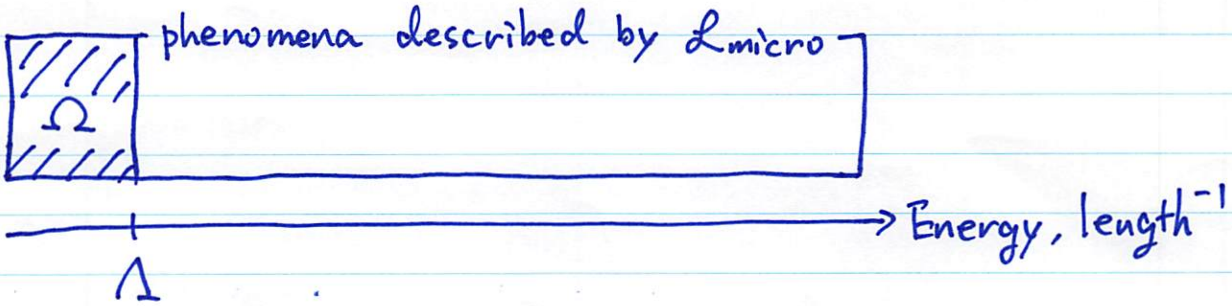
§ 2. Effective field theory

2.1 Basic idea

Consider a microscopic/fundamental quantum field theory $\mathcal{L}_{\rm micro}$ (e.g. QCD)



Ω : low-energy / long-wavelength phenomena

Λ : ultraviolet cutoff, below which the EFT is applicable.

Effective field theory (EFT)

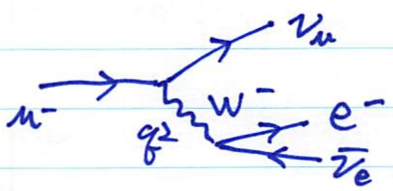
- describes Ω as $\mathcal{L}_{\rm micro}$ does
- can be systematically improved

• Example Weinberg-Salam theory and Fermi-Breit theory

$\mathcal{L}_{\rm micro} = \mathcal{L}_{WS}$; leptons, $W^\pm$, Z bosons

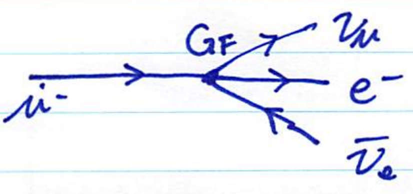
$\mathcal{L}_{\rm eff} = \mathcal{L}_{FB}$; leptons   heavy d.o.f. are "integrated out"

μ^- decay



$$\propto \frac{g^2}{q^2 - M_W^2} = -\frac{g^2}{M_W^2}\left(1 + O\left(\frac{q^2}{m_W^2}\right)\right)$$

$(0.1\ {\rm GeV})^2$ $(100\ {\rm GeV})^2$



$\propto G_F$

QCD case : two difficulties.

- $\mathcal{L}_{QCD}$ at low energy is nonperturbative (not calculable)
- relevant d.o.f. are different : hadrons / quarks, gluons

"theorem" (Weinberg 1979)

> Most general $\mathcal{L}_{eff}$, which is consistent with symmetries of $\mathcal{L}_{micro}$, gives an effective description of $\mathcal{L}_{micro}$.

$$\mathcal{L}_{QCD}(q, A_\mu^a) \xrightarrow{\text{symmetry}} \mathcal{L}_{ChPT}(M, B)$$

Most general Lagrangian contains infinitely many terms.

→ Terms are sorted out by the hierarchy of the importance

$$\mathcal{L}_{ChPT} = \underbrace{\underbrace{\mathcal{L}^{(LO)}} + \mathcal{L}^{(NLO)}}_{\text{systematic improvement}} + \cdots\cdots$$

(more in §3)

2.2 A simple EFT (Zero-Range Model)

- System with large scattering length a_0 compared with interaction range R
- Scattering amplitude $f(p) = \dfrac{1}{-1/a_0 - ip}$ (only s-wave)
  - Nucleons $a_0({}^1S_0) \simeq 20$ fm, $a_0({}^3S_1) \simeq -4$ fm $\gg R \sim \dfrac{1}{m_\pi} \sim 1$ fm

    $\mathcal{L}_{micro} = \mathcal{L}_{NN}$ (or $\mathcal{L}_{QCD}$)
  - ^{4}He atom $a_0 \simeq 200$ [Bohr radius] $\gg R \sim 10$ [Bohr radius]

    $\mathcal{L}_{micro} = \mathcal{L}_{vdW}$

• Lagrangian

$$\mathcal{L}_{\text{eff}} = \underbrace{\psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m}\right)\psi}_{\text{kinetic term}} - \underbrace{\frac{\lambda_0}{4}(\psi^\dagger\psi)^2}_{\text{interaction}}$$

$\psi(t, \mathbb{x})$: Boson field (N or ^{4}He)

m : mass of the boson

λ_0 : coupling constant

$\mathcal{L}_{\text{int}} \sim -\mathcal{H}_{\text{int}} \sim$ energy

$\lambda_0 > 0$: increase energy $\Rightarrow$ repulsive

$\lambda_0 < 0$: decrease energy $\Rightarrow$ attractive

Lagrangian QFT primer

1) Derive Feynman rules (diagram $\leftrightarrow$ equation)

2) Sum up all the diagrams (usually impossible, but this case possible)

2′) Perform perturbation theory

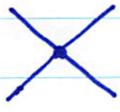
• Feynman rules

• propagator

$$\xrightarrow{\omega,\, \mathbb{k}} = iG(\omega, \mathbb{k}) = \frac{1}{\omega - \mathbb{k}^2/2m + i0+}$$

no negative energy → forward going

• vertex

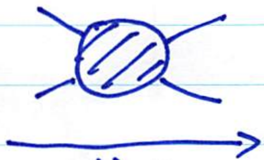


$= -i\lambda_0$ constant in momentum space
→ δ fn in coordinate space

Ex. 7) Confirm that G is the inverse of the kinetic term operator.
Where has the factor 1/4 gone in the coupling?

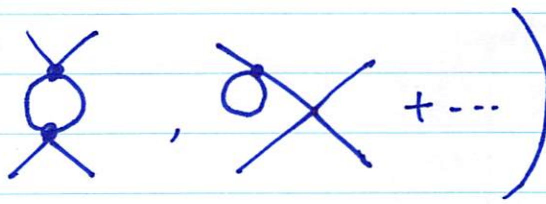
• two-body scattering

- 4-point function (2 in, 2 out)

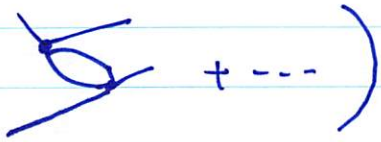


time

- possible diagrams

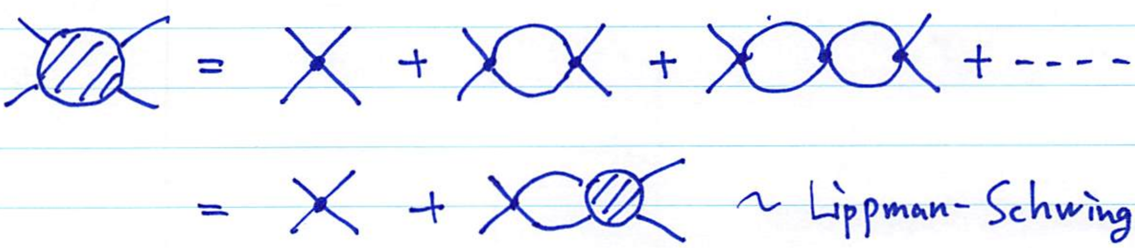


not allowed, because propagator is only forward going



not allowed, because vertex should have 2-in, 2-out.

⇒

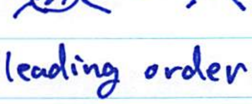


~ Lippman-Schwinger eq.

- This is the <u>exact</u> 4-point function in this EFT. No approximation.
  ← phase symmetry (number conservation) + nonrelativistic kinematics.

- Different powers of $(\lambda_0)^n$ are infinitely summed : <u>nonperturbative</u>

- If we perform perturbation theory for small λ_0,

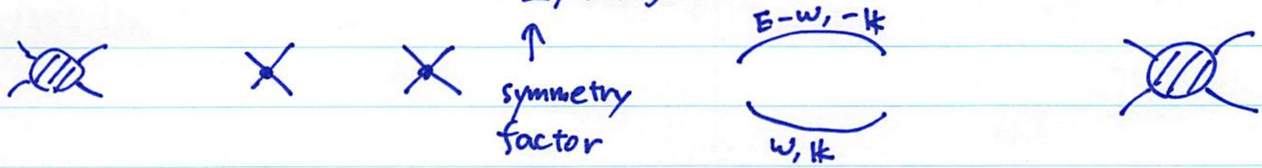


(leading order)

• Evaluation of the amplitude

Amplitude in the center of mass frame with energy E

$$i\mathcal{A}(E) = -i\lambda_0 - i\lambda_0 \frac{1}{2}\int \frac{d\omega\, d^3\mathbb{k}}{(2\pi)^4}\, iG(\omega,\mathbb{k})\, iG(E-\omega,-\mathbb{k})\, i\mathcal{A}(E)$$



Ex. 8) Evaluate $\mathcal{A}(E)$ with a sharp cutoff Λ for $|\mathbb{k}|$ integration to obtain

$$\mathcal{A}(E) = -\left[\frac{1}{\lambda_0} + \frac{m}{4\pi^2}\left(\Lambda - \sqrt{-mE-i0^+}\arctan\frac{\Lambda}{\sqrt{-mE-i0^+}}\right)\right]^{-1}$$

In the sufficiently low energy region compared with the cutoff,

$$\Lambda \gg |\sqrt{-mE}| \implies \arctan\frac{\Lambda}{\sqrt{-mE-i0^+}} \sim \frac{\pi}{2} + \mathcal{O}\left(\frac{\sqrt{-mE+i0^+}}{\Lambda}\right)$$

• two-body relative momentum $p = \sqrt{2\mu E}$

reduced mass $\mu = \frac{mm}{m+m} = \frac{m}{2}$ (two particles with the same mass)

$$\implies \sqrt{-mE-i0^+} = ip$$

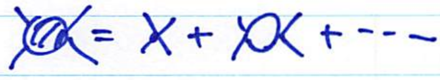
Finally we obtain

$$\mathcal{A}(E) = -\left[\frac{1}{\lambda_0} + \frac{m}{4\pi^2}\Lambda - ip\,\frac{m}{8\pi}\right]^{-1}$$

$$f(E_p) = -\frac{m}{8\pi}\mathcal{A}(E_p) = \frac{1}{\underbrace{-\frac{8\pi}{m}\left(\frac{1}{\lambda_0} + \frac{m}{4\pi^2}\Lambda\right)}_{\text{scattering length } \frac{1}{a_0}} - ip}$$

In the Zero-Range Model, two-body sector is exactly solvable, and the scattering amplitude is determined by a_0

·Unitarity

$$f(E_p) = \frac{1}{-\frac{1}{a_0} - ip}$$ (non perturbative) 

$$f(E_p) = \frac{m}{8\pi}\lambda_0$$ (perturbative, E_p independent) 

Ex. 9) Show that the nonperturbative amplitude satisfies unitarity $SS^* = 1$ while the perturbative one does not.

⇒ Perturbation theory (Born approx.) does not respect unitarity.
Non perturbative resummation (L.S. eq.) is necessary.

2.3 Advanced topics

• Systematic improvement

To increase the applicable region of EFT, one can include next term:

$$\mathcal{L}_{int} = -\frac{1}{4}\lambda_0(\psi^+\psi)^2 - \frac{1}{4}\rho_0 \nabla(\psi^+\psi)\cdot\nabla(\psi^+\psi)$$

new constant (ρ_0), derivative coupling

→ Amplitude contains the effective range term $\frac{r_e}{2}p^2$.

• Renormalization group

If a_0 is fixed (by experiments), we can determine λ_0 for a given cutoff

$$\lambda_0(\Lambda) = \left(1 - \frac{2}{\pi}a_0\Lambda\right)^{-1}\frac{8\pi}{m}a_0$$ ← Renormalization; absorb change of Λ by change of λ_0

Renormalization group flow has two fixed points.

- trivial $a_0 = 0$ (non-interacting)
- nontrivial $a_0 = \pm\infty$ (unitary limit)