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1 Problem of Combat

We start from the linear equation of Z-force combat:

$$\partial_t \begin{pmatrix} n_1(t) \\ \vdots \\ n_Z(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & \cdots & a_{Z1}(t) \\ \vdots & & \vdots \\ a_{1Z}(t) & \cdots & a_{ZZ}(t) \end{pmatrix} \begin{pmatrix} n_1(t) \\ \vdots \\ n_Z(t) \end{pmatrix} \equiv \hat{\mathcal{A}}(t) \cdot \vec{N}(t). \tag{1}$$

Here $\hat{\mathcal{A}}$ and $\vec{N}$ are generally time-dependent. $n_i(t)$ is the man-power of i-th force in time. There is not an analytic solution in general, and one needs computational (numerical) simulations.

If the combat matrix, $\hat{\mathcal{A}}$, can be (i) time-independent ($\hat{\mathcal{A}} \bowtie t$), and (ii) also diagonalized, Eq.(1) is solved as follows.

$$\hat{\mathcal{A}} \longrightarrow \hat{\mathcal{U}}^{-1}\hat{\mathcal{A}}\hat{\mathcal{U}} \equiv \hat{\mathcal{B}} = \begin{pmatrix} b_{11} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & b_{ZZ} \end{pmatrix}. \tag{2}$$

Using this diagonalized matrix, we obtain,

$$\begin{aligned} & \partial_t \vec{N}(t) = \hat{\mathcal{A}} \cdot \vec{N}(t) \\ \Longrightarrow \quad & \partial_t \vec{X}(t) = \hat{\mathcal{B}} \cdot \vec{X}(t), \text{ where } \vec{X} \equiv \hat{\mathcal{U}}^{-1} \cdot \vec{N}, \qquad (3) \\ \Longrightarrow \quad & \dot{x}_i = b_{ii} x_i \\ & x_i(t) = e^{t b_{ii}} x_i(0). \qquad (4) \end{aligned}$$

Finally,

$$\vec{N}(t) = \hat{\mathcal{U}}\vec{X}(t). \tag{5}$$

Here, "solvability" is thanks to the conditions (i) and (ii). Notice that this statement is valid even when the combat matrix is complex. In that case, $\hat{\mathcal{A}}$ should be Hermitian (?), because $\{b_{ii}\}$ should be real.

1.1 Case with $Z = 2$ and $a_{11} = a_{22} = 0$

This case actually corresponds to *the second (square) law of Lanchester* [1]. In this case, we concern the combat of two forces, where

$$\hat{\mathcal{A}} = \begin{pmatrix} 0 & -b \\ -c & 0 \end{pmatrix} \longleftrightarrow \begin{cases} \dot{n}_1(t) = -b n_2(t), \\ \dot{n}_2(t) = -c n_1(t). \end{cases} \tag{6}$$

In the following, we assume that factors (b, c) are positive, corresponding to that man-powers $(n_1(t), n_2(t))$ only decrease in combat. If $b < c$, force-1 is stronger than force-2: damage per a unit of time (DPT) by force-2 $<$ DPT by force-1. For example, $(b, c) = (3, 5)$ means "one of force-1 kills 5 of force-2, while one of force-2 kills 3 of force-1".

The solution can be obtained analytically. Obviously the eigenvalues are $\lambda = \pm\sqrt{bc}$, and diagonalizing process yields,

$$\hat{\mathcal{U}} = \begin{pmatrix} 1 & 1 \\ \sqrt{c/b} & -\sqrt{c/b} \end{pmatrix}, \ \hat{\mathcal{U}}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{b/c} \\ 1 & -\sqrt{b/c} \end{pmatrix}, \tag{7}$$

$$\Longrightarrow \ \hat{\mathcal{B}} = \hat{\mathcal{U}}^{-1}\hat{\mathcal{A}}\hat{\mathcal{U}} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{b/c} \\ 1 & -\sqrt{b/c} \end{pmatrix}\begin{pmatrix} 0 & -b \\ -c & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \sqrt{c/b} & -\sqrt{c/b} \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{b/c} \\ 1 & -\sqrt{b/c} \end{pmatrix}\begin{pmatrix} -\sqrt{bc} & \sqrt{bc} \\ -c & -c \end{pmatrix} = \begin{pmatrix} -\sqrt{bc} & 0 \\ 0 & \sqrt{bc} \end{pmatrix}. \tag{8}$$

(Remember that $\hat{\mathcal{A}}$ was not definitely the orthogonal matrix.) Thus,

$$\partial_t \vec{X}(t) = \hat{\mathcal{B}}\vec{X}(t) \longrightarrow \vec{X}(t) = \begin{pmatrix} x_1(0)e^{-t\sqrt{bc}} \\ x_2(0)e^{t\sqrt{bc}} \end{pmatrix}. \tag{9}$$

Note that the initial values, $x_i(0)$, are given as

$$\vec{X}(0) \equiv \hat{\mathcal{U}}^{-1}\vec{N}(0) = \begin{cases} x_1(0) = \frac{1}{2}\left[n_1(0) + \sqrt{\frac{b}{c}}n_2(0)\right], \\ \\ x_2(0) = \frac{1}{2}\left[n_1(0) - \sqrt{\frac{b}{c}}n_2(0)\right], \end{cases} \tag{10}$$

where $n_1(0)$ and $n_2(0)$ should be positive. *(What negative number of soldiers means ??)* For later discussions, we emphasize that

$$-1 < \frac{x_2(0)}{x_1(0)} < 1. \tag{11}$$

Finally,

$$\vec{N}(t) = \hat{\mathcal{U}}\vec{X}(t) = \begin{cases} n_1(t) = & x_1(t) + x_2(t), \\ n_2(t) = & \sqrt{\frac{c}{b}}x_1(t) - \sqrt{\frac{c}{b}}x_2(t). \end{cases} \tag{12}$$

Eqs.(10) and (12) provides the core result of this problem. Also, we can find that $x_2(0)$ determines "win or lose". Details are given in the following.

1.1.1 condition to win or lose

Let's concern the "time to kill out". Under a certain condition determined with $(n_1(0), n_2(0))$ and (b, c), force-1 can win by killing all the soldiers of force-2, at certain time, t_{w1}. Namely,

$$\begin{aligned} n_2(t_{w1}) = \sqrt{\frac{c}{b}}\left[x_1(t_{w1}) - x_2(t_{w1})\right] &= 0, \\ x_1(t_{w1}) &= x_2(t_{w1}) \\ e^{-2t_{w1}\sqrt{bc}} &= \frac{x_2(0)}{x_1(0)} \\ t_{w1} &= \frac{-1}{2\sqrt{bc}}\ln\left(\frac{x_2(0)}{x_1(0)}\right). \end{aligned} \tag{13}$$

Thus, if t_{w1} can be positive finite, force-1 wins. This condition is trivially equivalent to that $0 < x_2(0)$. Remember also that $x_2(0) < x_1(0)$ is always satisfied. At $t = t_{w1}$, the number of remaining soldiers of force-1 is,

$$\begin{aligned} n_1(t_{w1}) = x_1(t_{w1}) + x_2(t_{w1}) &= x_1(0)e^{\frac{1}{2}\ln\left(\frac{x_2(0)}{x_1(0)}\right)} + x_2(0)e^{\frac{-1}{2}\ln\left(\frac{x_2(0)}{x_1(0)}\right)} \\ &= x_1(0)\sqrt{\frac{x_2(0)}{x_1(0)}} + x_2(0)\sqrt{\frac{x_1(0)}{x_2(0)}} = 2\sqrt{x_1(0)x_2(0)} \\ &= \sqrt{n_1(0)^2 - \left(\frac{b}{c}\right) n_2(0)^2}. \end{aligned} \tag{14}$$

In contrast, the condition of force-2 to win can be given as that $0 < t_{w2} < +\infty$, where

$$\begin{aligned} n_1(t_{w2}) = x_1(t_{w2}) + x_2(t_{w2}) &= 0, \\ &\vdots \\ t_{w2} &= \frac{-1}{2\sqrt{bc}} \ln\left(\frac{-x_2(0)}{x_1(0)}\right). \end{aligned} \tag{15}$$

Consequently, force-2 must keep $x_2(0) < 0$ to win.

From Eqs.(11), (13) and (15), we find that t_{w1} and t_{w2} are di-lemma quantities: if one is real, another should be imaginary. Namely, there are no possibilities of "win-win" case in this problem.

If $x_2(0) = +0$ or -0, as well as $(t_{w1}, t_{w2}) \longrightarrow (+\infty, imag)$ or $(imag, +\infty)$, respectively, the combat never ends. This happens when force-1 and 2 satisfy the par (Gokaku) condition, given as

$$x_2(0) = 0 \leftrightarrow n_1(0) = \sqrt{\frac{b}{c}} n_2(0). \tag{16}$$

In this case, $n_1(t)/n_2(t) = \sqrt{b/c} = const$ during the time-evolution.

1.1.2 case study

In the following, we describe some examples. For this purpose, we employ the software *gnuplot* and its script as follows.

```
# (d/dt) n1(t) = -b * n2(t)
# (d/dt) n2(t) = -c * n1(t)

#--- Input-1: initial numbers of soldiers (powers). Default=(a)
n1_0 = 5.0 ; n2_0 = 3.0

#--- Input-2: proficiency factors, (b,c). Default=(a)
p = 1.0
b = 1.0*p ; c = 1.0*p
#b = p*(n1_0/n2_0)**2 ; c = p ### factors for "never-ending combat".

#--- Results:
a = sqrt(b*c) ; d = sqrt(b/c)

f0   = (n1_0 + d*n2_0) * 0.5 ; g0   = (n1_0 - d*n2_0) * 0.5
f(x) = f0 * exp(-a*x)              ; g(x) = g0 * exp( a*x)
```

```
w1 = -0.5*log( g0/f0) / a ### Time when "n_2(w1)=0".
#w2 = -0.5*log(-g0/f0) / a ### Time when "n_1(w2)=0".

n1(x) =  f(x) + g(x)
n2(x) = (f(x) - g(x)) / d
s(x) = n1(x)-n2(x)
t(x) = n1(w1)
z(x) = 0

set size 0.6, 0.6
set xlabel "Time, t" ; set ylabel "Power"

xm = 1.3*w1 ; ym = f0*2.5
set arrow from w1,-2 to w1,n1(w1) nohead lt 2
set xtics (0, w1) ; set label 1 at w1,-0.5 "t_{w1}"

p[0:xm][-2:ym] \
n1(x) w lp lt 1 ti "n_1(t)", \
n2(x) w lp lt 3 ti "n_2(t)", \
s(x)  w lp lt 7 ti "n_1(t)-n_2(t)", \
t(x)  w l  lt 8 ti "n_1(t_{w1})", \
z(x)  w l  lt 0 ti ""

#pause -1
#unset label ; unset arrow #; reset
```

(See also the corresponding panel in Fig.1 for each case.)

- (a): case with $(n_1(0), n_2(0)) = (5, 3)$ and $(b, c) = (1, 1)$. This is a typical study of the second (square) law of Lanchester, and its result is well known as [1],

$$n_1(t_{w1}) = \sqrt{n_1(0)^2 - n_2(0)^2} = \sqrt{5^2 - 3^2} = 4. \tag{17}$$

  In Fig.1-(a), we plot $n_1(t)$ and $n_2(t)$. At $t_{w1} \simeq 0.7$, $n_1(t_{w1}) = 4$ is exactly reproduced.

- (b): $(n_1(0), n_2(0)) = (5, 3)$ and $(b, c) = (25/9, 1)$. Here we use $(b/c) = (n_1(0)/n_2(0))^2$ in order to satisfy Eq.(16). In Fig.1-(b), both powers decrease but never become zero.

- (c): $(n_1(0), n_2(0)) = (5, 5)$ and $(b, c) = (0.16, 0.25)$. In this case, at $t = 0$, force-1 and 2 have the equal numbers of soldiers. However, due to the different proficiency factors, force-1 eventually wins. Utilizing Eq.(14), the remaining force-1 soldiers at the end of combat is $\sqrt{5^2 - (\frac{0.16}{0.25})5^2} = 3$.

- (d): $(n_1(0), n_2(0)) = (4, 5)$ and $(b, c) = (0.3, 0.5)$. In this case, force-1 eventually wins even though it had the less soldiers than force-2 at $t = 0$. Again, the remaining force-1 soldiers at the end of combat is $\sqrt{4^2 - (\frac{0.3}{0.5})5^2} = 1$.

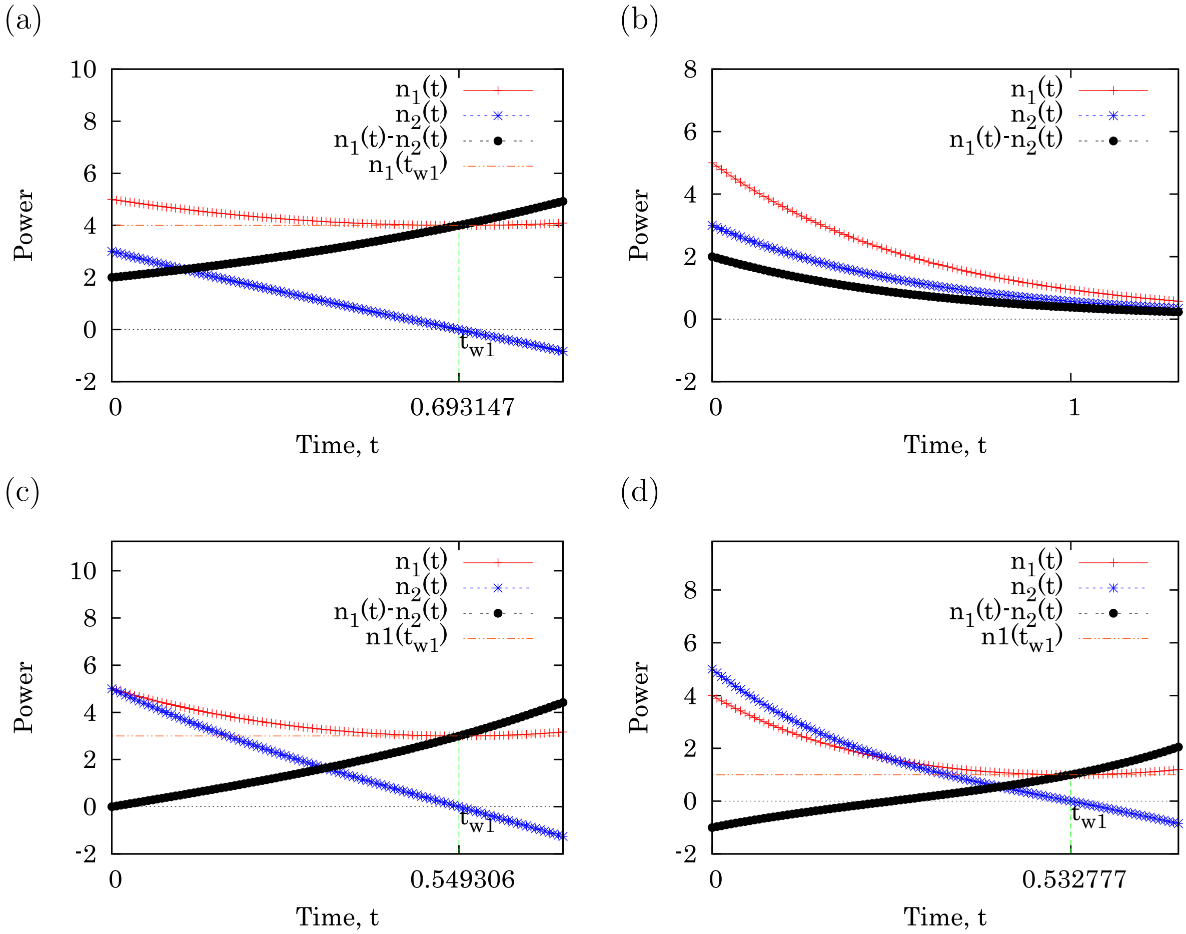

Figure 1: Plots for several sets of parameters, whose details are in text. Time and power are plotted in arbitrary units.

References

[1] Wikipedia "Lanchester's laws" (https://en.wikipedia.org/wiki/Lanchester's_laws).