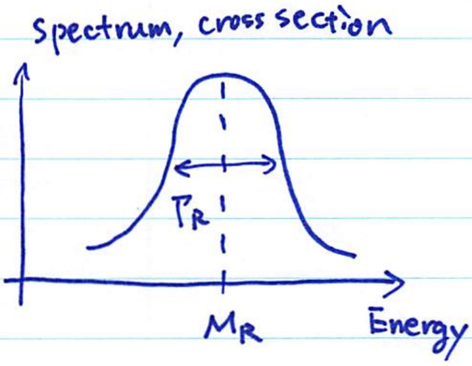
§ 1. Scattering theory

1. 1. Resonance as an eigenstate of Hamiltonian



(Breit-Wigner)

$$E = M_R - \frac{i}{2}\Gamma_R \quad : \text{complex}$$

mass $\qquad$ width

$$\boxed{\text{Resonance is a discrete eigenstate of Hamiltonian with a \underline{complex} eigenenergy}}$$

- Hermite operator (e.g. $\hat{H}$) should have a real eigenvalue. —— ①

"Hermiticity" is defined, not only by the operator $\hat{H}$ but also by the space $\{|\psi_n\rangle\}$ on which $\hat{H}$ acts.

def. of $^+$: $\langle \psi_n | \hat{H} \psi_m \rangle \equiv \langle \hat{H}^+ \psi_n | \psi_m \rangle$

① is true when $\{|\psi_n\rangle\}$ are square integrable (Hilbert space)

$\int |\psi_n|^2 dr < \infty$, c.f. bound state

Resonance w.f. is <u>not</u> square integrable (Rigged Hilbert space)

$$\left(\begin{array}{l} \text{c.f. plane wave} \quad \psi(r) \sim e^{\pm ipr} \quad \text{with } p \in \mathbb{R} \\ \qquad \Rightarrow \int_{-\infty}^{\infty} |\psi|^2 dr = \int_{-\infty}^{\infty} 1\, dr = \infty \end{array} \right)$$

• **Example** potential problem in 1d quantum mechanics

$\hbar = 1$, $m = 1$ unit

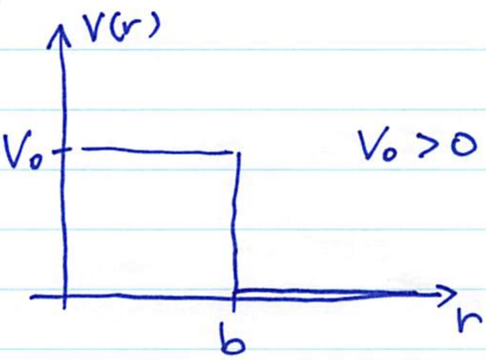
Ex. 1) Show that all the quantities ($E, p, \cdots$) are measurable by the units of length, but the dim. of E and p are different.

Schrödinger equation

$$\left(-\frac{1}{2}\frac{d^2}{dr^2} + V(r)\right)\psi(r) = E\,\psi(r)$$

- Rectangular repulsive potential

$$V(r) = \begin{cases} \infty & r \le 0 \\ V_0 & 0 < r \le b \\ 0 & b < r \end{cases}$$



$V_0 > 0$

- General solution (continuum spectrum)

• $\psi(r) \propto e^{\pm ipr}$ for $b < r$, $p = \sqrt{2E}$

• $\psi(r) \propto e^{\pm ikr}$ for $0 < r \le b$, $k = \sqrt{2(E - V_0)}$

• boundary condition : $\psi(0) = 0$

$$\Rightarrow \psi(r) = \begin{cases} \sin(kr) & 0 < r \le b \\ A(p)\, e^{-ipr} + B(p)\, e^{ipr} & b < r \end{cases}$$

- magnitude in $r < b$ is normalized to unity

- $A(p), B(p) \leftarrow$ continuity of $\psi(r)$ and $\psi'(r)$ at $r = b$

• Discrete solution

- boundary condition at $r \to \infty$

$$\left(\text{c.f. bound state : } p = ik \Rightarrow \cancel{e^{+kr}},\ e^{-kr} \Rightarrow A(p) = 0 \right)$$

- We require the outgoing (e^{+ipr}) boundary condition $A(p)=0$ but with allowing complex solutions for p.

Ex. 2) Calculate $A(p)$ and show that $A(p)=0$ gives

$$\tan\left[\sqrt{p^2-2V_0}\cdot b\right] = -i\frac{\sqrt{p^2-2V_0}}{p}$$

- No bound solution ($p = ik$) ← repulsive interaction.

- Complex solutions for $V_0 = 100\ b^{-2}$

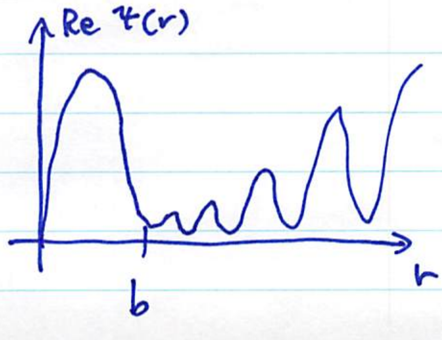
← potential heigt

	$p\ [b^{-1}]$	$E_p - 100\ [b^{-2}]$
lowest	$14.5 - 0.05i$	$4.9 - 0.7i$
1st	$15.5 - 0.2i$	$19.5 - 2.7i$
2nd	$17.0 - 0.3i$	$43.9 - 5.9i$

Eigenmomentum has a negative imaginary part

$p = P_R - iP_I \quad P_R, P_I > 0$

$$\psi(r) \to e^{ipr} = \underbrace{e^{iP_R r}}_{\text{oscillation}}\ \underbrace{e^{+P_I r}}_{\text{increasing}}$$



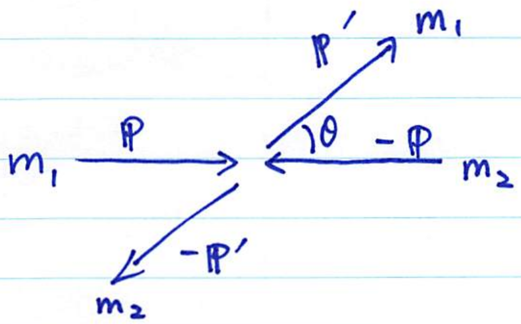
1.2. S-matrix, phase shift, and scattering amplitude

• System ($\hbar = 1$ unit)

- two-body scattering of spinless particles, distinguishable
- elastic scattering (initial state = final state)
- three spatial dimensions, non-relativistic kinematics
- Short range interaction ($V(\mathbf{r})$ vanishes sufficiently rapidly for $|\mathbf{r}| \to \infty$)
- rotational invariance $\Leftrightarrow$ spherical potential $V(|\mathbf{r}|)$

$$\Leftrightarrow [\hat{H}, \hat{\mathbf{L}}] = 0$$

angular momentum



$$|\mathbf{p}| = |\mathbf{p}'| \equiv p \quad \Leftrightarrow \quad E_p = \frac{p^2}{2\mu}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{reduced mass}$$

$$\cos\theta = \mathbf{p} \cdot \mathbf{p}' / p^2$$

- Scattering process is specified by E_p (or p) and θ

- State vectors

momentum representation $|\mathbf{p}\rangle$ $\qquad \langle \mathbf{p} | \mathbf{p}' \rangle = \delta^3(\mathbf{p}' - \mathbf{p})$

angular momentum representation $|E, \ell, m\rangle$ $\qquad \langle E', \ell', m' | E, \ell, m \rangle = \delta(E' - E)\,\delta_{\ell\ell'}\,\delta_{mm'}$

$$|\underset{\uparrow\atop \hat{H}}{E}, \underset{\uparrow\atop \hat{\mathbf{L}}^2}{\ell}, \underset{\uparrow\atop \hat{L}_z}{m}\rangle$$

$$\langle \mathbf{p} | E, \ell, m \rangle = \frac{1}{\sqrt{\mu p}}\, \delta(E_p - E)\, Y_\ell^m(\hat{\mathbf{p}})$$

- Definition

operator S-matrix element

S-matrix $\langle E', \ell', m' | \hat{S} | E, \ell, m \rangle = \delta(E'-E)\,\delta_{\ell'\ell}\,\delta_{m'm}\, S_\ell(E)$

phase shift $S_\ell(E) = e^{2i\delta_\ell(E)}$

scattering amplitude $\langle \mathbf{p}' | (\hat{S} - 1) | \mathbf{p} \rangle = \frac{i}{2\pi\mu}\delta(E_{p'} - E_p)\, f(E_p, \theta)$

Ex.3) Express $f(E_p,\theta)$ in terms of $S_\ell(E_p)$ and confirm

$$f(E_p,\theta) = \frac{1}{2ip}\sum_{\ell=0}^{\infty}(2\ell+1)\left[S_\ell(E_p) - 1\right]P_\ell(\cos\theta)$$

$\hookrightarrow \sum_\ell (2\ell+1) f_\ell(E_p) P_\ell(\cos\theta)$: partial wave decomposition

$$\Rightarrow f_\ell(E_p) = \frac{S_\ell(E_p) - 1}{2ip} = \frac{e^{i\delta_\ell(E_p)}\sin\delta_\ell(E_p)}{p}$$

For each partial wave ℓ, scattering amplitude is a function of E

$\leftarrow$ 1d radial Schrödinger equation for spherical potential

- Unitarity

S-matrix : time evolution of the system

To conserve the probability (norm of the state), $\hat{S}$ must be unitary for $E \in \mathbb{R}$

$$\hat{S}^\dagger \hat{S} = \hat{1} \Leftrightarrow S_\ell(E) S_\ell^*(E) = 1 \Leftrightarrow \exp\left[2i(\delta_\ell(E) - \delta_\ell^*(E))\right] = 1$$

$$\Rightarrow \delta_\ell(E) \in \mathbb{R} \text{ or } |S_\ell(E)| = 1$$

- **Cross section**

$$\frac{d\sigma(E)}{d\Omega} = |f(E,\theta)|^2$$

$$\sigma(E) = \int d\Omega\, |f(E,\theta)|^2$$

Ex. 4) Show that $\sigma(E) = \sum_\ell \sigma_\ell(E)$ with

$$\sigma_\ell(E) = 4\pi(2\ell+1)\,|f_\ell(E)|^2$$

$$= \frac{8\pi\mu(2\ell+1)}{E} \sin^2\delta_\ell(E)$$

Unitarity $\Rightarrow$ $\delta_\ell(E) \in \mathbb{R}$

$\Rightarrow$ $|\sin\delta_\ell(E)|^2 \leq 1$

$\Rightarrow$ $\sigma_\ell(E) \leq \dfrac{8\pi\mu(2\ell+1)}{E}$

"unitarity bound"

1.3 Resonance as a pole of scattering amplitude

• Jost function

$\simeq$ amplitude of the incoming component of the wave function

$\phi_{\ell,p}(r)$: (regular) solution of radial Schrödinger equation for angular momentum ℓ and momentum p

- Asymptotic ($r \to \infty$ s.t. potential vanishes) behavior

$$\phi_{\ell,p}(r) \to \frac{i}{2}\left[\underline{\underline{f_\ell(p)}}\,\hat{h}_\ell^-(pr) - f_\ell(-p)\,\hat{h}_\ell^+(pr)\right]$$

Jost fn. Riccati-Hankel fn.

c.f. $\ell=0$: $\hat{h}_0^-(pr) \sim e^{-ipr}$ (incoming), $\hat{h}_0^+(pr) \sim e^{+ipr}$ (outgoing)

- Small p expansion

$$f_\ell(p) = 1 + \underbrace{\left[\alpha_\ell + \beta_\ell p^2 + \mathcal{O}(p^4)\right]}_{\text{even powers of } p} + i\underbrace{\left[\gamma_\ell\, p^{2\ell+1} + \mathcal{O}(p^{2\ell+3})\right]}_{\text{odd powers}} \quad \text{— (i)}$$

$$\Rightarrow [f_\ell(p)]^* = f_\ell(-p^*) \quad \text{— (ii)}$$

• S-matrix, scattering amplitude

$$S_\ell(p) = \frac{f_\ell(-p)}{f_\ell(p)} \sim \text{ratio of } \frac{\text{outgoing}}{\text{incoming}}$$

Ex. 5) Show that $S_\ell^*(p) = S_\ell(p)^{-1}$ (unitarity) for $p \in \mathbb{R}$ and

$$f_\ell(p) = \frac{f_\ell(-p) - f_\ell(p)}{2ip\, f_\ell(p)}$$

Recall 1d potential problem in 1.1.

$$\psi_{out}(r) = A(p)e^{-ipr} + B(p)e^{ipr}$$

discrete eigenstate $\leftarrow$ $A(p)=0$, only outgoing wave

Jost function vanishes at eigenmomentum $f_\ell(p)=0$

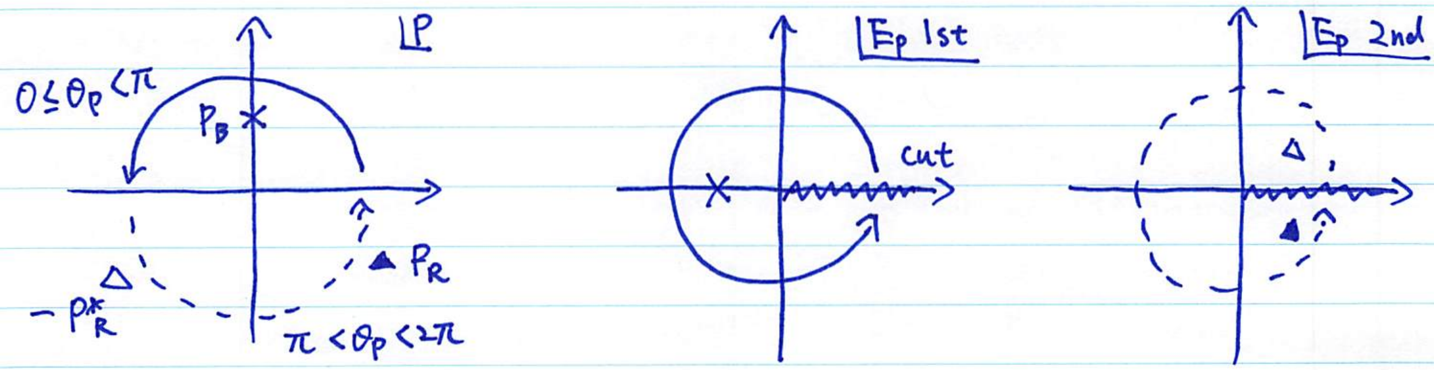
$\Rightarrow$ Eigenmomentum : (complex) pole of analytically continued S-matrix $S_\ell(p)$ and amplitude $f_\ell(p)$

- Eigenenergy and Riemann sheet

$$E_p = \frac{p^2}{2\mu}, \quad p = |p|e^{i\theta_p} \rightarrow E_p = \frac{|p|^2}{2\mu}e^{2i\theta_p}$$

p and $-p$ (θ_p and $\theta_p+\pi$) give the same E_p

A (meromorphic) function of p : a function of E_p with two Riemann sheets.



From (ii) eigenstate at p $\Rightarrow$ eigenstate at $-p^*$

case ① pure imaginary $p=-p^*$

× bound state (1st sheet) $\leftarrow$ square integrable w.f.

case ② pair of $(p, -p^*)$: symmetric w.r.t. imaginary axis

▲ resonance (2nd sheet), △ (anti-resonance)

• Effective range expansion

Ex. 6) Express the scattering amplitude by $\delta_\ell(p)$ to obtain

$$f_\ell(p) = \frac{1}{p\cot\delta_\ell(p) - ip} = \frac{p^{2\ell}}{p^{2\ell+1}\cot\delta_\ell(p) - i\,p^{2\ell+1}}$$

Next, show that $p^{2\ell+1}\cot\delta_\ell(p)$ is a function of p^2 and behaves as $\mathcal{O}(p^0)$ for $p \to 0$ (use (i)).

- Effective range expansion: Taylor series of $p^{2\ell+1}\cot\delta_\ell(p)$ for $p \to 0$

$$p^{2\ell+1}\cot\delta_\ell(p) = -\frac{1}{a_\ell} + \frac{r_e}{2}p^2 + \mathcal{O}(p^4)$$

For s-wave ($\ell = 0$)

$$f_0(p) = \frac{1}{-\frac{1}{a_0} + \frac{r_e}{2}p^2 + \mathcal{O}(p^4) - ip}$$

effective range ($\sim$ interaction range)

scattering length ($\sim$ spatial extent of w.f.)

Note! sign convention of a_0 is opposite in hadron physics

In the low-energy ($p \to 0$) scattering, the higher p^n terms are irrelevant

• leading order ($a_0 \gg r_e$)

$$f_0(p) = \frac{1}{-\frac{1}{a_0} - ip} \Rightarrow \text{pole at } p = \frac{i}{a_0}$$

for $a_0 > 0$, bound state exists with

$$B = \frac{1}{2\mu a_0^2}$$