Note for relativistic Dirac formalism

Tomohiro Oishi

1 Convention

See TABLE 1 for basic conventions.

TABLE 1: Conventional rules in this note.

Name	Quantity	Definition
flat metric	$g^{\mu\nu} = g_{\mu\nu}$	$= diag(+,-,-,-)$
4D coordinate	$x^\mu = (x^0, x^1, x^2, x^3)$	$= (ct, x, y, z)$
	$x_\mu = (x_0, x_1, x_2, x_3)$	$= (ct, -x, -y, -z)$
4D derivative	$\partial^\mu = \frac{\partial}{\partial x_\mu}$	$= \left(\frac{\partial}{c\partial t}, \ -\vec{\nabla}\right)$
	$\partial_\mu = \frac{\partial}{\partial x^\mu} = g_{\mu\nu}\partial^\nu$	$= \left(\frac{\partial}{c\partial t}, \ \vec{\nabla}\right)$
reduced derivative	$\gamma^\mu \partial_\mu = \gamma_\mu \partial^\mu$	$= \beta\partial_{ct} + \vec{\gamma}\cdot\vec{\nabla}$
4D momentum	$p^\mu = (p^0, p^1, p^2, p^3) = i\hbar\partial^\mu$	$= \left(\frac{E}{c}, \vec{p}\right)$
	$p_\mu = g_{\mu\nu}p^\nu$	$= \left(\frac{E}{c}, -\vec{p}\right)$

1.1 spin algebra

Pauli's sigma matricies read

$$\hat{s}_x \equiv \frac{\sigma_1}{2}, \quad \hat{s}_y \equiv \frac{\sigma_2}{2}, \quad \hat{s}_z \equiv \frac{\sigma_3}{2}, \quad \text{where} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1)$$

These satisfy $\sigma_i\sigma_j = \delta_{ij} + i\epsilon^{ijk}\sigma_k$, and thus,

$$\begin{aligned} &\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}, \\ &\sigma_i\sigma_j - \sigma_j\sigma_i = 2i\epsilon^{ijk}\sigma_k \Longleftrightarrow [\hat{s}_i, \hat{s}_j] = i\epsilon^{ijk}\hat{s}_k. \end{aligned} \quad (2)$$

It is also worthwhile to define $\sigma_{0,\pm 1}$:

$$\sigma_0 = \sigma_{3(z)}, \quad \sigma_\pm \equiv \frac{1}{\sqrt{2}}\left(\sigma_{1(x)} \pm i\sigma_{2(y)}\right). \quad (3)$$

1.2 gamma matrices

In Dirac's representation, the (4×4) gamma matrices are defined as

$$\gamma^\mu = (\gamma^0, \vec{\gamma}) \equiv (\beta, \beta\vec{\alpha}), \quad (4)$$

where

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \longleftrightarrow \gamma^k = (\beta\vec{\alpha})^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}. \tag{5}$$

Note that $\gamma_0 = \gamma^0$. The following matrices are also useful:

$$\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]. \tag{6}$$

Dirac's conjugate:

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0. \tag{7}$$

Thus,

$$\bar{\psi}_a(x)\psi_b(x) = F_a^*(x)F_b(x) - G_a^*(x)G_b(x), \quad \bar{\psi}_a(x)\gamma^0\psi_b(x) = F_a^*(x)F_b(x) + G_a^*(x)G_b(x). \tag{8}$$

1.3 Dirac spinor

Dirac spinor for the spherical system is generally given as

$$\psi_N(\boldsymbol{r}) = \psi_{nljm}(\boldsymbol{r}) = \begin{pmatrix} iF_N(\boldsymbol{r}) \\ \\ G_N(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} if_{nlj}(r)\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) \\ \\ g_{nlj}(r)\frac{\vec{\sigma}\cdot\boldsymbol{r}}{r}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) \end{pmatrix}, \tag{9}$$

where

$$\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) \equiv \sum_{v=\pm 1/2} \mathcal{C}_{h,v}^{(j,m)l,\frac{1}{2}} Y_{l,h=m-v}(\bar{\boldsymbol{r}}) \cdot \chi_v. \tag{10}$$

Of course, $\hat{s}_z\chi_{\pm\frac{1}{2}} = \pm\frac{1}{2}\chi_{\pm\frac{1}{2}}$. Note also that

$$\frac{\vec{\sigma}\cdot\boldsymbol{r}}{r}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) = \mathcal{Y}_{\ell jm}(\bar{\boldsymbol{r}}), \tag{11}$$

where $\ell = l \mp 1$ when $l = j \pm \frac{1}{2}$. Thus, the Dirac spinor can be reformulated as

$$\psi_{nljm}(\boldsymbol{r}) = \begin{pmatrix} if_{nlj}(r)\mathcal{Y}_{(l=j\pm 1/2)jm}(\bar{\boldsymbol{r}}) \\ g_{nlj}(r)\mathcal{Y}_{(\ell=j\mp 1/2)jm}(\bar{\boldsymbol{r}}) \end{pmatrix}. \tag{12}$$

Remember also that $(\vec{\sigma}\cdot\boldsymbol{r}/r)^2 = r^2/r^2 = 1$. For example, when the larger component has the $d_{5/2}$ ($l = 2$) charactor, the corresponding smaller component has the $f_{5/2}$ ($\ell = 3$) charactor. Table 2 lists some sets of (l, ℓ).

1.4 angular-momentum convenstion

Clebsch-Gordan (CG) coefficient:

$$\mathcal{C}_{m_1,m_2}^{(J,M)j_1,j_2} \equiv \langle j_1, m_1; j_2, m_2 \mid (j_1 j_2)J, M\rangle \iff |J, M\rangle = \sum_{m_1,m_2} \mathcal{C}_{m_1,m_2}^{(J,M)j_1,j_2} |j_1, m_1\rangle |j_2, m_2\rangle. \tag{13}$$

Note EQs. (3.5.14) and (3.5.17) in Edmonds's textbook [**?**]:

$$\mathcal{C}_{m_2,m_1}^{(J,M)j_2,j_1} = P\mathcal{C}_{m_1,m_2}^{(J,M)j_1,j_2}, \quad \mathcal{C}_{-m_1,-m_2}^{(J,-M)j_1,j_2} = P\mathcal{C}_{m_1,m_2}^{(J,M)j_1,j_2}, \tag{14}$$

where $P = (-)^{j_1+j_2-J}$. CG coefficients can be defined as REAL in any case.

TABLE 2: Angular quantum numbers for Dirac spinors.

larger	smaller	(l,ℓ)
$s_{1/2}$	$p_{1/2}$	(0,1)
$p_{3/2}$	$d_{3/2}$	(1,2)
$p_{1/2}$	$s_{1/2}$	(1,0)
$d_{5/2}$	$f_{5/2}$	(2,3)
$d_{3/2}$	$p_{3/2}$	(2,1)

3j symbol as in EQ. (3.7.3) in Edmonds's textbook [?]:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \equiv \frac{(-)^{j_1-j_2+m_3}}{\sqrt{2j_3+1}} \mathcal{C}^{(j_3,m_3)j_1,j_2}_{m_1,m_2} = \begin{pmatrix} j_3 & j_1 & j_2 \\ -m_3 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & -m_3 & m_1 \end{pmatrix}. \tag{15}$$

Note that, for the 3j-symbol, an even permutation of any two columns keeps it identical, whereas an odd permutation yields the factor $(-)^{j_1+j_2+j_3}$ as in Eq. (3.7.5) in Ref. [?].

Double-bar matrix element (DBME) or reduced matrix element as in EQ. (5.4.1) in Ref. [?]:

$$\begin{aligned} \left\langle j', m' \mid \hat{T}_{K,M} \mid j, m \right\rangle &= (-)^{j'-m'} \begin{pmatrix} j' & K & j \\ -m' & M & m \end{pmatrix} \left\langle j' \mid\mid \hat{T}_K \mid\mid j \right\rangle \\ &= \frac{(-)^{j'+K-j}}{\sqrt{2j'+1}} \mathcal{C}^{(j',m')K,j}_{M,m} \left\langle j' \mid\mid \hat{T}_K \mid\mid j \right\rangle = \frac{(-)^{j-m}}{\sqrt{2K+1}} \mathcal{C}^{(K,M)j',j}_{m',-m} \left\langle ... \right\rangle . \end{aligned} \tag{16}$$

2 Dirac equation with spherical potential(s)

We assume the $(1+3)$-dimensional time and space. In the MKSA or CGS-Gauss system of units, except the electro-magnetic terms, the Dirac equation is given as

$$i\hbar c\frac{\partial}{\partial(ct)}\psi(t,\boldsymbol{r}) = \left[-i\hbar c\beta\vec{\gamma}\cdot\vec{\nabla} + \beta Mc^2 + \beta S(r) + W(r)\right]\psi(t,\boldsymbol{r}), \tag{17}$$

where $S(r)$ and $W(r)$ are the spherical, scalar and vector potentials, respectively, given in the unit of energy (e.g., MeV). From $\beta\beta = I$ and $\gamma^\mu\partial_\mu = \beta\partial_{ct} + \vec{\gamma}\cdot\vec{\nabla}$, it is also expressed as

$$\left[i\hbar c\gamma^\mu\partial_\mu - Mc^2 - S(r) - \beta W(r)\right]\psi(t,\boldsymbol{r}) = 0. \tag{18}$$

The Lagangian density, which works as the source of this equation, reads

$$\mathcal{L} = \bar{\psi}\left[i\hbar c\gamma^\mu\partial_\mu - Mc^2 - S(r) - \beta W(r)\right]\psi(x), \tag{19}$$

where $\bar{\psi} \equiv \psi^\dagger\beta$. Note that, in the meson-exchange model for atomic nuclei, the potential terms are obtained from the sigma and omega meson fields. That is, $S(r) = g_\sigma\sigma(r)$ and $W(r) = g_\omega\omega(r)$ with $\omega_\mu = \delta_{\mu 0}\omega(r)$, respectively. In numerical calculations, these meson fields need to be solved self-consistently to the fermion field.

2.1 note for dimension

Note that, because the Lagrangian $L \equiv \int d^3\boldsymbol{r}\mathcal{L}$ and Mc^2 have the dimension of energy, $\bar{\psi}\psi$ is in the unit of fm^{-3}. As coincidence, if some interaction term(s) has the form,

$$\mathcal{L}_{\mathrm{I}} = \bar{\psi}X\psi(x), \tag{20}$$

then this wild-card part X must have the dimension of energy, e.g. in MeV. This knowledge may help us, for example, to infer the unit of the coupling constant.

2.2 large and small components

For the time-independent solution of Eq. (17), that is, $i\hbar\partial_t\psi = E_N\psi$, the Dirac equation reads

$$\left[-i\hbar c\beta\vec{\gamma}\cdot\vec{\nabla} + \beta Mc^2 + \beta S(r) + W(r)\right]\psi_N(t,\boldsymbol{r}) = E_N\psi_N(t,\boldsymbol{r}). \tag{21}$$

Dirac spinor for the spherical system is generally given as

$$\psi_N(\boldsymbol{r}) = \psi_{nljm}(\boldsymbol{r}) = \begin{pmatrix} iF_N(\boldsymbol{r}) \\ \\ G_N(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} i\frac{a_{nlj}(r)}{r}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) \\ \\ \frac{b_{nlj}(r)}{r}\frac{\vec{\sigma}\cdot\boldsymbol{r}}{r}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) \end{pmatrix}, \tag{22}$$

where

$$\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) \equiv \sum_{v=\pm 1/2} \mathcal{C}^{(j,m)l,\frac{1}{2}}_{h,v} Y_{l,h=m-v}(\bar{\boldsymbol{r}})\cdot\chi_v, \quad \text{with} \quad \hat{s}_z\chi_{\pm\frac{1}{2}} = \pm\frac{1}{2}\chi_{\pm\frac{1}{2}}. \tag{23}$$

Technique - Note that

$$\frac{\vec{\sigma}\cdot\boldsymbol{r}}{r}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) = \mathcal{Y}_{\ell jm}(\bar{\boldsymbol{r}}), \tag{24}$$

where $\ell = l \mp 1$ when $l = j \pm \frac{1}{2}$.

By using this ansatz, the Eq. (21) is transformed as

$$\begin{aligned} -i\hbar c\vec{\sigma}\cdot\vec{\nabla}G_N(\boldsymbol{r}) + \left[Mc^2 + S(r) + W(r)\right] iF_N(\boldsymbol{r}) &= E_N iF_N(\boldsymbol{r}), \\ -i\hbar c\vec{\sigma}\cdot\vec{\nabla}iF_N(\boldsymbol{r}) + \left[-Mc^2 - S(r) + W(r)\right] G_N(\boldsymbol{r}) &= E_N G_N(\boldsymbol{r}). \end{aligned} \tag{25}$$

Before going to the further calculations, now we focus on the $\vec{\sigma}\cdot\vec{\nabla}$ term. By using,

$$\left(\vec{\sigma}\cdot\vec{A}\right)\left(\vec{\sigma}\cdot\vec{B}\right) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot\left(\vec{A}\times\vec{B}\right) \ , \tag{26}$$

then the operator $\vec{\sigma}\cdot\vec{\nabla}$ becomes

$$\begin{aligned} \vec{\sigma}\cdot\vec{\nabla} &= \frac{(\vec{\sigma}\cdot\vec{r})^2}{r^2}\vec{\sigma}\cdot\vec{\nabla} = \frac{\vec{\sigma}\cdot\vec{r}}{r^2}(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{\nabla}) \\ &= \frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left[\vec{r}\cdot\vec{\nabla} + i\vec{\sigma}\cdot(\vec{r}\times\vec{\nabla})\right] \\ &= \frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left[\vec{r}\cdot\vec{\nabla} - \vec{\sigma}\cdot\vec{L}/\hbar\right] = \frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left[r\frac{d}{dr} - \frac{2\vec{S}\cdot\vec{L}}{\hbar^2}\right], \end{aligned} \tag{27}$$

where we have used $\vec{\sigma} = 2\vec{S}/\hbar$, $i\vec{\nabla} = -\vec{p}/\hbar$, and $\vec{L} = \vec{r}\times\vec{p}$. Namely, the spin-orbit coupling is naturally concluded from "kinetic term" in the Dirac formalism. If the gap of potentials, $S(r) - W(r)$, is constant, this spin-orbit term vanishes, as we see in the following.

2.3 spin-orbit coupling and Darwin term

From Eq. (25),

$$G_N(\boldsymbol{r}) = \frac{-i\hbar c}{E_N + Mc^2 + S(r) - W(r)}\vec{\sigma}\cdot\vec{\nabla}iF_N(\boldsymbol{r}). \tag{28}$$

Thus, the corresponding large component reads

$$-(\hbar c)^2\vec{\sigma}\cdot\vec{\nabla}\frac{\vec{\sigma}\cdot\nabla i\vec{F_N}(\boldsymbol{r})}{E_N + Mc^2 + S(r) - W(r)} + \left[Mc^2 + S(r) + W(r)\right] iF_N(\boldsymbol{r}) = E_N iF_N(\boldsymbol{r}). \tag{29}$$

We use $\epsilon_N(r) \equiv E_N + Mc^2 + S(r) - W(r)$ and $iF_N \longrightarrow F_N$ in the following. Since $(\vec{\sigma}\cdot\vec{\nabla})^2 = \nabla^2$, it becomes

$$\begin{aligned} -\frac{(\hbar c)^2}{\epsilon_N(r)}\nabla^2 F_N(\boldsymbol{r}) - (\hbar c)^2\left(\vec{\sigma}\cdot\vec{\nabla}\frac{1}{\epsilon_N(r)}\right)\left(\vec{\sigma}\cdot\vec{\nabla}F_N(\boldsymbol{r})\right) & \\ + \left[Mc^2 + S(r) + W(r)\right] F_N(\boldsymbol{r}) &= E_N F_N(\boldsymbol{r}). \end{aligned} \tag{30}$$

Next, for the second term, please notice that

$$\begin{aligned} \left(\vec{\sigma}\cdot\vec{\nabla}\frac{1}{\epsilon_N(r)}\right) &= \frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left[r\frac{d\epsilon_N^{-1}(r)}{dr} - \left(\vec{\sigma}\cdot\vec{L}\frac{1}{\epsilon_N(r)}\right)\right] = \frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left[r\frac{d\epsilon_N^{-1}(r)}{dr} - 0\right], \\ \left(\vec{\sigma}\cdot\vec{\nabla}F_N(\boldsymbol{r})\right) &= \frac{\vec{\sigma}\cdot\vec{r}}{r^2}\left[r\frac{d}{dr} - \frac{2\vec{S}\cdot\vec{L}}{\hbar^2}\right]F_N(\boldsymbol{r}). \end{aligned} \tag{31}$$

Thus, by using $(\vec{\sigma}\cdot\vec{r})^2/r^4 = 1/r^2$, the Eq. (30) is transformed as

$$\begin{aligned}
-\frac{(\hbar c)^2}{\epsilon_N(r)}\nabla^2 F_N(\boldsymbol{r}) - \frac{(\hbar c)^2}{r^2}\left[r\frac{d\epsilon_N^{-1}(r)}{dr}\right]\left[r\frac{d}{dr} - \frac{2\vec{S}\cdot\vec{L}}{\hbar^2}\right] F_N(\boldsymbol{r}) & \\
+\left[Mc^2 + S(r) + W(r)\right] F_N(\boldsymbol{r}) &= E_N F_N(\boldsymbol{r}), \\
\Longrightarrow \left[-\frac{(\hbar c)^2}{\epsilon_N(r)}\nabla^2 - (\hbar c)^2\frac{(-)\epsilon'_N(r)}{\epsilon_N^2(r)}\frac{d}{dr} + \frac{(\hbar c)^2}{r}\frac{(-)\epsilon'_N(r)}{\epsilon_N^2(r)}\frac{2\vec{S}\cdot\vec{L}}{\hbar^2}\right. & \\
\left. +S(r) + W(r)\right] F_N(\boldsymbol{r}) &= (E_N - Mc^2)F_N(\boldsymbol{r}), \qquad (32)
\end{aligned}$$

where the 1st term in the LHS corresponds to the kinetic energy, the 2nd term is so-called Darwin term, and the 3rd term indicates the spin-orbit coupling. These Darwin and spin-orbit terms can be naturally concluded from the Dirac equation, whereas those were just introduced as "phenomenology" in the Schroedinger equation.

It is convenient to find that,

- the total potential is given as $S(r) + W(r)$, whereas,
- the spin-orbit and Darwin terms depend on the $\epsilon'_N(r) = S'(r) - W'(r)$.

Thus, even though the total potential is zero or very small, it does not guarantee the free condition for fermions. Remember also that, for the spin-orbit coupling term,

$$2\vec{S}\cdot\vec{L}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) = \hbar^2 K_{lj}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}), \qquad (33)$$

where

$$\begin{aligned}
K_{lj} = j(j+1) - l(l+1) - \frac{3}{4} &= l, \quad \text{when } j = l + \frac{1}{2}, \\
&= -l - 1, \quad \text{when } j = l - \frac{1}{2}. \qquad (34)
\end{aligned}$$

It is also convenient to note that,

$$\begin{aligned}
2\vec{S}\cdot\vec{L}\frac{\vec{\sigma}\cdot\boldsymbol{r}}{r}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) &= 2\vec{S}\cdot\vec{L}\mathcal{Y}_{\ell,jm}(\bar{\boldsymbol{r}}), \quad \text{with } \ell = l \pm 1 \text{ for } j = l \pm \frac{1}{2}, \\
&= \hbar^2 Q_{lj}\mathcal{Y}_{\ell,jm}(\bar{\boldsymbol{r}}), \qquad (35)
\end{aligned}$$

where

$$\begin{aligned}
Q_{lj} = j(j+1) - \ell(\ell+1) - \frac{3}{4} &= -l - 2, \quad \text{when } j = l + \frac{1}{2}, \\
&= l - 1, \quad \text{when } j = l - \frac{1}{2}. \qquad (36)
\end{aligned}$$

2.4 from Dirac to Schroedinger equations

The correspondence between the Eq. (32) and the Schroedinger equation is obtained as follows. First (i) we assume $S(r) = 0$, namely, only the vector-type potential is finite. Notice that, e.g. the Coulomb potential mediated by the photon (vector-gauge field) is consistent to this assumption. Then (ii) in the non-relativistic limit, $E_N - W(r) \cong Mc^2$, and thus, $\epsilon_N(r) \cong 2Mc^2$. Note also that $\epsilon'_N(r) = -W'(r)$. Therefore, the Eq. (32) is approximated as

$$\left[-\frac{\hbar^2}{2M}\nabla^2 - (\hbar c)^2\frac{W'(r)}{4M^2c^4}\frac{d}{dr} + \frac{(\hbar c)^2}{r}\frac{W'(r)}{4M^2c^4}\frac{2\vec{S}\cdot\vec{L}}{\hbar^2} + W(r)\right] F_N(\boldsymbol{r}) = (E_N - Mc^2)F_N(\boldsymbol{r}). \qquad (37)$$

The 1st and 4th terms are well-known kinetic and potential terms in the Schroedinger equation, respectively.

2.5 Solution of free Dirac equation

For $E = +\sqrt{c^2\vec{p}^2 + c^4M^2} > 0$, there are two solutions with $p_0 = +E$ and $p_0 = -E$:

$$\begin{aligned}
\psi_{[+E,\ +\vec{p},\ +s]}(x) &= \exp\left[-i\frac{p^\mu x_\mu}{\hbar}\right]\sqrt{\frac{E+M}{2E}}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{M+E}\end{pmatrix}\chi_{+s}\ , \\
\psi_{[-E,\ -\vec{p},\ -s]}(x) &= \exp\left[+i\frac{p^\mu x_\mu}{\hbar}\right]\sqrt{\frac{E+M}{2E}}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{M+E} \\ 1\end{pmatrix}\chi_{-s}\ .
\end{aligned} \tag{38}$$

3 Numerical solution of spherical Dirac equation

Our goal in this section is to summarize the numerical method for the spherical Dirac equation. We start again from Eq. (25),

$$\begin{aligned}
-i\hbar c\vec{\sigma}\cdot\vec{\nabla}G_N(\boldsymbol{r}) + \left[Mc^2 + S(r) + W(r)\right]iF_N(\boldsymbol{r}) &= E_N iF_N(\boldsymbol{r}), \\
-i\hbar c\vec{\sigma}\cdot\vec{\nabla}iF_N(\boldsymbol{r}) + \left[-Mc^2 - S(r) + W(r)\right]G_N(\boldsymbol{r}) &= E_N G_N(\boldsymbol{r}).
\end{aligned} \tag{39}$$

and Eq.(27),

$$\vec{\sigma}\cdot\vec{\nabla} = \frac{(\vec{\sigma}\cdot\vec{r})^2}{r^2}\vec{\sigma}\cdot\vec{\nabla} = \frac{\vec{\sigma}\cdot\boldsymbol{r}}{r^2}\left[\vec{r}\cdot\vec{\nabla} - \vec{\sigma}\cdot\vec{L}/\hbar\right] = \frac{\vec{\sigma}\cdot\boldsymbol{r}}{r^2}\left[r\frac{d}{dr} - \frac{2\vec{S}\cdot\vec{L}}{\hbar^2}\right], \tag{40}$$

where we have used $\vec{\sigma} = 2\vec{S}/\hbar$, $i\vec{\nabla} = -\vec{p}/\hbar$, and $\vec{L} = \vec{r}\times\vec{p}$. Using the label K_{lj} and Q_{lj}, which are determined as $2\vec{S}\cdot\vec{L}\mathcal{Y}_{ljm} = \hbar^2 K_{lj}\mathcal{Y}_{ljm}$ and $2\vec{S}\cdot\vec{L}\frac{\vec{\sigma}\cdot\boldsymbol{r}}{r}\mathcal{Y}_{ljm} = \hbar^2 Q_{lj}\frac{\vec{\sigma}\cdot\boldsymbol{r}}{r}\mathcal{Y}_{ljm}$, one finds that

$$\begin{aligned}
\vec{\sigma}\cdot\vec{\nabla}iF_N(\boldsymbol{r}) &= \vec{\sigma}\cdot\vec{\nabla}iF_{nlj}(r)\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) = \frac{\vec{\sigma}\cdot\boldsymbol{r}}{r^2}i\left[r\frac{dF_{nlj}(r)}{dr} - K_{lj}F_{nlj}(r)\right]\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}), \\
&= i\left[\frac{dF_{nlj}(r)}{dr} - \frac{K_{lj}}{r}F_{nlj}(r)\right]\frac{\vec{\sigma}\cdot\boldsymbol{r}}{r}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}),
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
\vec{\sigma}\cdot\vec{\nabla}G_N(\boldsymbol{r}) &= \vec{\sigma}\cdot\vec{\nabla}G_{nlj}(r)\frac{\vec{\sigma}\cdot\boldsymbol{r}}{r}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) = \frac{\vec{\sigma}\cdot\boldsymbol{r}}{r^2}\left[r\frac{dG_{nlj}(r)}{dr} - Q_{lj}G_{nlj}(r)\right]\frac{\vec{\sigma}\cdot\boldsymbol{r}}{r}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) \\
&= \left[\frac{dG_{nlj}(r)}{dr} - \frac{Q_{lj}}{r}G_{nlj}(r)\right]\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}).
\end{aligned} \tag{42}$$

Therefore, Eq. (39) is transformed as

$$\begin{aligned}
-i\hbar c\left[\frac{dG_{nlj}(r)}{dr} - \frac{Q_{lj}}{r}G_{nlj}(r)\right]\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) &= \left[E_N - W(r) - S(r) - Mc^2\right]iF_{nlj}(r)\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}), \\
-i\hbar c\cdot i\left[\frac{dF_{nlj}(r)}{dr} - \frac{K_{lj}}{r}F_{nlj}(r)\right]\frac{\vec{\sigma}\cdot\boldsymbol{r}}{r}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}) &= \left[E_N - W(r) + S(r) + Mc^2\right] \\
&\qquad G_{nlj}(r)\frac{\vec{\sigma}\cdot\boldsymbol{r}}{r}\mathcal{Y}_{ljm}(\bar{\boldsymbol{r}}).
\end{aligned} \tag{43}$$

Thus,

$$\begin{aligned}
\frac{dF_{nlj}}{dr} &= \frac{K_{lj}}{r}F_{nlj}(r) + \frac{Mc^2 + S(r) + E_N - W(r)}{\hbar c}G_{nlj}(r), \\
\frac{dG_{nlj}}{dr} &= \frac{Mc^2 + S(r) - E_N + W(r)}{\hbar c}F_{nlj}(r) + \frac{Q_{lj}}{r}G_{nlj}(r).
\end{aligned} \tag{44}$$

For another representation with $F_{nlj}(r) \equiv \frac{a_{nlj}(r)}{r}$ and $G_{nlj}(r) \equiv \frac{b_{nlj}(r)}{r}$, these equations change as

$$\begin{aligned} \frac{da_{nlj}}{dr} &= \frac{K_{lj}+1}{r}a_{nlj}(r) + \frac{Mc^2+S(r)+E_N-W(r)}{\hbar c}b_{nlj}(r), \\ \frac{db_{nlj}}{dr} &= \frac{Mc^2+S(r)-E_N+W(r)}{\hbar c}a_{nlj}(r) + \frac{Q_{lj}+1}{r}b_{nlj}(r). \end{aligned} \quad (45)$$

Here, one can use a trick: $K_{lj}+1 = -Q_{lj}-1$ for whatever $j = l \pm 1/2$. Thus, using

$$\begin{aligned} \kappa_{lj} \equiv K_{lj}+1 = -Q_{lj}-1 &= l+1 \text{ for } j = l+1/2, \\ &= -l \text{ for } j = l-1/2, \end{aligned} \quad (46)$$

then one finally gets

$$\begin{aligned} \frac{da_{nlj}}{dr} &= \frac{\kappa_{lj}}{r}a_{nlj}(r) + \frac{Mc^2+S(r)+E_N-W(r)}{\hbar c}b_{nlj}(r), \\ \frac{db_{nlj}}{dr} &= \frac{Mc^2+S(r)-E_N+W(r)}{\hbar c}a_{nlj}(r) + \frac{-\kappa_{lj}}{r}b_{nlj}(r). \end{aligned} \quad (47)$$

In the following, we introduce the new symbols as

$$s(r) \equiv Mc^2 + S(r), \quad v(r) \equiv E_N - W(r), \quad \epsilon_N(r) \equiv s(r) + v(r).$$

Then the last equations for $\{a_{nlj}(r), b_{nlj}(r)\}$ read

$$\frac{d}{dr}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{\kappa}{r} & \frac{s+v}{\hbar c} \\ \frac{s-v}{\hbar c} & \frac{-\kappa}{r} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}. \quad (48)$$

3.1 large component $a(r)$

First, we eliminate $b(r)$:

$$\begin{aligned} b(r) &= \frac{\hbar c}{\epsilon_N(r)}\left(a'(r) - \frac{\kappa}{r}a(r)\right), \\ b'(r) &= \hbar c\left\{(-)\frac{\epsilon_N'}{\epsilon_N^2}\left(a'(r) - \frac{\kappa}{r}a(r)\right) + \frac{1}{\epsilon_N}\left(a''(r) - \frac{\kappa}{r}a'(r) + \frac{\kappa}{r^2}a(r)\right)\right\} \\ &= (\text{from EOM...}) = \frac{s(r)-v(r)}{\hbar c}a(r) - \frac{\kappa}{r}\frac{\hbar c}{\epsilon_N}\left(a'(r) - \frac{\kappa}{r}a(r)\right). \end{aligned} \quad (49)$$

By some calculations,

$$\begin{aligned} &\Longrightarrow a''(r) - \frac{\kappa}{r}a'(r) + \frac{\kappa}{r^2}a(r) - \frac{\epsilon_N'}{\epsilon_N(r)}\left(a'(r) - \frac{\kappa}{r}a(r)\right) = \frac{s^2-v^2}{(\hbar c)^2}a(r) - \frac{\kappa}{r}\left(a'(r) - \frac{\kappa}{r}a(r)\right) \\ &\Longrightarrow a''(r) - \frac{\epsilon_N'}{\epsilon_N}a'(r) + \left(\frac{\kappa}{r^2} + \frac{\epsilon_N'(r)}{\epsilon_N(r)}\cdot\frac{\kappa}{r} - \frac{s^2-v^2}{(\hbar c)^2} - \frac{\kappa^2}{r^2}\right)a(r) = 0 \\ &\quad\;\; a''(r) - \frac{\epsilon_N'}{\epsilon_N}a'(r) + \left(-\frac{l(l+1)}{r^2} + \frac{\epsilon_N'}{\epsilon_N}\cdot\frac{\kappa}{r} - \frac{s^2-v^2}{(\hbar c)^2}\right)a(r) = 0, \end{aligned} \quad (50)$$

where we have used $\kappa_{lj}(\kappa_{lj}-1) = l(l+1)$ for whatever $j = l \pm 1/2$. Or equivalently,

$$\begin{aligned} &-\frac{(\hbar c)^2}{\epsilon_N(r)}a''(r) + \frac{(\hbar c)^2\epsilon_N'(r)}{\epsilon_N^2(r)}a'(r) + \left[\frac{(\hbar c)^2}{\epsilon_N(r)}\frac{l(l+1)}{r^2} - \frac{(\hbar c)^2\epsilon_N'(r)}{\epsilon_N^2(r)}\frac{\kappa}{r} + s(r) - v(r)\right]a(r) = 0, \\ &\left\{-\frac{(\hbar c)^2}{\epsilon_N(r)}\frac{d^2}{dr^2} + \frac{(\hbar c)^2\epsilon_N'(r)}{\epsilon_N^2(r)}\frac{d}{dr} + \left[\frac{(\hbar c)^2}{\epsilon_N(r)}\frac{l(l+1)}{r^2} - \frac{(\hbar c)^2\epsilon_N'(r)}{\epsilon_N^2(r)}\frac{\kappa}{r} + S(r) + W(r)\right]\right\}a(r) \\ &= \left(E_N - Mc^2\right)a(r). \end{aligned} \quad (51)$$

Then, in the non-relativistic limit, this equation becomes the Schroedinger equation with the potential $S(r) + W(r)$.

3.2 small component $b(r)$

Next we focus on $b_{nlj}(r)$. By introducing $\zeta_N \equiv s(r) - v(r) = Mc^2 + S(r) - E_N + W(r)$,

$$\begin{aligned} a(r) &= \frac{\hbar c}{\zeta_N(r)} \left(b'(r) + \frac{\kappa}{r} b(r) \right), \\ a'(r) &= \hbar c \left\{ (-)\frac{\zeta_N'}{\zeta_N^2} \left(b'(r) + \frac{\kappa}{r} b(r) \right) + \frac{1}{\zeta_N(r)} \left(b''(r) + \frac{\kappa}{r} b'(r) - \frac{\kappa}{r^2} b(r) \right) \right\} \\ &= (\text{from EOM...}) = \frac{\kappa}{r} \frac{\hbar c}{\zeta_N(r)} \left(b'(r) + \frac{\kappa}{r} b(r) \right) + \frac{s(r)+v(r)}{\hbar c} b(r) \end{aligned} \tag{52}$$

By some calculations,

$$\begin{aligned} &\Longrightarrow \; b''(r) + \frac{\kappa}{r} b'(r) - \frac{\kappa}{r^2} b(r) - \frac{\zeta_N'}{\zeta_N} \left(b'(r) + \frac{\kappa}{r} b(r) \right) = \frac{\kappa}{r} \left(b'(r) + \frac{\kappa}{r} b(r) \right) + \frac{s^2 - v^2}{(\hbar c)^2} b(r) \\ &\Longrightarrow \; b''(r) - \frac{\zeta_N'}{\zeta_N} b'(r) + \left(-\frac{\kappa(\kappa+1)}{r^2} - \frac{\zeta_N'}{\zeta_N} \cdot \frac{\kappa}{r} - \frac{s^2 - v^2}{(\hbar c)^2} \right) b(r) = 0. \end{aligned} \tag{53}$$

Deviding this equation by $-\epsilon_N(r)/(\hbar c)^2$ where $\epsilon_N(r) = s(r) + v(r)$, one finds

$$\begin{aligned} &\left\{ -\frac{(\hbar c)^2}{\epsilon_N(r)} \frac{d^2}{dr^2} + \frac{(\hbar c)^2 \zeta_N'(r)}{\epsilon_N(r)\zeta_N(r)} \frac{d}{dr} + \left[\frac{(\hbar c)^2}{\epsilon_N(r)} \frac{\kappa(\kappa+1)}{r^2} - \frac{(\hbar c)^2 \epsilon_N'(r)}{\epsilon_N(r)\zeta_N(r)} \frac{\kappa}{r} + S(r) + W(r) \right] \right\} b(r) \\ &= \left(E_N - Mc^2 \right) b(r). \end{aligned} \tag{54}$$

3.3 asymptotic form at $r \cong 0$ with $W'(r) = S'(r) = 0$

Within this assumption, the Eq. (50) is approximated as

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - C(r) \right] a_{nlj}(r) \cong 0, \quad C(r) \equiv \frac{s^2(r) - v^2(r)}{(\hbar c)^2}, \quad \frac{d}{dr} C(r \cong 0) = 0. \tag{55}$$

(i) Because this Eq. keeps the same for $r \longrightarrow -r$, the asymptotic form must be $a(r) \cong \sum_n r^{2n+1}$ or $\cong \sum_n r^{2n}$, in its expanded form. (ii) By considering the special case with $S(r) = W(r) \equiv 0$, namely $C(r) = const.$, the possible form can be limited as $a(r) \cong r^{l+1} + \mathcal{O}(r^{l+3})$. (iii) Assuming $a(r) \cong r^{l+1} + \chi C(r) r^{l+3} + \mathcal{O}(r^{l+5})$, the factor χ must satisfy that,

$$0 \cdot \frac{r^{l+1}}{r^2} + \frac{r^{l+3}}{r^2} \left\{ \chi(l+3)(l+2) - \chi(l+1)l - 1 \right\} C(r) + \mathcal{O}(r^{l+5-2}) \cong 0 \;\longrightarrow\; \chi = \frac{1}{4l+6}. \tag{56}$$

Therefore, without the normalization,

$$a_{nlj}(r \cong 0) = r^{l+1} + \frac{C(r)}{4l+6} r^{l+3} + \mathcal{O}(r^{l+5}). \tag{57}$$

The corresponding $b_{nlj}(r)$ can be computed from the Dirac equation:

$$b_{nlj}(r \cong 0) = \frac{\hbar c}{s(r) + v(r)} \left[\frac{da_{nlj}}{dr} - \frac{\kappa_{lj}}{r} a_{nlj}(r) \right]. \tag{58}$$

4 Isoscalar-scalar interaction

IS-S interaction:

$$\begin{aligned} \mathcal{L}_{int.}(\boldsymbol{r}) &= \frac{-\alpha_S[\rho(\boldsymbol{r})]}{2}\left[\bar{\psi}(\boldsymbol{r})\psi(\boldsymbol{r})\right]\left[\bar{\psi}(\boldsymbol{r})\psi(\boldsymbol{r})\right], \\ V_{int.} &= \int d\boldsymbol{r}\mathcal{H}_{int.}(\boldsymbol{r}) = -\int d\boldsymbol{r}\mathcal{L}_{int.}(\boldsymbol{r}) \\ &= \frac{1}{2}\int d\boldsymbol{r}\alpha_S[\rho(\boldsymbol{r})]\left[\bar{\psi}(\boldsymbol{r})\psi(\boldsymbol{r})\right]\left[\bar{\psi}(\boldsymbol{r})\psi(\boldsymbol{r})\right]. \end{aligned} \tag{59}$$

Matrix element (ME) via basis states $\langle 12| = \langle ab|$ and $|12\rangle = |cd\rangle$ reads

$$\langle ab \mid V_{int.} \mid cd\rangle = \frac{1}{2}\int d\boldsymbol{r}_1\alpha_S[\rho(\boldsymbol{r}_1)]\bar{\psi}_a(x_1)\bar{\psi}_b(x_1)\psi_c(x_1)\psi_d(x_1), \tag{60}$$

where $x_i = (\boldsymbol{r}_i, \vec{s}_i)$, and $\psi_d(x_i) \equiv \langle x_i \mid d\rangle$ is the SP basis in the d orbit. By employing $\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$, this can be represented as the ME of the point-coupling interaction:

$$\langle ab \mid V_{int.} \mid cd\rangle = \frac{1}{2}\int d\boldsymbol{r}_1\int d\boldsymbol{r}_2\alpha_S[\rho(\boldsymbol{r}_*)]\bar{\psi}_a(x_1)\bar{\psi}_b(x_2)\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\psi_c(x_1)\psi_d(x_2). \tag{61}$$

Note that

$$\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \frac{\delta(r_1 - r_2)}{r_1 r_2}\sum_{\lambda=0}^{\infty}\sum_{\zeta=-\lambda}^{\lambda} Y_{\lambda,\zeta}(\bar{\boldsymbol{r}}_1)Y^*_{\lambda,\zeta}(\bar{\boldsymbol{r}}_2). \tag{62}$$

4.1 calculation

In EQ. (61),

$$\begin{aligned} \bar{\psi}_a(x_1)\psi_c(x_1) &= \begin{pmatrix} if_{nlj(a)}(r_1)\mathcal{Y}_{(l=j\pm1/2)jm(a)}(\bar{\boldsymbol{r}}_1) \\ g_{nlj(a)}(r_1)\mathcal{Y}_{(\ell=j\mp1/2)jm(a)}(\bar{\boldsymbol{r}}_1) \end{pmatrix}^{\dagger}\gamma_0\begin{pmatrix} if_{nlj(c)}(r_1)\mathcal{Y}_{(l=j\pm1/2)jm(c)}(\bar{\boldsymbol{r}}_1) \\ g_{nlj(c)}(r_1)\mathcal{Y}_{(\ell=j\mp1/2)jm(c)}(\bar{\boldsymbol{r}}_1) \end{pmatrix} \\ &= f^*_{(a)}\mathcal{Y}^*_{(l)jm(a)}\cdot f_{(c)}(r_1)\mathcal{Y}_{(l)jm(c)}(\bar{\boldsymbol{r}}_1) - g^*_{(a)}\mathcal{Y}^*_{(\ell)jm(a)}\cdot g_{(c)}(r_1)\mathcal{Y}_{(\ell)jm(c)}(\bar{\boldsymbol{r}}_1), \end{aligned} \tag{63}$$

as well as

$$\begin{aligned} \bar{\psi}_b(x_2)\psi_d(x_2) &= \begin{pmatrix} if_{nlj(b)}(r_2)\mathcal{Y}_{(l=j\pm1/2)jm(b)}(\bar{\boldsymbol{r}}_2) \\ g_{nlj(b)}(r_2)\mathcal{Y}_{(\ell=j\mp1/2)jm(b)}(\bar{\boldsymbol{r}}_2) \end{pmatrix}^{\dagger}\gamma_0\begin{pmatrix} if_{nlj(d)}(r_2)\mathcal{Y}_{(l=j\pm1/2)jm(d)}(\bar{\boldsymbol{r}}_2) \\ g_{nlj(d)}(r_2)\mathcal{Y}_{(\ell=j\mp1/2)jm(d)}(\bar{\boldsymbol{r}}_2) \end{pmatrix} \\ &= f^*_{(b)}\mathcal{Y}^*_{(l)jm(b)}\cdot f_{(d)}(r_2)\mathcal{Y}_{(l)jm(d)}(\bar{\boldsymbol{r}}_2) - g^*_{(b)}\mathcal{Y}^*_{(\ell)jm(b)}\cdot g_{(d)}(r_2)\mathcal{Y}_{(\ell)jm(d)}(\bar{\boldsymbol{r}}_2), \end{aligned} \tag{64}$$

are included. Thus, for the computation, one must evaluate four terms. By utilizing the formula (62), the ME, $\langle ab \mid V_{int.} \mid cd\rangle$, can be decomposed into the radial-integration and angle-integration parts. That is,

$$\begin{aligned} \langle ab \mid V_{int.} \mid cd\rangle &= R^{ff,ff}_{ab,cd}\cdot S_{[ljm]_a[ljm]_b,[ljm]_c[ljm]_d} - R^{gf,gf}_{ab,cd}\cdot S_{[\ell jm]_a[ljm]_b,[\ell jm]_c[ljm]_d} \\ &\quad - R^{fg,fg}_{ab,cd}\cdot S_{[ljm]_a[\ell jm]_b,[ljm]_c[\ell jm]_d} + R^{gg,gg}_{ab,cd}\cdot S_{[\ell jm]_a[\ell jm]_b,[\ell jm]_c[\ell jm]_d}. \end{aligned} \tag{65}$$

Notice the minus sign for (fg, fg) and (gf, gf) terms. Here the radial-integration part reads

$$\begin{aligned} R^{ff,ff}_{ab,cd} &\equiv \int r_1^2 dr_1\int r_2^2 dr_2\frac{\alpha_S[\rho(r_1)]}{2}f^*_{(a)}(r_1)f^*_{(b)}(r_2)\frac{\delta(r_1 - r_2)}{r_1 r_2}f_{(c)}(r_1)f_{(d)}(r_2), \\ R^{gf,gf}_{ab,cd} &\equiv \int r_1^2 dr_1\int r_2^2 dr_2\frac{\alpha_S[\rho(r_1)]}{2}g^*_{(a)}(r_1)f^*_{(b)}(r_2)\frac{\delta(r_1 - r_2)}{r_1 r_2}g_{(c)}(r_1)f_{(d)}(r_2), \end{aligned} \tag{66}$$

etc., where we assumed that $\rho(r)$ is spherical. The angular-integration part, on the other hand, is given as

$$\begin{aligned} S_{[pjm]_1[pjm]_2,[pjm]_3[pjm]_4} &\equiv \int d\bar{\boldsymbol{r}}_1 \int d\bar{\boldsymbol{r}}_2 \mathcal{Y}^*_{(p_1)j_1m_1}(\bar{\boldsymbol{r}}_1)\mathcal{Y}^*_{(p_2)j_2m_2}(\bar{\boldsymbol{r}}_2) \\ &\quad \left[\sum_\lambda \sum_\zeta Y_{\lambda,\zeta}(\bar{\boldsymbol{r}}_1)Y^*_{\lambda,\zeta}(\bar{\boldsymbol{r}}_2)\right] \mathcal{Y}_{(p_3)j_3m_3}(\bar{\boldsymbol{r}}_1)\mathcal{Y}_{(p_4)j_4m_4}(\bar{\boldsymbol{r}}_2), \\ &= \sum_\lambda \sum_\zeta \langle \mathcal{Y}_{[pjm]_1} \mid Y_{\lambda,\zeta} \mid \mathcal{Y}_{[pjm]_3}\rangle \langle \mathcal{Y}_{[pjm]_2} \mid Y^*_{\lambda,\zeta} \mid \mathcal{Y}_{[pjm]_4}\rangle, \end{aligned} \tag{67}$$

where $p = l$ or ℓ depending on each term.

4.2 angular-momentum projection

See also section D.2.1 in Ref. [2]. For a direct term, after the angular-momentum projection, the (J, M)-projected ME is given as

$$\langle ab \mid V_{int.} \mid cd\rangle^{(J,M)} = \sum_{m_a,m_b,m_c,m_d} (-)^{j_c-m_c}\mathcal{C}^{(J,M)j_a,j_c}_{m_a,-m_c} \cdot (-)^{j_b-m_b}\mathcal{C}^{(J,M)j_d,j_b}_{m_d,-m_b} \langle ab \mid V_{int.} \mid cd\rangle \tag{68}$$

Because the angular part of $\langle ab \mid V_{int.} \mid cd\rangle$ is separable from the radial part, we have

$$S^{(J,M)}_{[pj]_1[pj]_2,[pj]_3[pj]_4} = \sum_{(all\ m)} (-)^{j_3-m_3}\mathcal{C}^{(J,M)j_1,j_3}_{m_1,-m_3} \cdot (-)^{j_2-m_2}\mathcal{C}^{(J,M)j_4,j_2}_{m_4,-m_2} S_{[pjm]_1[pjm]_2,[pjm]_3[pjm]_4}, \tag{69}$$

for the (J, M)-fixed ME. Now it is worthwhile to use (*Is the following argument correct?*)

$$\langle \mathcal{Y}_{[pjm]_2} \mid Y^*_{\lambda,\zeta} \mid \mathcal{Y}_{[pjm]_4}\rangle = \left[\langle \mathcal{Y}_{[pjm]_4} \mid Y_{\lambda,\zeta} \mid \mathcal{Y}_{[pjm]_2}\rangle\right]^* = \frac{(-)^{j_2-m_2}}{\sqrt{2\lambda+1}}\mathcal{C}^{(\lambda,\zeta)j_4,j_2}_{m_4,-m_2} \langle \mathcal{Y}_{[pj]_4} \| Y_\lambda \| \mathcal{Y}_{[pj]_2}\rangle^* . \tag{70}$$

Notice that CG coefficients are defined as REAL. Thus,

$$\begin{aligned} S^{(J,M)}_{[pj]_1[pj]_2,[pj]_3[pj]_4} = \sum_{(all\ m)} &(-)^{j_3-m_3}\mathcal{C}^{(J,M)j_1,j_3}_{m_1,-m_3} \cdot (-)^{j_2-m_2}\mathcal{C}^{(J,M)j_4,j_2}_{m_4,-m_2} \\ &\sum_\lambda \sum_\zeta \frac{(-)^{j_3-m_3}}{\sqrt{2\lambda+1}}\mathcal{C}^{(\lambda,\zeta)j_1,j_3}_{m_1,-m_3} \langle \mathcal{Y}_{[pj]_1} \| Y_\lambda \| \mathcal{Y}_{[pj]_3}\rangle \\ &\quad \frac{(-)^{j_2-m_2}}{\sqrt{2\lambda+1}}\mathcal{C}^{(\lambda,\zeta)j_4,j_2}_{m_4,-m_2} \langle \mathcal{Y}_{[pj]_4} \| Y_\lambda \| \mathcal{Y}_{[pj]_2}\rangle^* . \end{aligned} \tag{71}$$

From the orthogonality of CG coefficients, we can get

$$\begin{aligned} S^{(J,M)}_{[pj]_1[pj]_2,[pj]_3[pj]_4} &= \sum_\lambda \sum_\zeta \frac{\delta_{J,\lambda}\delta_{M,\zeta}}{\sqrt{2\lambda+1}} \cdot \frac{\delta_{J,\lambda}\delta_{M,\zeta}}{\sqrt{2\lambda+1}} \langle \mathcal{Y}_{[pj]_1} \| Y_\lambda \| \mathcal{Y}_{[pj]_3}\rangle \langle \mathcal{Y}_{[pj]_4} \| Y_\lambda \| \mathcal{Y}_{[pj]_2}\rangle \\ &= \frac{1}{2J+1} \langle \mathcal{Y}_{[pj]_1} \| Y_\lambda \| \mathcal{Y}_{[pj]_3}\rangle \langle \mathcal{Y}_{[pj]_4} \| Y_\lambda \| \mathcal{Y}_{[pj]_2}\rangle^* . \end{aligned} \tag{72}$$

5 Isoscalar-vector interaction

IS-V interaction;

$$\begin{aligned}
\mathcal{L}_{int.}(\boldsymbol{r}) &= \frac{-\alpha_V[\rho(\boldsymbol{r})]}{2}\left[\bar{\psi}(\boldsymbol{r})\gamma_\mu\psi(\boldsymbol{r})\right]\left[\bar{\psi}(\boldsymbol{r})\gamma^\mu\psi(\boldsymbol{r})\right], \\
V_{int.} &= \int d\boldsymbol{r}\mathcal{H}_{int.}(\boldsymbol{r}) = -\int d\boldsymbol{r}\mathcal{L}_{int.}(\boldsymbol{r}) \\
&= \frac{1}{2}\int d\boldsymbol{r}\alpha_V[\rho(\boldsymbol{r})]\left[\bar{\psi}(\boldsymbol{r})\gamma_\mu\psi(\boldsymbol{r})\right]\left[\bar{\psi}(\boldsymbol{r})\gamma^\mu\psi(\boldsymbol{r})\right] \\
&= \frac{1}{2}\int d\boldsymbol{r}_1\int d\boldsymbol{r}_2\alpha_V[\rho(\boldsymbol{r}_1)]\left[\bar{\psi}(\boldsymbol{r}_1)\gamma_\mu\psi(\boldsymbol{r}_1)\right]\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\left[\bar{\psi}(\boldsymbol{r}_2)\gamma^\mu\psi(\boldsymbol{r}_2)\right]. \qquad (73)
\end{aligned}$$

Note that

$$\gamma^\mu = \{\beta, \beta\vec{\alpha}\} = \left\{ \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \right\}, \quad \gamma_\mu = \left\{ \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, -\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \right\}. \qquad (74)$$

It is also useful to note this formula:

$$\begin{aligned}
\bar{A}_1\gamma_\mu(1)C_1\cdot\bar{B}_2\gamma^\mu(2)D_2 &= A^\dagger{}_1C_1\cdot B^\dagger{}_2D_2 \\
&\quad -A^\dagger{}_1\begin{pmatrix} 0 & \vec{\sigma}(1) \\ -\vec{\sigma}(1) & 0 \end{pmatrix}C_1\cdot B^\dagger{}_2\begin{pmatrix} 0 & \vec{\sigma}(2) \\ -\vec{\sigma}(2) & 0 \end{pmatrix}D_2, \qquad (75)
\end{aligned}$$

where we have utilized $\beta^2(i) = \gamma^0(i)\gamma_0(i) = \hat{1}(i)$.

5.1 time-space decomposition

The ME of $V_{int.}$ via basis states $\langle 12| = \langle ab|$ and $|12\rangle = |cd\rangle$ is decomposed into "timelike" and "spacelike" terms. That is,

$$\langle ab \mid V_{int.} \mid cd\rangle = \langle ab \mid V_{int.} \mid cd\rangle_T - \langle ab \mid V_{int.} \mid cd\rangle_S, \qquad (76)$$

where

$$\langle ab \mid V_{int.} \mid cd\rangle_T = \int d\boldsymbol{r}_1\int d\boldsymbol{r}_2\frac{\alpha_V(\rho(\boldsymbol{r}_1))}{2}\psi_a^\dagger(x_1)\psi_c(x_1)\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\psi_b^\dagger(x_2)\psi_d(x_2), \qquad (77)$$

and

$$\begin{aligned}
\langle ab \mid V_{int.} \mid cd\rangle_S &= \int d\boldsymbol{r}_1\int d\boldsymbol{r}_2\frac{\alpha_V(\rho(\boldsymbol{r}_1))}{2}\delta(\boldsymbol{r}_1-\boldsymbol{r}_2) \\
&\quad \psi_a^\dagger(x_1)\begin{pmatrix} 0 & \vec{\sigma}(1) \\ -\vec{\sigma}(1) & 0 \end{pmatrix}\psi_c(x_1)\cdot\psi_b^\dagger(x_2)\begin{pmatrix} 0 & \vec{\sigma}(2) \\ -\vec{\sigma}(2) & 0 \end{pmatrix}\psi_d(x_2). \qquad (78)
\end{aligned}$$

5.2 timelike term

By utilizing the formula (62), the timelike ME, $\langle ab \mid V_{int.} \mid cd\rangle_T$, can be decomposed into the radial-integration and angle-integration parts. That is,

$$\begin{aligned}
\langle ab \mid V_{int.} \mid cd\rangle_T &= R^{ff,ff}_{ab,cd}\cdot S_{[ljm]_a[ljm]_b,[ljm]_c[ljm]_d} + R^{gf,gf}_{ab,cd}\cdot S_{[\ell jm]_a[ljm]_b,[\ell jm]_c[ljm]_d} \\
&\quad + R^{fg,fg}_{ab,cd}\cdot S_{[ljm]_a[\ell jm]_b,[ljm]_c[\ell jm]_d} + R^{gg,gg}_{ab,cd}\cdot S_{[\ell jm]_a[\ell jm]_b,[\ell jm]_c[\ell jm]_d}. \qquad (79)
\end{aligned}$$

Here the radial-integration part reads the same as EQ. (66), but replacing $\alpha_S \to \alpha_V$, where we assumed that $\rho(r)$ is spherical again. On the other side, the angle-integration part, $S_{[ljm]_a[ljm]_b,[ljm]_c[ljm]_d}$ etc., can be idential to EQ. (67).

5.3 spacelike term

First we note that

$$\begin{aligned}
\psi_a^\dagger(x_1)\begin{pmatrix} 0 & \vec{\sigma}(1) \\ -\vec{\sigma}(1) & 0 \end{pmatrix}\psi_c(x_1) &= -iF_a^*\vec{\sigma}(1)G_c(x_1) - G_a^*\vec{\sigma}(1)iF_c(x_1) \equiv \vec{P}_{a,c}, \\
\psi_b^\dagger(x_2)\begin{pmatrix} 0 & \vec{\sigma}(2) \\ -\vec{\sigma}(2) & 0 \end{pmatrix}\psi_d(x_2) &= -iF_b^*\vec{\sigma}(2)G_d(x_2) - G_b^*\vec{\sigma}(2)iF_d(x_2) \equiv \vec{P}_{b,d},
\end{aligned}$$

and thus, their product is represented as

$$\begin{aligned}
\vec{P}_{a,c}\cdot\vec{P}_{b,d} &= F_a^*(1)F_b^*(2)\hat{T}(12)G_c(1)G_d(2) + F_a^*(1)G_b^*(2)\hat{T}(12)G_c(1)F_d(2) \\
&+ G_a^*(1)F_b^*(2)\hat{T}(12)F_c(1)G_d(2) + G_a^*(1)G_b^*(2)\hat{T}(12)F_c(1)F_d(2), \quad \text{with} \qquad (80) \\
\hat{T}(12) &= \vec{\sigma}(1)\cdot\vec{\sigma}(2).
\end{aligned}$$

Thus, for the evaluation of $\langle ab \mid V_{int.} \mid cd\rangle_S$, one must compute the integration of these four terms multiplied by $\alpha_V[\rho]\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)/2$. Each term can be again decomposed into the radial and angular parts. That is,

$$\begin{aligned}
\langle ab \mid V_{int.} \mid cd\rangle_S &= R^{ff,gg}_{ab,cd}\cdot T_{[ljm]_a[ljm]_b,[\ell jm]_c[\ell jm]_d} + R^{fg,gf}_{ab,cd}\cdot T_{[ljm]_a[\ell jm]_b,[\ell jm]_c[ljm]_d} \\
&+R^{gf,fg}_{ab,cd}\cdot T_{[\ell jm]_a[ljm]_b,[ljm]_c[\ell jm]_d} + R^{gg,ff}_{ab,cd}\cdot T_{[\ell jm]_a[\ell jm]_b,[ljm]_c[ljm]_d}. \qquad (81)
\end{aligned}$$

Here the radial part can be the same to EQ. (66), but for different sets of f and g. The angular part, $T_{[pjm]_1[pjm]_2,[pjm]_3[pjm]_4}$, reads

$$\begin{aligned}
T_{[pjm]_1[pjm]_2,[pjm]_3[pjm]_4} &\equiv \int d\bar{\boldsymbol{r}}_1\int d\bar{\boldsymbol{r}}_2\mathcal{Y}^*_{(p_1)j_1m_1}(\bar{\boldsymbol{r}}_1)\mathcal{Y}^*_{(p_2)j_2m_2}(\bar{\boldsymbol{r}}_2) \\
&\quad\left[\sum_\lambda\sum_\varsigma Y_{\lambda,\varsigma}(\bar{\boldsymbol{r}}_1)Y^*_{\lambda,\varsigma}(\bar{\boldsymbol{r}}_2)\right]\hat{T}(12)\mathcal{Y}_{(p_3)j_3m_3}(\bar{\boldsymbol{r}}_1)\mathcal{Y}_{(p_4)j_4m_4}(\bar{\boldsymbol{r}}_2), \\
&= \sum_\lambda\sum_\varsigma\left\langle\mathcal{Y}_{[pjm]_1}\mid Y_{\lambda,\varsigma}\vec{\sigma}\mid\mathcal{Y}_{[pjm]_3}\right\rangle\cdot\left\langle\mathcal{Y}_{[pjm]_2}\mid Y^*_{\lambda,\varsigma}\vec{\sigma}\mid\mathcal{Y}_{[pjm]_4}\right\rangle \\
&= \sum_\lambda\sum_\varsigma\sum_{v=0,\pm1}\left\langle\mathcal{Y}_{[pjm]_1}\mid Y_{\lambda,\varsigma}\sigma_v\mid\mathcal{Y}_{[pjm]_3}\right\rangle\left\langle\mathcal{Y}_{[.]_2}\mid (Y_{\lambda,\varsigma}\sigma_v)^\dagger\mid\mathcal{Y}_{[.]_4}\right\rangle \\
&= \sum_{\lambda,\varsigma,v}\left\langle\mathcal{Y}_{[pjm]_1}\mid Y_{\lambda,\varsigma}\sigma_v\mid\mathcal{Y}_{[pjm]_3}\right\rangle\left\langle\mathcal{Y}_{[pjm]_4}\mid Y_{\lambda,\varsigma}\sigma_v\mid\mathcal{Y}_{[pjm]_2}\right\rangle^*, \qquad (82)
\end{aligned}$$

where we have utilized $\vec{\sigma}\cdot\vec{\sigma} = \sum_v \sigma_v\sigma_{-v}$ and $\sigma_v^\dagger = \sigma_{-v}$. In addition, by utilizing

$$[Y_\lambda\otimes\sigma]^{(T,K)} = \sum_{\varsigma',v'}\mathcal{C}^{(T,K)\lambda,1}_{\varsigma',v'}Y_{\lambda,\varsigma'}\sigma_{v'} \quad\Leftrightarrow\quad Y_{\lambda,\varsigma}\sigma_v = \sum_{T,K}\mathcal{C}^{(T,K)\lambda,1*}_{\varsigma,v}[Y_\lambda\otimes\sigma]^{(T,K)}, \qquad (83)$$

this angular part is represented as

$$\begin{aligned}
T_{[pjm]_1[pjm]_2,[pjm]_3[pjm]_4} &= \sum_{\lambda,\varsigma,v}\left\{\sum_{T_1,K_1}\mathcal{C}^{(T_1,K_1)\lambda,1*}_{\varsigma,v}\left\langle\mathcal{Y}_{[pjm]_1}\mid[Y_\lambda\otimes\sigma]^{(T_1,K_1)}\mid\mathcal{Y}_{[pjm]_3}\right\rangle\right\} \\
&\quad\left\{\sum_{T_2,K_2}\mathcal{C}^{(T_2,K_2)\lambda,1*}_{\varsigma,v}\left\langle\mathcal{Y}_{[pjm]_4}\mid[Y_\lambda\otimes\sigma]^{(T_2,K_2)}\mid\mathcal{Y}_{[pjm]_2}\right\rangle\right\}^*. \qquad (84)
\end{aligned}$$

For the ME $\left\langle \mathcal{Y}_{[pjm]_1} \mid [Y_\lambda \otimes \sigma]^{(T_1,K_1)} \mid \mathcal{Y}_{[pjm]_3} \right\rangle$, notice that

$$\begin{aligned}
\left\langle \mathcal{Y}_{[pjm]_1} \mid [Y_\lambda \otimes \sigma]^{(T_1,K_1)} \mid \mathcal{Y}_{[pjm]_3} \right\rangle &= \frac{(-)^{j_3-m_3}}{\sqrt{2T_1+1}} \mathcal{C}^{(T_1,K_1)j_1,j_3}_{m_1,-m_3} \left\langle \mathcal{Y}_{[pj]_1} \left\| [Y_\lambda \sigma]^{(T_1)} \right\| \mathcal{Y}_{[pj]_3} \right\rangle, \\
\left\langle \mathcal{Y}_{[pjm]_4} \mid [Y_\lambda \otimes \sigma]^{(T_2,K_2)} \mid \mathcal{Y}_{[pjm]_2} \right\rangle &= \frac{(-)^{j_2-m_2}}{\sqrt{2T_2+1}} \mathcal{C}^{(T_2,K_2)j_4,j_2}_{m_4,-m_2} \left\langle \mathcal{Y}_{[pj]_4} \left\| [Y_\lambda \sigma]^{(T_2)} \right\| \mathcal{Y}_{[pj]_2} \right\rangle. \quad (85)
\end{aligned}$$

5.4 angular-momentum projection of IS-V interaction

The timelike term of ME, $\langle ab \mid V_{int.} \mid cd \rangle_T$, can be easily projected to the well-defined (J, M) with the same technique in the IS-S case.

For the spacelike term, $\langle ab \mid V_{int.} \mid cd \rangle_S^{(J,M)}$, one must evaluate the angular part after the (J, M) projection. That is,

$$T^{(J,M)}_{[pj]_1[pj]_2,[pj]_3[pj]_4} = \sum_{(all\ m)} (-)^{j_3-m_3} \mathcal{C}^{(J,M)j_1,j_3}_{m_1,-m_3} \cdot (-)^{j_2-m_2} \mathcal{C}^{(J,M)j_4,j_2}_{m_4,-m_2} T_{[pjm]_1[pjm]_2,[pjm]_3[pjm]_4}. \quad (86)$$

By combining EQs. (84) and (85), that is

$$\begin{aligned}
T^{(J,M)}_{[pj]_1[pj]_2,[pj]_3[pj]_4} = &\sum_{(all\ m)} (-)^{j_3-m_3} \mathcal{C}^{(J,M)j_1,j_3}_{m_1,-m_3} \cdot (-)^{j_2-m_2} \mathcal{C}^{(J,M)j_4,j_2}_{m_4,-m_2} \\
&\sum_{\lambda,\zeta,v} \sum_{T_1,K_1} \mathcal{C}^{(T_1,K_1)\lambda,1*}_{\zeta,v} \frac{(-)^{j_3-m_3}}{\sqrt{2T_1+1}} \mathcal{C}^{(T_1,K_1)j_1,j_3}_{m_1,-m_3} \left\langle \mathcal{Y}_{[pj]_1} \left\| [Y_\lambda \sigma]^{(T_1)} \right\| \mathcal{Y}_{[pj]_3} \right\rangle \\
&\sum_{T_2,K_2} \mathcal{C}^{(T_2,K_2)\lambda,1}_{\zeta,v} \frac{(-)^{j_2-m_2}}{\sqrt{2T_2+1}} \mathcal{C}^{(T_2,K_2)j_4,j_2*}_{m_4,-m_2} \left\langle \mathcal{Y}_{[pj]_4} \left\| [Y_\lambda \sigma]^{(T_2)} \right\| \mathcal{Y}_{[pj]_2} \right\rangle^*. \quad (87)
\end{aligned}$$

Since $\sum_{m_1,m_3} \mathcal{C}^{(J,M)j_1,j_3}_{m_1,-m_3} \mathcal{C}^{(T_1,K_1)j_1,j_3}_{m_1,-m_3} = \delta_{J,T_1}\delta_{M,K_1}$, this quantity can be reduced as

$$\begin{aligned}
T^{(J,M)}_{[pj]_1[pj]_2,[pj]_3[pj]_4} &= \sum_{\lambda,\zeta,v} \sum_{T_1,K_1} \delta_{J,T_1}\delta_{M,K_1} \frac{1}{\sqrt{2T_1+1}} \sum_{T_2,K_2} \delta_{J,T_2}\delta_{M,K_2} \frac{1}{\sqrt{2T_2+1}} \\
&\quad \mathcal{C}^{(T_1,K_1)\lambda,1*}_{\zeta,v} \left\langle \mathcal{Y}_{[pj]_1} \left\| [Y_\lambda \sigma]^{(T_1)} \right\| \mathcal{Y}_{[pj]_3} \right\rangle \mathcal{C}^{(T_2,K_2)\lambda,1}_{\zeta,v} \left\langle \mathcal{Y}_{[pj]_4} \left\| [Y_\lambda \sigma]^{(T_2)} \right\| \mathcal{Y}_{[pj]_2} \right\rangle^* \\
&= \frac{1}{2J+1} \sum_{\lambda} \sum_{\zeta,v} \left| \mathcal{C}^{(J,M)\lambda,1}_{\zeta,v} \right|^2 \left\langle \mathcal{Y}_{[pj]_1} \left\| [Y_\lambda \sigma]^{(J)} \right\| \mathcal{Y}_{[pj]_3} \right\rangle \left\langle \mathcal{Y}_{[pj]_4} \left\| [Y_\lambda \sigma]^{(J)} \right\| \mathcal{Y}_{[pj]_2} \right\rangle^* \\
&= \frac{1}{2J+1} \sum_{\lambda=|J-1|}^{J+1} \left\langle \mathcal{Y}_{[pj]_1} \left\| [Y_\lambda \sigma]^{(J)} \right\| \mathcal{Y}_{[pj]_3} \right\rangle \left\langle \mathcal{Y}_{[pj]_4} \left\| [Y_\lambda \sigma]^{(J)} \right\| \mathcal{Y}_{[pj]_2} \right\rangle^*, \quad (88)
\end{aligned}$$

where it can be independent of M. Remember that $p = l$ or ℓ. As shown in EQ. (J.20) in [2],

$$\begin{aligned}
\left\langle \mathcal{Y}_{[pj]_4} \left\| [Y_\lambda \sigma]^{(J)} \right\| \mathcal{Y}_{[pj]_2} \right\rangle^* &= \left\langle \mathcal{Y}_{[pj]_2} \left\| \left\{ [Y_\lambda \sigma]^{(J)} \right\}^\dagger \right\| \mathcal{Y}_{[pj]_4} \right\rangle = \left\langle \mathcal{Y}_{[pj]_2} \left\| [Y_\lambda \sigma]^{(J)} \right\| \mathcal{Y}_{[pj]_4} \right\rangle \\
&= (-)^{j_2+j_4+J+\lambda} \left\langle \mathcal{Y}_{[pj]_4} \left\| [Y_\lambda \sigma]^{(J)} \right\| \mathcal{Y}_{[pj]_2} \right\rangle, \quad (89)
\end{aligned}$$

where we also utilized that, for the DBME, $\langle f \| A^\dagger \| i \rangle = \langle f \| A \| i \rangle$.

Final result reads

$$\begin{aligned}
\langle ab \mid V_{int.} \mid cd \rangle_S^{(J,M)} &= R^{ff,gg}_{ab,cd} \cdot T^{(J,M)}_{[lj]_a[lj]_b,[\ell j]_c[\ell j]_d} + R^{fg,gf}_{ab,cd} \cdot T^{(J,M)}_{[lj]_a[\ell j]_b,[\ell j]_c[lj]_d} \\
&+ R^{gf,fg}_{ab,cd} \cdot T^{(J,M)}_{[\ell j]_a[lj]_b,[lj]_c[\ell j]_d} + R^{gg,ff}_{ab,cd} \cdot T^{(J,M)}_{[\ell j]_a[\ell j]_b,[lj]_c[lj]_d}. \quad (90)
\end{aligned}$$

6 Isoscalar-pseudovector interaction

IS-PV interaction is given by the following Lagrangian density.

$$\begin{aligned}
\mathcal{L}_{int.}(\boldsymbol{r}) &= \frac{-\alpha_{\mathrm{IS-PV}}}{2}\left[\bar{\psi}(\boldsymbol{r})\gamma_5\gamma_\mu\psi(\boldsymbol{r})\right]\left[\bar{\psi}(\boldsymbol{r})\gamma_5\gamma^\mu\psi(\boldsymbol{r})\right], \\
V_{int.} &= \int d\boldsymbol{r}\mathcal{H}_{int.}(\boldsymbol{r}) = -\int d\boldsymbol{r}\mathcal{L}_{int.}(\boldsymbol{r}) \\
&= \frac{\alpha_{\mathrm{IS-PV}}}{2}\int d\boldsymbol{r}\left[\bar{\psi}(\boldsymbol{r})\gamma_5\gamma_\mu\psi(\boldsymbol{r})\right]\left[\bar{\psi}(\boldsymbol{r})\gamma_5\gamma^\mu\psi(\boldsymbol{r})\right] \\
&= \frac{\alpha_{\mathrm{IS-PV}}}{2}\int d\boldsymbol{r}_1\int d\boldsymbol{r}_2\left[\psi^\dagger(\boldsymbol{r}_1)\gamma^0\gamma_5\gamma_\mu\psi(\boldsymbol{r}_1)\right]\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\left[\psi^\dagger(\boldsymbol{r}_2)\gamma^0\gamma_5\gamma^\mu\psi(\boldsymbol{r}_2)\right]. \quad (91)
\end{aligned}$$

Note that the coupling coefficient must keep that $\alpha_{\mathrm{IS-PV}} > 0$ for the consistency with the one-pion-exchange model. Note also that,

$$\begin{aligned}
&\gamma^0\gamma_5\gamma^\mu = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\{\beta, \beta\vec{\alpha}\} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\left\{\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}\right\} \\
&= \left\{\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}\right\} = \left\{-\gamma_5, -\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}\right\}, \\
&\gamma^0\gamma_5\gamma_\mu = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\{\beta, -\beta\vec{\alpha}\} = \left\{-\gamma_5, \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}\right\}. \quad (92)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&V_{int.} = \frac{\alpha_{\mathrm{IS-PV}}}{2}\int d\boldsymbol{r}_1\int d\boldsymbol{r}_2\psi^\dagger(\boldsymbol{r}_1)\psi^\dagger(\boldsymbol{r}_2)\left[\gamma_5(1)\cdot\gamma_5(2)-\omega(1,2)\right]\psi(\boldsymbol{r}_1)\psi(\boldsymbol{r}_2)\delta(\boldsymbol{r}_1-\boldsymbol{r}_2), \\
&\omega(1,2) = \begin{pmatrix} \boldsymbol{\sigma}(1)\cdot\boldsymbol{\sigma}(2) & 0 \\ 0 & \boldsymbol{\sigma}(1)\cdot\boldsymbol{\sigma}(2) \end{pmatrix}. \quad (93)
\end{aligned}$$

7 Isovector-pseudovector interaction

IV-PV interaction is given by the following Lagrangian density.

$$\mathcal{L}_{int.}(\boldsymbol{r}) = \frac{-\alpha_{\mathrm{IV-PV}}}{2}\sum_{a=1}^{3}\left[\bar{\psi}(\boldsymbol{r})\hat{\tau}_a\gamma_5\gamma_\mu\psi(\boldsymbol{r})\right]\left[\bar{\psi}(\boldsymbol{r})\hat{\tau}_a\gamma_5\gamma^\mu\psi(\boldsymbol{r})\right], \quad (94)$$

where $\hat{\tau}_a$ is the isospin operator. The corresponding Hamiltonian (interaction) reads

$$\begin{aligned}
V_{int.} &= \int d\boldsymbol{r}\mathcal{H}_{int.}(\boldsymbol{r}) = -\int d\boldsymbol{r}\mathcal{L}_{int.}(\boldsymbol{r}) \\
&= \frac{\alpha_{\mathrm{IV-PV}}}{2}\int d\boldsymbol{r}\sum_{a=1}^{3}\left[\bar{\psi}(\boldsymbol{r})\hat{\tau}_a\gamma_5\gamma_\mu\psi(\boldsymbol{r})\right]\left[\bar{\psi}(\boldsymbol{r})\hat{\tau}_a\gamma_5\gamma^\mu\psi(\boldsymbol{r})\right] \\
&= \frac{\alpha_{\mathrm{IV-PV}}}{2}\int d\boldsymbol{r}_1\int d\boldsymbol{r}_2\sum_{a=1}^{3}\left[\bar{\psi}(\boldsymbol{r}_1)\hat{\tau}_a\gamma_5\gamma_\mu\psi(\boldsymbol{r}_1)\right]\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\left[\bar{\psi}(\boldsymbol{r}_2)\hat{\tau}_a\gamma_5\gamma^\mu\psi(\boldsymbol{r}_2)\right]. \quad (95)
\end{aligned}$$

Note that the coupling coefficient must keep that $\alpha_{\mathrm{IV-PV}} > 0$ for the consistency with the one-pion-exchange model. By using the formulas,

$$\begin{aligned}
\gamma^0\gamma_5\gamma^\mu &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\{\beta, -\beta\vec{\alpha}\} = \left\{-\gamma_5, -\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}\right\}, \\
\gamma^0\gamma_5\gamma_\mu &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\{\beta, \beta\vec{\alpha}\} = \left\{-\gamma_5, \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}\right\}, \quad (96)
\end{aligned}$$

the IV-PV interaction can be represented as

$$V_{int.} = \frac{\alpha_{\mathrm{IS-PV}}}{2}\int d\boldsymbol{r}_1 \int d\boldsymbol{r}_2 \psi^\dagger(\boldsymbol{r}_1)\psi^\dagger(\boldsymbol{r}_2)\left[\gamma_5(1)\cdot\gamma_5(2) - \omega(1,2)\right]\vec{\tau}(1)\cdot\vec{\tau}(2)\psi(\boldsymbol{r}_1)\psi(\boldsymbol{r}_2)\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2),$$
$$\omega(1,2) = \begin{pmatrix} \boldsymbol{\sigma}(1)\cdot\boldsymbol{\sigma}(2) & 0 \\ 0 & \boldsymbol{\sigma}(1)\cdot\boldsymbol{\sigma}(2) \end{pmatrix}. \tag{97}$$

8 appendix for movie

Schroedinger equation for electrons in the atom:

$$\hat{H}_0 |\psi_n\rangle = E_n |\psi_n\rangle$$
$$\text{where} \;\; E_n = -\mu c^2 \frac{Z^2\alpha^2}{2n^2}, \quad \alpha = \frac{e^2}{4\pi\epsilon_0 \cdot \hbar c} \cong \frac{1}{137}, \quad \mu = \frac{m_e m_{\text{Nucl.}}}{m_e + m_{\text{Nucl.}}} \cong m_e \; . \tag{98}$$

$$\hat{H}_0 = -\frac{\hbar^2}{2\mu}\nabla^2 + V(\boldsymbol{r}) \tag{99}$$

Other equations:

$$\hat{H}_0 \longrightarrow \hat{H} = \hat{H}_0 + \hat{H}_{\text{LS}} + \hat{H}_{\text{Darwin}} + \hat{T}_{\text{Rela}} \tag{100}$$

$$\begin{aligned} &\hat{H}_{\text{LS}} \cdots \text{spin}-\text{orbit coupling} \\ &\hat{H}_{\text{Darwin}} \cdots \text{Darwin term} \\ &\hat{T}_{\text{Rela}} \cdots \text{relativistic kinetic correction} \end{aligned} \tag{101}$$

$$\hat{H}_{\text{LS}} = X(\boldsymbol{r})\vec{l}\cdot\vec{s} \quad \text{with} \quad \vec{l}\cdot\vec{s} = \frac{\vec{j}^2 - \vec{s}^2 - \vec{l}^2}{2} \tag{102}$$

$$V(\boldsymbol{r}) = V_{\text{Central}}(\boldsymbol{r}) + V_{\text{LS}}(\boldsymbol{r})\vec{l}\cdot\vec{s} \tag{103}$$

$$\hat{H}\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \;, \text{ where}$$
$$\hat{H} = -\frac{\hbar^2}{2\mu}\nabla^2 + V(\boldsymbol{r}) + \hat{H}_{\text{LS}} + \hat{H}_{\text{Darwin}} + \hat{T}_{\text{Rela}} \; . \tag{104}$$

Time-independent Dirac equation:

$$\left[-i\hbar c\beta\vec{\gamma}\cdot\vec{\nabla} + \beta Mc^2 + V(\boldsymbol{r})\right]\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \; . \tag{105}$$

$$\psi(t,\boldsymbol{r}) = \exp\left[-it\frac{E}{\hbar}\right]\psi(\boldsymbol{r}) \; . \tag{106}$$

References

[1] T. Oishi, Doctoral Thesis (2014).

[2] M. Serra, Doctoral Thesis (2001).

[3] “Density Functional Theory”, E. Engel and R. M. Dreizler.

[4] “The Nuclear Many-Body Problems”, P. Ring and P. Schuck.