Basic Formalism of Relativistic Nuclear Many-body Theory

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1 Convention and basic formulas

See TABLE 1 for basic conventions.

TABLE 1: Conventional rules in this note.

Name	Quantity	Definition
flat metric	$g^{\mu\nu} = g_{\mu\nu}$	$= diag(+,-,-,-)$
4D coordinate	$x^\mu = (x^0, x^1, x^2, x^3)$	$= (ct, x, y, z)$
	$x_\mu = (x_0, x_1, x_2, x_3)$	$= (ct, -x, -y, -z)$
4D derivative	$\partial^\mu = \frac{\partial}{\partial x_\mu}$	$= \left(\frac{\partial}{c\partial t}, \ -\vec{\nabla} \right)$
	$\partial_\mu = \frac{\partial}{\partial x^\mu} = g_{\mu\nu}\partial^\nu$	$= \left(\frac{\partial}{c\partial t}, \ \vec{\nabla} \right)$
4D momentum	$p^\mu = (p^0, p^1, p^2, p^3) = i\hbar\partial^\mu$	$= \left(\frac{E}{c}, \vec{p} \right)$
	$p_\mu = g_{\mu\nu} p^\nu$	$= \left(\frac{E}{c}, -\vec{p} \right)$
gamma matrices	$\gamma^\mu = (\gamma^0, \vec{\gamma})$	$= (\beta, \beta\vec{\gamma})$
reduced derivative	$\gamma^\mu \partial_\mu = \gamma_\mu \partial^\mu$	$= \gamma^0 \partial_{ct} + \vec{\gamma} \cdot \vec{\nabla}$

2 Units

We assume the $(1+3)$-dimensional time and space. In the MKSA or CGS-Gauss system of units, except the electro-magnetic terms, the Dirac equation is given as

$$i\hbar \frac{\partial}{\partial t} \psi(t, \boldsymbol{r}) = \left[-i\hbar c \beta \vec{\gamma} \cdot \vec{\nabla} + \beta M c^2 + W \right] \psi(t, \boldsymbol{r}), \tag{1}$$

where W is some external potential in the unit of energy (e.g., MeV). From $\beta\beta = I$ and $\gamma^\mu \partial_\mu = \beta \partial_{ct} + \vec{\gamma} \cdot \vec{\nabla}$, it is also expressed as

$$\left[i\hbar c \gamma^\mu \partial_\mu - M c^2 - \beta W \right] \psi(t, \boldsymbol{r}) = 0. \tag{2}$$

The Lagangian density, which works as the source of this equation, reads

$$\mathcal{L} = \bar{\psi} \left[i\hbar c \gamma^\mu \partial_\mu - M c^2 - \beta W \right] \psi(x), \tag{3}$$

where $\bar{\psi} \equiv \psi^\dagger \beta$. Note that, because the Lagrangian $L \equiv \int d^3\boldsymbol{r} \mathcal{L}$ and Mc^2 have the dimension of energy, $\bar{\psi}\psi$ is in the unit of fm^{-3}. As coincidence, if some interaction term(s) has the form,

$$\mathcal{L}_{\rm I} = \bar{\psi} X \psi(x), \tag{4}$$

then this wild-card part X must have the dimension of energy, e.g. in MeV. This knowledge may help us, for example, to infer the unit of the coupling constant.

For dimensional analyis, the action follows $[S]_D = \left[\int dt \int d^3\boldsymbol{r}\mathcal{L}\right]_D = ET$, since Lagrangian (as well as Hamiltonian) keeps the dimension of energy, $[d^3\boldsymbol{r}\mathcal{L}]_D = E = ML^2T^{-2}$. Thus, Lagrangian density has $[\mathcal{L}]_D = EL^{-3}$. Note that, in the MKSA or CGS-Gauss system of units, the dimensional analysis concludes that,

$$[c^2 \cdot \text{mass}]_D = [\text{energy}]_D = \left[\frac{\hbar c}{\text{length}}\right]_D = \left[\frac{\hbar}{\text{time}}\right]_D = E. \tag{5}$$

2.1 Plank's natural system of units

In the Plank's natural system of units, we assume that $\hbar \equiv 1$ and $c \equiv 1$. With this assumption, dimensions of mass, energy, length, and time can be related as

$$[\text{mass}]_D = [\text{energy}]_D = \left[\frac{1}{\text{length}}\right]_D = \left[\frac{1}{\text{time}}\right]_D = M^{+1}. \tag{6}$$

TABLE 2: Dimensional numbers of some quantities, $[\text{Quantity}]_D$.

Quantity	In MKSA or CGS-Gauss	In Plank's natural
mass	M	M^{+1}
time and length	T and L	M^{-1}
energy	$E = ML^2T^{-2}$	M^{+1}
$\mathcal{L}$ or $\mathcal{H}$	EL^{-3}	M^{+4}
$\bar{\psi}\psi(x)$	L^{-3}	M^{+3}
$\phi^2(x)$ (scalar boson)	$E^{-1}L^{-3}$	M^{+2}
$A^\mu A_\mu(x)$ (vector boson)	$E^{-1}L^{-3}$	M^{+2}

3 Lagrangian

In the relativistic nuclear theory (RNT), nucleon is described by a Dirac spinor $\psi(x)$, where $x = \{\boldsymbol{r}, \boldsymbol{s}, \vec{\tau}\}$. The phenomenological Lagrangian density reads

$$\mathcal{L} = \bar{\psi}(x)[i\gamma_\mu\partial^\mu - M]\psi(x) + \mathcal{L}_\text{M} + \mathcal{L}_\text{I}. \tag{7}$$

Here $\mathcal{L}_\text{M}$ is the kinetic and self-interaction part of mesons in the model. The interaction part, $\mathcal{L}_\text{I}$, on the other hand, includes all the possible terms of interactions. See TABLEs 3 and 4 for details.

For meson terms $\mathcal{L}_M$,

$$\begin{aligned}\mathcal{L}_M &= \frac{1}{2}\left[\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2\right] + U(\sigma) - \frac{1}{2}\left[\Omega_{\mu\nu}\Omega^{\mu\nu} - m_\omega^2\omega_\mu\omega^\mu\right] - \frac{1}{2}\left[\vec{\Upsilon}_{\mu\nu}\vec{\Upsilon}^{\mu\nu} - m_\rho^2\vec{\rho}_\mu\vec{\rho}^\mu\right] \\ &+\frac{1}{2}\left[\partial_\mu\vec{\pi}\partial^\mu\vec{\pi} - m_\sigma^2\vec{\pi}\vec{\pi}\right] - \frac{1}{2}F_{\mu\nu}F^{\mu\nu}.\end{aligned} \tag{8}$$

TABLE 3: Kinetic and self-interaction terms included in $\mathcal{L}_{\rm M}$. Label (i) indicates isoscalar (IS) or isovector (IV). Label (ii) indicates scalar (S), vector (V), pseudo-scalar (PS) or pseudo-vector (PV).

(i)	(ii)	(T, J^{π})	Meson	
IS	S	$(0, 0^+)$	σ	$+\frac{1}{2}\left[\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2\right] + U(\sigma)$
	V	$(0, 1^-)$	ω^μ	$-\frac{1}{2}\left[\Omega_{\mu\nu}\Omega^{\mu\nu} - m_\omega^2\omega_\mu\omega^\mu\right]$
				with $\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$
	PS	$(0, 0^-)$	$\times$	
	PV	$(0, 1^+)$	$\times$	
IV	S	$(1, 0^+)$	$\times$	
	V	$(1, 1^-)$	$\vec{\rho}^\mu$	$-\frac{1}{2}\left[\vec{\Upsilon}_{\mu\nu}\vec{\Upsilon}^{\mu\nu} - m_\rho^2\vec{\rho}_\mu\vec{\rho}^\mu\right]$
				with $\vec{\Upsilon}_{\mu\nu} = \partial_\mu\vec{\rho}_\nu - \partial_\nu\vec{\rho}_\mu$
	PS	$(1, 0^-)$	$\vec{\pi}$	$+\frac{1}{2}\left[\partial_\mu\vec{\pi}\partial^\mu\vec{\pi} - m_\sigma^2\vec{\pi}\vec{\pi}\right]$
	PV	$(1, 1^+)$	$\times$	
Coulomb				$-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$
				with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

In the meson-exchange model,

$$\begin{aligned}\mathcal{L}_I &= -g_\sigma\bar{\psi}\sigma\psi - g_\omega[\bar{\psi}\gamma_\mu\omega^\mu\psi] - g_\rho[\bar{\psi}\gamma_\mu(\vec{\tau}\vec{\rho}^\mu)\psi] - ig_\pi[\bar{\psi}\gamma_5(\vec{\tau}\vec{\pi})\psi] - \frac{f_\pi}{m_\pi}[\bar{\psi}\gamma_5\gamma_\mu\partial^\mu(\vec{\tau}\vec{\pi})\psi] \\ &\quad -e\bar{\psi}\gamma_\mu A^\mu\left(\frac{1-\hat{\tau}_3}{2}\right)\psi(x). \end{aligned} \tag{9}$$

In the point-coupling model,

$$\begin{aligned}\mathcal{L}_I = &-\frac{\alpha_{\rm IS-S}(\rho)}{2}[\bar{\psi}\psi][\bar{\psi}\psi] - \frac{\alpha_{\rm IS-V}(\rho)}{2}[\bar{\psi}\gamma_\mu\psi][\bar{\psi}\gamma^\mu\psi] - \frac{\alpha_{\rm IV-V}(\rho)}{2}[\bar{\psi}\gamma_\mu\vec{\tau}\psi][\bar{\psi}\gamma^\mu\vec{\tau}\psi] \\ &- \frac{\alpha_{\rm IV-PS}(\rho)}{2}[\bar{\psi}\gamma_5\vec{\tau}\psi][\bar{\psi}\gamma_5\vec{\tau}\psi] - \frac{\alpha_{\rm IV-PV}(\rho)}{2}[\bar{\psi}\gamma_5\gamma_\mu\vec{\tau}\psi][\bar{\psi}\gamma_5\gamma^\mu\vec{\tau}\psi] \\ &- e\bar{\psi}\gamma_\mu A^\mu\left(\frac{1-\hat{\tau}_3}{2}\right)\psi(x). \end{aligned} \tag{10}$$

4 Equation of Motion

The euqation of motion (EOM) reads

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu q_i)}\right) - \frac{\partial\mathcal{L}}{\partial q_i} = 0. \tag{11}$$

Here $q_i(x)$ is generally utilized for $\psi(x)$, $A_\mu(x)$, or meson fields.

TABLE 4: Interaction terms included in $\mathcal{L}_{\mathrm{I}}$. Label (i) indicates isoscalar (IS) or isovector (IV). Label (ii) indicates scalar (S), vector (V), pseudo-scalar (PS) or pseudo-vector (PV).

(i)	(ii)	(T, J^{π})	Meson	Meson-exchange	Point-coupling
IS	S	$(0, 0^{+})$	σ	$-g_{\sigma}\bar{\psi}\sigma\psi$	$-\alpha_{\mathrm{IS-S}}(\rho)[\bar{\psi}\psi][\bar{\psi}\psi]/2$
					$-\delta_{\mathrm{IS-S}}(\rho)\partial_{\mu}[\bar{\psi}\psi]\partial^{\mu}[\bar{\psi}\psi]/2$
	V	$(0, 1^{-})$	ω^{μ}	$-g_{\omega}[\bar{\psi}\gamma_{\mu}\omega^{\mu}\psi]$	$-\alpha_{\mathrm{IS-V}}(\rho)[\bar{\psi}\gamma_{\mu}\psi][\bar{\psi}\gamma^{\mu}\psi]/2$
	PS	$(0, 0^{-})$	×	×	×
	PV	$(0, 1^{+})$	×	×	×
IV	S	$(1, 0^{+})$	×	×	×
	V	$(1, 1^{-})$	$\vec{\rho}^{\mu}$	$-g_{\rho}[\bar{\psi}\gamma_{\mu}(\vec{\tau}\vec{\rho}^{\mu})\psi]$	$-\alpha_{\mathrm{IV-V}}(\rho)[\bar{\psi}\gamma_{\mu}\vec{\tau}\psi][\bar{\psi}\gamma^{\mu}\vec{\tau}\psi]/2$
	PS	$(1, 0^{-})$	$\vec{\pi}$	$-ig_{\pi}[\bar{\psi}\gamma_{5}(\vec{\tau}\vec{\pi})\psi]$	$-\alpha_{\mathrm{IV-PS}}(\rho)[\bar{\psi}\gamma_{5}\vec{\tau}\psi][\bar{\psi}\gamma_{5}\vec{\tau}\psi]/2$
	PV	$(1, 1^{+})$	$\partial_{\mu}\vec{\pi}$	$-\frac{f_{\pi}}{m_{\pi}}[\bar{\psi}\gamma_{5}\gamma_{\mu}\partial^{\mu}(\vec{\tau}\vec{\pi})\psi]$	$-\alpha_{\mathrm{IV-PV}}(\rho)[\bar{\psi}\gamma_{5}\gamma_{\mu}\vec{\tau}\psi][\bar{\psi}\gamma_{5}\gamma^{\mu}\vec{\tau}\psi]/2$
Coulomb					$-e\bar{\psi}\gamma_{\mu}A^{\mu}\left(\frac{1-\hat{\tau}_{3}}{2}\right)\psi$

4.1 IV-PS (pion-nucleon) coupling

Focusing on the isovector-pseudoscalar (IV-PS) coupling, the pion-exchange model is defined with the following interaction:

$$\begin{aligned}\mathcal{L}_{\mathrm{ME}} &= \bar{\psi}\left[i\gamma_{\mu}\partial^{\mu} - M\right]\psi(x) \\ &\quad -ig_{\mathrm{IV-PS}}\left[\bar{\psi}\gamma_{5}(\vec{\tau}\vec{\pi})\psi(x)\right] + \frac{1}{2}\left[\partial_{\mu}\vec{\pi}\partial^{\mu}\vec{\pi} - m^{2}\vec{\pi}\vec{\pi}\right].\end{aligned} \tag{12}$$

On the other side, based on the point-coupling (zero-range) model, it is defined as

$$\begin{aligned}\mathcal{L}_{\mathrm{PC}} &= \bar{\psi}\left[i\gamma_{\mu}\partial^{\mu} - M\right]\psi(x) \\ &\quad -\frac{\alpha_{\mathrm{IV-PS}}}{2}\left(\bar{\psi}\gamma_{5}\vec{\tau}\psi\right)\left(\bar{\psi}\gamma_{5}\vec{\tau}\psi\right) + \left(\frac{1}{2}\left[\partial_{\mu}\vec{\pi}\partial^{\mu}\vec{\pi} - m^{2}\vec{\pi}\vec{\pi}\right]\right)_{\mathrm{neglectable}}.\end{aligned} \tag{13}$$

Thus, roughly speaking, these two models can be related as

$$-ig_{\mathrm{IV-PS}}\vec{\pi}(x) \longleftrightarrow -\frac{\alpha_{\mathrm{IV-PS}}}{2}\gamma_{5}\vec{\tau}\bar{\psi}\psi(x). \tag{14}$$

In the following, we explain the background of this analogy. The factor 1/2 is indeed not correct, but the derivative of the square term works instead.

First we note the equation of motion for $\psi(x)$ of the pion-exchange model. That is,

$$\begin{aligned}&\partial_{\mu}\frac{\delta\mathcal{L}_{\mathrm{ME}}}{\delta(\partial_{\mu}\psi^{\dagger})} - \frac{\delta\mathcal{L}_{\mathrm{ME}}}{\delta\psi^{\dagger}} = 0, \\ &0 - \gamma_{0}\left[i\gamma_{\mu}\partial^{\mu} - M\right]\psi(x) + ig_{\mathrm{IV-PS}}\gamma_{0}\gamma_{5}(\vec{\tau}\vec{\pi})\psi(x) = 0, \\ &\left[i\gamma_{\mu}\partial^{\mu} - M\right]\psi(x) = ig_{\mathrm{IV-PS}}\vec{\pi}\vec{\tau}\gamma_{5}\psi(x).\end{aligned} \tag{15}$$

Second, from the equation of motion for $\psi(x)$ of the pion-exchange model,

$$\begin{aligned}
&\partial_\mu \frac{\delta \mathcal{L}_{\rm ME}}{\delta(\partial_\mu \pi_a)} - \frac{\delta \mathcal{L}_{\rm ME}}{\delta \pi_a} = 0, \\
&\partial_\mu \partial^\mu \pi_a - \left[-m^2 \pi_a(x) - i g_{\rm IV-PS} \gamma_5 \tau_a \bar{\psi}\psi(x) \right] = 0, \\
&\left[\partial_\mu \partial^\mu + m^2 \right] \pi_a(x) = -i g_{\rm IV-PS} \gamma_5 \tau_a \bar{\psi}\psi(x).
\end{aligned} \tag{16}$$

Then, we suppose the heavy-pion limit. In this case, we can naively approximate as

$$\pi_a(x) \simeq \frac{-i g_{\rm IV-PS}}{m^2} \gamma_5 \tau_a \bar{\psi}\psi(x). \tag{17}$$

By substituting this into the EOM of $\psi(x)$, we find that

$$\left[i\gamma_\mu \partial^\mu - M \right] \psi(x) \simeq -\alpha_{\rm IV-PS} \left(\gamma_5 \tau_a \bar{\psi}\psi \right) \gamma_5 \tau_a \psi(x), \tag{18}$$

where $-\alpha_{\rm IV-PS} = (-i g_{\rm IV-PS})^2/m^2$. This is indeed the EOM but obtained from the other, point-coupling Lagrangian. Notice also that, for the correspondence of two models, $\alpha_{\rm IV-PS} > 0$. Its unit must be in, e.g., MeV·fm^3, since $\mathcal{L}_{\rm PC}$ and $\bar{\psi}\psi(x)$ have the units of MeV·fm^{-3} and fm^{-3}, respectively.

4.2 IV-PV coupling

Focusing on the IV-PV coupling, the pion-exchange model reads

$$\begin{aligned}
\mathcal{L}_{\rm ME} &= \bar{\psi} \left[i\gamma_\mu \partial^\mu - M \right] \psi(x) \\
&\quad - g_{\rm IV-PV} \left(\bar{\psi} \gamma_\mu \gamma_5 \psi \right) \vec{\tau} \cdot \partial^\mu \vec{\pi} + \frac{1}{2} \left[\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} - m^2 \vec{\pi} \cdot \vec{\pi} \right].
\end{aligned} \tag{19}$$

On the other side, based on the point-coupling model, it is usually given as

$$\begin{aligned}
\mathcal{L}_{\rm PC} &= \bar{\psi} \left[i\gamma_\mu \partial^\mu - M \right] \psi(x) \\
&\quad - \frac{\alpha_{\rm IV-PV}}{2} \left(\bar{\psi} \gamma_\mu \gamma_5 \psi \right) \vec{\tau} \cdot \left(\bar{\psi} \gamma^\mu \gamma_5 \psi \right) \vec{\tau} + \left(\frac{1}{2} \left[\partial_\mu \vec{\pi} \partial^\mu \vec{\pi} - m^2 \vec{\pi}\vec{\pi} \right] \right)_{\rm neglectable}.
\end{aligned} \tag{20}$$

Thus, roughly speaking, these two models can be related as

$$-g_{\rm IV-PV} \partial_\mu \vec{\pi}(x) \longleftrightarrow -\frac{\alpha_{\rm IV-PV}}{2} \left(\bar{\psi} \gamma_\mu \gamma_5 \psi \right) \vec{\tau}. \tag{21}$$

In the following, we explain the background of this analogy.

- The equation of motion for $\psi(x)$ from $\mathcal{L}_{\rm ME}$:

$$\begin{aligned}
&\partial_\mu \frac{\delta \mathcal{L}_{\rm ME}}{\delta(\partial_\mu \psi^\dagger)} - \frac{\delta \mathcal{L}_{\rm ME}}{\delta \psi^\dagger} = 0, \\
&\left[i\gamma_\mu \partial^\mu - M \right] \psi(x) = -g \partial_\mu \vec{\pi} \cdot \vec{\tau} \gamma^\mu \gamma_5 \psi(x).
\end{aligned} \tag{22}$$

- The equation of motion for $\pi_a(x)$ from $\mathcal{L}_{\rm ME}$:

$$\begin{aligned}
&\partial_\mu \frac{\delta \mathcal{L}_{\rm ME}}{\delta(\partial_\mu \pi_a)} - \frac{\delta \mathcal{L}_{\rm ME}}{\delta \pi_a} = 0, \\
&\partial_\mu \left[\partial^\mu \pi_a - g \tau_a \cdot \bar{\psi} \gamma^\mu \gamma_5 \psi(x) \right] + m^2 \pi_a(x) = 0, \\
&\left[\partial_\mu \partial^\mu + m^2 \right] \pi_a(x) = g \tau_a \partial_\mu \left(\bar{\psi} \gamma^\mu \gamma_5 \psi \right).
\end{aligned} \tag{23}$$

By using the free-meson Green function (propagator),

$$\Delta_\pi(x-y) = \int \frac{d^4p}{16\pi^4}\frac{e^{-ip(x-y)}}{p^2-m^2} \implies \left[\partial_\mu\partial^\mu + m^2\right]\Delta_\pi(x-y) = \delta(x-y), \tag{24}$$

then the pion field can be formally solved as

$$\pi_a(x) = g\tau_a \int dy \Delta_\pi(x-y)\cdot\left[\partial_\mu\left(\bar{\psi}\gamma^\mu\gamma_5\psi\right)\right]_{(y)}. \tag{25}$$

With the partial-integration technique combined with the vanishing-flux condition, one finds that

$$\begin{aligned}\pi_a(x) &= 0 - g\tau_a\int dy\left[\partial_\mu^{(y)}\Delta_\pi(x-y)\right]\cdot\left(\bar{\psi}\gamma^\mu\gamma_5\psi\right)_{(y)}, \\ \partial_\nu^{(x)}\pi_a(x) &= -g\tau_a\int dy\left[\partial_\nu^{(x)}\partial_\mu^{(y)}\Delta_\pi(x-y)\right]\cdot\left(\bar{\psi}\gamma^\mu\gamma_5\psi\right)_{(y)}.\end{aligned} \tag{26}$$

In the heavy-pion limit, $\partial_\nu^{(x)}\partial_\mu^{(y)}\Delta_\pi(x-y) \simeq -g_{\nu\mu}\delta(x-y)/m^2$. Thus,

$$\partial_\nu\vec{\pi} \simeq \frac{g}{m^2}\vec{\tau}\left(\bar{\psi}\gamma^\mu\gamma_5\psi\right). \tag{27}$$

Therefore, the EOM for $\psi(x)$ is approximated as

$$\left[i\gamma_\mu\partial^\mu - M\right]\psi(x) \simeq -\frac{g^2}{m^2}\vec{\tau}\left(\bar{\psi}\gamma^\mu\gamma_5\psi\right)\cdot\vec{\tau}\gamma^\mu\gamma_5\psi(x). \tag{28}$$

This equation is the same to that obtained from $\mathcal{L}_{\rm PC}$, with a relation,

$$-\frac{g^2}{m^2} = -\alpha_{\rm IV-PV}. \tag{29}$$

Notice that, for the correspondence of two models, $\alpha_{\rm IV-PV} > 0$. Its unit must be in, e.g., MeV$\cdot$fm^3, since $\mathcal{L}_{\rm PC}$ and $\bar{\psi}\psi(x)$ have the units of MeV$\cdot$fm^{-3} and fm^{-3}, respectively.

5 Quantization of Dirac spinor

In general, the spinor field consists of particle states with $E > 0$ and anti-particle states with $-E < 0$. Thus, it can be formally expanded as

$$\begin{aligned}\psi(x) &= \sum_s \psi_s(x), \\ \psi_s(x) &\equiv \langle x \mid s\rangle = \int_{E>0} dE\left[u_{s,E}(x)c_{s,E} + v_{s,-E}(x)b_{s,-E}\right],\end{aligned} \tag{30}$$

as well as,

$$\psi_r^\dagger(x) = \int_{E>0} dE\left[u_{r,E}^\dagger(x)c_{r,E}^\dagger + v_{r,-E}^\dagger(x)b_{r,-E}^\dagger\right], \tag{31}$$

where $u_{s,E}(x) \equiv \langle x \mid s, E\rangle$ and $v_{s,-E}(x) \equiv \langle x \mid s, -E\rangle$. Here the index s indicates the spin component, whereas $E > 0$ means the eigenvalue for certain Dirac's Hamiltonian. Assuming this Hamiltonian as $\hat{h}$, these basic states satisfy that,

$$\hat{h}u_{s,E}(x) = Eu_{s,E}(x), \quad \hat{h}v_{s,-E}(x) = -Ev_{s,-E}(x). \tag{32}$$

In the following, we assume that $\hat{h}$ does not depend on time apparently. Thus, from Dirac equation, $i\hbar\partial_t u(x) = \hat{h}u(x)$, it is represented as

$$u_{s,E}(x) = e^{-itE/\hbar} u_{s,E}(\boldsymbol{r}), \quad v_{s,-E}(x) = e^{itE/\hbar} v_{s,-E}(\boldsymbol{r}). \tag{33}$$

Note the following points.

- Completeness of basis:

$$\hat{1} = \sum_s \int_{E>0} dE \Big[|s,E\rangle \langle s,E| + |s,-E\rangle \langle s,-E| \Big]. \tag{34}$$

  Thus, from the overlap of y and x,

$$\langle y \mid x \rangle = \sum_s \int dE \left[u^\dagger_{s,E}(y) u_{s,E}(x) + v^\dagger_{s,-E}(y) v_{s,-E}(x) \right] = \delta(y - x). \tag{35}$$

  From Eq. (33), it is also concluded as

$$\sum_s \int dE \left[u^\dagger_{s,E}(\boldsymbol{y}) u_{s,E}(\boldsymbol{x}) + v^\dagger_{s,-E}(\boldsymbol{y}) v_{s,-E}(\boldsymbol{x}) \right] = \delta(\boldsymbol{y} - \boldsymbol{x}). \tag{36}$$

- Orthogonality of basis:

$$\langle r, E' \mid s, E \rangle \equiv \delta(E' - E)\delta_{rs}, \quad \langle r, -E' \mid s, -E \rangle \equiv \delta(E' - E)\delta_{rs}.$$
$$\longrightarrow \int d^3\boldsymbol{r} u^\dagger_{r,E'}(\boldsymbol{r}) u_{s,E}(\boldsymbol{r}) = \int d^3\boldsymbol{r} v^\dagger_{r,-E'}(\boldsymbol{r}) v_{s,-E}(\boldsymbol{r}) = \delta(E' - E)\delta_{rs}. \tag{37}$$

  Also, remembering $E',\ E > 0$,

$$\begin{aligned} \langle r, -E' \mid s, E \rangle &= \int d^3\boldsymbol{r} v^\dagger_{r,-E'}(x) u_{s,E}(x) = 0, \\ \langle r, E' \mid s, -E \rangle &= \int d^3\boldsymbol{r} u^\dagger_{r,E'}(x) v_{s,-E}(x) = 0. \end{aligned} \tag{38}$$

- Spinor field must satisfy the anti-commutation relation at the same time:

$$\left\{ \psi^\dagger_r(y),\ \psi_s(x) \right\}_{y_0=x_0} = \delta(\boldsymbol{y} - \boldsymbol{x})\delta_{rs}, \tag{39}$$
$$\left\{ \psi_r(y),\ \psi_s(x) \right\}_{y_0=x_0} = \left\{ \psi^\dagger_r(y),\ \psi^\dagger_s(x) \right\}_{y_0=x_0} = 0. \tag{40}$$

  For the first relation, we find that,

$$\begin{aligned} \left\{ \psi^\dagger_r(y),\ \psi_s(x) \right\}_{y_0=x_0} &= \sum_{r,s} \int dE' \int dE \\ &\Big[u^\dagger_{r,E'}(y) u_{s,E}(x) \left\{ c^\dagger_{r,E'},\ c_{s,E} \right\} + v^\dagger_{r,-E'}(y) v_{s,-E}(x) \left\{ b^\dagger_{r,-E'},\ b_{s,-E} \right\} \\ &+ v^\dagger_{r,-E'}(y) u_{s,-E}(x) \left\{ b^\dagger_{r,-E'},\ c_{s,E} \right\} + u^\dagger_{r,-E'}(y) v_{s,-E}(x) \left\{ c^\dagger_{r,E'},\ b_{s,-E} \right\} \Big]_{y_0=x_0}. \end{aligned} \tag{41}$$

  Therefore, to keep consistency with Eqs. (36) and (39), the operators must satisfy that

$$\left\{ c^\dagger_{r,E'},\ c_{s,E} \right\} = \left\{ b^\dagger_{r,-E'},\ b_{s,-E} \right\} = \delta_{rs}\delta(E' - E), \quad \{others\} = 0. \tag{42}$$

Notice that above formulas can work even in the case with general interaction(s) included in the Lagrangian density.

5.1 Hamiltonian

In general, Lagrangian density is written as $\mathcal{L} = \bar{\psi}\left(i\not\partial - M\right)\psi(x) + \bar{\psi}X\psi(x)$. The corresponding Hamiltonian density reads

$$\mathcal{H}(x) \equiv \left(\partial_0\psi^\dagger\right)\frac{\delta\mathcal{L}}{\delta\left(\partial_0\psi^\dagger\right)} + \frac{\delta\mathcal{L}}{\delta\left(\partial_0\psi\right)}\left(\partial_0\psi\right) - \mathcal{L} \tag{43}$$

$$= 0 + \bar{\psi}i\gamma^0\left(\partial_0\psi\right) - \bar{\psi}i\left[\gamma^0\partial_0 + \gamma^k\partial_k\right]\psi(x) + M\bar{\psi}\psi(x) - \bar{\psi}X\psi(x)$$

$$= \psi^\dagger\left[-i\vec{\alpha}\cdot\vec{\nabla} + \beta M - \beta X\right]\psi(x) \equiv \psi^\dagger\hat{h}_D\psi(x), \tag{44}$$

where $\hat{h}_D$ indicates the Dirac single-field Hamiltonian. Note that, however, here I neglect the exchange (Fock) terms, which could appear from the interactions $\bar{\psi}X\psi(x)$. The proper Hamiltonian is then given as $H(t) = \int d^3\boldsymbol{r}\mathcal{H}(x)$.

By employing the basis expansion introduced above, it can be represented as

$$H(t) = \sum_{r,s}\int dE'\int dE\int d^3\boldsymbol{r}$$

$$\left[u^\dagger_{r,E'}(x)c^\dagger_{r,E'} + v^\dagger_{r,-E'}(x)b^\dagger_{r,-E'}\right]E\left[u_{s,E}(x)c_{s,E} - v_{s,-E}(x)b_{s,-E}\right], \tag{45}$$

where we have used $\hat{h}_D u_{s,E} = Eu_{s,E}$ and $\hat{h}_D v_{s,-E} = -Ev_{s,-E}$. From Eqs. (37) and (38), one can find that only the $c^\dagger_* c_*$ and $b^\dagger_* b_*$ terms survive. That is,

$$H = \sum_{r,s}\int dE'\int dE\left[e^{it(E'-E)/\hbar}c^\dagger_{r,E'}c_{s,E} - e^{-it(E'-E)/\hbar}b^\dagger_{r,-E'}b_{s,-E}\right]E\delta(E'-E)\delta_{rs} + 0$$

$$= \sum_{s}\int dE\left[c^\dagger_{s,E}c_{s,E} - b^\dagger_{s,-E}b_{s,-E}\right]E. \tag{46}$$

This equation almost looks as the proper form for the total energy. However, the second term means that $b^\dagger_*$ creates the negative-energy particle. To remedy this wired property, the anti-particle states are re-defined as $a_* \equiv b^\dagger_*$ and $a^\dagger_* \equiv b_*$. By this procedure, finally we can find that

$$H = \sum_{s}\int dE\left[c^\dagger_{s,E}c_{s,E} + a^\dagger_{s,-E}a_{s,-E}\right]E - const. \tag{47}$$

The vacuum is then defined as the state to become zero for c_* and a_*.

5.2 general representation with basis

In practical calculations, the Hamiltonian is represented with the choosen basic states. That is,

$$H(t) = \sum_{r,s}\int dE'\int dE\int d^3\boldsymbol{r}$$

$$\left[u^\dagger_{r,E'}(x)c^\dagger_{r,E'} + v^\dagger_{r,-E'}(x)b^\dagger_{r,-E'}\right]\hat{h}_D\left[u_{s,E}(x)c_{s,E} + v_{s,-E}(x)b_{s,-E}\right], \tag{48}$$

where $u(x)$ and $v(x)$ are, however, NOT the eigenstates of $\hat{h}_D$ anymore. Thus, the labels E and E' are now general ones: those are not definitely for energies. By using the matrix elements,

$$h^{(pp)}_{r,E',s,E}(t) \equiv \int d^3\boldsymbol{r}u^\dagger_{r,E'}(x)\hat{h}_D u_{s,E}(x), \quad h^{(ap)}_{r,-E',s,E}(t) \equiv \int d^3\boldsymbol{r}v^\dagger_{r,-E'}(x)\hat{h}_D u_{s,E}(x), \quad \text{etc.}, \tag{49}$$

then it can be formally given as

$$\begin{aligned} H(t) &= \sum_{r,s} \int dE' \int dE \Big[h^{(pp)}_{r,E',s,E}(t) c^{\dagger}_{r,E'} c_{s,E} + h^{(ap)}_{r,-E',s,E}(t) b^{\dagger}_{r,-E'} c_{s,E} \\ & \quad h^{(pa)}_{r,E',s,-E}(t) c^{\dagger}_{r,E'} b_{s,-E} + h^{(aa)}_{r,-E',s,-E}(t) b^{\dagger}_{r,-E'} b_{s,-E} \Big]. \end{aligned} \tag{50}$$

Within the no-sea approximation, we neglect the anti-particle components, namely the 2nd to 4th terms in the Hamiltonian. In this case, one finds the usual form,

$$H(t) \simeq \sum_{k,l} h^{(pp)}_{k,l}(t) c^{\dagger}_{k} c_{l}, \tag{51}$$

where the simplified labels $k = \{r, E'\}$ and $l = \{s, E\}$ are employed. The vacuum-expectation value $\langle H(t) \rangle_{\Phi}$ is then a functional of several densities, similarly in the non-relativistic multi-fermion models. The Bogoliubov transformation can be also determined for $c^{\dagger}_{*}$ and c_{*} operators.

Basic Formalism for Meanfiled, HFB, and QRPA Methods

Tomohiro Oishi

6 Operators

Creation and Annihilation Operators;

$$\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta} \tag{52}$$
$$\{c_\alpha, c_\beta\} = \{c_\alpha^\dagger, c_\beta^\dagger\} = 0 \tag{53}$$

$$c_\alpha = \int dx \, \langle \alpha \,|\, x \rangle \, \hat{\psi}(x), \qquad c_\alpha^\dagger = \int dx \, \langle x \,|\, \alpha \rangle \, \hat{\psi}^\dagger(x) \qquad (\langle x \,|\, \alpha \rangle = \phi_\alpha(x)) \tag{54}$$
$$\hat{\psi}(x) = \sum_\alpha \langle x \,|\, \alpha \rangle \, c_\alpha, \qquad \hat{\psi}^\dagger(x) = \sum_\alpha \langle \alpha \,|\, x \rangle \, c_\alpha^\dagger \tag{55}$$

$$\{\hat{\psi}(x), \hat{\psi}^\dagger(y)\} = \delta(x, y) \tag{56}$$
$$\{\hat{\psi}(x), \hat{\psi}(y)\} = \{\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(y)\} = 0 \tag{57}$$

7 States

Vacuum;

$$|\alpha_1\rangle = c_{\alpha_1}^\dagger \,|0\rangle \tag{58}$$
$$c_\alpha \,|0\rangle = 0, \qquad \langle 0|\, c_\alpha^\dagger = 0 \tag{59}$$

$$|x\rangle = \hat{\psi}^\dagger(x) \,|0\rangle \tag{60}$$
$$\hat{\psi}(x) \,|0\rangle = 0, \qquad \langle 0|\, \hat{\psi}^\dagger(x) = 0 \tag{61}$$

Slater Determinant;

$$|\alpha_1, \alpha_2, \cdots, \alpha_N\rangle = \frac{1}{\sqrt{N!}} \sum_{P \in S_N} (-)^P \,|\alpha_{P_1}\rangle \,|\alpha_{P_2}\rangle \cdots |\alpha_{P_N}\rangle \tag{62}$$
$$= c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger \cdots c_{\alpha_N}^\dagger \,|0\rangle \tag{63}$$

$$c_\beta^\dagger \,|\alpha_1, \alpha_2, \cdots, \alpha_N\rangle = |\beta, \alpha_1, \alpha_2, \cdots, \alpha_N\rangle \tag{64}$$

Completeness;

$$\frac{1}{N!} \sum_{\alpha_1 \cdots \alpha_N} |\alpha_1 \cdots \alpha_N\rangle \langle \alpha_1 \cdots \alpha_N| = \hat{1}_{\mathcal{F}_N} \tag{65}$$
$$\sum_{\alpha_1 < \alpha_2 < \cdots < \alpha_N} |\alpha_1 \cdots \alpha_N\rangle \langle \alpha_1 \cdots \alpha_N| = \hat{1}_{\mathcal{F}_N} \tag{66}$$

$$|0\rangle \langle 0| + \sum_{N=1}^{\infty} \left[\frac{1}{N!} \sum_{\alpha_1 \cdots \alpha_N} |\alpha_1 \cdots \alpha_N\rangle \langle \alpha_1 \cdots \alpha_N| \right] = \hat{1}_{\mathcal{F}} \tag{67}$$
$$\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_N \oplus \cdots \tag{68}$$

General A-body State;

$$
\begin{aligned}
|\Psi_A\rangle &= \hat{\Psi}_A^\dagger |0\rangle = \left[\frac{1}{\sqrt{A!}}\sum_{\{\alpha_k\}} D(\alpha_1\cdots\alpha_A) c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger \cdots c_{\alpha_A}^\dagger\right]|0\rangle && (69)\\
&= \frac{1}{\sqrt{A!}}\sum_{\{\alpha_k\}} D(\alpha_1\cdots\alpha_A)\,|\alpha_1\cdots\alpha_A\rangle && (70)\\
&= \sum_{\alpha_1<\alpha_2<\cdots<\alpha_A} D(\alpha_1\cdots\alpha_A)\,|\alpha_1\cdots\alpha_A\rangle && (71)\\
D(\cdots\alpha_k\cdots\alpha_l\cdots) &= (-)D(\cdots\alpha_l\cdots\alpha_k\cdots) && (72)
\end{aligned}
$$

Normalization;

$$
\begin{aligned}
\langle\Psi_A\,|\,\Psi_A\rangle &= \frac{1}{A!}\sum_{\{\alpha_k\}}\sum_{\{\beta_k\}} D^*(\alpha_1\cdots\alpha_A)D(\beta_1\cdots\beta_A)\,\langle\alpha_1\cdots\alpha_A\,|\,\beta_1\cdots\beta_A\rangle && (73)\\
&= \frac{1}{A!}\sum_{\{\alpha_k\}} |D(\alpha_1\cdots\alpha_A)|^2 = \sum_{\alpha_1<\alpha_2<\cdots<\alpha_A} |D(\alpha_1\cdots\alpha_A)|^2 \equiv 1 && (74)
\end{aligned}
$$

8 Density Matrix

Several Representations (even with Time-Dependence);

$$
\begin{aligned}
\hat{\rho}^{(\alpha\beta)} &\equiv c_\beta^\dagger c_\alpha, && (75)\\
\hat{\rho}^{(xy)} &\equiv \hat{\psi}^\dagger(y)\hat{\psi}(x) && (76)
\end{aligned}
$$

$$
\begin{aligned}
\rho^{(\alpha\beta)}_{\Psi'_A(t),\Psi_A(t)} &= \langle\Psi'_A(t)\,|\,\hat{\rho}^{(\alpha\beta)}\,|\,\Psi_A(t)\rangle = \left\langle\Psi'_A(t)\,\middle|\,c_\beta^\dagger c_\alpha\,\middle|\,\Psi_A(t)\right\rangle, && (77)\\
&= \langle\alpha\,|\,\hat{\rho}_{\Psi'_A(t),\Psi_A(t)}\,|\,\beta\rangle && (78)\\
\rho^{(xy)}_{\Psi'_A(t),\Psi_A(t)} &= \langle\Psi'_A(t)\,|\,\hat{\rho}^{(xy)}\,|\,\Psi_A(t)\rangle = \left\langle\Psi'_A(t)\,\middle|\,\hat{\psi}^\dagger(y)\hat{\psi}(x)\,\middle|\,\Psi_A(t)\right\rangle, && (79)\\
&= \langle x\,|\,\hat{\rho}_{\Psi'_A(t),\Psi_A(t)}\,|\,y\rangle && (80)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho}_{\Psi'_A(t),\Psi_A(t)} &= \sum_\alpha\sum_\beta |\alpha\rangle\,\rho^{(\alpha\beta)}_{\Psi'_A(t),\Psi_A(t)}\,\langle\beta|, && (81)\\
&= \int dx\int dy\,|x\rangle\,\rho^{(xy)}_{\Psi'_A(t),\Psi_A(t)}\,\langle y| && (82)
\end{aligned}
$$

Density Matrix of $|\Psi_A\rangle$;

$$
\begin{aligned}
\rho^{(\alpha\beta)}_{\Psi_A} &= \left\langle\Psi_A\,\middle|\,c_\beta^\dagger c_\alpha\,\middle|\,\Psi_A\right\rangle (= \langle\alpha\,|\,\hat{\rho}_{\Psi_A,\Psi_A}\,|\,\beta\rangle) && (83)\\
&= \frac{1}{A!}\sum_{\{\beta_k\}}\sum_{\{\alpha_k\}} D^*(\beta_1\cdots\beta_A)\,\langle 0|\,(c_{\beta_A}\cdots c_{\beta_1})\,c_\beta^\dagger c_\alpha\left(c_{\alpha_1}^\dagger\cdots c_{\alpha_A}^\dagger\right)|0\rangle\,D(\alpha_1\cdots\alpha_A) && (84)\\
&= \frac{(A-1)!}{A!}\sum_{\gamma_2\cdots\gamma_A}\left[D^*(\beta,\gamma_2,\cdots)+(-)D^*(\gamma_2,\beta,\cdots)+\cdots\right]_{(A\ terms)}\\
&\qquad\qquad \times\left[D(\alpha,\gamma_2,\cdots)+(-)D(\gamma_2,\alpha,\cdots)+\cdots\right]_{(A\ terms)} && (85)\\
&= A\sum_{\gamma_2\cdots\gamma_A} D^*(\beta,\gamma_2\cdots\gamma_A)D(\alpha,\gamma_2\cdots\gamma_A) && (86)\\
&= (A!)\sum_{\gamma_2<\cdots<\gamma_A} D^*(\beta,\gamma_2\cdots\gamma_A)D(\alpha,\gamma_2\cdots\gamma_A) && (87)
\end{aligned}
$$

$$\rho_{\Psi_A}^{(xy)} = \left\langle \Psi_A \middle| \hat{\psi}^\dagger(y)\hat{\psi}(x) \middle| \Psi_A \right\rangle (= \langle x \,|\, \hat{\rho}_{\Psi_A,\Psi_A} \,|\, y \rangle) \tag{88}$$

$$= \sum_\beta \sum_\alpha \phi_\beta^*(y) \left\langle \Psi_A \middle| c_\beta^\dagger c_\alpha \middle| \Psi_A \right\rangle \phi_\alpha(x) \tag{89}$$

$$= \sum_\beta \sum_\alpha \phi_\beta^*(y) \rho_{\Psi_A}^{(\alpha\beta)} \phi_\alpha(x) = \langle x| \left[\sum_\alpha \sum_\beta |\alpha\rangle \, \rho_{\Psi_A}^{(\alpha\beta)} \langle \beta| \right] |y\rangle \tag{90}$$

Density Matrix of the Single Slater Determinant;

$$\rho^{(\alpha\beta)} = \left\langle \alpha_1 \cdots \alpha_A \middle| c_\beta^\dagger c_\alpha \middle| \alpha_1 \cdots \alpha_A \right\rangle \propto \delta_{\alpha\beta}, \tag{91}$$

$$\rho^{(\alpha\alpha)} = \begin{cases} 0 & (\alpha > \alpha_A) \\ 1 & (\alpha \leq \alpha_A) \end{cases} \tag{92}$$

$$\hat{\rho} = \sum_\alpha \sum_\beta |\alpha\rangle \, \rho^{(\alpha\beta)} \langle \beta| = \sum_{j \leq \alpha_A} |j\rangle \langle j| \,, \tag{93}$$

$$\hat{\rho}^2 = \hat{\rho} \tag{94}$$

9 Hamiltonian

Basis Representation;

$$H = T + V = \sum_{\alpha\gamma} t_{\alpha,\gamma} c_\alpha^\dagger c_\gamma + \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \tilde{v}_{\alpha\beta,\gamma\delta} \left(c_\beta c_\alpha \right)^\dagger \left(c_\delta c_\gamma \right) \tag{95}$$

$$t_{\alpha,\gamma} = \langle \alpha \,|\, T \,|\, \gamma \rangle \tag{96}$$

$$\tilde{v}_{\alpha\beta,\gamma\delta} = \frac{1}{2} \left[\langle \alpha\beta \,|\, V \,|\, \gamma\delta \rangle - (\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) + (\alpha \leftrightarrow \beta \;\&\; \gamma \leftrightarrow \delta) \right] \tag{97}$$

Local Force Assumption;

$$\langle x'y' \,|\, V \,|\, xy \rangle = \delta(x', x)\delta(y', y)V(x, y) \tag{98}$$

Expectational Value via Slater Determinant;

$$E_0 \equiv \langle \alpha_1 \cdots \alpha_N \,|\, H \,|\, \alpha_1 \cdots \alpha_N \rangle = \langle \alpha_1 \cdots \alpha_N \,|\, T + V \,|\, \alpha_1 \cdots \alpha_N \rangle \tag{99}$$

$$= \sum_{\rho=1}^{\infty} \theta_\rho t_{\rho,\rho} + \frac{1}{4} \sum_{(\mu\neq\nu)=1}^{\infty} \theta_\mu \theta_\nu \left[\tilde{v}_{\mu\nu,\mu\nu} - \tilde{v}_{\mu\nu,\nu\mu} \right] \tag{100}$$

$$= \sum_{k=1}^{N} t_{k,k} + \frac{1}{2} \sum_{(i\neq j)=1}^{N} \tilde{v}_{ij,ij} \tag{101}$$

$$= \sum_i \left[\langle i \,|\, T \,|\, i \rangle + \frac{1}{2} \sum_{j(\neq i)} \left(\langle ij \,|\, V \,|\, ij \rangle - \langle ij \,|\, V \,|\, ji \rangle \right) \right] \tag{102}$$

$$\langle i \,|\, T \,|\, k \rangle = \int dx' \int dx \phi_i^*(x') \langle x' \,|\, T \,|\, x \rangle \, \phi_k(x) \tag{103}$$

$$\langle ij \,|\, V \,|\, kl \rangle = \int^{(2)} dx'dy' \int^{(2)} dxdy \phi_i^*(x')\phi_j^*(y') \langle x'y' \,|\, V \,|\, xy \rangle \, \phi_k(x)\phi_l(y) \tag{104}$$

$$= \int^{(2)} dxdy \phi_i^*(x)\phi_j^*(y)V(x, y)\phi_k(x)\phi_l(y) \tag{105}$$

Coordinate Representation;

$$T = \int dx' \int dx \hat{\psi}^\dagger(x') \langle x' \,|\, T \,|\, x \rangle \hat{\psi}(x) \tag{106}$$

$$V = \frac{1}{2} \int^{(2)} dx'dy' \int^{(2)} dxdy \hat{\psi}^\dagger(x')\hat{\psi}^\dagger(y') \langle x'y' \,|\, V \,|\, xy \rangle \hat{\psi}(x)\hat{\psi}(y) \tag{107}$$

$$= \frac{1}{2} \int^{(2)} dxdy \hat{\psi}^\dagger(x)\hat{\psi}^\dagger(y) V(x,y) \hat{\psi}(x)\hat{\psi}(y) \tag{108}$$

External Local Field;

$$V_{ext} = \sum_{\alpha\gamma} \langle \alpha \,|\, V_{ext} \,|\, \gamma \rangle c^\dagger_\alpha c_\gamma \tag{109}$$

$$= \int dx' \int dx \sum_{\alpha\gamma} \phi^*_\alpha(x') \langle x' \,|\, V_{ext} \,|\, x \rangle \phi_\gamma(x) c^\dagger_\alpha c_\gamma \tag{110}$$

$$= \int dx' \int dx \hat{\psi}^\dagger(x') \langle x' \,|\, V_{ext} \,|\, x \rangle \hat{\psi}(x) \tag{111}$$

$$= \int dx \hat{\psi}^\dagger(x) V_{ext}(x) \hat{\psi}(x) \tag{112}$$

$$\langle \alpha_1 \cdots \alpha_N \,|\, V_{ext} \,|\, \alpha_1 \cdots \alpha_N \rangle = \sum_i \int dx \phi^*_i(x) V_{ext}(x) \phi_i(x) \tag{113}$$

10 Hartree-Fock Method

HF-Ground State as a Slater Determinant;

$$|\Psi\rangle = |\alpha_1 \cdots \alpha_N\rangle, \qquad E_0 \equiv \langle \Psi \,|\, H \,|\, \Psi \rangle \qquad (H = T + V + V_{ext}) \tag{114}$$

Variational Principle;

$$\frac{\delta \left(E_0 - \sum_\beta e_\beta \langle \phi_\beta \,|\, \phi_\beta \rangle \right)}{\delta \phi^*_\alpha(w)} = 0 \tag{115}$$

Self-Consistent Equation;

$$\left[-\frac{\hbar^2}{2m} \nabla^2_w + V_H(w) + V_{ext}(w) \right] \phi_\alpha(w) - \int dy V_F(w,y) \phi_\alpha(y) = e_\alpha \phi_\alpha(w) \tag{116}$$

$$V_H(w) = \sum_{\beta(\neq\alpha)=1}^{N} \int dy \phi^*_\beta(y) V(w,y) \phi_\beta(y) \tag{117}$$

$$V_F(w,y) = \sum_{\beta(\neq\alpha)=1}^{N} \phi^*_\beta(y) V(w,y) \phi_\beta(w) \tag{118}$$

1-Body Density;

$$\hat{\rho}(\boldsymbol{r}) = \sum_s \hat{\psi}^\dagger(\boldsymbol{r}s)\hat{\psi}(\boldsymbol{r}s) = \sum_s \sum_\alpha \langle \alpha \,|\, \boldsymbol{r}s \rangle c^\dagger_\alpha \sum_\beta \langle \beta \,|\, \boldsymbol{r}s \rangle c_\beta \tag{119}$$

$$|\Psi[\rho]\rangle \leftrightarrow \rho(\boldsymbol{r}) = \langle \Psi \,|\, \hat{\rho}(\boldsymbol{r}) \,|\, \Psi \rangle \tag{120}$$

$$= \left\langle \alpha_1 \cdots \alpha_N \left| \sum_s \sum_\alpha \phi^*_\alpha(\boldsymbol{r}s) c^\dagger_\alpha \sum_\beta \phi_\beta(\boldsymbol{r}s) c_\beta \right| \alpha_1 \cdots \alpha_N \right\rangle \tag{121}$$

$$= \sum_s \sum_\alpha \sum_\beta \phi^*_\alpha(\boldsymbol{r}s)\phi_\beta(\boldsymbol{r}s)\rho^{(\beta\alpha)} = \sum_s \sum_{\alpha=\alpha_1}^{\alpha_N} |\phi_\alpha(\boldsymbol{r}s)|^2 \tag{122}$$

Energy Density Functional (at HF-level);

$$\mathcal{E}_{\mathrm{HF}}[\rho] = \langle \Psi[\rho] \,|\, H \,|\, \Psi[\rho] \rangle = T[\rho] + E_H[\rho] + E_F[\rho] + E_{ext}[\rho] \tag{123}$$

$$T[\rho] = -\frac{\hbar^2}{2m} \sum_{k=1}^{N} |\nabla \phi_k(x)|^2 \qquad (x = \boldsymbol{r}s) \tag{124}$$

$$E_H[\rho] = \frac{1}{2} \int dx \int dx' V(x', x) \rho(x') \rho(x) \tag{125}$$

$$E_F[\rho] = ? \tag{126}$$

$$E_{ext}[\rho] = ? \tag{127}$$

11 HF + Bogoliubov Method

Hamiltonian (with Einstein's Rule);

$$H' = H - \lambda N = (t_{k,m} - \lambda \delta_{k,m}) c^\dagger_k c_m + \frac{1}{4} \tilde{v}_{kl,mn} (c_l c_k)^\dagger (c_n c_m) \tag{128}$$

Bogoliubov Transformation;

$$\left\{ \begin{array}{l} b_k = U^*_{ik} c_i + V^*_{ik} c^\dagger_i \\ b^\dagger_k = U_{ik} c^\dagger_i + V_{ik} c_i \end{array} \right. \leftrightarrow \left\{ \begin{array}{l} c_i = U_{im} b_m + V^*_{im} b^\dagger_m \\ c^\dagger_i = U^*_{im} b^\dagger_m + V_{im} b_m \end{array} \right. \tag{129}$$

$$\begin{pmatrix} b \\ b^\dagger \end{pmatrix} = \mathcal{W} \begin{pmatrix} c \\ c^\dagger \end{pmatrix} = \begin{pmatrix} U^\dagger & V^\dagger \\ V^T & U^T \end{pmatrix} \begin{pmatrix} c \\ c^\dagger \end{pmatrix} \leftrightarrow \begin{pmatrix} c \\ c^\dagger \end{pmatrix} = \begin{pmatrix} U & V^* \\ V & U^* \end{pmatrix} \begin{pmatrix} b \\ b^\dagger \end{pmatrix} \tag{130}$$

Unitarity;

$$\mathcal{W}\mathcal{W}^\dagger = \begin{pmatrix} U^\dagger U + V^\dagger V & U^\dagger V^* + V^\dagger U^* \\ V^T U + U^T V & V^T V^* + U^T U^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \left\{ \begin{array}{l} U^\dagger U + V^\dagger V = 1 \\ V^T U + U^T V = 0 \end{array} \right. \tag{131}$$

$$\mathcal{W}^\dagger\mathcal{W} = \begin{pmatrix} U U^\dagger + V^* V^T & U V^\dagger + V^* U^T \\ V U^\dagger + U^* V^T & V V^\dagger + U^* U^T \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \left\{ \begin{array}{l} U U^\dagger + V^* V^T = 1 \\ U V^\dagger + V^* U^T = 0 \end{array} \right. \tag{132}$$

HFB Vacuum and Normal Ordering;

$$b_k \,|.\rangle = 0, \qquad \langle .|\, b^\dagger_k = 0 \tag{133}$$

$$\big\langle . \,\big|\, : X(b^\dagger, b) : \,\big|\, . \big\rangle = 0 \tag{134}$$

Density-Tensor and Pair-Tensor;

$$\rho_{kl} \equiv \left\langle . \left| c^\dagger_l c_k \right| . \right\rangle = V_{lm} V^*_{km} \quad , \quad \kappa_{kl} \equiv \langle . \,|\, c_l c_k \,|\, . \rangle = (-)\kappa_{lk} = U_{lm} V^*_{km} \tag{135}$$

$$\Longleftrightarrow \rho = V^* V^T = 1 - U U^\dagger = \rho^\dagger \quad , \quad \kappa = (-)\kappa^T = V^* U^T = -U V^\dagger \tag{136}$$

Meanfield and Pairing Potentials;

$$\Gamma_{kl} = \tilde{v}_{km,ln}\rho_{nl}, \qquad \Delta_{kl} = \frac{1}{2}\tilde{v}_{kl,mn}\kappa_{mn} \leftrightarrow \Delta^*_{kl} = \frac{1}{2}\tilde{v}_{mn,kl}\kappa^*_{mn} \tag{137}$$

Quasiparticle Representation of $H' = H - \lambda N$;

$$\begin{aligned}
H' &= H^{(0)} + H^{(2)} + H^{(4)} && (138)\\
H^{(0)} &= \langle . \,|\, H' \,|\, . \rangle = tr\left[(t-\lambda 1)\rho + \frac{1}{2}\Gamma\rho - \frac{1}{2}\Delta\kappa^*\right] && (139)\\
H^{(2)} &= \frac{1}{2} : \begin{pmatrix} c^\dagger & c \end{pmatrix} \begin{pmatrix} h-\lambda 1 & \Delta \\ -\Delta^* & -h^*+\lambda 1 \end{pmatrix} \begin{pmatrix} c \\ c^\dagger \end{pmatrix} : \qquad (h = t + \Gamma) && (140)\\
&= H^{(11)} + H^{(20)} + h.c. && (141)\\
H^{(11)} &= b^\dagger_m b_n \left[\begin{pmatrix} U^\dagger & V^\dagger \end{pmatrix} \begin{pmatrix} h-\lambda 1 & \Delta \\ -\Delta^* & -h^*+\lambda 1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \right]_{mn} && (142)\\
H^{(20)} + h.c. &= \frac{1}{2} b^\dagger_m b^\dagger_n \left[\begin{pmatrix} U^\dagger & V^\dagger \end{pmatrix} \begin{pmatrix} h-\lambda 1 & \Delta \\ -\Delta^* & -h^*+\lambda 1 \end{pmatrix} \begin{pmatrix} V^* \\ U^* \end{pmatrix} \right]_{mn} + h.c. && (143)\\
H^{(4)} &= \frac{1}{4}\tilde{v}_{kl,mn} : (c_l c_k)^\dagger (c_n c_m) : && (144)
\end{aligned}$$

HFB Solution (U, V);

$$H' = H^{(0)} + \sum_k E_k b^\dagger_k b_k + 0 + H^{(4)} \tag{145}$$

Energy Density Functional (at HFB-level);

$$\begin{aligned}
\mathcal{E}[\rho, \kappa, \kappa^*] &= \langle . \,|\, H' \,|\, . \rangle = \mathcal{E}_{\rm HF}[\rho, \kappa, \kappa^*] + \mathcal{E}_{\rm pair}[\rho, \kappa, \kappa^*] && (146)\\
&= H^{(0)} = tr\left[(t-\lambda 1)\rho + \frac{1}{2}\Gamma\rho - \frac{1}{2}\Delta\kappa^*\right] && (147)
\end{aligned}$$

$$h_{kl}[\rho, \kappa, \kappa^*] = \frac{\partial \mathcal{E}}{\partial \rho_{lk}} = (t-\lambda 1)_{kl} + \Gamma_{kl}, \qquad \Delta_{kl}[\rho, \kappa, \kappa^*] = \frac{\partial \mathcal{E}}{\partial \kappa^*_{kl}} \tag{148}$$

12 Finite Amplitude Method

FAM Linear Responce Equation;

$$A\vec{x} = \vec{f} \tag{149}$$

$$\vec{x} = \begin{pmatrix} X_{\mu\nu}(\omega) \\ Y_{\mu\nu}(\omega) \end{pmatrix} \qquad \vec{f} = \begin{pmatrix} F^{20}_{\mu\nu}(\omega) \\ F^{02}_{\mu\nu}(\omega) \end{pmatrix} \tag{150}$$

$$A\vec{x} = \begin{pmatrix} (E_\mu + E_\nu - \omega)X_{\mu\nu}(\omega) + \delta H^{20}_{\mu\nu}(\omega) \\ (E_\mu + E_\nu + \omega)Y_{\mu\nu}(\omega) + \delta H^{02}_{\mu\nu}(\omega) \end{pmatrix} \tag{151}$$

$$\begin{aligned}
\delta H^{20}_{\mu\nu}(\omega) &= +U^\dagger \delta h V^* - V^\dagger \delta\Delta^{(-)*} V^* + U^\dagger \delta\Delta^{(+)} U^* - V^\dagger \delta h^T U^* && (152)\\
\delta H^{02}_{\mu\nu}(\omega) &= -V^T \delta h U + U^T \delta\Delta^{(-)*} U - V^T \delta\Delta^{(+)} V + U^T \delta h^T V && (153)
\end{aligned}$$

13 Wick's Theorem

Philosophy;

$$\hat{\mathcal{T}}\left[O\right] = \langle - \,|\, O \,|\, - \rangle + \hat{\mathcal{N}}\left[O\right] \tag{154}$$

Formulas;

$$\begin{aligned}
\hat{\mathcal{T}}\left[\psi(x_1)\cdots\psi(x_n)\right] \;=\;& :\psi(x_1)\cdots\psi(x_n): \\
&+\sum_{P\in S_n}\langle - \,|\, \psi(x_{P_1})\psi(x_{P_2}) \,|\, - \rangle :\psi(x_{P_3})\cdots\psi(x_{P_n}): \\
&+\sum_{P\in S_n}\langle - \,|\, \psi(x_{P_1})\psi(x_{P_2}) \,|\, - \rangle \langle - \,|\, \psi(x_{P_3})\psi(x_{P_4}) \,|\, - \rangle :\psi(x_{P_5})\cdots\psi(x_{P_n}): \\
&\;\vdots \\
&+\sum_{P\in S_n}\langle - \,|\, \psi(x_{P_1})\psi(x_{P_2}) \,|\, - \rangle \cdots \left\langle - \,\middle|\, \psi(x_{P_{n-1}})\psi(x_{P_n}) \,\middle|\, - \right\rangle
\end{aligned} \tag{155}$$

$$\begin{aligned}
d_1\cdots d_n \;=\;& :d_1\cdots d_n: \qquad (d_k \leftarrow c_k, c_k^\dagger) \\
&+\sum_{P\in S_n}\langle - \,|\, d_{P_1}d_{P_2} \,|\, - \rangle :d_{P_3}\cdots d_{P_n}: \\
&+\sum_{P\in S_n}\langle - \,|\, d_{P_1}d_{P_2} \,|\, - \rangle \langle - \,|\, d_{P_3}d_{P_4} \,|\, - \rangle :d_{P_5}\cdots d_{P_n}: \\
&\;\vdots \\
&+\sum_{P\in S_n}\langle - \,|\, d_{P_1}d_{P_2} \,|\, - \rangle \cdots \left\langle - \,\middle|\, d_{P_{n-1}}d_{P_n} \,\middle|\, - \right\rangle
\end{aligned} \tag{156}$$

Dirac HFB equation from the sigma-nucleon model

14 Introduction

After reading Ref. [2], here I try to formalize the HFB equation from the relativistic Lagrangian (density), which is, however, simpler than the original version. Namely, it contains only the nucleon $\psi(x)$ and the sigma meson $\sigma(x)$[1]. That is,

$$\begin{aligned}\mathcal{L}\left[\psi, \psi^{*}, \sigma\right] &= \bar{\psi}\left(i \gamma^{\mu} \partial_{\mu}-m\right) \psi(x)+\frac{1}{2}\left(\partial^{\mu} \sigma \partial_{\mu} \sigma-\mu^{2} \sigma^{2}\right) \\ & -g_{\sigma} \bar{\psi} \sigma \psi(x), \end{aligned} \tag{157}$$

where $\partial^{\mu}=(\frac{\partial}{\partial t},-\vec{\nabla})$ and $\partial_{\mu}=(\frac{\partial}{\partial t},+\vec{\nabla})$. Thus, conjugate fields are

$$\begin{aligned}\Pi_{\psi}(x) &= \frac{\partial \mathcal{L}}{\partial(\partial_{0} \psi)}=i \bar{\psi}(x) \gamma^{0}=i \psi^{\dagger}(x), \\ \Pi_{\psi *}(x) &= 0, \\ \Pi_{\sigma}(x) &= \frac{\partial \mathcal{L}}{\partial(\partial_{0} \sigma)}=\partial^{0} \sigma(x)=\dot{\sigma}(x). \end{aligned} \tag{158}$$

Therefore, the Hamiltonian (density) is given as

$$\begin{aligned}\mathcal{H} &= \Pi_{\psi} \dot{\psi}+\Pi_{\sigma} \dot{\sigma}-\mathcal{L} \\ &= i \bar{\psi}(x) \gamma^{0}(\partial_{0} \psi)+\partial^{0} \sigma(x) \cdot \partial_{0} \sigma(x)-\mathcal{L} \\ &= 0+i \bar{\psi}\left[\gamma^{1} \partial_{1}+\gamma^{2} \partial_{2}+\gamma^{3} \partial_{3}+m\right] \psi(x) \\ & +\frac{1}{2}\left[\dot{\sigma}^{2}+(\vec{\nabla} \sigma)^{2}+\mu^{2} \sigma^{2}(x)\right]+g_{\sigma} \bar{\psi} \sigma \psi, \\ \Longrightarrow H &= \int d V_{x} \mathcal{H}=H_{N}+H_{M}+H_{I}, \end{aligned} \tag{159}$$

where

$$\begin{aligned}H_{N} &= \int d V_{x} \psi^{\dagger}(x)\left[\vec{\alpha} \cdot \boldsymbol{p}+\beta m\right] \psi(x), \\ H_{M} &= \int d V_{x} \frac{1}{2}\left[\Pi_{\sigma}^{2}+(\vec{\nabla} \sigma)^{2}+\mu^{2} \sigma^{2}(x)\right], \\ H_{I} &= \int d V_{x} g_{\sigma} \bar{\psi}(x) \sigma(x) \psi(x). \end{aligned} \tag{160}$$

Remember that $\beta=\gamma^{0}$, $\alpha_{n}=\gamma^{0} \gamma^{n}$, and $p_{n}=i \partial_{n}$ $(n=1,2,3)$.

15 Equations of Interacting Fields

15.1 sigma meson

First we solve the σ-meson field. Klein-Gordon equation for $\sigma(x)$ reads

$$\begin{aligned} & \partial^{\mu} \frac{\partial \mathcal{L}}{\partial(\partial^{\mu} \sigma)}-\frac{\partial \mathcal{L}}{\partial \sigma}=0 \\ & \Longrightarrow\left(\partial^{\mu} \partial_{\mu}+\mu^{2}\right) \sigma(x)=-g_{\sigma} \bar{\psi}(x) \psi(x). \end{aligned} \tag{161}$$

[1]This model is indeed Yukawa model as written in Eq. (4.112) in the textbook [1].

By using the Green function D_σ, which satisfies

$$\left(\partial^\mu \partial_\mu + \mu^2\right)_x D_\sigma(x-y) = \delta(x-y), \tag{162}$$

the formal solution is given as

$$\sigma(x) = \int dy D_\sigma(x-y) \cdot (-g_\sigma)\bar{\psi}(y)\psi(y). \tag{163}$$

As well known, this Green function is indeed *propagator* of the scalar field:

$$D_\sigma(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{(-)}{k^2 - \mu^2 + i\epsilon} e^{-ik(x-y)}. \tag{164}$$

15.2 nucleon

Next we consider the nucleon field. Within the Heisenberg representation $\psi(x) = e^{itH}\psi(0, \boldsymbol{x})e^{-itH}$, the field operator follows the time-development equation,

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(x) &= [\psi(x), H] = [\psi(x), H_N + H_I] \\ &= \int dV_w \left[\psi(x), \psi^\dagger(w) \left\{\vec{\alpha} \cdot \boldsymbol{p} + \beta m + g_\sigma \gamma_0 \sigma(w)\right\}_w \psi(w)\right]. \end{aligned} \tag{165}$$

The first plus second terms yield the usual formula:

$$\begin{aligned} [\psi(x), H_N] &= \int dV_w \psi(x)\psi^\dagger(w) \left\{\vec{\alpha} \cdot \boldsymbol{p} + \beta m\right\}_w \psi(w) \\ &\quad - \int dV_w \psi^\dagger(w) \left\{\vec{\alpha} \cdot \boldsymbol{p} + \beta m\right\}_w \psi(w)\psi(x) \\ &= \int dV_w \delta(x-w) \left\{\vec{\alpha} \cdot \boldsymbol{p} + \beta m\right\}_w \psi(w) + 0 \\ &= \left\{\vec{\alpha} \cdot \boldsymbol{p} + \beta m\right\}_x \psi(x), \end{aligned} \tag{166}$$

from $\psi(x)\psi^\dagger(w) = \delta(x-w) - \psi^\dagger(w)\psi(x)$. This result is consistent to the free Dirac equation. From the similar calculation, the third term yields

$$[\psi(x), H_I] = g_\sigma \gamma^0 \sigma \psi(x), \tag{167}$$

consistently to the interaction term in the Dirac equation. Note that its conjugate version follows the similar form. Summarizing these results, we have obtained

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(x) = [\psi(x), H] \quad &\Leftrightarrow \quad \left[i\partial_t - \left\{\vec{\alpha} \cdot \boldsymbol{p} + \beta m\right\}_x\right]\psi(x) = g_\sigma \gamma^0 \sigma \psi(x), \\ i\frac{\partial}{\partial t}\psi^\dagger(x) = \left[\psi^\dagger(x), H\right] \quad &\Leftrightarrow \quad \psi^\dagger(x)\left[i\partial_t - \left\{\vec{\alpha} \cdot \boldsymbol{p} + \beta m\right\}_x\right] = g_\sigma \bar{\psi}(x)\sigma(x), \end{aligned} \tag{168}$$

where the source term $g_\sigma \gamma_0 \sigma(x)$ shows up. It is useful to note that

$$\begin{aligned} \hat{T}\left[\psi(x)\bar{\psi}(y)\right] &= S_F(x,y) + \hat{N}_0\left[\psi(x)\bar{\psi}(y)\right] = S_F(x,y) + (-)\bar{\psi}(y)\psi(x), \\ S_F(x,y) &= \left\langle 0 \mid \hat{T}\left[\psi(x)\bar{\psi}(y)\right] \mid 0\right\rangle, \end{aligned} \tag{169}$$

from the Wick's theorem[2], where $\hat{N}_0$ means the normal ordering with respect to the free vacuum: $\left\langle 0 \mid \hat{N}_0 [...] \mid 0 \right\rangle$. The $S_F(x,y)$ is the Feynman propagator of the free fermion, satisfying

$$\begin{aligned} [i\gamma^\nu \partial_\nu - m]_x S_F(x,y) = \delta(x,y) \quad &\Leftrightarrow \quad \left[i\gamma^0 \partial_0 - \vec{\gamma}\cdot\boldsymbol{p} - m\right]_x S_F(x,y) = \delta(x-y) \\ &\Leftrightarrow \quad [i\partial_t - \{\vec{\alpha}\cdot\boldsymbol{p} + \beta m\}]_x S_F(x,y) = \gamma^0 \delta(x-y). \end{aligned} \tag{170}$$

Using this S_F, the fermion (nucleon) field can be formally solved as

$$\psi(x) = \int dy S_F(x,y) g_\sigma \sigma(y) \psi(y). \tag{171}$$

We can also follows the time-development of the fermion propagator. That is

$$\begin{aligned} G(x,y) &= \left\langle A \mid \hat{T}\psi(x)\bar{\psi}(y) \mid A \right\rangle \\ &= S_F(x,y) + \left\langle A \mid (-)\bar{\psi}(y)\psi(x) \mid A \right\rangle. \end{aligned} \tag{172}$$

This $G(x,y)$ can be also interpreted as the *density tensor*, $\rho_{xy} = G(x,y)$, in the usual meanfield framework. For this propagator, one finds

$$\begin{aligned} [i\partial_t - \{\vec{\alpha}\cdot\boldsymbol{p} + \beta m\}]_x G(x,y) &= \gamma^0 \delta(x-y) - \left\langle A \mid \bar{\psi}(y) [i\partial_t - \{\vec{\alpha}\cdot\boldsymbol{p} + \beta m\}]_x \psi(x) \mid A \right\rangle \\ &= \gamma^0 \delta(x-y) - \left\langle A \mid \bar{\psi}(y) g_\sigma \gamma^0 \sigma(x) \psi(x) \mid A \right\rangle \\ [i\gamma^\nu \partial_\nu - m]_x G(x,y) &= \delta(x-y) - g_\sigma \left\langle A \mid \bar{\psi}(y) \sigma(x) \psi(x) \mid A \right\rangle. \end{aligned} \tag{173}$$

From Eq. (163), $\sigma(x)$ can be eliminated:

$$[i\gamma^\nu \partial_\nu - m]_x G(x,y) = \delta(x-y) + g_\sigma^2 \left\langle A \mid \bar{\psi}(y) \int dw D_\sigma(x-w) \bar{\psi}(w)\psi(w)\psi(x) \mid A \right\rangle. \tag{174}$$

Notice that the quadratic term of $\bar{\psi}\bar{\psi}\psi\psi$ appears in the RHS.

References

[1] M. E. Peskin and D. V. Schroeder, *"An Introduction to Quantum Field Theory.*

[2] H. Kucharek and P. Ring, Zeitschrift für Physik A, 339. 23-35 (1991).

[2]See Eq. (4.107) in Ref. [1].