} \frac{\partial}{\partial z^p} \, , \tag{C.22b}$$

$$\widetilde{\Gamma}^{\overline{p}}{}_{nm} \ = \ 0 \, , \quad \widetilde{\Gamma}^p{}_{\overline{n}\overline{m}} \ = \ 0 \, , \quad \widetilde{\Gamma}^p{}_{\overline{n}m} \ = \ 0 \, , \quad \widetilde{\Gamma}^{\overline{p}}{}_{n\overline{m}} \ = \ 0 \, , \tag{C.22c}$$

where the above affine connections, called the hermitian connections, can contain torsion. Actually, the vanishing affine connection reduces the structure group (or the holonomy group) from $SO(2d)$ to $U(d)$ (or $SU(d)$). Note that $\widetilde{\nabla}_m \partial/\partial z^n$ behaves as a tensor. This hermitian connection itself is given by the metricity condition:

$$0 \ = \ \widetilde{\nabla}_m \widetilde{g}_{n\overline{p}} \ = \ \partial_m \widetilde{g}_{n\overline{p}} - \widetilde{\Gamma}^q{}_{nm} \widetilde{g}_{q\overline{p}} - \widetilde{\Gamma}^{\overline{q}}{}_{\overline{p}m} \widetilde{g}_{n\overline{q}} \, . \tag{C.23}$$

The Riemann tensor associated with the hermitian connection is also restricted compared to the one on a generic Riemannian manifold. The definition of the Riemann tensor on a generic Riemannian manifold is (see also (A.13))

$$R^P{}_{QMN}(\Gamma) \ = \ \partial_M \Gamma^P{}_{QN} - \partial_N \Gamma^P{}_{QM} + \Gamma^P{}_{RM} \Gamma^R{}_{QN} - \Gamma^P{}_{RN} \Gamma^R{}_{QM} \, , \tag{C.24}$$

where the affine connection $\Gamma^P{}_{MN}$ can contain torsion. Applying the hermitian condition on the affine connection (C.22) to the Riemann tensor, the expression becomes quite simple:

$$\widetilde{R}^p{}_{qMN}(\widetilde{\Gamma}) \ = \ \partial_M \widetilde{\Gamma}^p{}_{qN} - \partial_N \widetilde{\Gamma}^p{}_{qM} + \widetilde{\Gamma}^p{}_{RM} \widetilde{\Gamma}^R{}_{qN} - \widetilde{\Gamma}^p{}_{RN} \widetilde{\Gamma}^R{}_{qM} \ = \ -\widetilde{R}^p{}_{qNM}(\widetilde{\Gamma}) \, , \tag{C.25a}$$

$$\widetilde{R}^p{}_{\overline{q}MN}(\widetilde{\Gamma}) \;=\; \partial_M \widetilde{\Gamma}^p{}_{\overline{q}N} - \partial_N \widetilde{\Gamma}^p{}_{\overline{q}M} + \widetilde{\Gamma}^p{}_{RM} \widetilde{\Gamma}^R{}_{\overline{q}N} - \widetilde{\Gamma}^p{}_{RN} \widetilde{\Gamma}^R{}_{\overline{q}M} \;=\; 0 \,, \tag{C.25b}$$

$$\widetilde{R}^{\overline{p}}{}_{qMN}(\widetilde{\Gamma}) \;=\; \partial_M \widetilde{\Gamma}^{\overline{p}}{}_{qN} - \partial_N \widetilde{\Gamma}^{\overline{p}}{}_{qM} + \widetilde{\Gamma}^{\overline{p}}{}_{RM} \widetilde{\Gamma}^R{}_{qN} - \widetilde{\Gamma}^{\overline{p}}{}_{RN} \widetilde{\Gamma}^R{}_{qM} \;=\; 0 \,, \tag{C.25c}$$

$$\widetilde{R}^{\overline{p}}{}_{\overline{q}MN}(\widetilde{\Gamma}) \;=\; \partial_M \widetilde{\Gamma}^{\overline{p}}{}_{\overline{q}N} - \partial_N \widetilde{\Gamma}^{\overline{p}}{}_{\overline{q}M} + \widetilde{\Gamma}^{\overline{p}}{}_{RM} \widetilde{\Gamma}^R{}_{\overline{q}N} - \widetilde{\Gamma}^{\overline{p}}{}_{RN} \widetilde{\Gamma}^R{}_{\overline{q}M} \;=\; -\widetilde{R}^{\overline{p}}{}_{\overline{q}NM}(\widetilde{\Gamma}) \,, \tag{C.25d}$$

where the capital indices M and N run both holomorphic and anti-holomorphic directions.

The hermitian connection is decomposed into the symmetric and the anti-symmetric hermitian connections, whose explict forms are, for instance, given as

$$\widetilde{\Gamma}^m{}_{np} \;=\; \widetilde{\Gamma}^m{}_{(np)} + \widetilde{\Gamma}^m{}_{[np]} \,, \tag{C.26a}$$

$$\widetilde{\Gamma}^m{}_{(np)} \;\equiv\; \frac{1}{2}\Big(\widetilde{\Gamma}^m{}_{np} + \widetilde{\Gamma}^m{}_{pn} \Big) \,, \qquad T^m{}_{np} \;\equiv\; \widetilde{\Gamma}^m{}_{[np]} \;=\; \frac{1}{2}\Big(\widetilde{\Gamma}^m{}_{np} - \widetilde{\Gamma}^m{}_{pn} \Big) \,. \tag{C.26b}$$

Furthermore, we can decompose the symmetric part of the hermitian connection into the Levi-Civita connection[5] $\widetilde{\Gamma}_{(\mathrm{L})}$ and the terms depending on the torsion:

$$\widetilde{\Gamma}^m{}_{(np)} \;=\; \widetilde{\Gamma}^m_{(\mathrm{L})np} - \Big(T_n{}^m{}_p + T_p{}^m{}_n \Big) \,, \tag{C.27a}$$

$$\therefore \quad \widetilde{\Gamma}^m{}_{np} \;=\; \widetilde{\Gamma}^m_{(\mathrm{L})np} + K^m{}_{np} \,, \qquad K^m{}_{np} \;\equiv\; T^m{}_{np} - T_n{}^m{}_p - T_p{}^m{}_n \,, \tag{C.27b}$$

where $K^m{}_{np}$ is called the contorsion. Here let us explicitly list up all components:

$$\widetilde{\Gamma}^m{}_{np} \;=\; \widetilde{\Gamma}^m_{(\mathrm{L})np} + K^m{}_{np} \,, \qquad \widetilde{\Gamma}^{\overline{m}}{}_{\overline{n}\overline{p}} \;=\; \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})\overline{n}\overline{p}} + K^{\overline{m}}{}_{\overline{n}\overline{p}} \,, \tag{C.28a}$$

$$\widetilde{\Gamma}^m{}_{n\overline{p}} \;=\; \widetilde{\Gamma}^m_{(\mathrm{L})n\overline{p}} + K^m{}_{n\overline{p}} \,, \qquad \widetilde{\Gamma}^{\overline{m}}{}_{\overline{n}p} \;=\; \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})\overline{n}p} + K^{\overline{m}}{}_{\overline{n}p} \,, \tag{C.28b}$$

$$0 \;=\; \widetilde{\Gamma}^{\overline{m}}{}_{np} \;=\; \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})np} + K^{\overline{m}}{}_{np} \,, \qquad 0 \;=\; \widetilde{\Gamma}^m{}_{\overline{n}\overline{p}} \;=\; \widetilde{\Gamma}^m_{(\mathrm{L})\overline{n}\overline{p}} + K^m{}_{\overline{n}\overline{p}} \,, \tag{C.28c}$$

$$0 \;=\; \widetilde{\Gamma}^m{}_{\overline{n}p} \;=\; \widetilde{\Gamma}^m_{(\mathrm{L})\overline{n}p} + K^m{}_{\overline{n}p} \,, \qquad 0 \;=\; \widetilde{\Gamma}^{\overline{m}}{}_{n\overline{p}} \;=\; \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})n\overline{p}} + K^{\overline{m}}{}_{n\overline{p}} \,. \tag{C.28d}$$

It is worth discussing the explicit forms of the Levi-Civita connection $\widetilde{\Gamma}_{(\mathrm{L})}$ on the hermitian manifold:

$$\widetilde{\Gamma}^m_{(\mathrm{L})np} \;=\; \frac{1}{2} \widetilde{g}^{m\overline{q}} \Big(\partial_n \widetilde{g}_{\overline{q}p} + \partial_p \widetilde{g}_{n\overline{q}} \Big) \;=\; \widetilde{\Gamma}^m_{(\mathrm{L})pn} \,, \quad \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})\overline{n}\overline{p}} \;=\; \frac{1}{2} \widetilde{g}^{\overline{m}q} \Big(\partial_{\overline{n}} \widetilde{g}_{q\overline{p}} + \partial_{\overline{p}} \widetilde{g}_{\overline{n}q} \Big) \;=\; \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})\overline{p}\overline{n}} \,, \tag{C.29a}$$

$$\widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})np} \;=\; 0 \,, \qquad \widetilde{\Gamma}^m_{(\mathrm{L})\overline{n}\overline{p}} \;=\; 0 \,, \tag{C.29b}$$

$$\widetilde{\Gamma}^m_{(\mathrm{L})n\overline{p}} \;=\; \frac{1}{2} \widetilde{g}^{m\overline{q}} \Big(\partial_{\overline{p}} \widetilde{g}_{n\overline{q}} - \partial_{\overline{q}} \widetilde{g}_{n\overline{p}} \Big) \;=\; \widetilde{\Gamma}^m_{(\mathrm{L})\overline{p}n} \,, \quad \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})\overline{n}p} \;=\; \frac{1}{2} \widetilde{g}^{\overline{m}q} \Big(\partial_p \widetilde{g}_{\overline{n}q} - \partial_q \widetilde{g}_{\overline{n}p} \Big) \;=\; \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})p\overline{n}} \,. \tag{C.29c}$$

Here we should point out that the Levi-Civita connections $\widetilde{\Gamma}^m_{(\mathrm{L})\overline{n}p}$ and $\widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})n\overline{p}}$ have non-trivial values whereas the hermitian connections of same type vanish: $\widetilde{\Gamma}^m{}_{\overline{n}p} = \widetilde{\Gamma}^{\overline{m}}{}_{n\overline{p}} = 0$, see (C.22). This

[5] Strictly speaking, the Levi-Civita connection is defined on a torsionless manifold. Due to this, the metric in the Levi-Civita connection should be given as the metric on the torsionless geometry.

indicates that the contorsion of certain types are equal to the Levi-Civita connections in the vanishing hermitian connections:

$$0 \;=\; \widetilde{\Gamma}^m{}_{\overline{n}p} \;=\; \widetilde{\Gamma}^m_{(\mathrm{L})\overline{n}p} + K^m{}_{\overline{n}p} \;, \quad \therefore \;\; K^m{}_{\overline{n}p} \;=\; -\widetilde{\Gamma}^m_{(\mathrm{L})\overline{n}p} \;=\; -\frac{1}{2}\widetilde{g}^{m\overline{q}}\Big(\partial_{\overline{n}}\widetilde{g}_{p\overline{q}} - \partial_{\overline{q}}\widetilde{g}_{p\overline{n}}\Big) \;, \tag{C.30a}$$

$$0 \;=\; \widetilde{\Gamma}^{\overline{m}}{}_{n\overline{p}} \;=\; \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})n\overline{p}} + K^{\overline{m}}{}_{n\overline{p}} \;, \quad \therefore \;\; K^{\overline{m}}{}_{n\overline{p}} \;=\; -\widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})n\overline{p}} \;=\; -\frac{1}{2}\widetilde{g}^{\overline{m}q}\Big(\partial_{n}\widetilde{g}_{\overline{p}q} - \partial_{q}\widetilde{g}_{\overline{p}n}\Big) \;. \tag{C.30b}$$

We should notice that the Riemann tensor $\widetilde{R}^P{}_{QMN}(\widetilde{\Gamma}_{(\mathrm{L})})$ should have same properties as the ones on the Riemann tensor $R^P{}_{QMN}(\Gamma_0)$ on a generic real manifold such as

$$\widetilde{R}_{PQMN}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; -\widetilde{R}_{PQNM}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; -\widetilde{R}_{QPMN}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; \widetilde{R}_{MNPQ}(\widetilde{\Gamma}_{(\mathrm{L})}) \;, \tag{C.31a}$$

$$\widetilde{R}_{P[QMN]}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; 0 \;, \quad \widetilde{\nabla}_{[M}\widetilde{R}^N_{|P|QR]}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; 0 \;, \tag{C.31b}$$

where the equations (C.31b) follow the identities on the Riemann tensor associated with the Levi-Civita connection on a generic real manifold (A.19b) and (A.20). Let us further investigate to find vanishing components (see, for detail, in appendix A.6):

$$\widetilde{R}_{pqmn}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; -\widetilde{R}_{pqnm}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; -\widetilde{R}_{qpmn}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; \widetilde{R}_{mnpq}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; 0 \;, \tag{C.31c}$$

$$\widetilde{R}_{\overline{pqmn}}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; -\widetilde{R}_{\overline{pqnm}}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; -\widetilde{R}_{\overline{qpmn}}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; \widetilde{R}_{\overline{mnpq}}(\widetilde{\Gamma}_{(\mathrm{L})}) \;=\; 0 \;. \tag{C.31d}$$

Let us also analyze the reduction of the spin connection. To do so, let us define the vielbein and its inverse in the complex coordinate frame:

$$\widetilde{g}_{m\overline{n}} \;\equiv\; \delta_{AB}\,\widetilde{e}_m{}^A\,\widetilde{e}_{\overline{n}}{}^B \;, \quad \widetilde{g}_{mn} \;=\; 0 \;=\; \delta_{AB}\widetilde{e}_m{}^A\,\widetilde{e}_n{}^B \;, \quad \widetilde{g}_{\overline{mn}} \;=\; 0 \;=\; \delta_{AB}\widetilde{e}_{\overline{m}}{}^A\,\widetilde{e}_{\overline{n}}{}^B \;, \tag{C.32a}$$

$$\widetilde{e}_m{}^A\,\widetilde{E}_A{}^n \;=\; \delta^n_m \;, \quad \widetilde{e}_{\overline{m}}{}^A\,\widetilde{E}_A{}^{\overline{n}} \;=\; \delta^{\overline{n}}_{\overline{m}} \;, \quad \delta^A_B \;=\; \widetilde{e}_p{}^A\widetilde{E}_B{}^p + \widetilde{e}_{\overline{p}}{}^A\widetilde{E}_B{}^{\overline{p}} \;. \tag{C.32b}$$

Furthermore, since the local Lorentz coordinates coincide with the curved space coordinates in the flat limit, we can set

$$\delta_{a\overline{b}} \;=\; \delta_{\overline{b}a} \;, \quad \delta_{ab} \;=\; 0 \;=\; \delta_{\overline{ab}} \;. \tag{C.32c}$$

Here we analyze the degrees of freedom of the metrics $\widetilde{g}_{MN}$. First, without any constraints, it has $(2d)^2 \;=\; 4d^2$ degrees of freedom. Using the constraints $\widetilde{g}_{mn} \;=\; 0 \;=\; \widetilde{g}_{\overline{mn}}$, it is reduced to $4d^2 - 2\times d^2 \;=\; 2d^2$. Furthermore, the symmetries $\widetilde{g}_{m\overline{n}} \;=\; \widetilde{g}_{\overline{n}m}$ halves the degrees to d^2, which coincides with the physical degrees of freedom of $\widetilde{g}_{m\overline{n}}$. The same reduction is also applied to the metric δ_{AB}, which gives d^2 degrees of freedom on $\delta_{a\overline{b}}$. The vielbein $e_M{}^A$ should carries the same number of degrees of freedom as the metrics $\widetilde{g}_{MN}$ and δ_{AB}, which we will consider. Originally, if there are no constraints, $e_M{}^A$ has $(2d)^2 = 4d^2$ degrees. The constraints

$$0 \;=\; \widetilde{g}_{mn} \;=\; \delta_{a\overline{b}}\,\widetilde{e}_m{}^a\widetilde{e}_n{}^{\overline{b}} + \delta_{\overline{a}b}\,\widetilde{e}_m{}^{\overline{a}}\widetilde{e}_n{}^b \;, \quad 0 \;=\; \widetilde{g}_{\overline{mn}} \;=\; \delta_{a\overline{b}}\,\widetilde{e}_{\overline{m}}{}^a\widetilde{e}_{\overline{n}}{}^{\overline{b}} + \delta_{\overline{a}b}\,\widetilde{e}_{\overline{m}}{}^{\overline{a}}\widetilde{e}_{\overline{n}}{}^b \tag{C.32d}$$

impose $2 \times d^2$ constraints on the vielbein. (The symmetric conditions on $\widetilde{g}_{m\overline{n}}$ and $\delta_{a\overline{b}}$ do not yield any constraints.) Then, the vielbein has only $4d^2 - 2d^2 = 2d^2$ degrees of freeom, which coincides with the metrics $\widetilde{g}_{m\overline{n}}$ and $\delta_{a\overline{b}}$. To realize such a reduction, we set $2d^2$ components of the vielbein to be zero. Although, there are actually many ways to do it, the following setting is much useful:

$$\widetilde{e}_m{}^{\overline{a}} \equiv 0 \, , \quad \widetilde{e}_{\overline{m}}{}^{a} \equiv 0 \, . \tag{C.32e}$$

We can confirm the matching of degrees of freedom of the metric and the vielbein. The metric has $2 \times d^2$ degrees of freedom (i.e., both $\widetilde{g}_{m\overline{n}}$ and $\widetilde{g}_{\overline{m}n}$ have d^2 degrees individually), while the non-trivial vielbein components $\{\widetilde{e}_m{}^a, \widetilde{e}_{\overline{n}}{}^{\overline{a}}\}$ have $2 \times d^2$ degrees of freedom. Later we will use the fixing condition (C.32e).

By using the vielbein and its inverse and (C.20), we find the relation between the hermitian connection and the spin connection. The non-vanishing hermitian connection give rise to the equations:

$$\widetilde{\Gamma}^p{}_{mn} = \widetilde{E}_a{}^p \Big(\partial_n \widetilde{e}_m{}^a + \widetilde{\omega}_n{}^a{}_b \, \widetilde{e}_m{}^b \Big) \, , \qquad \widetilde{\Gamma}^{\overline{p}}{}_{\overline{m}\overline{n}} = \widetilde{E}_{\overline{a}}{}^{\overline{p}} \Big(\partial_{\overline{n}} \widetilde{e}_{\overline{m}}{}^{\overline{a}} + \widetilde{\omega}_{\overline{n}}{}^{\overline{a}}{}_{\overline{b}} \, \widetilde{e}_{\overline{m}}{}^{\overline{b}} \Big) \, , \tag{C.33a}$$

$$\widetilde{\Gamma}^p{}_{m\overline{n}} = \widetilde{E}_a{}^p \Big(\partial_{\overline{n}} \widetilde{e}_m{}^a + \widetilde{\omega}_{\overline{n}}{}^a{}_b \, \widetilde{e}_m{}^b \Big) \, , \qquad \widetilde{\Gamma}^{\overline{p}}{}_{\overline{m}n} = \widetilde{E}_{\overline{a}}{}^{\overline{p}} \Big(\partial_n \widetilde{e}_{\overline{m}}{}^{\overline{a}} + \widetilde{\omega}_n{}^{\overline{a}}{}_{\overline{b}} \, \widetilde{e}_{\overline{m}}{}^{\overline{b}} \Big) \, , \tag{C.33b}$$

while the vanshing condition of the hermitian connection gives

$$0 = \widetilde{\Gamma}^p{}_{\overline{m}n} = \widetilde{E}_A{}^p \Big(\partial_n \widetilde{e}_{\overline{m}}{}^A + \widetilde{\omega}_n{}^A{}_B \, \widetilde{e}_{\overline{m}}{}^B \Big) = \widetilde{\omega}_n{}^a{}_{\overline{b}} \, \widetilde{e}_{\overline{m}}{}^{\overline{b}} \widetilde{E}_a{}^p \, , \tag{C.33c}$$

$$0 = \widetilde{\Gamma}^{\overline{p}}{}_{m\overline{n}} = \widetilde{E}_A{}^{\overline{p}} \Big(\partial_{\overline{n}} \widetilde{e}_m{}^A + \widetilde{\omega}_{\overline{n}}{}^A{}_B \, \widetilde{e}_m{}^B \Big) = \widetilde{\omega}_{\overline{n}}{}^{\overline{a}}{}_b \, \widetilde{e}_m{}^b \widetilde{E}_{\overline{a}}{}^{\overline{p}} \, , \tag{C.33d}$$

$$0 = \widetilde{\Gamma}^{\overline{p}}{}_{mn} = \widetilde{E}_A{}^{\overline{p}} \Big(\partial_n \widetilde{e}_m{}^A + \widetilde{\omega}_n{}^A{}_B \, \widetilde{e}_m{}^B \Big) = \widetilde{\omega}_n{}^{\overline{a}}{}_b \, \widetilde{e}_m{}^b \widetilde{E}_{\overline{a}}{}^{\overline{p}} \, , \tag{C.33e}$$

$$0 = \widetilde{\Gamma}^p{}_{\overline{m}\overline{n}} = \widetilde{E}_A{}^p \Big(\partial_{\overline{n}} \widetilde{e}_{\overline{m}}{}^A + \widetilde{\omega}_{\overline{n}}{}^A{}_B \, \widetilde{e}_{\overline{m}}{}^B \Big) = \widetilde{\omega}_{\overline{n}}{}^a{}_{\overline{b}} \, \widetilde{e}_{\overline{m}}{}^{\overline{b}} \widetilde{E}_a{}^p \, . \tag{C.33f}$$

Equivalently, we can also discuss the spin connection via (C.21):

$$\widetilde{\omega}_m{}^a{}_b = -\widetilde{E}_b{}^n \Big(\partial_m \widetilde{e}_n{}^a - \widetilde{\Gamma}^p{}_{nm} \, \widetilde{e}_p{}^a \Big) \, , \tag{C.34a}$$

$$\widetilde{\omega}_m{}^{\overline{a}}{}_{\overline{b}} = -\widetilde{E}_{\overline{b}}{}^{\overline{n}} \Big(\partial_m \widetilde{e}_{\overline{n}}{}^{\overline{a}} - \widetilde{\Gamma}^{\overline{p}}{}_{\overline{n}m} \, \widetilde{e}_{\overline{p}}{}^{\overline{a}} \Big) \, , \tag{C.34b}$$

$$\widetilde{\omega}_{\overline{m}}{}^a{}_b = -\widetilde{E}_b{}^n \Big(\partial_{\overline{m}} \widetilde{e}_n{}^a - \widetilde{\Gamma}^p{}_{n\overline{m}} \, \widetilde{e}_p{}^a \Big) \, , \tag{C.34c}$$

$$\widetilde{\omega}_{\overline{m}}{}^{\overline{a}}{}_{\overline{b}} = -\widetilde{E}_{\overline{b}}{}^{\overline{n}} \Big(\partial_{\overline{m}} \widetilde{e}_{\overline{n}}{}^{\overline{a}} - \widetilde{\Gamma}^{\overline{p}}{}_{\overline{n}\overline{m}} \, \widetilde{e}_{\overline{p}}{}^{\overline{a}} \Big) \, , \tag{C.34d}$$

$$\widetilde{\omega}_m{}^{\overline{a}}{}_b = -\widetilde{E}_b{}^N \Big(\partial_m \widetilde{e}_N{}^{\overline{a}} - \widetilde{\Gamma}^{\overline{p}}{}_{Nm} \, \widetilde{e}_{\overline{p}}{}^{\overline{a}} \Big) = \widetilde{\Gamma}^{\overline{p}}{}_{nm} \, \widetilde{e}_{\overline{p}}{}^{\overline{a}} \widetilde{E}_b{}^n = 0 \, , \tag{C.34e}$$

$$\widetilde{\omega}_{\overline{m}}{}^{\overline{a}}{}_b = -\widetilde{E}_b{}^N \Big(\partial_{\overline{m}} \widetilde{e}_N{}^{\overline{a}} - \widetilde{\Gamma}^{\overline{p}}{}_{N\overline{m}} \, \widetilde{e}_{\overline{p}}{}^{\overline{a}} \Big) = \widetilde{\Gamma}^{\overline{p}}{}_{n\overline{m}} \, \widetilde{e}_{\overline{p}}{}^{\overline{a}} \widetilde{E}_b{}^n = 0 \, , \tag{C.34f}$$

$$\widetilde{\omega}_m{}^a{}_{\overline{b}} = -\widetilde{E}_{\overline{b}}{}^N \Big(\partial_m \widetilde{e}_N{}^a - \widetilde{\Gamma}^p{}_{Nm} \, \widetilde{e}_p{}^a \Big) = \widetilde{\Gamma}^p{}_{\overline{n}m} \, \widetilde{e}_p{}^a \widetilde{E}_{\overline{b}}{}^{\overline{n}} = 0 \, , \tag{C.34g}$$

$$\widetilde{\omega}_{\overline{m}}{}^{a}{}_{\overline{b}} \ = \ -\widetilde{E}_{\overline{b}}{}^{N}\Big(\partial_{\overline{m}}\widetilde{e}_{N}{}^{a} - \widetilde{\Gamma}^{p}{}_{N\overline{m}}\,\widetilde{e}_{p}{}^{a}\Big) \ = \ \widetilde{\Gamma}^{p}{}_{\overline{n}\overline{m}}\,\widetilde{e}_{p}{}^{a}\widetilde{E}_{\overline{b}}{}^{\overline{n}} \ = \ 0 \; . \tag{C.34h}$$

This indicates that the hermitian spin connections of "pure type" such as $\widetilde{\omega}_{Mab}$ and $\widetilde{\omega}_{M\overline{a}\overline{b}}$ vanish on the hermitian manifold. Later we will analyze the spin connection more. The Riemann tensor of the spin connection is given in the following way:

$$[D_M(\widetilde{\omega}), D_N(\widetilde{\omega})] \ = \ -\frac{i}{2}\widetilde{R}_{ABMN}(\widetilde{\omega})\,\Sigma^{AB} \; , \tag{C.35a}$$

$$\widetilde{R}^{A}{}_{BMN}(\widetilde{\omega}) \ = \ \partial_M\widetilde{\omega}_N{}^{A}{}_{B} - \partial_N\widetilde{\omega}_M{}^{A}{}_{B} + \widetilde{\omega}_M{}^{A}{}_{C}\,\widetilde{\omega}_N{}^{C}{}_{B} - \widetilde{\omega}_N{}^{A}{}_{C}\,\widetilde{\omega}_M{}^{C}{}_{B} \; . \tag{C.35b}$$

Here let us explicitly describe the Riemann tensor:

$$\begin{aligned}
\widetilde{R}_{\overline{a}bMN}(\widetilde{\omega}) \ &= \ \partial_M\widetilde{\omega}_{N\overline{a}b} - \partial_N\widetilde{\omega}_{M\overline{a}b} + \widetilde{\omega}_{M\overline{a}c}\,\widetilde{\omega}_N{}^{c}{}_{b} - \widetilde{\omega}_{N\overline{a}c}\,\widetilde{\omega}_M{}^{c}{}_{b} \\
&= \ -\widetilde{R}_{\overline{a}bNM}(\widetilde{\omega}) \ = \ -\widetilde{R}_{b\overline{a}MN}(\widetilde{\omega}) \; ,
\end{aligned} \tag{C.35c}$$

$$\begin{aligned}
\widetilde{R}_{a\overline{b}MN}(\widetilde{\omega}) \ &= \ \partial_M\widetilde{\omega}_{Na\overline{b}} - \partial_N\widetilde{\omega}_{Ma\overline{b}} + \widetilde{\omega}_{Ma\overline{c}}\,\widetilde{\omega}_N{}^{\overline{c}}{}_{\overline{b}} - \widetilde{\omega}_{Na\overline{c}}\,\widetilde{\omega}_M{}^{\overline{c}}{}_{\overline{b}} \\
&= \ -\widetilde{R}_{a\overline{b}NM}(\widetilde{\omega}) \ = \ -\widetilde{R}_{\overline{b}aMN}(\widetilde{\omega}) \; ,
\end{aligned} \tag{C.35d}$$

$$\widetilde{R}_{abMN}(\widetilde{\omega}) \ = \ 0 \; , \quad \widetilde{R}_{\overline{a}\overline{b}MN}(\widetilde{\omega}) \ = \ 0 \; . \tag{C.35e}$$

Notice that the Riemann tensor is antisymmetric under the exchange between the latter two indices by definition. We also notice that it is also antisymmetric under the exchange between the **former** two indices via the antisymmetry of the spin connection $\widetilde{\omega}_{MAB} = -\widetilde{\omega}_{MBA}$ (A.29).

Here let us again discuss the properties on the Riemann tensor of the hermitian affine connection $\widetilde{R}_{p\overline{q}MN}(\widetilde{\Gamma})$ given in (C.25), i.e., tt is worth investigating whether the Riemann tensor $\widetilde{R}_{p\overline{q}MN}(\widetilde{\Gamma})$ has (anti)symmetries under exchanging of indices. Notice that since the hermitian manifold has, in general, a torsion, then the Riemann tensor might not has all the properties in (A.19b). Fortunately, however, it is related to the Riemann tensor of the spin connection in such a way as $\widetilde{R}_{p\overline{q}MN}(\widetilde{\Gamma}) = \widetilde{R}_{\overline{a}bMN}(\widetilde{\omega})\,\widetilde{e}_{p\overline{a}}\widetilde{e}_{\overline{q}b}$. Then we derive the followings:

$$\begin{aligned}
\widetilde{R}_{p\overline{q}MN}(\widetilde{\Gamma}) \ &= \ \widetilde{R}_{\overline{a}bMN}(\widetilde{\omega})\,\widetilde{e}_{p\overline{a}}\widetilde{e}_{\overline{q}b} \ = \ -\widetilde{R}_{\overline{a}bNM}(\widetilde{\omega})\,\widetilde{e}_{p\overline{a}}\widetilde{e}_{\overline{q}b} \ = \ \widetilde{R}_{b\overline{a}MN}(\widetilde{\omega})\,\widetilde{e}_{p\overline{a}}\widetilde{e}_{\overline{q}b} \\
&= \ -\widetilde{R}_{p\overline{q}NM}(\widetilde{\Gamma}) \ = \ -\widetilde{R}_{\overline{q}pMN}(\widetilde{\Gamma}) \; .
\end{aligned} \tag{C.36}$$

In the same way as the affine connection (C.28), the spin connection $\widetilde{\omega}_{MAB}$ should also be decomposed into the Levi-Civita part and the contorsion part via (C.34):

$$\widetilde{\omega}_{m\overline{a}b} \ = \ -\widetilde{E}_{b}{}^{n}\Big(\partial_m\widetilde{e}_{n\overline{a}} - \big\{\widetilde{\Gamma}^{p}_{(\mathrm{L})nm} + K^{p}{}_{nm}\big\}\widetilde{e}_{p\overline{a}}\Big) \ \equiv \ \widetilde{\omega}^{(\mathrm{L})}_{m\overline{a}b} + K_{\overline{a}bm} \; , \tag{C.37a}$$

$$\widetilde{\omega}_{ma\overline{b}} \ = \ -\widetilde{E}_{\overline{b}}{}^{\overline{n}}\Big(\partial_m\widetilde{e}_{\overline{n}a} - \big\{\widetilde{\Gamma}^{\overline{p}}_{(\mathrm{L})\overline{n}m} + K^{\overline{p}}{}_{\overline{n}m}\big\}\widetilde{e}_{\overline{p}a}\Big) \ \equiv \ \widetilde{\omega}^{(\mathrm{L})}_{ma\overline{b}} + K_{a\overline{b}m} \; , \tag{C.37b}$$

$$\widetilde{\omega}_{\overline{m}a\overline{b}} \ = \ -\widetilde{E}_{\overline{b}}{}^{\overline{n}}\Big(\partial_{\overline{m}}\widetilde{e}_{\overline{n}a} - \big\{\widetilde{\Gamma}^{\overline{p}}_{(\mathrm{L})\overline{n}\overline{m}} + K^{\overline{p}}{}_{\overline{n}\overline{m}}\big\}\widetilde{e}_{\overline{p}a}\Big) \ \equiv \ \widetilde{\omega}^{(\mathrm{L})}_{\overline{m}a\overline{b}} + K_{a\overline{b}\overline{m}} \; , \tag{C.37c}$$

$$\widetilde{\omega}_{\overline{m}ab} \ = \ -\widetilde{E}_b{}^n \Big(\partial_{\overline{m}} \widetilde{e}_{n\overline{a}} - \big\{ \widetilde{\Gamma}^p_{(\mathrm{L})n\overline{m}} + K^p{}_{n\overline{m}} \big\} \widetilde{e}_{p\overline{a}} \Big) \ \equiv \ \widetilde{\omega}^{(\mathrm{L})}_{\overline{m}ab} + K_{\overline{a}b\overline{m}} \ , \tag{C.37d}$$

$$0 \ = \ \widetilde{\omega}_{mab} \ = \ \big\{ \widetilde{\Gamma}^{\overline{p}}_{(\mathrm{L})nm} + K^{\overline{p}}{}_{nm} \big\} \widetilde{e}_{\overline{p}a} \widetilde{E}_b{}^n \ = \ K_{abm} \ , \tag{C.37e}$$

$$0 \ = \ \widetilde{\omega}_{\overline{m}ab} \ = \ \big\{ \widetilde{\Gamma}^{\overline{p}}_{(\mathrm{L})n\overline{m}} + K^{\overline{p}}{}_{n\overline{m}} \big\} \widetilde{e}_{\overline{p}a} \widetilde{E}_b{}^n \ \equiv \ \widetilde{\omega}^{(\mathrm{L})}_{\overline{m}ab} + K_{ab\overline{m}} \ , \tag{C.37f}$$

$$0 \ = \ \widetilde{\omega}_{m\overline{a}\overline{b}} \ = \ \big\{ \widetilde{\Gamma}^p{}_{(\mathrm{L})\overline{n}m} + K^p{}_{\overline{n}m} \big\} \widetilde{e}_{p\overline{a}} \widetilde{E}_{\overline{b}}{}^{\overline{n}} \ \equiv \ \widetilde{\omega}^{(\mathrm{L})}_{m\overline{a}\overline{b}} + K_{\overline{a}\overline{b}m} \ , \tag{C.37g}$$

$$0 \ = \ \widetilde{\omega}_{\overline{m}\overline{a}\overline{b}} \ = \ \big\{ \widetilde{\Gamma}^p_{(\mathrm{L})\overline{n}\overline{m}} + K^p{}_{\overline{n}\overline{m}} \big\} \widetilde{e}_{p\overline{a}} \widetilde{E}_{\overline{b}}{}^{\overline{n}} \ = \ K_{\overline{a}\overline{b}\overline{m}} \ . \tag{C.37h}$$

C.5 Additional constraint: Kähler manifold

So far, we have imposed the manifold is hermitian. Now we also introduce the Kähler form Ω

$$\Omega \ \equiv \ i\widetilde{g}_{m\overline{n}}\, \mathrm{d}z^m \wedge \mathrm{d}\overline{z}^{\overline{n}} \ = \ J_{m\overline{n}}\, \mathrm{d}z^m \wedge \mathrm{d}\overline{z}^{\overline{n}} \ \equiv \ J \ , \tag{C.38}$$

where the component of this two-form $J_{m\overline{n}} = J_m{}^p\, \widetilde{g}_{p\overline{n}}$ is given by the complex structure $J_m{}^n$ [13]. Due to this relation, the fundamental two-form J is also interpreted as the Kähler form. The closed condition of the Kähler form, $\mathrm{d}\Omega = 0$, is the definition of the Kähler manifold. This constraint is equivalent to a constraint on the hermitian metric:

$$\mathrm{d}\Omega \ = \ 0 \quad \leftrightarrow \quad \partial_m \widetilde{g}_{n\overline{p}} \ = \ \partial_n \widetilde{g}_{m\overline{p}} \quad \text{and} \quad \partial_{\overline{m}} \widetilde{g}_{n\overline{p}} \ = \ \partial_{\overline{p}} \widetilde{g}_{n\overline{m}} \ . \tag{C.39}$$

This is called the Kähler metric. If the metric is Kähler (or equivalently, if there is no (con)torsion), the hermitian connection (C.22) exactly coincides with the Levi-Civita connection of pure type, i.e., the affine connection of mixed type vanishes and that the exchange of the two subscripts of the hermitian connection becomes symmetric, see (C.29):

$$\widetilde{\Gamma}^M{}_{NP} \ = \ \widetilde{\Gamma}^M_{0NP} + K^M{}_{NP} \ = \ \widetilde{\Gamma}^M_{0NP} \ , \tag{C.40a}$$

$$\widetilde{\Gamma}^m_{0np} \ = \ \widetilde{\Gamma}^m_{0pn} \ = \ \widetilde{g}^{m\overline{q}} \partial_n \widetilde{g}_{p\overline{q}} \ , \quad \widetilde{\Gamma}^{\overline{m}}_{0\overline{n}\overline{p}} \ = \ \widetilde{\Gamma}^{\overline{m}}_{0\overline{p}\overline{n}} \ = \ \widetilde{g}^{\overline{m}q} \partial_{\overline{n}} \widetilde{g}_{q\overline{p}} \ , \tag{C.40b}$$

$$\widetilde{\Gamma}^{\overline{m}}_{0np} \ = \ \widetilde{\Gamma}^m_{0\overline{n}\overline{p}} \ = \ \widetilde{\Gamma}^m_{0n\overline{p}} \ = \ \widetilde{\Gamma}^{\overline{m}}_{0\overline{n}p} \ = \ 0 \ . \tag{C.40c}$$

This indicates that the metric on the hermitian manifold is affected by the torsion. Actually, the Levi-Civita connection $\widetilde{\Gamma}^m_{(\mathrm{L})n\overline{p}}$ on the hermitian manifold vanishes when the torsion disappears on the Kähler manifold. This is nothing but the evidence of the torsion back reaction on the metric! In addition, when we go back to the real coordinate frame, we find the equality conditions

$$\Gamma^P{}_{nm} \ = \ -\Gamma^P{}_{n'm'} \ , \quad \Gamma^P{}_{n'm} \ = \ \Gamma^P{}_{nm'} \ . \tag{C.41}$$

Futhermore, the closed condition $\mathrm{d}\Omega = 0$ also indicates that there is a conserved charge and we can introduce a Kähler potential $\mathcal{K}(z,\overline{z})$, which yields the metric satisfying the relation (C.39):

$$\widetilde{g}_{m\overline{n}} \ \equiv \ \partial_m \partial_{\overline{n}} \mathcal{K}(z,\overline{z}) \ . \tag{C.42}$$

Note that a hermitian manifold is a Kähler manifold if and only if the hermitian connection $\widetilde{\Gamma}^m{}_{np}$ is torsion free (see p.180 of [12]). Since the hermitian connection on the Kähler manifold is the Kähler Levi-Civita connection $\widetilde{\Gamma}^m_{0np}$ (C.40), the Riemann tensor on the Kähler manifold is more symmetric than the one on a generic hermitian manifold. Compared to (C.25), we can see

$$\widetilde{R}^m{}_{np\overline{q}}(\widetilde{\Gamma}_0) \ = \ -\partial_{\overline{q}} \widetilde{\Gamma}^m_{0np} \ = \ -\partial_{\overline{q}} \Big(\widetilde{g}^{m\overline{r}} \partial_p \widetilde{g}_{n\overline{r}} \Big) \ = \ -\partial_{\overline{q}} \Big(\widetilde{g}^{m\overline{r}} \partial_n \widetilde{g}_{p\overline{r}} \Big) \ = \ \widetilde{R}^m{}_{pn\overline{q}}(\widetilde{\Gamma}_0) \, . \tag{C.43a}$$

Combining it with the properties (C.25), we also obtain

$$\widetilde{R}_{m\overline{n}p\overline{q}}(\widetilde{\Gamma}_0) \ = \ -\widetilde{R}_{m\overline{n}\overline{q}p}(\widetilde{\Gamma}_0) \ = \ -\widetilde{R}_{\overline{n}mp\overline{q}}(\widetilde{\Gamma}_0) \ = \ \widetilde{R}_{\overline{n}m\overline{q}p}(\widetilde{\Gamma}_0) \, , \tag{C.43b}$$

$$\widetilde{R}^{\overline{m}}{}_{\overline{n}\overline{p}q}(\widetilde{\Gamma}_0) \ = \ \widetilde{R}^{\overline{m}}{}_{\overline{p}\overline{n}q}(\widetilde{\Gamma}_0) \, , \quad \widetilde{R}^m{}_{n\overline{p}q}(\widetilde{\Gamma}_0) \ = \ \widetilde{R}^m{}_{q\overline{p}n}(\widetilde{\Gamma}_0) \, , \quad \widetilde{R}^{\overline{m}}{}_{\overline{n}q\overline{p}}(\widetilde{\Gamma}_0) \ = \ \widetilde{R}^{\overline{m}}{}_{\overline{p}q\overline{n}}(\widetilde{\Gamma}_0) \, . \tag{C.43c}$$

The spin connection on the Kähler manifold is also more restricted than the one on the hermitian manifold, i.e., the torsionless spin connection $\widetilde{\omega}_0$ associated with the Kähler affine connection (C.40) via the relation (C.17a):

$$0 \ = \ D_m(\widetilde{\omega}_0, \widetilde{\Gamma}_0) \widetilde{e}_n{}^a \ = \ \partial_m \widetilde{e}_n{}^a - \widetilde{\Gamma}^p_{0nm} \, \widetilde{e}_p{}^a + \widetilde{\omega}_{0m}{}^a{}_b \, \widetilde{e}_n{}^b \, , \tag{C.44a}$$

$$0 \ = \ D_m(\widetilde{\omega}_0, \widetilde{\Gamma}_0) \widetilde{e}_{\overline{n}}{}^{\overline{a}} \ = \ \partial_m \widetilde{e}_{\overline{n}}{}^{\overline{a}} + \widetilde{\omega}_{0m}{}^{\overline{a}}{}_{\overline{b}} \, \widetilde{e}_{\overline{n}}{}^{\overline{b}} \, , \tag{C.44b}$$

$$0 \ = \ D_{\overline{m}}(\widetilde{\omega}_0, \widetilde{\Gamma}_0) \widetilde{e}_n{}^a \ = \ \partial_{\overline{m}} \widetilde{e}_n{}^a + \widetilde{\omega}_{0\overline{m}}{}^a{}_b \, \widetilde{e}_n{}^b \, , \tag{C.44c}$$

$$0 \ = \ D_{\overline{m}}(\widetilde{\omega}_0, \widetilde{\Gamma}_0) \widetilde{e}_{\overline{n}}{}^{\overline{a}} \ = \ \partial_{\overline{m}} \widetilde{e}_{\overline{n}}{}^{\overline{a}} - \widetilde{\Gamma}^{\overline{p}}_{0\overline{n}\overline{m}} \, \widetilde{e}_{\overline{p}}{}^{\overline{a}} + \widetilde{\omega}_{0\overline{m}}{}^{\overline{a}}{}_{\overline{b}} \, \widetilde{e}_{\overline{n}}{}^{\overline{b}} \, . \tag{C.44d}$$

First, the explicit forms of the spin connection are given from (C.44):

$$\widetilde{\omega}_{0m}{}^a{}_b \ = \ -\widetilde{E}_b{}^n \partial_m \widetilde{e}_n{}^a + \widetilde{\Gamma}^p_{0nm} \, \widetilde{e}_p{}^a \widetilde{E}_b{}^n \, , \qquad \widetilde{\omega}_{0m}{}^{\overline{a}}{}_{\overline{b}} \ = \ -\widetilde{E}_{\overline{b}}{}^{\overline{n}} \partial_m \widetilde{e}_{\overline{n}}{}^{\overline{a}} \, , \tag{C.45a}$$

$$\widetilde{\omega}_{0\overline{m}}{}^a{}_b \ = \ -\widetilde{E}_b{}^n \partial_m \widetilde{e}_n{}^a \, , \qquad \widetilde{\omega}_{0\overline{m}}{}^{\overline{a}}{}_{\overline{b}} \ = \ -\widetilde{E}_{\overline{b}}{}^{\overline{n}} \partial_{\overline{m}} \widetilde{e}_{\overline{n}}{}^{\overline{a}} + \widetilde{\Gamma}^{\overline{p}}_{0\overline{n}\overline{m}} \, \widetilde{e}_{\overline{p}}{}^{\overline{a}} \widetilde{E}_{\overline{b}}{}^{\overline{n}} \, . \tag{C.45b}$$

These are related to each other via the antisymmetry $\widetilde{\omega}_{MAB} = -\widetilde{\omega}_{MBA}$ in the following way:

$$\begin{aligned}
\widetilde{\omega}_{0m}{}^a{}_b \ &= \ -\widetilde{E}_b{}^n \partial_m \widetilde{e}_n{}^a + \widetilde{\Gamma}^p_{0nm} \, \widetilde{e}_p{}^a \widetilde{E}_b{}^n \ = \ -\widetilde{E}_b{}^n \partial_m \Big(\widetilde{g}_{n\overline{q}} \, \widetilde{e}^{\overline{q}a} \Big) + \Big(\widetilde{g}^{p\overline{q}} \partial_m \widetilde{g}_{n\overline{q}} \Big) \widetilde{e}_p{}^a \widetilde{E}_b{}^n \\
&= \ -\widetilde{E}_b{}^n \Big(\partial_m \widetilde{g}_{n\overline{q}} \Big) \widetilde{e}^{\overline{q}a} - \widetilde{E}_{b\overline{q}} \partial_m \widetilde{e}^{\overline{q}a} + \partial_m \widetilde{g}_{n\overline{q}} \, \widetilde{e}^{\overline{q}a} \widetilde{E}_b{}^n \\
&= \ \widetilde{e}^{\overline{q}a} \partial_m \widetilde{E}_{b\overline{q}} \ = \ \widetilde{E}^{a\overline{q}} \partial_m \widetilde{e}_{\overline{q}b} \ = \ \delta^{a\overline{c}} \delta_{b\overline{d}} \, \widetilde{E}_{\overline{c}}{}^{\overline{q}} \partial_m \widetilde{e}_{\overline{q}}{}^{\overline{d}} \\
&= \ -\delta^{a\overline{c}} \delta_{b\overline{d}} \, \widetilde{\omega}_{0m}{}^{\overline{d}}{}_{\overline{c}} \, ,
\end{aligned} \tag{C.46a}$$

$$\begin{aligned}
\widetilde{\omega}_{0\overline{m}}{}^{\overline{a}}{}_{\overline{b}} \ &= \ -\widetilde{E}_{\overline{b}}{}^{\overline{n}} \partial_{\overline{m}} \widetilde{e}_{\overline{n}}{}^{\overline{a}} + \widetilde{\Gamma}^{\overline{p}}_{0\overline{n}\overline{m}} \, \widetilde{e}_{\overline{p}}{}^{\overline{a}} \widetilde{E}_{\overline{b}}{}^{\overline{n}} \ = \ -\widetilde{E}_{\overline{b}}{}^{\overline{n}} \partial_{\overline{m}} \Big(\widetilde{g}_{\overline{n}q} \widetilde{e}^{q\overline{a}} \Big) + \Big(\widetilde{g}^{\overline{p}q} \partial_{\overline{m}} \widetilde{g}_{\overline{n}q} \Big) \widetilde{e}_{\overline{p}}{}^{\overline{a}} \widetilde{E}_{\overline{b}}{}^{\overline{n}} \\
&= \ -\widetilde{E}_{\overline{b}}{}^{\overline{n}} \Big(\partial_{\overline{m}} \widetilde{g}_{\overline{n}q} \Big) \widetilde{e}^{q\overline{a}} - \widetilde{E}_{\overline{b}q} \partial_{\overline{m}} \widetilde{e}^{q\overline{a}} + \partial_{\overline{m}} \widetilde{g}_{\overline{n}q} \, \widetilde{e}^{q\overline{a}} \widetilde{E}_{\overline{b}}{}^{\overline{n}} \\
&= \ \widetilde{e}^{q\overline{a}} \partial_{\overline{m}} \widetilde{E}_{\overline{b}q} \ = \ \widetilde{E}^{\overline{a}q} \partial_{\overline{m}} \widetilde{e}_{q\overline{b}} \ = \ \delta^{\overline{a}c} \delta_{\overline{b}d} \, \widetilde{E}_c{}^q \partial_{\overline{m}} \widetilde{e}_q{}^d \\
&= \ -\delta^{\overline{a}c} \delta_{\overline{b}d} \, \widetilde{\omega}_{0\overline{m}}{}^d{}_c \, ,
\end{aligned} \tag{C.46b}$$

where we used $\widetilde{E}_{b\overline{q}} = \widetilde{e}_{\overline{q}b}$ given by (A.24). Thus, the following forms are much useful to analyze the Riemann tensor:

$$\widetilde{\omega}_{0m}{}^{a}{}_{b} \;=\; -\widetilde{\omega}_{0mb}{}^{a} \;=\; -\delta_{b\overline{d}}\delta^{a\overline{c}}\,\widetilde{\omega}_{0m}{}^{\overline{d}}{}_{\overline{c}} \;=\; \widetilde{E}^{a\overline{q}}\partial_m \widetilde{e}_{\overline{q}b}\;, \tag{C.47a}$$

$$\widetilde{\omega}_{0\overline{m}}{}^{\overline{a}}{}_{\overline{b}} \;=\; -\widetilde{\omega}_{0\overline{m}\overline{b}}{}^{\overline{a}} \;=\; -\delta^{\overline{a}c}\delta_{\overline{b}d}\,\widetilde{\omega}_{0\overline{m}}{}^{d}{}_{c} \;=\; \widetilde{E}^{\overline{a}q}\partial_{\overline{m}} \widetilde{e}_{q\overline{b}}\;. \tag{C.47b}$$

Following these expressions, it turns out the following:

$$\begin{aligned}
2\widetilde{\omega}_{0[m}{}^{a}{}_{|c|}\,\widetilde{\omega}_{0n]}{}^{c}{}_{b} \;&=\; \Big(\widetilde{E}^{a\overline{p}}\partial_m \widetilde{e}_{\overline{p}c}\Big)\Big(\widetilde{E}^{c\overline{q}}\partial_n \widetilde{e}_{\overline{q}b}\Big) - \Big(\widetilde{E}^{a\overline{p}}\partial_n \widetilde{e}_{\overline{p}c}\Big)\Big(\widetilde{E}^{c\overline{q}}\partial_m \widetilde{e}_{\overline{q}b}\Big) \\
&=\; \partial_m\Big(\widetilde{E}^{a\overline{q}}\partial_n \widetilde{e}_{\overline{q}b}\Big) - 2\partial_m \widetilde{E}^{a\overline{q}}\partial_n \widetilde{e}_{\overline{q}b} - \widetilde{E}^{a\overline{q}}\partial_m\partial_n \widetilde{e}_{\overline{q}b} \\
&\quad - \partial_n\Big(\widetilde{E}^{a\overline{q}}\partial_m \widetilde{e}_{\overline{q}b}\Big) + 2\partial_n \widetilde{E}^{a\overline{q}}\partial_m \widetilde{e}_{\overline{q}b} + \widetilde{E}^{a\overline{q}}\partial_m\partial_n \widetilde{e}_{\overline{q}b} \\
&=\; \partial_m\Big(\widetilde{E}^{a\overline{q}}\partial_n \widetilde{e}_{\overline{q}b}\Big) - 2\partial_n\Big(\widetilde{e}_{\overline{q}b}\partial_m \widetilde{E}^{a\overline{q}}\Big) + 2\widetilde{e}_{\overline{q}b}\partial_m\partial_n \widetilde{E}^{a\overline{q}} \\
&\quad - \partial_n\Big(\widetilde{E}^{a\overline{q}}\partial_m \widetilde{e}_{\overline{q}b}\Big) + 2\partial_m\Big(\widetilde{e}_{\overline{q}b}\partial_n \widetilde{E}^{a\overline{q}}\Big) - 2\widetilde{e}_{\overline{q}b}\partial_m\partial_n \widetilde{E}^{a\overline{q}} \\
&=\; -\partial_m\Big(\widetilde{E}^{a\overline{q}}\partial_n \widetilde{e}_{\overline{q}b}\Big) + \partial_n\Big(\widetilde{E}^{a\overline{q}}\partial_m \widetilde{e}_{\overline{q}b}\Big) \;=\; -\partial_m \widetilde{\omega}_{0n}{}^{a}{}_{b} + \partial_n \widetilde{\omega}_{0m}{}^{a}{}_{b} \\
&=\; -2\partial_{[m}\widetilde{\omega}_{0n]}{}^{a}{}_{b}\;.
\end{aligned} \tag{C.48}$$

By using (C.48), we obtain the following remarkable property on the Riemann tensor of the spin connection as well as the one of the affine connection (C.43):

$$\widetilde{R}^{a}{}_{bmn}(\widetilde{\omega}_0) \;=\; 2\partial_{[m}\widetilde{\omega}_{0n]}{}^{a}{}_{b} + 2\widetilde{\omega}_{0[m}{}^{a}{}_{|c|}\,\widetilde{\omega}_{0n]}{}^{c}{}_{b} \;=\; 0\;, \tag{C.49a}$$

$$\widetilde{R}^{\overline{a}}{}_{\overline{b}mn}(\widetilde{\omega}_0) \;=\; 0\;, \qquad \widetilde{R}^{a}{}_{b\overline{m}\overline{n}}(\widetilde{\omega}_0) \;=\; 0\;, \qquad \widetilde{R}^{\overline{a}}{}_{\overline{b}\overline{m}\overline{n}}(\widetilde{\omega}_0) \;=\; 0\;. \tag{C.49b}$$

Furthermore, since the Riemann tensor of the spin connection is related to the Riemann tensor of the affine connection, we find that the non-trivial component of the Riemann tensor is given as

$$\widetilde{R}_{a\overline{b}m\overline{n}}(\widetilde{\omega}_0) \;=\; \widetilde{E}_{a}{}^{p}\widetilde{E}_{\overline{b}}{}^{\overline{q}}\,\widetilde{R}_{p\overline{q}m\overline{n}}(\widetilde{\Gamma}_0)\;. \tag{C.49c}$$

D Introducing torsion

We have already known that the NS-NS three-form flux H plays as a totally anti-symmetric **contorsion** on a compactified six-dimensional manifold in heterotic string theory [9]. In particular, this manifold is a conformally balanced manifold whose contorsion is given as the Bismut torsion:

$$H \;=\; \frac{i}{2}\big(\partial - \overline{\partial}\big) J \;=\; -\frac{1}{2} \mathrm{d}^c J \; , \qquad J \;=\; i \widetilde{g}_{m\overline{n}} \, \mathrm{d}z^m \wedge \mathrm{d}\overline{z}^{\overline{n}} \; , \tag{D.1}$$

where J is the fundamental two-form on the manifold, whose component is given by the complex structure satisfying the covariantly constant condition $D_M(\Gamma_-) J_N{}^P = 0$, where $\Gamma_- = \Gamma_{(\mathrm{L})} - H$. Notice that the fundamental two-form is not closed $\mathrm{d}J \neq 0$. See for the detail in appendix A.7. Due to this, the three-form is decomposed into two parts; the (2,1)-form $H^{(2,1)} = \frac{i}{2}\partial J$ and the $(1,2)$-form $H^{(1,2)} = -\frac{i}{2}\overline{\partial} J$, whose components are given as

$$\begin{aligned}
H^{(2,1)} \;&=\; \frac{i}{2} \partial J \;=\; \frac{i}{2} \partial_m J_{n\overline{p}} \, \mathrm{d}z^m \wedge \mathrm{d}z^n \wedge \mathrm{d}\overline{z}^{\overline{p}} \; , && \text{(D.2a)} \\
H^{(1,2)} \;&=\; -\frac{i}{2} \overline{\partial} J \;=\; -\frac{i}{2} \partial_{\overline{m}} J_{\overline{n}p} \, \mathrm{d}\overline{z}^{\overline{m}} \wedge \mathrm{d}\overline{z}^{\overline{n}} \wedge \mathrm{d}z^p \; , && \text{(D.2b)} \\
H_{mn\overline{p}} \;&=\; \frac{i}{2} \Big(\partial_m J_{n\overline{p}} - \partial_n J_{m\overline{p}} \Big) \;=\; -\frac{1}{2} \Big(\partial_m \widetilde{g}_{n\overline{p}} - \partial_n \widetilde{g}_{m\overline{p}} \Big) \; , && \text{(D.2c)} \\
H_{\overline{m}\overline{n}p} \;&=\; -\frac{i}{2} \Big(\partial_{\overline{m}} J_{\overline{n}p} - \partial_{\overline{n}} J_{\overline{m}p} \Big) \;=\; -\frac{1}{2} \Big(\partial_{\overline{m}} \widetilde{g}_{\overline{n}p} - \partial_{\overline{n}} \widetilde{g}_{\overline{m}p} \Big) \; . && \text{(D.2d)}
\end{aligned}$$

Furthermore, the exterior derivative of the NS-NS three-form is given as

$$\mathrm{d}H \;=\; (\partial + \overline{\partial}) \frac{i}{2} (\partial - \overline{\partial}) J \;=\; -\frac{i}{2} \partial \overline{\partial} J + \frac{i}{2} \overline{\partial} \partial J \; . \tag{D.3}$$

Then we find that $\mathrm{d}H$ is $(2,2)$-form and there is neither $(3,1)$- nor $(1,3)$-form in the expansion. Notice that ∂ commutes with $\overline{\partial}$ if and only if the manifold is Kähler, which is not the present case. In addition, the NS-NS three-form (D.1) can also be given as

$$H^{(2,1)} \;=\; \frac{1}{2} H_{mp\overline{q}} \, \mathrm{d}z^m \wedge \mathrm{d}z^p \wedge \mathrm{d}\overline{z}^{\overline{q}} \; , \qquad H^{(1,2)} \;=\; \frac{1}{2} H_{m\overline{p}\overline{q}} \, \mathrm{d}z^m \wedge \mathrm{d}\overline{z}^{\overline{p}} \wedge \mathrm{d}\overline{z}^{\overline{q}} \; . \tag{D.4}$$

Actually, the NS-NS three-form H is the totally antisymmetric torsion, as well as the totally anti-symmetric contorsion on the hermitian manifold (C.26b) with different sign; i.e., $K = -H$. The difference of sign is from the definition of the complex structure in the supergravity [11] in such a way as $0 = D_M(\Gamma_-) J_N{}^P$. Following to (C.28) and (C.37), we again describe the hermitian affine connection and the spin connection:

$$\begin{aligned}
\widetilde{\Gamma}{}^m_{-np} \;&\equiv\; \widetilde{\Gamma}{}^m_{(\mathrm{L})np} - H^m{}_{np} \;=\; \frac{1}{2} \widetilde{g}^{m\overline{q}} \Big(\partial_n \widetilde{g}_{p\overline{q}} + \partial_p \widetilde{g}_{n\overline{q}} \Big) + \frac{1}{2} \widetilde{g}^{m\overline{q}} \Big(\partial_n \widetilde{g}_{p\overline{q}} - \partial_p \widetilde{g}_{n\overline{q}} \Big) \; , && \text{(D.5a)} \\
\widetilde{\Gamma}{}^{\overline{m}}_{-\overline{n}\overline{p}} \;&\equiv\; \widetilde{\Gamma}{}^{\overline{m}}_{(\mathrm{L})\overline{n}\overline{p}} - H^{\overline{m}}{}_{\overline{n}\overline{p}} \;=\; \frac{1}{2} \widetilde{g}^{\overline{m}q} \Big(\partial_{\overline{n}} \widetilde{g}_{\overline{p}q} + \partial_{\overline{p}} \widetilde{g}_{\overline{n}q} \Big) + \frac{1}{2} \widetilde{g}^{\overline{m}q} \Big(\partial_{\overline{n}} \widetilde{g}_{\overline{p}q} - \partial_{\overline{p}} \widetilde{g}_{\overline{n}q} \Big) \; , && \text{(D.5b)}
\end{aligned}$$

$$\widetilde{\Gamma}^m_{-n\overline{p}} \equiv \widetilde{\Gamma}^m_{(\mathrm{L})n\overline{p}} - H^m{}_{n\overline{p}} = \frac{1}{2}\widetilde{g}^{m\overline{q}}\Big(\partial_{\overline{p}}\widetilde{g}_{n\overline{q}} - \partial_{\overline{q}}\widetilde{g}_{n\overline{p}}\Big) - \frac{1}{2}\widetilde{g}^{m\overline{q}}\Big(\partial_{\overline{q}}\widetilde{g}_{\overline{p}n} - \partial_{\overline{p}}\widetilde{g}_{\overline{q}n}\Big) , \quad \text{(D.5c)}$$

$$\widetilde{\Gamma}^{\overline{m}}_{-\overline{n}p} \equiv \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})\overline{n}p} - H^{\overline{m}}{}_{\overline{n}p} = \frac{1}{2}\widetilde{g}^{\overline{m}q}\Big(\partial_p\widetilde{g}_{\overline{n}q} - \partial_q\widetilde{g}_{\overline{n}p}\Big) - \frac{1}{2}\widetilde{g}^{\overline{m}q}\Big(\partial_q\widetilde{g}_{p\overline{n}} - \partial_p\widetilde{g}_{q\overline{n}}\Big) , \quad \text{(D.5d)}$$

$$0 = \widetilde{\Gamma}^{\overline{m}}_{-np} = \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})np} - H^{\overline{m}}{}_{np} = 0 - 0 , \quad \text{(D.5e)}$$

$$0 = \widetilde{\Gamma}^m_{-\overline{n}\overline{p}} = \widetilde{\Gamma}^m_{(\mathrm{L})\overline{n}\overline{p}} - H^m{}_{\overline{n}\overline{p}} = 0 - 0 , \quad \text{(D.5f)}$$

$$0 = \widetilde{\Gamma}^m_{-\overline{n}p} = \widetilde{\Gamma}^m_{(\mathrm{L})\overline{n}p} - H^m{}_{\overline{n}p} = \frac{1}{2}\widetilde{g}^{m\overline{q}}\Big(\partial_{\overline{n}}\widetilde{g}_{\overline{q}p} - \partial_{\overline{q}}\widetilde{g}_{\overline{n}p}\Big) - \frac{1}{2}\widetilde{g}^{m\overline{q}}\Big(\partial_{\overline{n}}\widetilde{g}_{\overline{q}p} - \partial_{\overline{q}}\widetilde{g}_{\overline{n}p}\Big) , \quad \text{(D.5g)}$$

$$0 = \widetilde{\Gamma}^{\overline{m}}_{-n\overline{p}} = \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})n\overline{p}} - H^{\overline{m}}{}_{n\overline{p}} = \frac{1}{2}\widetilde{g}^{\overline{m}q}\Big(\partial_n\widetilde{g}_{q\overline{p}} - \partial_q\widetilde{g}_{n\overline{p}}\Big) - \frac{1}{2}\widetilde{g}^{\overline{m}q}\Big(\partial_n\widetilde{g}_{q\overline{p}} - \partial_q\widetilde{g}_{n\overline{p}}\Big) , \quad \text{(D.5h)}$$

$$\widetilde{\omega}^{(-)}_{ma\overline{b}} = -\widetilde{E}_{\overline{b}}{}^{\overline{n}}\Big(\partial_m\widetilde{e}_{\overline{n}a} - \big\{\widetilde{\Gamma}^{\overline{p}}_{(\mathrm{L})\overline{n}m} - H^{\overline{p}}{}_{\overline{n}m}\big\}\widetilde{e}_{\overline{p}a}\Big) \equiv \widetilde{\omega}^{(\mathrm{L})}_{ma\overline{b}} - H_{ma\overline{b}} , \quad \text{(D.5i)}$$

$$\widetilde{\omega}^{(-)}_{m\overline{a}b} = -\widetilde{E}_b{}^n\Big(\partial_m\widetilde{e}_{n\overline{a}} - \big\{\widetilde{\Gamma}^p_{(\mathrm{L})nm} - H^p{}_{nm}\big\}\widetilde{e}_{p\overline{a}}\Big) \equiv \widetilde{\omega}^{(\mathrm{L})}_{m\overline{a}b} - H_{m\overline{a}b} , \quad \text{(D.5j)}$$

$$\widetilde{\omega}^{(-)}_{\overline{m}a\overline{b}} = -\widetilde{E}_{\overline{b}}{}^{\overline{n}}\Big(\partial_{\overline{m}}\widetilde{e}_{\overline{n}a} - \big\{\widetilde{\Gamma}^{\overline{p}}_{(\mathrm{L})\overline{n}\overline{m}} - H^{\overline{p}}{}_{\overline{n}\overline{m}}\big\}\widetilde{e}_{\overline{p}a}\Big) \equiv \widetilde{\omega}^{(\mathrm{L})}_{\overline{m}a\overline{b}} - H_{\overline{m}a\overline{b}} , \quad \text{(D.5k)}$$

$$\widetilde{\omega}^{(-)}_{\overline{m}\overline{a}b} = -\widetilde{E}_b{}^n\Big(\partial_{\overline{m}}\widetilde{e}_{n\overline{a}} - \big\{\widetilde{\Gamma}^p_{(\mathrm{L})n\overline{m}} - H^p{}_{n\overline{m}}\big\}\widetilde{e}_{p\overline{a}}\Big) \equiv \widetilde{\omega}^{(\mathrm{L})}_{\overline{m}\overline{a}b} - H_{\overline{m}\overline{a}b} , \quad \text{(D.5l)}$$

$$0 = \widetilde{\omega}^{(-)}_{mab} = \big\{\widetilde{\Gamma}^{\overline{p}}_{(\mathrm{L})nm} - H^{\overline{p}}{}_{nm}\big\}\widetilde{e}_{\overline{p}a}\widetilde{E}_b{}^n = 0 - 0 , \quad \text{(D.5m)}$$

$$0 = \widetilde{\omega}^{(-)}_{\overline{m}\overline{a}\overline{b}} = \big\{\widetilde{\Gamma}^p_{(\mathrm{L})\overline{n}\overline{m}} - H^p{}_{\overline{n}\overline{m}}\big\}\widetilde{e}_{p\overline{a}}\widetilde{E}_{\overline{b}}{}^{\overline{n}} = 0 - 0 , \quad \text{(D.5n)}$$

$$0 = \widetilde{\omega}^{(-)}_{\overline{m}ab} = \big\{\widetilde{\Gamma}^{\overline{p}}_{(\mathrm{L})n\overline{m}} - H^{\overline{p}}{}_{n\overline{m}}\big\}\widetilde{e}_{\overline{p}a}\widetilde{E}_b{}^n \equiv \widetilde{\omega}^{(\mathrm{L})}_{\overline{m}ab} - H_{\overline{m}ab} , \quad \text{(D.5o)}$$

$$0 = \widetilde{\omega}^{(-)}_{m\overline{a}\overline{b}} = \big\{\widetilde{\Gamma}^p{}_{(\mathrm{L})\overline{n}m} - H^p{}_{\overline{n}m}\big\}\widetilde{e}_{p\overline{a}}\widetilde{E}_{\overline{b}}{}^{\overline{n}} \equiv \widetilde{\omega}^{(\mathrm{L})}_{m\overline{a}\overline{b}} - H_{m\overline{a}\overline{b}} . \quad \text{(D.5p)}$$

Here we used (C.29) and (D.2). Notice that, under a certain transformation, the connection is not a tensor, while the torsion is. These are easily applied to more genetric spin connection such as

$$\widetilde{\Gamma}^m_{(\alpha)np} = \widetilde{\Gamma}^m_{(\mathrm{L})np} + \alpha H^m{}_{np} , \quad \widetilde{\Gamma}^{\overline{m}}_{(\alpha)\overline{n}\overline{p}} = \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})\overline{n}\overline{p}} + \alpha H^{\overline{m}}{}_{\overline{n}\overline{p}} , \quad \text{(D.6a)}$$

$$\widetilde{\Gamma}^m_{(\alpha)n\overline{p}} = \widetilde{\Gamma}^m_{(\mathrm{L})n\overline{p}} + \alpha H^m{}_{n\overline{p}} , \quad \widetilde{\Gamma}^{\overline{m}}_{(\alpha)\overline{n}p} = \widetilde{\Gamma}^{\overline{m}}_{(\mathrm{L})\overline{n}p} + \alpha H^{\overline{m}}{}_{\overline{n}p} , \quad \text{(D.6b)}$$

$$\widetilde{\Gamma}^{\overline{m}}_{(\alpha)np} = 0 , \quad \widetilde{\Gamma}^m_{(\alpha)\overline{n}\overline{p}} = 0 , \quad \text{(D.6c)}$$

$$\widetilde{\Gamma}^m_{(\alpha)\overline{n}p} = (1+\alpha)H^m{}_{\overline{n}p} , \quad \widetilde{\Gamma}^{\overline{m}}_{(\alpha)n\overline{p}} = (1+\alpha)H^{\overline{m}}{}_{n\overline{p}} , \quad \text{(D.6d)}$$

$$\widetilde{\omega}^{(\alpha)}_{ma\overline{b}} = \widetilde{\omega}^{(\mathrm{L})}_{ma\overline{b}} + \alpha H_{ma\overline{b}} = \widetilde{\omega}^{(-)}_{ma\overline{b}} + (1+\alpha)H_{ma\overline{b}} , \quad \text{(D.6e)}$$

$$\widetilde{\omega}^{(\alpha)}_{m\overline{a}b} = \widetilde{\omega}^{(\mathrm{L})}_{m\overline{a}b} + \alpha H_{m\overline{a}b} = \widetilde{\omega}^{(-)}_{m\overline{a}b} + (1+\alpha)H_{m\overline{a}b} , \quad \text{(D.6f)}$$

$$\widetilde{\omega}^{(\alpha)}_{\overline{m}a\overline{b}} = \widetilde{\omega}^{(\mathrm{L})}_{\overline{m}a\overline{b}} + \alpha H_{\overline{m}a\overline{b}} = \widetilde{\omega}^{(-)}_{\overline{m}a\overline{b}} + (1+\alpha)H_{\overline{m}a\overline{b}} , \quad \text{(D.6g)}$$

$$\widetilde{\omega}^{(\alpha)}_{\overline{m}\overline{a}b} = \widetilde{\omega}^{(\mathrm{L})}_{\overline{m}\overline{a}b} + \alpha H_{\overline{m}\overline{a}b} = \widetilde{\omega}^{(-)}_{\overline{m}\overline{a}b} + (1+\alpha)H_{\overline{m}\overline{a}b} , \quad \text{(D.6h)}$$

$$\widetilde{\omega}^{(\alpha)}_{mab} = 0 , \quad \widetilde{\omega}^{(\alpha)}_{\overline{m}\overline{a}\overline{b}} = 0 , \quad \text{(D.6i)}$$

$$\widetilde{\omega}^{(\alpha)}_{\overline{m}ab} = \widetilde{\omega}^{(\mathrm{L})}_{\overline{m}ab} + \alpha H_{\overline{m}ab} = (1+\alpha)H_{\overline{m}ab} , \quad \text{(D.6j)}$$

$$\widetilde{\omega}^{(\alpha)}_{m\overline{a}\overline{b}} = \widetilde{\omega}^{(\mathrm{L})}_{m\overline{a}\overline{b}} + \alpha H_{m\overline{a}\overline{b}} = (1+\alpha)H_{m\overline{a}\overline{b}} , \quad \text{(D.6k)}$$

where the factor α indicates the relation between the contorsion and the NS-NS three-form via $K = \alpha H$. We should again notice that, under a certain transformation, the connection is not a tensor, while the torsion is. These appear in the equations of motion for fermions in (heterotic) supergravity when $\alpha = -1/3$, while $\widetilde{\omega}^{(-)}_{MAB}$ (i.e., $\alpha = -1$), which is nothing but the hermitian spin connection with contorsion $K = -H$, appears in the supersymmetry variation of the gravitino in the same supergravity system [11]. Notice that only at the point $\alpha = 0$, we should set the eqs. (D.6a) are reduced to the Kähler affine connection, while the eqs. (D.6b) disappear, i.e., we should set $\{\widetilde{\Gamma}^m_{(\mathrm{L})np}, \widetilde{\Gamma}^m_{(\mathrm{L})n\overline{p}}, H, \widetilde{\omega}^{(\alpha)}\} = \{\widetilde{\Gamma}^m_{0np}, 0, 0, \widetilde{\omega}_0\}$, because the manifold becomes Kähler, whose metric satisfies (C.39). This point is isolated from the continuous one-parameter line α.

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