Basic Formalism of Relativistic Nuclear Many-body Theory
Tomohiro Oishi

1 Convention and basic formulas

See TABLE 1 for basic conventions.

<table>
<thead>
<tr>
<th>Name</th>
<th>Quantity</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>flat metric</td>
<td>$g^{\mu\nu} = g_{\mu\nu}$</td>
<td>$= \text{diag}(+,-,-,-)$</td>
</tr>
<tr>
<td>4D coordinate</td>
<td>$x^\mu = (x^0, x^1, x^2, x^3)$</td>
<td>$(ct, x, y, z)$</td>
</tr>
<tr>
<td></td>
<td>$x_\mu = (x_0, x_1, x_2, x_3)$</td>
<td>$(ct, -x, -y, -z)$</td>
</tr>
<tr>
<td>4D derivative</td>
<td>$\partial^\mu = \frac{\partial}{\partial x^\mu}$</td>
<td>$(\frac{\partial}{\partial ct}, -\vec{\nabla})$</td>
</tr>
<tr>
<td></td>
<td>$\partial_\mu = \frac{\partial}{\partial x_\mu} = g_{\mu\nu} \partial^\nu$</td>
<td>$(\frac{\partial}{\partial ct}, \vec{\nabla})$</td>
</tr>
<tr>
<td>4D momentum</td>
<td>$p^\mu = (p^0, p^1, p^2, p^3) = i\hbar \partial^\mu$</td>
<td>$(\frac{E}{c}, \vec{p})$</td>
</tr>
<tr>
<td></td>
<td>$p_\mu = g_{\mu\nu} p^\nu = i\hbar \partial_\mu$</td>
<td>$(\frac{E}{c}, -\vec{p})$</td>
</tr>
<tr>
<td>gamma matrices</td>
<td>$\gamma^\mu = (\gamma^0, \vec{\gamma})$</td>
<td>$(\beta, \beta \vec{\gamma})$</td>
</tr>
<tr>
<td>reduced derivative</td>
<td>$\gamma^\mu \partial_\mu = \gamma_\mu \partial^\mu$</td>
<td>$= \gamma^0 \partial ct + \vec{\gamma} \cdot \vec{\nabla}$</td>
</tr>
</tbody>
</table>

2 Units

We assume the $(1 + 3)$-dimensional time and space. In the MKSA or CGS-Gauss system of units, except the electro-magnetic terms, the Dirac equation is given as

$$i\hbar \frac{\partial}{\partial t} \psi(t, r) = \left[ -i\hbar c \beta \vec{\gamma} \cdot \vec{\nabla} + \beta Mc^2 + W \right] \psi(t, r),$$

(1)

where $W$ is some external potential in the unit of energy (e.g., MeV). From $\beta \beta = I$ and $\gamma^\mu \partial_\mu = \beta \partial ct + \vec{\gamma} \cdot \vec{\nabla}$, it is also expressed as

$$\left[ i\hbar c \gamma^\mu \partial_\mu - Mc^2 - \beta W \right] \psi(t, r) = 0.$$  

(2)

The Lagangian density, which works as the source of this equation, reads

$$\mathcal{L} = \bar{\psi} \left[ i\hbar c \gamma^\mu \partial_\mu - Mc^2 - \beta W \right] \psi(x),$$

(3)

where $\bar{\psi} \equiv \psi^\dagger \beta$. Note that, because the Lagrangian $L \equiv \int d^3 r \mathcal{L}$ and $Mc^2$ have the dimension of energy, $\bar{\psi} \psi$ is in the unit of fm$^{-3}$. As coincidence, if some interaction term(s) has the form,

$$\mathcal{L}_I = \bar{\psi} X \psi(x),$$

(4)
then this wild-card part \( X \) must have the dimension of energy, e.g. in MeV. This knowledge may help us, for example, to infer the unit of the coupling constant.

For dimensional analysis, the action follows \[ S \] since Lagrangian (as well as Hamiltonian) keeps the dimension of energy, \[ d^3 r \mathcal{L} \rvert_D = E = ML^2T^{-2} \]. Thus, Lagrangian density has \[ \mathcal{L} \rvert_D = EL^{-3} \]. Note that, in the MKSA or CGS-Gauss system of units, the dimensional analysis concludes that,

\[
[c^2 \cdot \text{mass}]_D = [\text{energy}]_D = \left[ \frac{hc}{\text{length}} \right]_D = \left[ \frac{h}{\text{time}} \right]_D = E.
\]

### 2.1 Plank’s natural system of units

In the Plank’s natural system of units, we assume that \( \hbar \equiv 1 \) and \( c \equiv 1 \). With this assumption, dimensions of mass, energy, length, and time can be related as

\[
[\text{mass}]_D = [\text{energy}]_D = \left[ \frac{1}{\text{length}} \right]_D = \left[ \frac{1}{\text{time}} \right]_D = M^{+1}.
\]

### Table 2: Dimensional numbers of some quantities, \([\text{Quantity}]_D\).

<table>
<thead>
<tr>
<th>Quantity</th>
<th>In MKSA or CGS-Gauss</th>
<th>In Plank’s natural</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass</td>
<td>( M )</td>
<td>( M^{+1} )</td>
</tr>
<tr>
<td>time and length</td>
<td>( T ) and ( L )</td>
<td>( M^{-1} )</td>
</tr>
<tr>
<td>energy</td>
<td>( E = ML^2T^{-2} )</td>
<td>( M^{+1} )</td>
</tr>
<tr>
<td>( \mathcal{L} ) or ( \mathcal{H} )</td>
<td>( EL^{-3} )</td>
<td>( M^{+4} )</td>
</tr>
<tr>
<td>( \bar{\psi}\psi(x) )</td>
<td>( L^{-3} )</td>
<td>( M^{+3} )</td>
</tr>
<tr>
<td>( \phi^2(x) ) (scalar boson)</td>
<td>( E^{-1}L^{-3} )</td>
<td>( M^{+2} )</td>
</tr>
<tr>
<td>( A^\mu A_\mu(x) ) (vector boson)</td>
<td>( E^{-1}L^{-3} )</td>
<td>( M^{+2} )</td>
</tr>
</tbody>
</table>

### 3 Lagrangian

In the relativistic nuclear theory (RNT), nucleon is described by a Dirac spinor \( \psi(x) \), where \( x = \{ r, s, \bar{r} \} \). The phenomenological Lagrangian density reads

\[
\mathcal{L} = \bar{\psi}(x) [i \gamma_\mu \partial^\mu - M] \psi(x) + \mathcal{L}_M + \mathcal{L}_I. \tag{7}
\]

Here \( \mathcal{L}_M \) is the kinetic and self-interaction part of mesons in the model. The interaction part, \( \mathcal{L}_I \), on the other hand, includes all the possible terms of interactions. See TABLEs 3 and 4 for details.

For meson terms \( \mathcal{L}_M \),

\[
\mathcal{L}_M = \frac{1}{2} \left[ \partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2 \right] + U(\sigma) - \frac{1}{2} \left[ \Omega_{\mu\nu} \Omega^{\mu\nu} - m_\omega^2 \omega_\mu \omega^\mu \right] - \frac{1}{2} \left[ \tilde{T}_{\mu\nu} \tilde{T}^{\mu\nu} - m_\rho^2 \tilde{\rho}_\mu \tilde{\rho}^\mu \right] + \frac{1}{2} \left[ \partial_\mu \tilde{\pi} \partial^\mu \tilde{\pi} - m_\pi^2 \tilde{\pi}^2 \right] - \frac{1}{2} F_{\mu\nu} F^{\mu\nu}. \tag{8}
\]
### Table 3: Kinetic and self-interaction terms included in $\mathcal{L}_M$. Label (i) indicates isoscalar (IS) or isovector (IV). Label (ii) indicates scalar (S), vector (V), pseudo-scalar (PS) or pseudo-vector (PV).

<table>
<thead>
<tr>
<th>(i)</th>
<th>(ii)</th>
<th>$(T, J^\pi)$</th>
<th>Meson</th>
</tr>
</thead>
<tbody>
<tr>
<td>IS</td>
<td>S</td>
<td>$(0, 0^+)$</td>
<td>$\sigma$</td>
</tr>
<tr>
<td></td>
<td>V</td>
<td>$(0, 1^-)$</td>
<td>$\omega^\mu$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PS</td>
<td>(0, 0)</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>PV</td>
<td>(0, 1)</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>S</td>
<td>$(1, 0^+)$</td>
<td>×</td>
</tr>
<tr>
<td></td>
<td>V</td>
<td>$(1, 1^-)$</td>
<td>$\vec{\rho}^\mu$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PS</td>
<td>(1, 0)</td>
<td>$\vec{\pi}$</td>
<td>$+\frac{1}{2} [\partial_\mu \vec{\pi} \partial^\mu \vec{\pi} - m_\pi^2 \vec{\pi} \cdot \vec{\pi}]$</td>
</tr>
<tr>
<td>PV</td>
<td>(1, 1)</td>
<td>×</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\vec{\Gamma} \cdot \vec{A}$</td>
<td>$-\frac{1}{2} F_{\mu\nu} F^{\mu\nu}$</td>
</tr>
<tr>
<td>Coulomb</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In the meson-exchange model, the Lagrangian $\mathcal{L}_I$ reads

$$\mathcal{L}_I = -g_\sigma \bar{\psi} \gamma_\mu \omega^\mu \psi - g_{\omega} [\bar{\psi} \gamma_\mu (\vec{\tau} \vec{p}^\mu) \psi] - g_{\rho} [\bar{\psi} \gamma_\mu (\vec{\tau} \vec{p}^\mu) \psi] - ig_\pi [\bar{\psi} \gamma_5 (\vec{\tau} \vec{\pi}) \psi] - \frac{f_\pi}{m_\pi} [\bar{\psi} \gamma_5 \gamma_\mu \partial^\mu (\vec{\tau} \vec{\pi}) \psi] - e \bar{\psi} \gamma_\mu A^\mu \left(1 - \frac{1}{2} \vec{\tau}_3 \right) \psi(x).$$  \(9\)

In the point-coupling model, the Lagrangian $\mathcal{L}_I$ reads

$$\mathcal{L}_I = -\frac{\alpha_{\text{IS-S}}(\rho)}{2} [\bar{\psi} \psi][\bar{\psi} \psi] - \frac{\alpha_{\text{IS-V}}(\rho)}{2} [\bar{\psi} \gamma_\mu \psi][\bar{\psi} \gamma_\mu \psi] - \frac{\alpha_{\text{IV-V}}(\rho)}{2} [\bar{\psi} \gamma_\mu \vec{\tau} \psi][\bar{\psi} \gamma_\mu \vec{\tau} \psi]$$

$$- \frac{\alpha_{\text{IV-PS}}(\rho)}{2} [\bar{\psi} \gamma_5 \vec{\tau} \psi][\bar{\psi} \gamma_5 \vec{\tau} \psi] - \frac{\alpha_{\text{IV-PV}}(\rho)}{2} [\bar{\psi} \gamma_5 \gamma_\mu \vec{\tau} \psi][\bar{\psi} \gamma_5 \gamma_\mu \vec{\tau} \psi]$$

$$- e \bar{\psi} \gamma_\mu A^\mu \left(1 - \frac{1}{2} \vec{\tau}_3 \right) \psi(x).$$  \(10\)

### 4 Equation of Motion

The equation of motion (EOM) reads

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu q_i)} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0.$$  \(11\)

Here $q_i(x)$ is generally utilized for $\psi(x)$, $A_\mu(x)$, or meson fields.
TABLE 4: Interaction terms included in $\mathcal{L}_i$. Label (i) indicates isoscalar (IS) or isovector (IV). Label (ii) indicates scalar (S), vector (V), pseudo-scalar (PS) or pseudo-vector (PV).

<table>
<thead>
<tr>
<th>(i)</th>
<th>(ii)</th>
<th>$(T, J^P)$</th>
<th>Meson</th>
<th>Meson-exchange</th>
<th>Point-coupling</th>
</tr>
</thead>
<tbody>
<tr>
<td>IS</td>
<td>S</td>
<td>$(0, 0^+)$</td>
<td>$\sigma$</td>
<td>$-g_\sigma \bar{\psi}\sigma\psi$</td>
<td>$-\alpha_{IS-S} (\rho) \bar{\psi}\psi/2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$-\delta_{IS-S} (\rho) \bar{\psi}\psi/2$</td>
<td>$-\delta_{IS-S} (\rho) \bar{\psi}\psi/2$</td>
</tr>
<tr>
<td>V</td>
<td>$(0, 1^-)$</td>
<td>$\omega^\mu$</td>
<td>$-g_\omega [\bar{\psi} \gamma_\mu \omega^\mu \psi]$</td>
<td>$-\alpha_{IS-V} (\rho) \bar{\psi}\gamma_\mu \psi/2$</td>
<td></td>
</tr>
<tr>
<td>PS</td>
<td>$(0, 0^-)$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>PV</td>
<td>$(0, 1^+)$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>IV</td>
<td>S</td>
<td>$(1, 0^+)$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>V</td>
<td>$(1, 1^-)$</td>
<td>$\vec{p}^\mu$</td>
<td>$-g_\rho [\bar{\psi} \gamma_\mu (\vec{p} \vec{p}) \psi]$</td>
<td>$-\alpha_{IV-V} (\rho) \bar{\psi}\gamma_\mu \psi/2$</td>
<td></td>
</tr>
<tr>
<td>PS</td>
<td>$(1, 0^-)$</td>
<td>$\vec{p}$</td>
<td>$-i g_\sigma [\bar{\psi} \gamma_5 (\vec{p} \vec{p}) \psi]$</td>
<td>$-\alpha_{IV-PS} (\rho) \bar{\psi}\gamma_5 \psi/2$</td>
<td></td>
</tr>
<tr>
<td>PV</td>
<td>$(1, 1^+)$</td>
<td>$\partial_\mu \vec{p}$</td>
<td>$-\frac{f_\pi}{m} [\bar{\psi} \gamma_5 \gamma_\mu \partial_\mu (\vec{p} \vec{p}) \psi]$</td>
<td>$-\alpha_{IV-PV} (\rho) \bar{\psi}\gamma_5 \gamma_\mu \psi/2$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Coulomb</td>
<td>$-e \bar{\psi}\gamma_\mu A^\mu (1 - \frac{\rho}{2}) \psi$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4.1 IV-PS (pion-nucleon) coupling

Focusing on the isovector-pseudoscalar (IV-PS) coupling, the pion-exchange model is defined with the following interaction:

$$\mathcal{L}_{ME} = \bar{\psi} [i \gamma_\mu \partial^\mu - M] \psi(x)$$

$$-i g_{IV-PS} [\bar{\psi} \gamma_5 (\vec{p} \vec{p}) \psi(x)] + \frac{1}{2} [\partial_\mu \vec{p} \partial^\mu \vec{p} - m^2 \vec{p} \vec{p}] .$$ (12)

On the other side, based on the point-coupling (zero-range) model, it is defined as

$$\mathcal{L}_{PC} = \bar{\psi} [i \gamma_\mu \partial^\mu - M] \psi(x)$$

$$- \frac{\alpha_{IV-PS}}{2} (\bar{\psi} \gamma_5 \vec{p} \psi) (\bar{\psi} \gamma_5 \vec{p} \psi) + \left( \frac{1}{2} [\partial_\mu \vec{p} \partial^\mu \vec{p} - m^2 \vec{p} \vec{p}] \right)_{\text{neglectable}} .$$ (13)

Thus, roughly speaking, these two models can be related as

$$-i g_{IV-PS} \vec{p} \psi(x) \leftrightarrow - \frac{\alpha_{IV-PS}}{2} \gamma_5 \vec{p} \psi(x).$$ (14)

In the following, we explain the background of this analogy. The factor $1/2$ is indeed not correct, but the derivative of the square term works instead.

First we note the equation of motion for $\psi(x)$ of the pion-exchange model. That is,

$$\partial_\mu \frac{\delta \mathcal{L}_{ME}}{\delta (\partial_\mu \psi)} - \frac{\delta \mathcal{L}_{ME}}{\delta \psi} = 0,$$

$$0 - \gamma_0 [i \gamma_\mu \partial^\mu - M] \psi(x) + i g_{IV-PS} \gamma_5 (\vec{p} \vec{p}) \psi(x) = 0,$$

$$[i \gamma_\mu \partial^\mu - M] \psi(x) = i g_{IV-PS} \vec{p} \vec{p} \gamma_5 \psi(x).$$ (15)
Second, from the equation of motion for $\psi(x)$ of the pion-exchange model,

$$\frac{\partial}{\partial \mu} \frac{\delta \mathcal{L}_{\text{ME}}}{\delta \partial_{\mu} \psi} - \frac{\delta \mathcal{L}_{\text{ME}}}{\delta \psi} = 0,$$

$$\partial_{\mu} \partial_{\mu} \pi_a - \left[ -m^2 \pi_a(x) - ig_{\text{IV-PV}} \gamma_5 \tau_a \psi(x) \right] = 0,$$

$$\left[ \partial_{\mu} \partial_{\mu} + m^2 \right] \pi_a(x) = -ig_{\text{IV-PV}} \gamma_5 \tau_a \psi(x). \quad (16)$$

Then, we suppose the heavy-pion limit. In this case, we can naively approximate as

$$\pi_a(x) \simeq \frac{-ig_{\text{IV-PV}}}{m^2} \gamma_5 \tau_a \psi(x). \quad (17)$$

By substituting this into the EOM of $\psi(x)$, we find that

$$[i \gamma_{\mu} \partial_{\mu} - M] \psi(x) \simeq -\alpha_{\text{IV-PV}} (\gamma_5 \tau_a \psi) \gamma_5 \tau_a \psi(x), \quad (18)$$

where $-\alpha_{\text{IV-PV}} = (-ig_{\text{IV-PV}})^2/m^2$. This is indeed the EOM but obtained from the other, point-coupling Lagrangian. Notice also that, for the correspondence of two models, $\alpha_{\text{IV-PV}} > 0$. Its unit must be in, e.g., MeV·fm$^3$, since $\mathcal{L}_{\text{PC}}$ and $\bar{\psi} \psi(x)$ have the units of MeV·fm$^{-3}$ and fm$^{-3}$, respectively.

### 4.2 IV-PV coupling

Focusing on the IV-PV coupling, the pion-exchange model reads

$$\mathcal{L}_{\text{ME}} = \bar{\psi} [i \gamma_{\mu} \partial_{\mu} - M] \psi(x) - g_{\text{IV-PV}} (\bar{\psi} \gamma_{\mu} \gamma_5 \psi) \frac{\bar{\tau}}{2} \cdot \partial_{\mu} \bar{\tau} + \frac{1}{2} \left[ \partial_{\mu} \bar{\tau} \cdot \partial_{\mu} \bar{\tau} - m^2 \bar{\tau} \cdot \bar{\tau} \right]. \quad (19)$$

On the other side, based on the point-coupling model, it is usually given as

$$\mathcal{L}_{\text{PC}} = \bar{\psi} [i \gamma_{\mu} \partial_{\mu} - M] \psi(x) - \frac{\alpha_{\text{IV-PV}}}{2} (\bar{\psi} \gamma_{\mu} \gamma_5 \psi) \frac{\bar{\tau}}{2} \cdot (\bar{\psi} \gamma_{\mu} \gamma_5 \psi) \frac{\bar{\tau}}{2} + \frac{1}{2} \left[ \partial_{\mu} \bar{\tau} \cdot \partial_{\mu} \bar{\tau} - m^2 \bar{\tau} \cdot \bar{\tau} \right] \text{neglectable}. \quad (20)$$

Thus, roughly speaking, these two models can be related as

$$-g_{\text{IV-PV}} \partial_{\mu} \bar{\tau}(x) \longleftrightarrow -\frac{\alpha_{\text{IV-PV}}}{2} (\bar{\psi} \gamma_{\mu} \gamma_5 \psi) \frac{\bar{\tau}}{2}. \quad (21)$$

In the following, we explain the background of this analogy.

- **The equation of motion for $\psi(x)$ from $\mathcal{L}_{\text{ME}}$**:

  $$\partial_{\mu} \frac{\delta \mathcal{L}_{\text{ME}}}{\delta \partial_{\mu} \psi} - \frac{\delta \mathcal{L}_{\text{ME}}}{\delta \psi} = 0,$$

  $$[i \gamma_{\mu} \partial_{\mu} - M] \psi(x) = -g \partial_{\mu} \bar{\tau} \cdot \bar{\tau} \gamma_{\mu} \gamma_5 \psi(x). \quad (22)$$

- **The equation of motion for $\pi_a(x)$ from $\mathcal{L}_{\text{ME}}$**:

  $$\partial_{\mu} \frac{\delta \mathcal{L}_{\text{ME}}}{\delta \partial_{\mu} \pi_a} - \frac{\delta \mathcal{L}_{\text{ME}}}{\delta \pi_a} = 0,$$

  $$\partial_{\mu} \left[ \partial_{\mu} \pi_a - g \gamma_a \cdot \bar{\psi} \gamma_{\mu} \gamma_5 \psi(x) \right] + m^2 \pi_a(x) = 0,$$

  $$\left[ \partial_{\mu} \partial_{\mu} + m^2 \right] \pi_a(x) = g \gamma_a \partial_{\mu} \left( \bar{\psi} \gamma_{\mu} \gamma_5 \psi \right). \quad (23)$$
By using the free-meson Green function (propagator),
\[ \Delta_\pi(x-y) = \int \frac{d^4 p}{16\pi^4} \frac{e^{-i(p\cdot(x-y))}}{p^2 - m^2} \implies [\partial_\mu \partial^\mu + m^2] \Delta_\pi(x-y) = \delta(x-y), \quad (24) \]
then the pion field can be formally solved as
\[ \pi_a(x) = g\tau_a \int dy \Delta_\pi(x-y) \cdot [\partial_\mu (\bar{\psi}\gamma^\mu \gamma_5 \psi)](y). \quad (25) \]
With the partial-integration technique combined with the vanishing-flux condition, one finds that
\[ \pi_a(x) = 0 - g\tau_a \int dy \left[ \partial^{(g)}_\mu \Delta_\pi(x-y) \right] \cdot (\bar{\psi}\gamma^\mu \gamma_5 \psi)(y), \quad (26) \]
In the heavy-pion limit, \( \partial^{(g)}_\mu \Delta_\pi(x-y) \simeq -g_{\nu\mu} \delta(x-y)/m^2 \). Thus,
\[ \partial_\nu \pi^\mu \simeq \frac{g}{m^2} \bar{\psi} \gamma^\mu \gamma_5 \psi. \quad (27) \]
Therefore, the EOM for \( \psi(x) \) is approximated as
\[ [i\gamma_\mu \partial^\mu - M] \psi(x) \simeq -\frac{g^2}{m^2} \bar{\psi} \gamma^\mu \gamma_5 \psi \cdot \bar{\tau} \gamma^\mu \gamma_5 \psi(x). \quad (28) \]
This equation is the same to that obtained from \( \mathcal{L}_{PC} \), with a relation,
\[ -\frac{g^2}{m^2} = -\alpha_{IV-PV}. \quad (29) \]
Notice that, for the correspondence of two models, \( \alpha_{IV-PV} > 0 \). Its unit must be in, e.g., MeV-fm\(^3\),
since \( \mathcal{L}_{PC} \) and \( \bar{\psi}\psi(x) \) have the units of MeV-fm\(^{-3}\) and fm\(^{-3}\), respectively.

## 5 Quantization of Dirac spinor

In general, the spinor field consists of particle states with \( E > 0 \) and anti-particle states with \( -E < 0 \). Thus, it can be formally expanded as
\[ \psi(x) = \sum_s \psi_s(x), \]
\[ \psi_s(x) \equiv \langle x \mid s \rangle = \int_{E>0} dE \left[ u_{s,E}(x)c_{s,E} + v_{s,-E}(x)b_{s,-E} \right], \quad (30) \]
as well as,
\[ \psi^\dagger_{r}(x) = \int_{E>0} dE \left[ u^\dagger_{r,E}(x)c^\dagger_{r,E} + v^\dagger_{r,-E}(x)b^\dagger_{r,-E} \right], \quad (31) \]
where \( u_{s,E}(x) \equiv \langle x \mid s, E \rangle \) and \( v_{s,-E}(x) \equiv \langle x \mid s, -E \rangle \). Here the index \( s \) indicates the spin component, whereas \( E > 0 \) means the eigenvalue for certain Dirac’s Hamiltonian. Assuming this Hamiltonian as \( \hat{h} \), these basic states satisfy that,
\[ \hat{h}u_{s,E}(x) = Eu_{s,E}(x), \quad \hat{h}v_{s,-E}(x) = -Ev_{s,-E}(x). \quad (32) \]
In the following, we assume that $\hat{h}$ does not depend on time apparently. Thus, from Dirac equation, $i\hbar\partial_t u(x) = \hat{h}u(x)$, it is represented as
\[
u_{s,E}(x) = e^{-itE/\hbar}u_{s,E}(r), \quad \nu_{s,-E}(x) = e^{itE/\hbar}v_{s,-E}(r).
\] (33)

Note the following points.

- Completeness of basis:
\[
\hat{1} = \sum_s \int_{E>0} dE \left[ |s,E\rangle \langle s,E| + |s,-E\rangle \langle s,-E| \right]. \tag{34}
\]

Thus, from the overlap of $y$ and $x$,
\[
\langle y | x \rangle = \sum_s \int dE \left[ u_{s,E}^\dagger(y)u_{s,E}(x) + v_{s,-E}^\dagger(y)v_{s,-E}(x) \right] = \delta(y - x). \tag{35}
\]

From Eq. (33), it is also concluded as
\[
\sum_s \int dE \left[ u_{s,E}^\dagger(y)u_{s,E}(x) + v_{s,-E}^\dagger(y)v_{s,-E}(x) \right] = \delta(y - x). \tag{36}
\]

- Orthogonality of basis:
\[
\langle r, E' | s, E \rangle \equiv \delta(E' - E)\delta_{rs}, \quad \langle r, -E' | s, -E \rangle \equiv \bar{\delta}(E' - E)\delta_{rs}.
\]

\[
\rightarrow \int d^3r u_{r,E'}^\dagger(r)u_{s,E}(r) = \int d^3r v_{r,-E'}^\dagger(r)v_{s,-E}(r) = \delta(E' - E)\delta_{rs}. \tag{37}
\]

Also, remembering $E', E > 0$,
\[
\langle r, -E' | s, E \rangle = \int d^3r u_{r,E'}^\dagger(x)u_{s,E}(x) = 0,
\]
\[
\langle r, E' | s, -E \rangle = \int d^3r v_{r,-E'}^\dagger(x)v_{s,-E}(x) = 0. \tag{38}
\]

- Spinor field must satisfy the anti-commutation relation at the same time:
\[
\{ \psi_r^\dagger(y), \psi_s(x) \}_{y_0=x_0} = \delta(y - x)\delta_{rs}, \tag{39}
\]
\[
\{ \psi_r(y), \psi_s(x) \}_{y_0=x_0} = \{ \psi_r^\dagger(y), \psi_s^\dagger(x) \}_{y_0=x_0} = 0. \tag{40}
\]

For the first relation, we find that,
\[
\{ \psi_r^\dagger(y), \psi_s(x) \}_{y_0=x_0} = \sum_{r,s} \int dE' \int dE \left[ u_{r,E'}^\dagger(y)u_{s,E}(x) \left\{ c_{r,E'}^\dagger, c_{s,E} \right\} + v_{r,-E'}^\dagger(y)v_{s,-E}(x) \left\{ b_{r,-E'}^\dagger, b_{s,-E} \right\} \right.
\]
\[
+ v_{r,-E'}^\dagger(y)u_{s,-E}(x) \left\{ b_{r,-E'}^\dagger, c_{s,E} \right\} + u_{r,-E'}^\dagger(y)v_{s,-E}(x) \left\{ c_{r,E'}^\dagger, b_{s,-E} \right\} \bigg]_{y_0=x_0}. \tag{41}
\]

Therefore, to keep consistency with Eqs. (36) and (39), the operators must satisfy that
\[
\{ \ c_{r,E'}, c_{s,E} \ = \ \delta_{rs}\delta(E' - E), \ \{ \text{others} \} = 0. \tag{42}
\]

Notice that above formulas can work even in the case with general interaction(s) included in the Lagrangian density.
5.1 Hamiltonian

In general, Lagrangian density is written as $\mathcal{L} = \bar{\psi} (i \not{\partial} - M) \psi(x) + \bar{\psi} X \psi(x)$. The corresponding Hamiltonian density reads

$$H(x) = \left( \partial_0 \psi \right) \frac{\delta \mathcal{L}}{\delta \left( \partial_0 \psi \right)^\dagger} + \frac{\delta \mathcal{L}}{\delta \psi} \left( \partial_0 \psi \right) - \mathcal{L} \tag{43}$$

$$= 0 + \bar{\psi} i \gamma^0 \left( \partial_0 \psi \right) - \bar{\psi} i \left[ \gamma^0 \partial_0 + \gamma^k \partial_k \right] \psi(x) + M \bar{\psi} \psi(x) - \bar{\psi} X \psi(x)$$

$$= \psi^\dagger \left[ -i \vec{\alpha} \cdot \vec{\nabla} + \beta M - \beta X \right] \psi(x) \equiv \psi^\dagger \hat{H}_D \psi(x), \tag{44}$$

where $\hat{H}_D$ indicates the Dirac single-field Hamiltonian. Note that, however, here I neglect the exchange (Fock) terms, which could appear from the interactions $\bar{\psi} X \psi(x)$. The proper Hamiltonian is then given as $H(t) = \int d^3 \mathbf{r} \mathcal{H}(x)$.

By employing the basis expansion introduced above, it can be represented as

$$H(t) = \sum_{r,s} \int dE' \int dE \int d^3 \mathbf{r}$$

$$\left[ u_{r,E'}^\dagger(x)c_{r,E}^\dagger + v_{r,-E'}^\dagger(x)b_{r,-E'}^\dagger \right] E \left[ u_{s,E}(x)c_{s,E} - v_{s,-E}(x)b_{s,-E} \right], \tag{45}$$

where we have used $\hat{h}_D u_{s,E} = E u_{s,E}$ and $\hat{h}_D v_{s,-E} = -E v_{s,-E}$. From Eqs. (37) and (38), one can find that only the $c_{s,E}^\dagger$ and $b_{s,-E}^\dagger$ terms survive. That is,

$$H = \sum_{r,s} \int dE' \int dE$$

$$\left[ e^{it(E'-E)/\hbar} c_{r,E}^\dagger c_{s,E} - e^{-it(E'-E)/\hbar} b_{r,-E}^\dagger b_{s,-E} \right] E \delta(E' - E) \delta_{rs} + 0$$

$$= \sum_s \int dE \left[ c_{s,E}^\dagger c_{s,E} - b_{s,-E}^\dagger b_{s,-E} \right] E. \tag{46}$$

This equation almost looks as the proper form for the total energy. However, the second term means that $b_{s}^\dagger$ creates the negative-energy particle. To remedy this wired property, the anti-particle states are re-defined as $a_s = b_s^\dagger$ and $a_s^\dagger = b_s$. By this procedure, finally we can find that

$$H = \sum_s \int dE \left[ c_{s,E}^\dagger c_{s,E} + a_{s,-E}^\dagger a_{s,-E} \right] E - \text{const}. \tag{47}$$

The vacuum is then defined as the state to become zero for $c_s$ and $a_s$.

5.2 general representation with basis

In practical calculations, the Hamiltonian is represented with the chosen basic states. That is,

$$H(t) = \sum_{r,s} \int dE' \int dE \int d^3 \mathbf{r}$$

$$\left[ u_{r,E'}^\dagger(x)c_{r,E}^\dagger + v_{r,-E'}^\dagger(x)b_{r,-E'}^\dagger \right] \hat{h}_D \left[ u_{s,E}(x)c_{s,E} + v_{s,-E}(x)b_{s,-E} \right], \tag{48}$$

where $u(x)$ and $v(x)$ are, however, NOT the eigenstates of $\hat{h}_D$ anymore. Thus, the labels $E$ and $E'$ are now general ones: those are not definitely for energies. By using the matrix elements,

$$\hat{h}_{r,E,s,E}^{(pp)}(t) \equiv \int d^3 \mathbf{r} u_{r,E}^\dagger(x) \hat{h}_D u_{s,E}(x), \quad \hat{h}_{r,-E,s,E}^{(ap)}(t) \equiv \int d^3 \mathbf{r} v_{r,-E}^\dagger(x) \hat{h}_D u_{s,E}(x), \quad \text{etc.}, \tag{49}$$

$$\hat{h}_{r,E,s,E}^{(pp)}(t) \equiv \int d^3 \mathbf{r} u_{r,E}^\dagger(x) \hat{h}_D u_{s,E}(x), \quad \hat{h}_{r,-E,s,E}^{(ap)}(t) \equiv \int d^3 \mathbf{r} v_{r,-E}^\dagger(x) \hat{h}_D u_{s,E}(x), \quad \text{etc.} \tag{49}$$
then it can be formally given as

\[
H(t) = \sum_{r,s} \int dE' \int dE \left[ h_{r,E',s,E}^{(pp)}(t)c_{r,E'}^\dagger c_{s,E} + h_{r,-E',s,E}^{(ap)}(t)b_{r,-E'}^\dagger c_{s,E} + h_{r,E',s,-E}^{(pa)}(t)c_{r,E'}^\dagger b_{s,-E} + h_{r,-E',s,-E}^{(aa)}(t)b_{r,-E'}^\dagger b_{s,-E} \right].
\] (50)

Within the no-sea approximation, we neglect the anti-particle components, namely the 2nd to 4th terms in the Hamiltonian. In this case, one finds the usual form,

\[
H(t) \simeq \sum_{k,l} h_{k,l}^{(pp)}(t)c_{k}^\dagger c_{l};
\] (51)

where the simplified labels \( k = \{r, E'\} \) and \( l = \{s, E\} \) are employed. The vacuum-expectation value \( \langle H(t) \rangle_\Phi \) is then a functional of several densities, similarly in the non-relativistic multi-fermion models. The Bogoliubov transformation can be also determined for \( c_{k}^\dagger \) and \( c_{l} \) operators.
6 Operators

Creation and Annihilation Operators;

\[ \{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta} \quad (52) \]
\[ \{c_\alpha, c_\beta\} = \{c_\beta^\dagger, c_\alpha^\dagger\} = 0 \quad (53) \]
\[ c_\alpha = \int dx \langle \alpha | x \rangle \hat{\psi}(x), \quad c_\alpha^\dagger = \int dx \langle x | \alpha \rangle \hat{\psi}^\dagger(x) \quad (\langle x | \alpha \rangle = \phi_\alpha(x)) \quad (54) \]
\[ \hat{\psi}(x) = \sum_\alpha \langle x | \alpha \rangle c_\alpha, \quad \hat{\psi}^\dagger(x) = \sum_\alpha \langle \alpha | x \rangle c_\alpha^\dagger \quad (55) \]
\[ \{\hat{\psi}(x), \hat{\psi}^\dagger(y)\} = \delta(x, y) \quad (56) \]
\[ \{\hat{\psi}(x), \hat{\psi}(y)\} = \{\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(y)\} = 0 \quad (57) \]

7 States

Vacuum;

\[ |\alpha_1\rangle = c_{\alpha_1}^\dagger |0\rangle \quad (58) \]
\[ c_\alpha |0\rangle = 0, \quad \langle 0 | c_\alpha^\dagger = 0 \quad (59) \]
\[ |x\rangle = \hat{\psi}^\dagger(x) |0\rangle \quad (60) \]
\[ \hat{\psi}(x) |0\rangle = 0, \quad \langle 0 | \hat{\psi}^\dagger(x) = 0 \quad (61) \]

Slater Determinant;

\[ |\alpha_1, \alpha_2, \ldots, \alpha_N\rangle = \frac{1}{\sqrt{N!}} \sum_{P \in S_N} (-)^P |\alpha_{P_1}\rangle |\alpha_{P_2}\rangle \cdots |\alpha_{P_N}\rangle \quad (62) \]
\[ = c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger \cdots c_{\alpha_N}^\dagger |0\rangle \quad (63) \]
\[ c_{\beta}^\dagger |\alpha_1, \alpha_2, \ldots, \alpha_N\rangle = |\beta, \alpha_1, \alpha_2, \ldots, \alpha_N\rangle \quad (64) \]

Completeness;

\[ \frac{1}{N!} \sum_{\alpha_1 \cdots \alpha_N} |\alpha_1 \cdots \alpha_N\rangle \langle \alpha_1 \cdots \alpha_N| = \hat{1}_F \quad (65) \]
\[ \sum_{\alpha_1 < \alpha_2 < \cdots < \alpha_N} |\alpha_1 \cdots \alpha_N\rangle \langle \alpha_1 \cdots \alpha_N| = \hat{1}_F \quad (66) \]
\[ |0\rangle \langle 0| + \sum_{N=1}^{\infty} \left[ \frac{1}{N!} \sum_{\alpha_1 \cdots \alpha_N} |\alpha_1 \cdots \alpha_N\rangle \langle \alpha_1 \cdots \alpha_N| \right] = \hat{1}_F \quad (67) \]
\[ \mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_N \oplus \cdots \quad (68) \]
General $A$-body State;

$$|\Psi_A\rangle = \hat{\Psi}_A^\dagger |0\rangle = \left[ \frac{1}{\sqrt{A!}} \sum \limits_\{\alpha_k\} D(\alpha_1 \cdots \alpha_A) c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger \cdots c_{\alpha_A}^\dagger \right] |0\rangle$$  \hspace{1cm} (69)

$$= \frac{1}{\sqrt{A!}} \sum \limits_\{\alpha_k\} D(\alpha_1 \cdots \alpha_A) |\alpha_1 \cdots \alpha_A\rangle$$  \hspace{1cm} (70)

$$= \sum \limits_{\alpha_1 < \alpha_2 < \cdots < \alpha_A} D(\alpha_1 \cdots \alpha_A) |\alpha_1 \cdots \alpha_A\rangle$$  \hspace{1cm} (71)

$$D(\cdots \alpha_k \cdots \alpha_l \cdots) = (-) D(\cdots \alpha_l \cdots \alpha_k \cdots)$$  \hspace{1cm} (72)

Normalization;

$$\langle \Psi_A | \Psi_A \rangle = \frac{1}{A!} \sum \limits_\{\alpha\} \sum \limits_\{\beta\} D^*(\alpha_1 \cdots \alpha_A) D(\beta_1 \cdots \beta_A) \langle \alpha_1 \cdots \alpha_A | \beta_1 \cdots \beta_A \rangle$$  \hspace{1cm} (73)

$$= \frac{1}{A!} \sum \limits_\{\alpha\} |D(\alpha_1 \cdots \alpha_A)|^2 = \sum \limits_{\alpha_1 < \alpha_2 < \cdots < \alpha_A} |D(\alpha_1 \cdots \alpha_A)|^2 \equiv 1$$  \hspace{1cm} (74)

8 Density Matrix

Several Representations (even with Time-Dependence);

$$\hat{\rho}^{(\alpha \beta)} \equiv c_{\beta}^\dagger c_\alpha,$$

$$\hat{\rho}^{(xy)} \equiv \hat{\psi}^\dagger(y)\hat{\psi}(x)$$  \hspace{1cm} (75)

$$\rho^{(\alpha \beta)}_{\Psi_A(t), \Psi_A(t)} = \langle \Psi'_A(t) | \hat{\rho}^{(\alpha \beta)} | \Psi_A(t) \rangle = \left\langle \Psi'_A(t) | c_{\beta}^\dagger c_\alpha | \Psi_A(t) \right\rangle,$$  \hspace{1cm} (77)

$$\rho^{(xy)}_{\Psi_A(t), \Psi_A(t)} = \langle \Psi'_A(t) | \hat{\rho}^{(xy)} | \Psi_A(t) \rangle = \left\langle \Psi'_A(t) | \hat{\psi}^\dagger(y)\hat{\psi}(x) | \Psi_A(t) \right\rangle,$$  \hspace{1cm} (79)

$$\hat{\rho}_{\Psi_A(t), \Psi_A(t)} = \sum \limits_\alpha \sum \limits_\beta |\alpha\rangle \rho^{(\alpha \beta)}_{\Psi_A(t), \Psi_A(t)} \langle \beta|,$$  \hspace{1cm} (81)

$$= \int dx \int dy |x\rangle \rho^{(xy)}_{\Psi_A(t), \Psi_A(t)} \langle y|$$  \hspace{1cm} (82)

Density Matrix of $|\Psi_A\rangle$;

$$\rho^{(\alpha \beta)}_{\Psi_A} = \langle \Psi_A | c_{\beta}^\dagger c_\alpha | \Psi_A \rangle = \langle \alpha | \hat{\rho}_{\Psi_A, \Psi_A} | \beta \rangle$$  \hspace{1cm} (83)

$$= \frac{1}{A!} \sum \limits_\{\alpha\} \sum \limits_\{\beta\} D^*(\beta_1 \cdots \beta_A) \langle 0 | c_{\beta_A}^\dagger \cdots c_{\beta_1}^\dagger c_{\alpha_1}^\dagger \cdots c_{\alpha_A}^\dagger | 0 \rangle D(\alpha_1 \cdots \alpha_A)$$  \hspace{1cm} (84)

$$= \frac{(A-1)!}{A!} \sum \limits_{\gamma_2 \cdots \gamma_A} \left[ D^*(\beta, \gamma_2, \cdots) + (-) D^*(\gamma_2, \beta, \cdots) + \cdots \right]_{(A \ \text{terms})} \times \left[ D(\alpha, \gamma_2, \cdots) + (-) D(\gamma_2, \alpha, \cdots) + \cdots \right]_{(A \ \text{terms})}$$  \hspace{1cm} (85)

$$= \sum \limits_{\gamma_2 \cdots \gamma_A} D^*(\beta, \gamma_2 \cdots \gamma_A) D(\alpha, \gamma_2 \cdots \gamma_A)$$  \hspace{1cm} (86)

$$= (A!) \sum \limits_{\gamma_2 < \cdots < \gamma_A} D^*(\beta, \gamma_2 \cdots \gamma_A) D(\alpha, \gamma_2 \cdots \gamma_A)$$  \hspace{1cm} (87)
\[ \rho^{(xy)}_{\Psi_A} = \langle \Psi_A | \hat{\psi}^\dagger(y) \hat{\psi}(x) | \Psi_A \rangle = \langle x | \hat{\rho}_{\Psi_A} | y \rangle \]  
(88)

\[ = \sum_\beta \sum_\alpha \phi_\beta^*(y) \langle \Psi_A | c_\beta^\dagger c_\alpha | \Psi_A \rangle \phi_\alpha(x) \]  
(89)

\[ = \sum_\beta \sum_\alpha \phi_\beta^*(y) \rho^{(\alpha\beta)}_{\Psi_A} \phi_\alpha(x) = \langle x | \left[ \sum_\alpha \sum_\beta |\alpha\rangle \rho^{(\alpha\beta)}_{\Psi_A} \langle \beta | \right] |y\rangle \]  
(90)

Density Matrix of the Single Slater Determinant;

\[ \rho^{(\alpha\beta)} = \langle \alpha_1 \cdots \alpha_A | c_\beta^\dagger c_\alpha | \alpha_1 \cdots \alpha_A \rangle \propto \delta_{\alpha\beta}, \]  
(91)

\[ \rho^{(\alpha\alpha)} = \begin{cases} 0 & (\alpha > \alpha_A) \\ 1 & (\alpha \leq \alpha_A) \end{cases} \]  
(92)

\[ \hat{\rho} = \sum_\alpha \sum_\beta |\alpha\rangle \rho^{(\alpha\beta)} \langle \beta | = \sum_{\beta \leq \alpha_A} |j\rangle \langle j| ; \]  
(93)

\[ \hat{\rho}^2 = \hat{\rho} \]  
(94)

9 Hamiltonian

Basis Representation;

\[ H = T + V = \sum_{\alpha\gamma} t_{\alpha\gamma} c_\alpha^\dagger c_\gamma + \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \tilde{v}_{\alpha\beta,\gamma\delta} (c_\beta^\dagger c_\alpha)^\dagger (c_\delta^\dagger c_\gamma) \]  
(95)

\[ t_{\alpha\gamma} = \langle \alpha | T | \gamma \rangle \]  
(96)

\[ \tilde{v}_{\alpha\beta,\gamma\delta} = \frac{1}{2} \langle (\alpha\beta) | V | \gamma\delta \rangle - \langle \alpha \leftrightarrow \beta \rangle - \langle \gamma \leftrightarrow \delta \rangle + \langle \alpha \leftrightarrow \beta \& \gamma \leftrightarrow \delta \rangle \]  
(97)

Local Force Assumption;

\[ \langle x' y' | V | xy \rangle = \delta(x', x) \delta(y', y) V(x, y) \]  
(98)

Expectational Value via Slater Determinant;

\[ E_0 = \langle \alpha_1 \cdots \alpha_N | H | \alpha_1 \cdots \alpha_N \rangle = \langle \alpha_1 \cdots \alpha_N | T + V | \alpha_1 \cdots \alpha_N \rangle \]  
(99)

\[ = \sum_{\rho=1}^{N} \theta_\rho t_{\rho,\rho} + \frac{1}{4} \sum_{\mu \neq \nu=1}^{\infty} \theta_\mu \theta_\nu \left[ \tilde{v}_{\mu\nu,\mu\nu} - \tilde{v}_{\mu\nu,\nu\mu} \right] \]  
(100)

\[ = \sum_{k=1}^{N} t_{k,k} + \frac{1}{2} \sum_{(i \neq j)=1}^{\infty} \tilde{v}_{ij,ij} \]  
(101)

\[ = \sum_{i} \left[ \langle i | T | i \rangle + \frac{1}{2} \sum_{(j \neq i)=1} \left( \langle ij | V | ij \rangle - \langle ij | V | ji \rangle \right) \right] \]  
(102)

\[ \langle i | T | k \rangle = \int dx' \int dx' \phi_i^*(x') \langle x' | T | x \rangle \phi_k(x) \]  
(103)

\[ \langle ij | V | kl \rangle = \int dx' dy' \int dx' dy' \phi_i^*(x') \phi_j^*(y') \langle x' y' | V | xy \rangle \phi_k(x) \phi_l(y) \]  
(104)

\[ = \int dx' dy' \phi_i^*(x') \phi_j^*(y') V(x, y) \phi_k(x) \phi_l(y) \]  
(105)
Coordinate Representation;

\[
T = \int dx' \int dx \hat{\psi}^\dagger(x') \langle x' | T | x \rangle \hat{\psi}(x) \tag{106}
\]

\[
V = \frac{1}{2} \int^{(2)} dx' dy' \int^{(2)} dx dy \hat{\psi}^\dagger(x') \hat{\psi}^\dagger(y') \langle x' y' | V | x y \rangle \hat{\psi}(x) \hat{\psi}(y) \tag{107}
\]

\[
= \frac{1}{2} \int^{(2)} dx dy \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(y) V(x, y) \hat{\psi}(x) \hat{\psi}(y) \tag{108}
\]

External Local Field;

\[
V_{\text{ext}} = \sum_{\alpha \gamma} \langle \alpha | V_{\text{ext}} | \gamma \rangle c^\dagger_{\alpha} c_{\gamma} \tag{109}
\]

\[
= \int dx' \int dx \sum_{\alpha \gamma} \phi^*_\alpha(x') \langle x' | V_{\text{ext}} | x \rangle \phi_\gamma(x) c^\dagger_{\alpha} c_{\gamma} \tag{110}
\]

\[
= \int dx' \int dx \hat{\psi}^\dagger(x') \langle x' | V_{\text{ext}} | x \rangle \hat{\psi}(x) \tag{111}
\]

\[
= \int dx \hat{\psi}^\dagger(x) V_{\text{ext}}(x) \hat{\psi}(x) \tag{112}
\]

\[
\langle \alpha_1 \cdots \alpha_N | V_{\text{ext}} | \alpha_1 \cdots \alpha_N \rangle = \sum_i \int dx \phi^*_i(x) V_{\text{ext}}(x) \phi_i(x) \tag{113}
\]

10 Hartree-Fock Method

HF-Ground State as a Slater Determinant;

\[
| \Psi \rangle = | \alpha_1 \cdots \alpha_N \rangle, \quad E_0 \equiv \langle \Psi | H | \Psi \rangle \quad (H = T + V + V_{\text{ext}}) \tag{114}
\]

Variational Principle;

\[
\delta \left( E_0 - \sum_{\beta} e_{\beta} \langle \phi_\beta | \phi_\beta \rangle \right) \over \delta \phi^*_\alpha(w) = 0 \tag{115}
\]

Self-Consistent Equation;

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2_w + V_H(w) + V_{\text{ext}}(w) \right] \phi_\alpha(w) - \int dy V_F(w, y) \phi_\alpha(y) = e_\alpha \phi_\alpha(w) \tag{116}
\]

\[
V_H(w) = \sum_{\beta(\neq \alpha)=1}^N \int dy \phi^*_\beta(y) V(w, y) \phi_\beta(y) \tag{117}
\]

\[
V_F(w, y) = \sum_{\beta(\neq \alpha)=1}^N \phi^*_\beta(y) V(w, y) \phi_\beta(w) \tag{118}
\]

1-Body Density;

\[
\hat{\rho}(r) = \sum_s \hat{\psi}^\dagger(rs) \hat{\psi}(rs) = \sum_s \sum_\alpha \langle \alpha | rs \rangle c^\dagger_\alpha \sum_\beta \langle \beta | rs \rangle c_\beta \tag{119}
\]
\[ |\Psi[\rho] \rangle \leftrightarrow \rho(r) = \langle \Psi | \hat{\rho}(r) | \Psi \rangle \]
\[ = \left\langle \alpha_1 \cdots \alpha_N \right| \sum_s \sum_\alpha \phi_\alpha^*(rs) c_\alpha^\dagger \sum_\beta \phi_\beta(rs) c_\beta | \alpha_1 \cdots \alpha_N \right\rangle \]
\[ = \sum_s \sum_\alpha \sum_\beta \phi_\alpha^*(rs) \phi_\beta(rs) \rho^{(\beta \alpha)} = \sum_s \sum_{\alpha=\alpha_1}^{\alpha_N} |\phi_\alpha(rs)|^2 \]

Energy Density Functional (at HF-level);
\[ \mathcal{E}_{\text{HF}}[\rho] = \langle \Psi[\rho] | H | \Psi[\rho] \rangle = T[\rho] + E_H[\rho] + E_F[\rho] + E_{\text{ext}}[\rho] \]
\[ T[\rho] = -\frac{\hbar^2}{2m} \sum_k^N |\nabla \phi_k(x)|^2 \quad (x = rs) \]
\[ E_H[\rho] = \frac{1}{2} \int dx \int dx' V(x', x) \rho(x') \rho(x) \]
\[ E_F[\rho] = ? \]
\[ E_{\text{ext}}[\rho] = ? \]

11 HF + Bogoliubov Method

Hamiltonian (with Einstein’s Rule);
\[ H' = H - \lambda N = (t_{k,m} - \lambda \delta_{k,m}) c_k^\dagger c_m + \frac{1}{4} \tilde{\nu}_{kl, mn} (c_l^\dagger c_k)(c_n c_m) \]

Bogoliubov Transformation;
\[ \left\{ \begin{array}{l} b_k = U_{ik} c_i + V_{ik} c_i^\dagger \\ b_k^\dagger = U_{ik}^\dagger c_i^\dagger + V_{ik}^\dagger c_i \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} c_i = U_{im} b_m + V_{im}^* b_m^\dagger \\ c_i^\dagger = U_{im}^\dagger b_m^\dagger + V_{im} b_m \end{array} \right\} \]
\[ \left( \begin{array}{c} b \\ b^\dagger \end{array} \right) = \mathcal{W} \left( \begin{array}{c} c \\ c^\dagger \end{array} \right) = \left( \begin{array}{cc} U & V^\dagger \\ V^T & U^\dagger \end{array} \right) \left( \begin{array}{c} c \\ c^\dagger \end{array} \right) \leftrightarrow \left( \begin{array}{c} c \\ c^\dagger \end{array} \right) = \left( \begin{array}{cc} U & V^* \\ V & U \end{array} \right) \left( \begin{array}{c} b \\ b^\dagger \end{array} \right) \]

Unitarity;
\[ \mathcal{W} \mathcal{W}^\dagger = \left( \begin{array}{cc} U^\dagger U + V^\dagger V & U^\dagger V^* + V^\dagger U^* \\ V^T U + U^T V & V^T V^* + U^T U^* \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \rightarrow \left\{ \begin{array}{l} U^\dagger U + V^\dagger V = 1 \\ V^T U + U^T V = 0 \end{array} \right\} \]
\[ \mathcal{W}^\dagger \mathcal{W} = \left( \begin{array}{cc} U^\dagger U + V^* V^T & U^\dagger V^T + V^* U^T \\ V^T U + U^* V^T & V^T V + U^* U^T \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \rightarrow \left\{ \begin{array}{l} U U^\dagger + V^* V^T = 1 \\ U V^\dagger + V^* U^T = 0 \end{array} \right\} \]

HFB Vacuum and Normal Ordering;
\[ b_k | . \rangle = 0, \quad | . \rangle | b_k^\dagger \rangle = 0 \]
\[ \langle . | : X(b^\dagger, b) : . \rangle = 0 \]

Density-Tensor and Pair-Tensor;
\[ \rho_{kl} \equiv \langle . | c_l^\dagger c_k | . \rangle = V_{lm} V^*_m, \quad \kappa_{kl} \equiv \langle . | c_l c_k | . \rangle = (-)^{l+k} V_{km} \]
\[ \leftrightarrow \rho = V^* V^T = 1 - U U^\dagger = \rho^\dagger, \quad \kappa = (-)^{l+k} V^* U^T = -U V^\dagger \]
Meanfield and Pairing Potentials;
\[ \Gamma_{kl} = \tilde{v}_{km,ln}, \quad \Delta_{kl} = \frac{1}{2} \tilde{v}_{kl, mn} \kappa_{mn} \leftrightarrow \Delta^*_{kl} = \frac{1}{2} \tilde{v}_{mn, kl} \kappa^*_{mn} \]  

(137)

Quasiparticle Representation of \( H' = H - \lambda N \);
\[ H' = H^{(0)} + H^{(2)} + H^{(4)} \]  

(138)

\[ H^{(0)} = \langle . | H' | . \rangle = tr \left[ (t - \lambda) \rho + \frac{1}{2} \Gamma \rho - \frac{1}{2} \Delta \kappa^* \right] \]  

(139)

\[ H^{(2)} = \frac{1}{2} \left( \begin{array}{ccc} c^\dagger & c \end{array} \right) \left( \begin{array}{ccc} h - \lambda & \frac{\Delta}{-\Delta^*} & -h^* + \lambda \end{array} \right) \left( \begin{array}{c} c \end{array} \right) \]  

(140)

\[ (h = t + \Gamma) \]

\[ H^{(11)} + H^{(20)} + h.c. \]  

(141)

\[ H^{(20)} + h.c. = \frac{1}{2} b^\dagger_m b_n \left[ \left( \begin{array}{cc} U^\dagger & V^\dagger \end{array} \right) \left( \begin{array}{c} \frac{h - \lambda}{-\Delta^*} & -h^* + \lambda \end{array} \right) \left( \begin{array}{c} U \end{array} \right) \right]_{mn} \]  

(142)

\[ H^{(4)} = \frac{1}{4} \tilde{v}_{kl, mn} : (c_l c_k) (c_n c_m) : \]  

(143)

HFB Solution \((U, V)\);
\[ H' = H^{(0)} + \sum \Delta_k b^\dagger_k b_k + 0 + H^{(4)} \]  

(144)

Energy Density Functional (at HFB-level);
\[ E[\rho, \kappa, \kappa^*] = \langle . | H' | . \rangle = E_{HF}[\rho, \kappa, \kappa^*] + E_{pair}[\rho, \kappa, \kappa^*] \]  

(145)

\[ H^{(0)} = tr \left[ (t - \lambda) \rho + \frac{1}{2} \Gamma \rho - \frac{1}{2} \Delta \kappa^* \right] \]  

(146)

\[ h_{kl}[\rho, \kappa, \kappa^*] = \frac{\partial E}{\partial \rho_{lk}} = (t - \lambda)_{kl} + \Gamma_{kl}, \quad \Delta_{kl}[\rho, \kappa, \kappa^*] = \frac{\partial E}{\partial \kappa_{kl}} \]  

(147)

12 Finite Amplitude Method

FAM Linear Responce Equation;
\[ A\ddot{x} = \ddot{f} \]  

(148)

\[ \ddot{f} = \left( \begin{array}{c} X_{\mu\nu}(\omega) \\ Y_{\mu\nu}(\omega) \end{array} \right) \]  

(149)

\[ A\ddot{x} = \left( \begin{array}{c} (E_{\mu} + E_{\nu} - \omega) X_{\mu\nu}(\omega) + \delta H^{20}_{\mu\nu}(\omega) \\ (E_{\mu} + E_{\nu} + \omega) Y_{\mu\nu}(\omega) + \delta H^{20}_{\mu\nu}(\omega) \end{array} \right) \]  

(150)

\[ \delta H^{20}_{\mu\nu}(\omega) = +U^\dagger \delta h V^* - V^\dagger \delta \Delta^{(-)+} V^* + U^\dagger \delta \Delta^{(-)+} V^* U^\dagger \delta h^T U^* \]  

(151)

\[ \delta H^{02}_{\mu\nu}(\omega) = -V^T \delta h U + U^T \delta \Delta^{(-)+} U - V^T \delta \Delta^{(-)+} V + U^T \delta h^T V \]  

(152)

(153)
13 Wick’s Theorem

Philosophy;
\[ \hat{T}[O] = \langle - | O | - \rangle + \hat{N}[O] \] (154)

Formulas;
\[ \hat{T}[\psi(x_1) \cdots \psi(x_n)] = :\psi(x_1) \cdots \psi(x_n) : + \sum_{P \in S_n} \langle - | \psi(x_{P_1})\psi(x_{P_2}) | - \rangle : \psi(x_{P_3}) \cdots \psi(x_{P_n}) : \\
+ \sum_{P \in S_n} \langle - | \psi(x_{P_1})\psi(x_{P_2}) | - \rangle \langle - | \psi(x_{P_3})\psi(x_{P_4}) | - \rangle : \psi(x_{P_5}) \cdots \psi(x_{P_n}) : \\
\vdots \\
+ \sum_{P \in S_n} \langle - | \psi(x_{P_1})\psi(x_{P_2}) | - \rangle \cdots \langle - | \psi(x_{P_{n-1}})\psi(x_{P_n}) | - \rangle \] (155)

\[ d_1 \cdots d_n = :d_1 \cdots d_n : \quad (d_k \leftarrow c_k, c_k^\dagger) \\
+ \sum_{P \in S_n} \langle - | d_{P_1}d_{P_2} | - \rangle : d_{P_3} \cdots d_{P_n} : \\
+ \sum_{P \in S_n} \langle - | d_{P_1}d_{P_2} | - \rangle \langle - | d_{P_3}d_{P_4} | - \rangle : d_{P_5} \cdots d_{P_n} : \\
\vdots \\
+ \sum_{P \in S_n} \langle - | d_{P_1}d_{P_2} | - \rangle \cdots \langle - | d_{P_{n-1}}d_{P_n} | - \rangle \] (156)
Dirac HFB equation from the sigma-nucleon model

14 Introduction

After reading Ref. [2], here I try to formalize the HFB equation from the relativistic Lagrangian (density), which is, however, simpler than the original version. Namely, it contains only the nucleon \(\psi(x)\) and the sigma meson \(\sigma(x)\). That is,

\[
\mathcal{L}[\psi, \psi^*, \sigma] = \bar{\psi}(i\gamma^\mu \partial_\mu - m) \psi(x) + \frac{1}{2} (\partial^\mu \sigma \partial_\mu \sigma - \mu^2 \sigma^2) - g_\sigma \bar{\psi} \sigma \psi(x),
\]

(157)

where \(\partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)\) and \(\partial_\mu = \left(\frac{\partial}{\partial t}, +\vec{\nabla}\right)\). Thus, conjugate fields are

\[
\Pi_\psi(x) = \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} = i\bar{\psi}(x)\gamma^0 = i\bar{\psi}^\dagger(x),
\]

\[
\Pi_{\psi^*}(x) = 0,
\]

\[
\Pi_\sigma(x) = \frac{\partial \mathcal{L}}{\partial (\partial_0 \sigma)} = \partial^0 \sigma(x) = \dot{\sigma}(x).
\]

(158)

Therefore, the Hamiltonian (density) is given as

\[
\mathcal{H} = \Pi_\psi \bar{\psi} + \Pi_\sigma \dot{\sigma} - \mathcal{L}
\]

\[
= i\bar{\psi}(x)\gamma^0 (\partial_0 \psi) + \partial^0 \sigma(x) \cdot \partial_0 \sigma(x) - \mathcal{L}
\]

\[
= 0 + i\bar{\psi} \left[\gamma^1 \partial_1 + \gamma^2 \partial_2 + \gamma^3 \partial_3 + m\right] \psi(x) + \frac{1}{2} \left[\sigma^2 + (\vec{\nabla} \sigma)^2 + \mu^2 \sigma^2(x)\right] + g_\sigma \bar{\psi} \sigma \psi(x),
\]

\[
\Rightarrow H = \int dV \mathcal{H} = H_N + H_M + H_I,
\]

(159)

where

\[
H_N = \int dV \bar{\psi}^\dagger(x) \left[\vec{\alpha} \cdot \vec{p} + \beta m\right] \psi(x),
\]

\[
H_M = \int dV \frac{1}{2} \left[\Pi_\sigma^2 + (\vec{\nabla} \sigma)^2 + \mu^2 \sigma^2(x)\right],
\]

\[
H_I = \int dV g_\sigma \bar{\psi}(x) \sigma(x) \psi(x).
\]

(160)

Remember that \(\beta = \gamma^0, \alpha_n = \gamma^0 \gamma^n\), and \(p_n = i\partial_n (n = 1, 2, 3)\).

15 Equations of Interacting Fields

15.1 sigma meson

First we solve the \(\sigma\)-meson field. Klein-Gordon equation for \(\sigma(x)\) reads

\[
\partial^\mu \frac{\partial \mathcal{L}}{\partial (\partial^\mu \sigma)} - \frac{\partial \mathcal{L}}{\partial \sigma} = 0
\]

\[
\Rightarrow (\partial^\mu \partial_\mu + \mu^2) \sigma(x) = -g_\sigma \bar{\psi}(x) \psi(x).
\]

(161)

\(\text{This model is indeed Yukawa model as written in Eq. (4.112) in the textbook [1].}\)
By using the Green function $D_\sigma$, which satisfies
\[
(\partial^\mu \partial_\mu + \mu^2)_{x} D_\sigma(x - y) = \delta(x - y),
\] (162)
the formal solution is given as
\[
\sigma(x) = \int dy D_\sigma(x - y) \cdot (-g_\sigma)\bar{\psi}(y)\psi(y).
\] (163)
As well known, this Green function is indeed propagator of the scalar field:
\[
D_\sigma(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{(-)}{k^2 - \mu^2 + i\epsilon} e^{-ik(x-y)}.
\] (164)

### 15.2 Nucleon

Next we consider the nucleon field. Within the Heisenberg representation $\psi(x) = e^{itH}\psi(0, x)e^{-itH}$, the field operator follows the time-development equation,
\[
i\frac{\partial}{\partial t} \psi(x) = [\psi(x), H] = [\psi(x), H_N + H_I]
\]  
\[
= \int dV_w [\psi(x), \psi^\dagger(w) \{\vec{\alpha} \cdot \vec{p} + \beta m + g_\sigma \gamma_0 \sigma(w)\}_w \psi(w)].
\] (165)
The first plus second terms yield the usual formula:
\[
[\psi(x), H_N] = \int dV_w \psi(x)\psi^\dagger(w) \{\vec{\alpha} \cdot \vec{p} + \beta m\}_w \psi(w)
\]  
\[
- \int dV_w \psi^\dagger(w) \{\vec{\alpha} \cdot \vec{p} + \beta m\}_w \psi(w)\psi(x)
\]  
\[
= \int dV_w \delta(x - w) \{\vec{\alpha} \cdot \vec{p} + \beta m\}_w \psi(w) + 0
\]  
\[
= \{\vec{\alpha} \cdot \vec{p} + \beta m\}_x \psi(x),
\] (166)
from $\psi(x)\psi^\dagger(w) = \delta(x - w) - \psi^\dagger(w)\psi(x)$. This result is consistent to the free Dirac equation. From the similar calculation, the third term yields
\[
[\psi(x), H_I] = g_\sigma \gamma^0 \sigma \psi(x),
\] (167)
consistently to the interaction term in the Dirac equation. Note that its conjugate version follows the similar form. Summarizing these results, we have obtained
\[
i\frac{\partial}{\partial t} \psi(x) = [\psi(x), H] \iff [i\partial_t - \{\vec{\alpha} \cdot \vec{p} + \beta m\}_x] \psi(x) = g_\sigma \gamma^0 \sigma \psi(x),
\] (168)
where the source term $g_\sigma \gamma_0 \sigma(x)$ shows up. It is useful to note that
\[
\hat{T} [\psi(x)\bar{\psi}(y)] = S_F(x, y) + \hat{N}_0 [\psi(x)\bar{\psi}(y)] = S_F(x, y) + (-)\bar{\psi}(y)\psi(x),
\]  
\[
S_F(x, y) = \left< 0 \left| \hat{T} [\psi(x)\bar{\psi}(y)] \right| 0 \right>,
\] (169)
from the Wick’s theorem\(^2\), where \(\hat{N}_0\) means the normal ordering with respect to the free vacuum: 
\[ \langle 0 | \hat{N}_0 [... ] | 0 \rangle . \] The \(S_F(x, y)\) is the Feynman propagator of the free fermion, satisfying

\[
[i \gamma^\nu \partial_\nu - m]_x S_F(x, y) = \delta(x, y) \iff [i \gamma^0 \partial_0 - \vec{\gamma} \cdot \vec{p} - m]_x S_F(x, y) = \delta(x - y) \\
\iff [i \partial_t - \{ \vec{\alpha} \cdot \vec{p} + \beta m \}]_x S_F(x, y) = \gamma^0 \delta(x - y). \tag{170}
\]

Using this \(S_F\), the fermion (nucleon) field can be formally solved as

\[
\psi(x) = \int dy S_F(x, y) g_\sigma \sigma(y) \psi(y). \tag{171}
\]

We can also follows the time-development of the fermion propagator. That is

\[
G(x, y) = \left\langle A | \hat{T} \psi(x) \bar{\psi}(y) | A \right\rangle \\
= S_F(x, y) + \left\langle A | (-) \bar{\psi}(y) \psi(x) | A \right\rangle. \tag{172}
\]

This \(G(x, y)\) can be also interpreted as the density tensor, \(\rho_{xy} = G(x, y)\), in the usual meanfield framework. For this propagator, one finds

\[
[i \partial_t - \{ \vec{\alpha} \cdot \vec{p} + \beta m \}]_x G(x, y) = \gamma^0 \delta(x - y) - \left\langle A | \bar{\psi}(y) [i \partial_t - \{ \vec{\alpha} \cdot \vec{p} + \beta m \}]_x \psi(x) | A \right\rangle \\
= \gamma^0 \delta(x - y) - \left\langle A | \bar{\psi}(y) g_\sigma \gamma^0 \sigma(x) \psi(x) | A \right\rangle \\
[i \gamma^\nu \partial_\nu - m]_x G(x, y) = \delta(x - y) - g_\sigma \left\langle A | \bar{\psi}(y) \sigma(x) \psi(x) | A \right\rangle. \tag{173}
\]

From Eq. (163), \(\sigma(x)\) can be eliminated:

\[
[i \gamma^\nu \partial_\nu - m]_x G(x, y) = \delta(x - y) + g_\sigma^2 \left\langle A | \bar{\psi}(y) \int dw D_\sigma(x - w) \bar{\psi}(w) \psi(w) \psi(x) | A \right\rangle. \tag{174}
\]

Notice that the quadratic term of \(\bar{\psi} \bar{\psi} \psi \psi\) appears in the RHS.

**References**


\(^2\text{See Eq. (4.107) in Ref. [1].}\)