Dual theories of vortex lattice

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Outline

- Vortex lattice in the lowest Landau level
- Non-commutative field theory of Tkachenko wave
- Symmetric tensor gauge theory
- Dual gravity theory of vortex lattice

Vortex lattice
In Lowest Landau Level limit

Two-fluid model

- Landau explained superfluid using the two-fluid model. **He-II** has local density $n = n_n + n_s$, the normal part has the velocity $\mathbf{v}_n$ and the superfluid part has the velocity $\mathbf{v}_s$
- In Landau's description the superfluid velocity is curl-free

$$\nabla \times \mathbf{v}_s = 0 \to \mathbf{v}_s = \frac{\hbar}{m_{\mathrm{He}}} \nabla \phi$$

- Because of the curl-free condition, Landau **believed** that if one rotates the superfluid, only the normal part can rotate with the velocity

$$\mathbf{v}_n = \mathbf{\Omega} \times \mathbf{r}, \quad \nabla \times \mathbf{v}_n = 2\mathbf{\Omega}$$

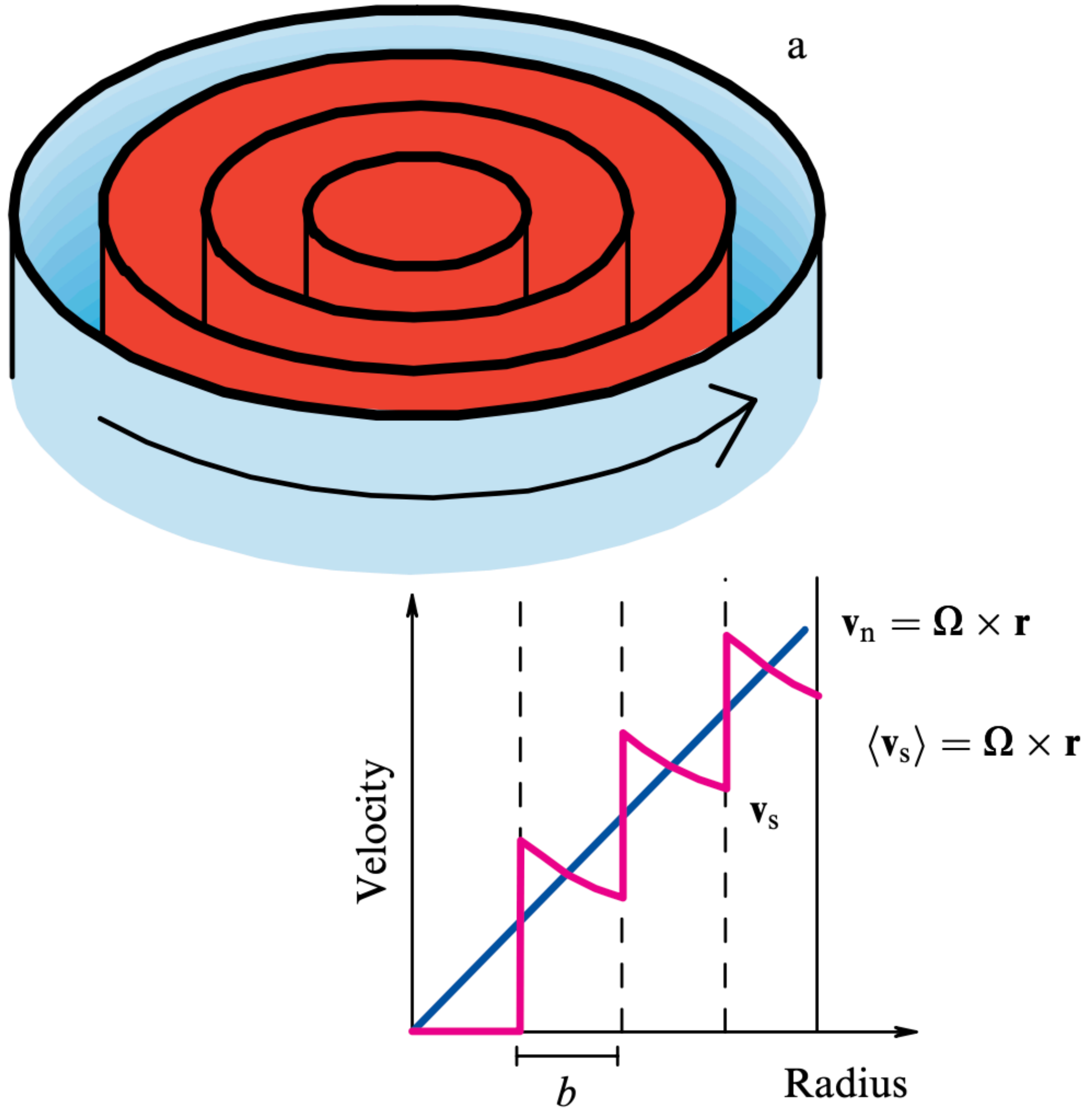
- The superfluid part can't rotate

Ω is the angular velocity

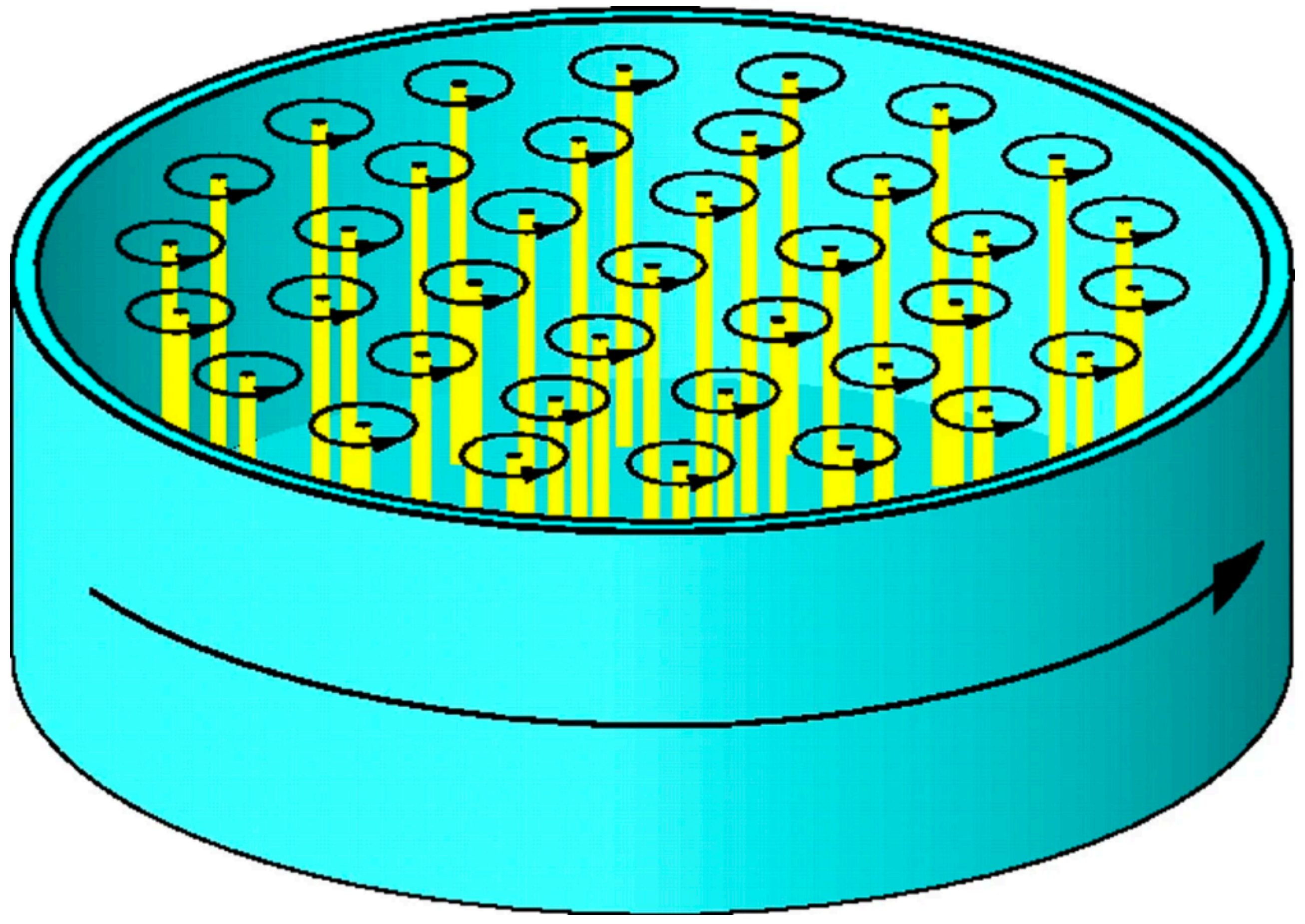
Vortex Lattice in a rotating superfluid

- Experiments demonstrated that the whole superfluid has to rotate to have the correct total angular momentum →The superfluid part rotates as the normal part → $\phi(\mathbf{x})$ (and $\mathbf{v}_s(\mathbf{x})$) has singularities

- **Landau and Lifshitz** proposed the layer-model, **Feynman** proposed the vortex lattice



Landau & Lifshitz 1955



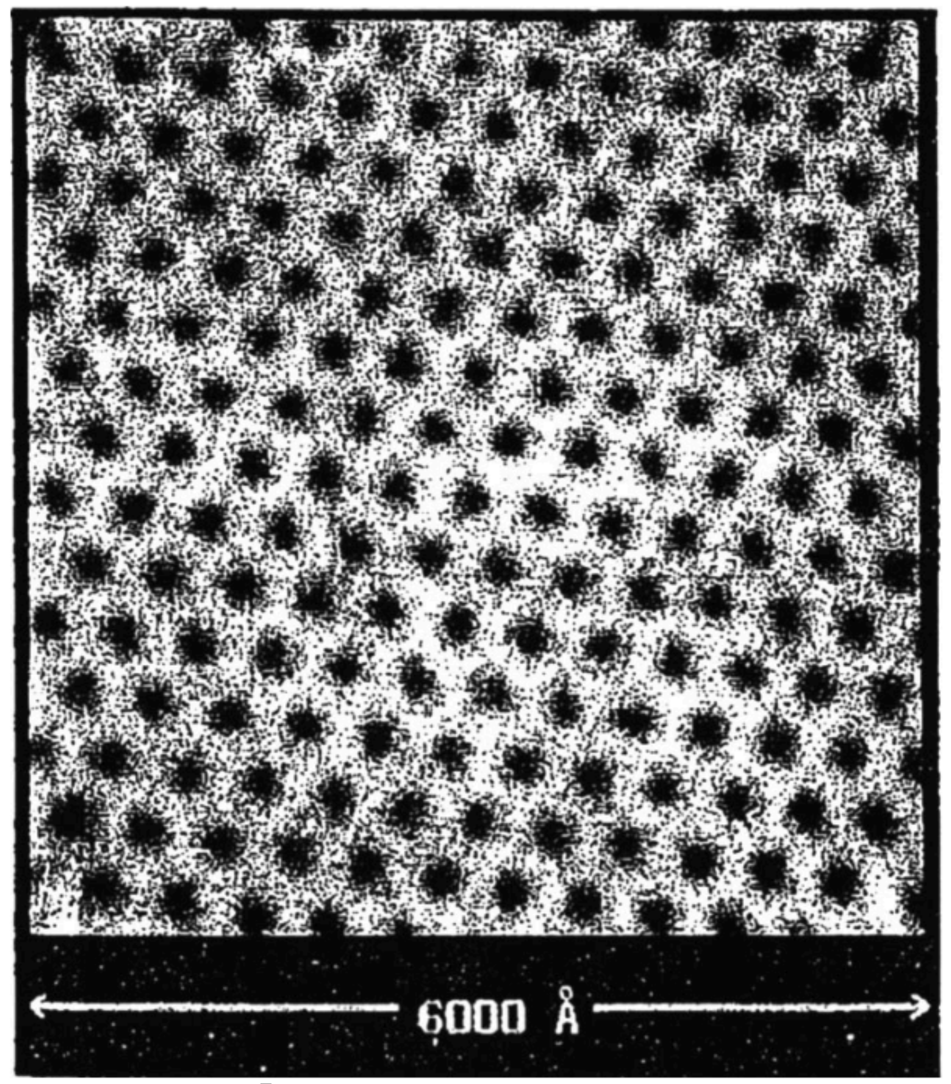
Circulation around a vortex line

$$\kappa = \oint d\mathbf{l} \cdot \mathbf{v}_s = \frac{\hbar}{m_{He}} \oint d\mathbf{l} \cdot \nabla \phi = \frac{h}{m_{He}}$$

Feynman 1955

Vortex lattice in superfluids and superconductors

Abrikosov vortex lattice in
type-II superconductor $NbSe_2$

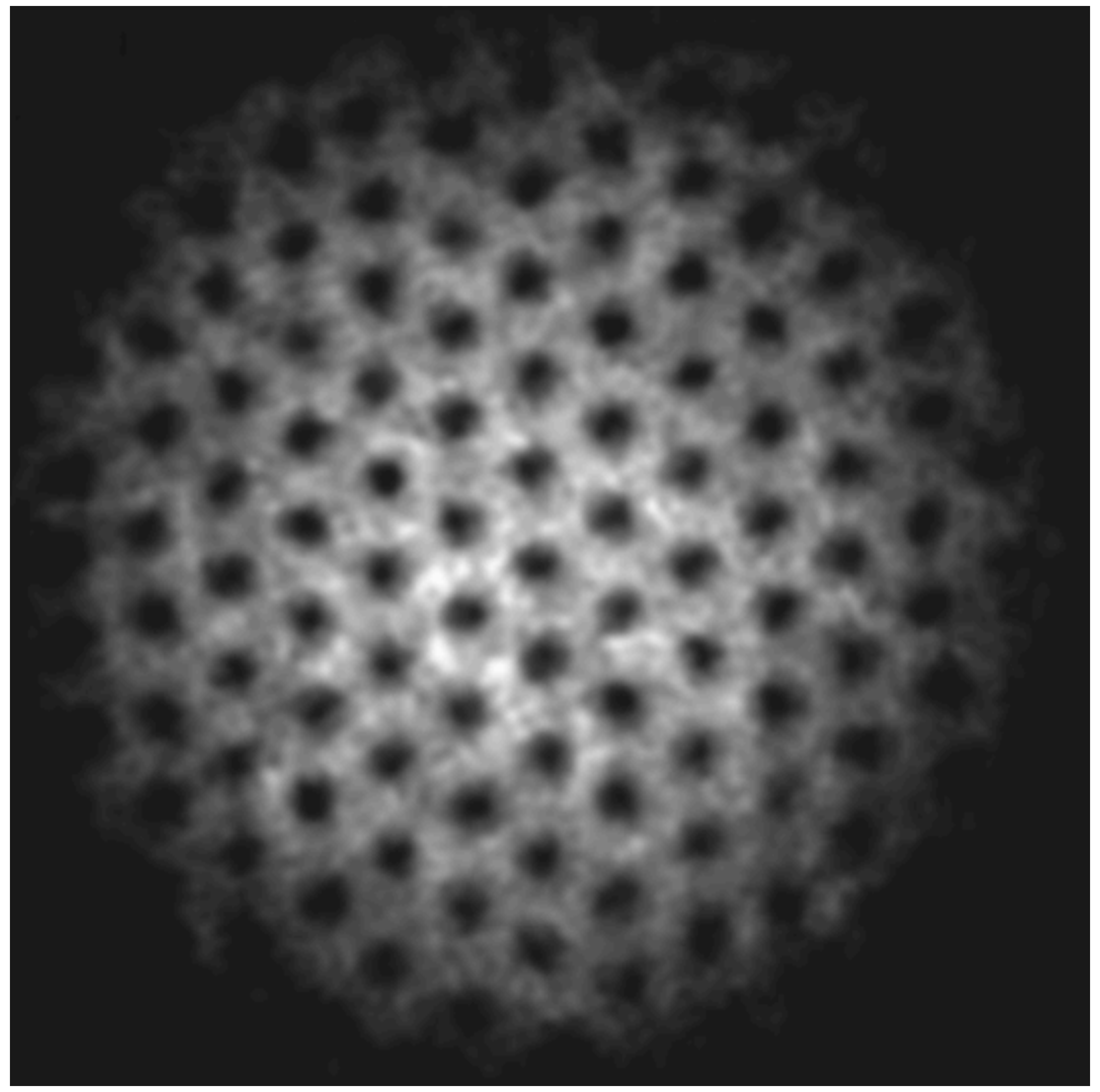


Superconductor in a magnetic field

$$\overrightarrow{F} = q\vec{v} \times \overrightarrow{B}$$

Lorentz Force

Vortex lattice in BEC of ^{87}Rb



Rotation of a neutral superfluid

$$\overrightarrow{F} = 2m\vec{v} \times \overrightarrow{\Omega}$$

Coriolis Force

- Vortex is the vorticity (singularity) of $\phi(\mathbf{r})$
- *Effective magnetic field* seen by superfluid $B = 2m\Omega$

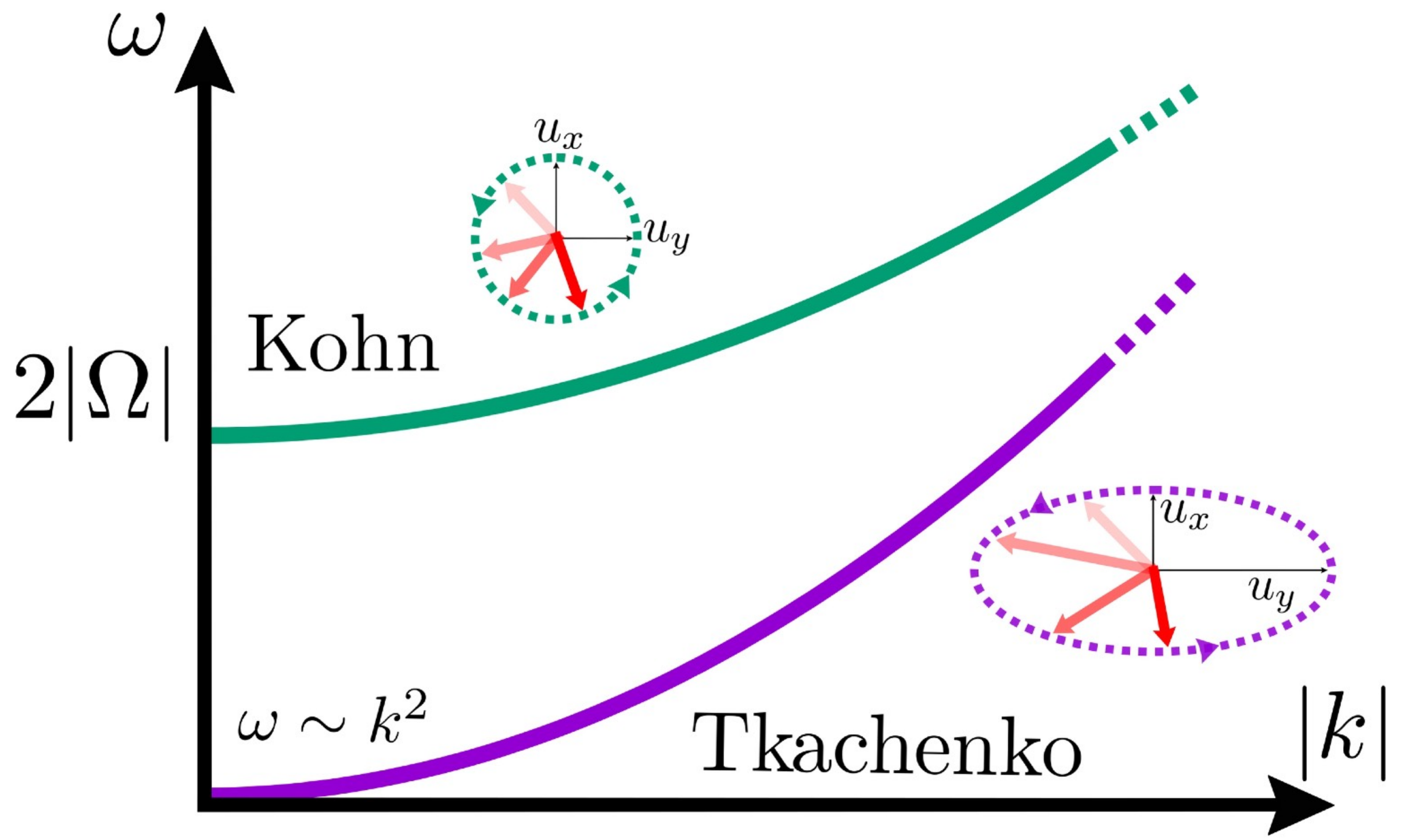
Tkachenko wave

- It was studied by Vladimir Tkachenko during 1965-1969
- **Tkachenko wave** was proposed as the vibration of the vortex lattice (phonon)
- It was observed in 2003 in BEC of ultracold atoms (^{87}Rb) Phys. Rev. Lett. 91, 100402 (2003)
- May play a role in the dynamics of Pulsars

RUDERMAN, M. *Long Period Oscillations in Rotating Neutron Stars*. ***Nature*** **225**, 619–620 **(1970).**

Strange behaviour of the Tkachenko wave

- It has quadratic dispersion $\omega = ck^2$ at small k
- Only 1 **polarisation**: transverse
- 1 **Nambu-Goldstone** boson for breaking of
  - 2 translations symmetry
  - 1 Rotation symmetry
  - $U(1)$ particle conservation (BEC)
- Landau cyclotron frequency $\omega_c = \dfrac{B}{m} = 2|\Omega|$
- Lowest Landau level (LLL) limit can be taken as $m \to 0$



Tkachenko wave as the shared Nambu Goldstone

- In the LLL, we ignore the Kohn mode, Tkachenko wave is the only low energy excitation
- Number operator $\hat{Q} = \int d^2\mathbf{x}\, \hat{n}(\mathbf{x})$ (ϵ^{ij} is Levi-Civita symbol)
- In the LLL ($m \to 0$) we ignore the first term of the momentum operator P^i

$$\hat{P}^i = \int d^2\mathbf{x} \left[m\,\hat{j}^i - \epsilon^{ij} x^j B\, \hat{n}(\mathbf{x}) \right] \to -\int d^2\mathbf{x}\; \epsilon^{ij} x^j B\, \hat{n}(\mathbf{x}) = \int d^2\mathbf{x}\; \hat{p}^i$$

- The **last term** is the contribution of *"Lorentz force"* to the variation of momentum

$$\dot{\hat{P}}^i = -\int d^2\mathbf{x}\; \epsilon^{ij} x^j B\, \dot{\hat{n}}(\mathbf{x}) \xrightarrow{(\dot{\hat{n}} = -\partial_i \hat{j}^i)} \int d^2\mathbf{x}\; \vec{\hat{j}} \times \vec{B}$$

- Total angular momentum

$$\hat{J} = \int d^2\mathbf{x}\, \epsilon^{ij} x^i \hat{p}^j = \int d^2\mathbf{x}\, B\, \mathbf{x}^2 \hat{n}(\mathbf{x})$$

All broken symmetry generators are dependent, we need only one NGB, which is the Tkachenko wave

Effective field theory

Boson-vortex duality

- Superfluid = dual photon
- In 2+1 D the conserved superfluid current $\partial_\mu j_s^\mu = 0$ can be satisfied as a Bianchi identity of the dual vector potential a_μ with the replacement

$$j_s^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$$

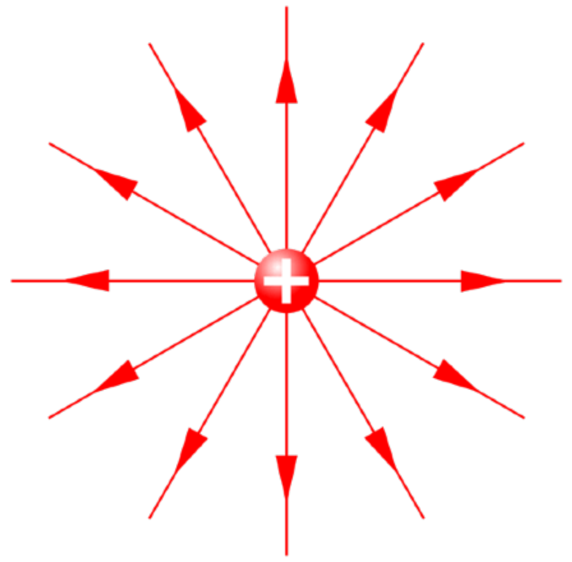
($\epsilon^{\mu\nu\lambda}$ is Levi-Civita symbol)

$$j_s^0 = n_s = \vec{\nabla} \times \vec{a} = b \qquad v_s^i = \frac{j_s^i}{n_s} = -\frac{\epsilon^{ij} e_j}{b} \qquad (e_i = \partial_t a_i - \partial_i a_0)$$

- In the **dual picture**, the vortex is a **dual point charge**
- Vortex lattice = lattice of charge particles couple with a dynamical dual gauge field a_μ

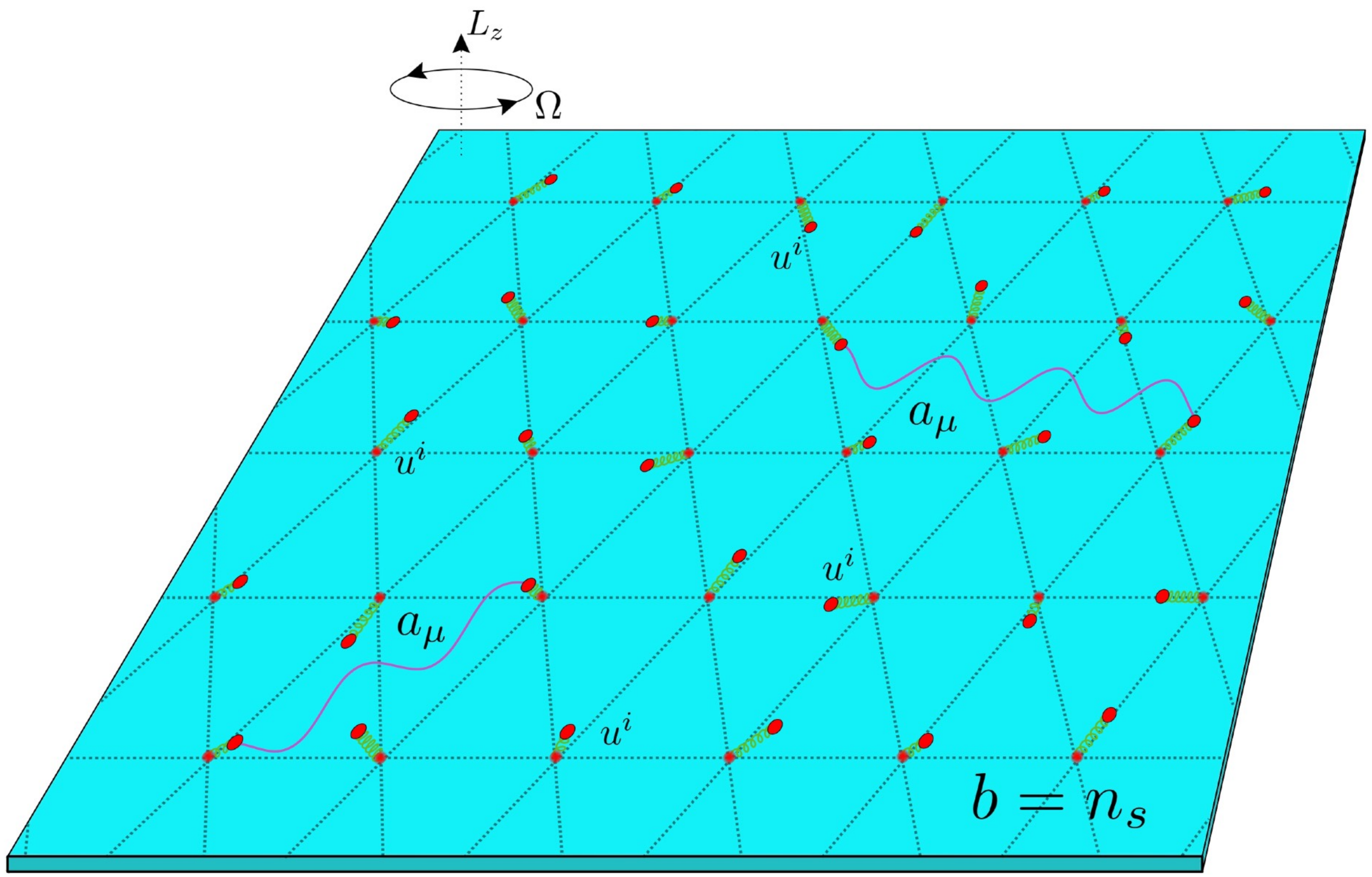
$$\text{Vortex at } \mathbf{x}_v \longrightarrow \nabla \times \mathbf{v}_s \sim \nabla \cdot \mathbf{e} = \delta(\mathbf{x} - \mathbf{x}_v)$$

(Gauss law)



Tkachenko as a special phonon

- u^i is the displacement (from the ground state positions) of vortices in the vortex lattice
- The movement of vortex (dual point charge) creates dual electromagnetic wave (dual photon)
- Tkachenko wave is the combination of the lattice vibration (phonon) and the dual photon
- Similar to polariton in photonics, which is the combination of photon and exciton



Linear theory

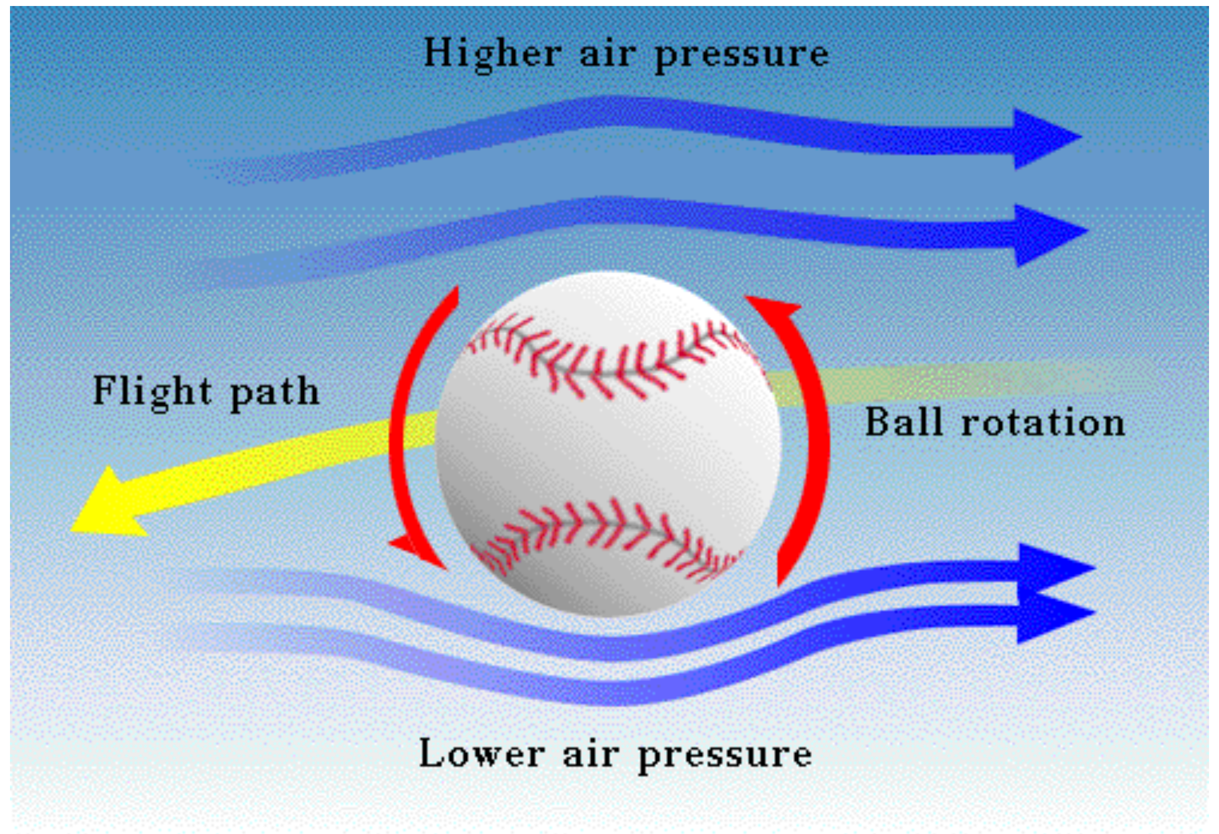
- The effective theory at linear order

$$\mathscr{L} = \frac{n_v}{2} n_s \epsilon^{ij} u_i \dot{u}_j + \dot{\mathbf{u}}^2 - G(\partial_i u_j + \partial_j u_i)^2 + \mathbf{u} \cdot \mathbf{e} + \frac{1}{2}\left(\frac{m}{\bar{b}} \mathbf{e}^2 + \lambda b^2 \right)$$

n_v is the average vortex density

Magnus force:
vortex moves in a background
Superfluid density n_s

Dual description of the superfluid dynamics
(Maxwell theory of a_μ)



Linear theory

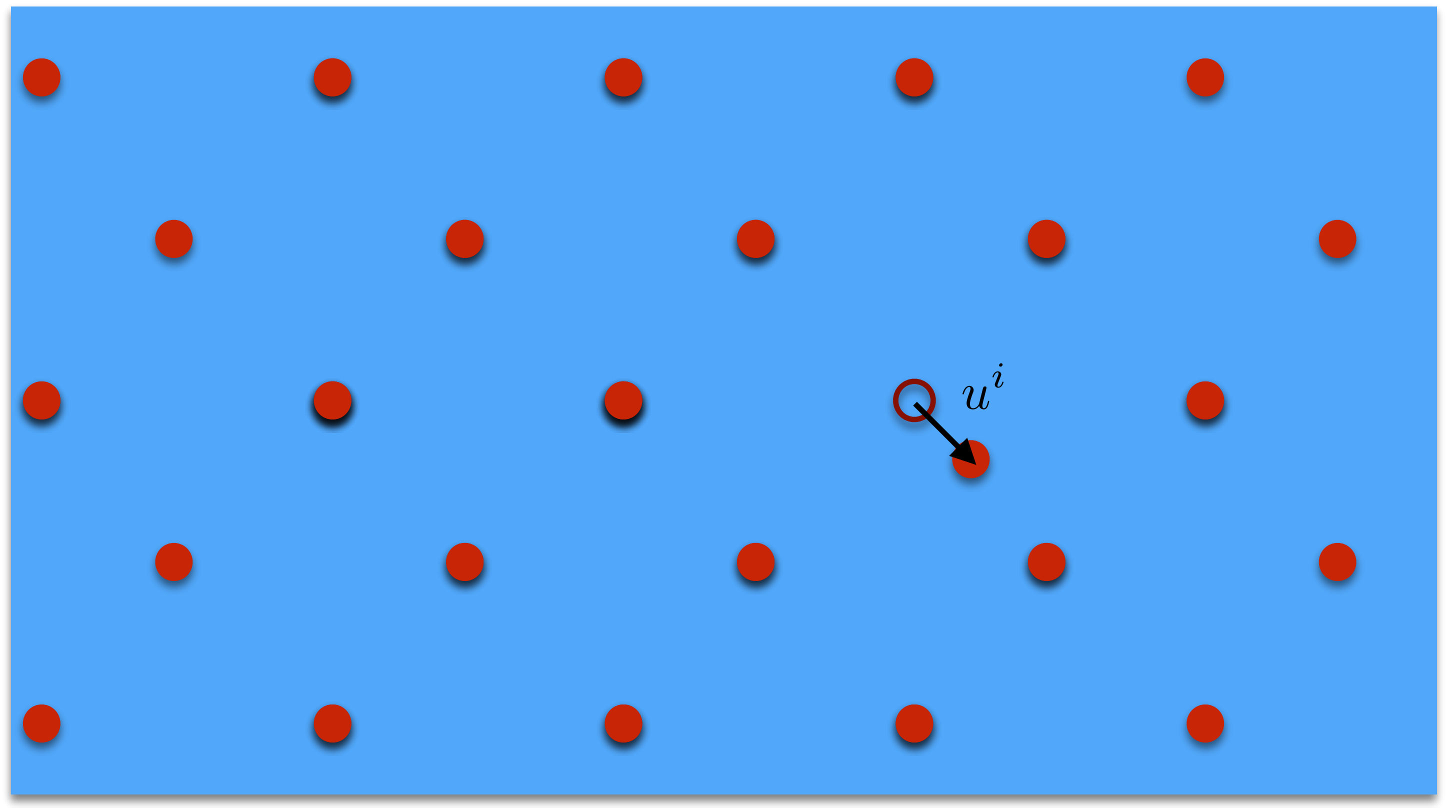
- The effective theory at linear order (the coefficients are not)

$$\mathscr{L} = \frac{n_v}{2} n_s \epsilon^{ij} u_i \dot{u}_j + \dot{\mathbf{u}}^2 - G(\partial_i u_j + \partial_j u_i)^2 + \mathbf{u}\cdot\mathbf{e} + \frac{1}{2}\left(\frac{m}{\bar{b}}\mathbf{e}^2 + \lambda b^2\right)$$

Kinetic energy of vortices

Elastic energy of
the vortex lattice
(Nonuniform displacement)

Dipole coupling
Displacement of charged particle = dipole



Linear theory

- The effective theory at linear order

$$\mathscr{L} = \frac{n_v}{2} n_s \epsilon^{ij} u_i \dot{u}_j + \cancel{\dot{\mathbf{u}}^2} - G(\partial_i u_j + \partial_j u_i)^2 + \mathbf{u} \cdot \mathbf{e} + \frac{1}{2}\left(\cancel{\frac{m}{b}\mathbf{e}^2} + \lambda b^2\right)$$

$(e_i = \partial_t a_i - \partial_i a_0)$

- Low energy dynamics $\omega \sim k^2 \ll k$
- Ignore terms with 2 time derivatives
- $\dfrac{\delta S}{\delta a_0} = \nabla \cdot \mathbf{u} = 0 \rightarrow u^i = \epsilon^{ij} \partial_j \phi$ $\quad \left(S = \int dt\, d^2\mathbf{x}\, \mathscr{L}\right)$
  - Only transverse phonon survives in the low energy
  - Compression mode costs too much energy
- Integrate out a_i, we arrive at the the Lifshitz model of scalar

$$\mathscr{L} = \dot{\phi}^2 - c^2 \left(\nabla^2 \phi\right)^2$$

Similar effective model appears at a critical (RK) point of quantum spin ices and quantum dimer models

Non-commutative field theory

Non-linear treatment of the vortex lattice

- A state of a solid is a map between external spatial coordinates and the frozen spatial coordinates (labeled by the balance position of vortices in the ground state)

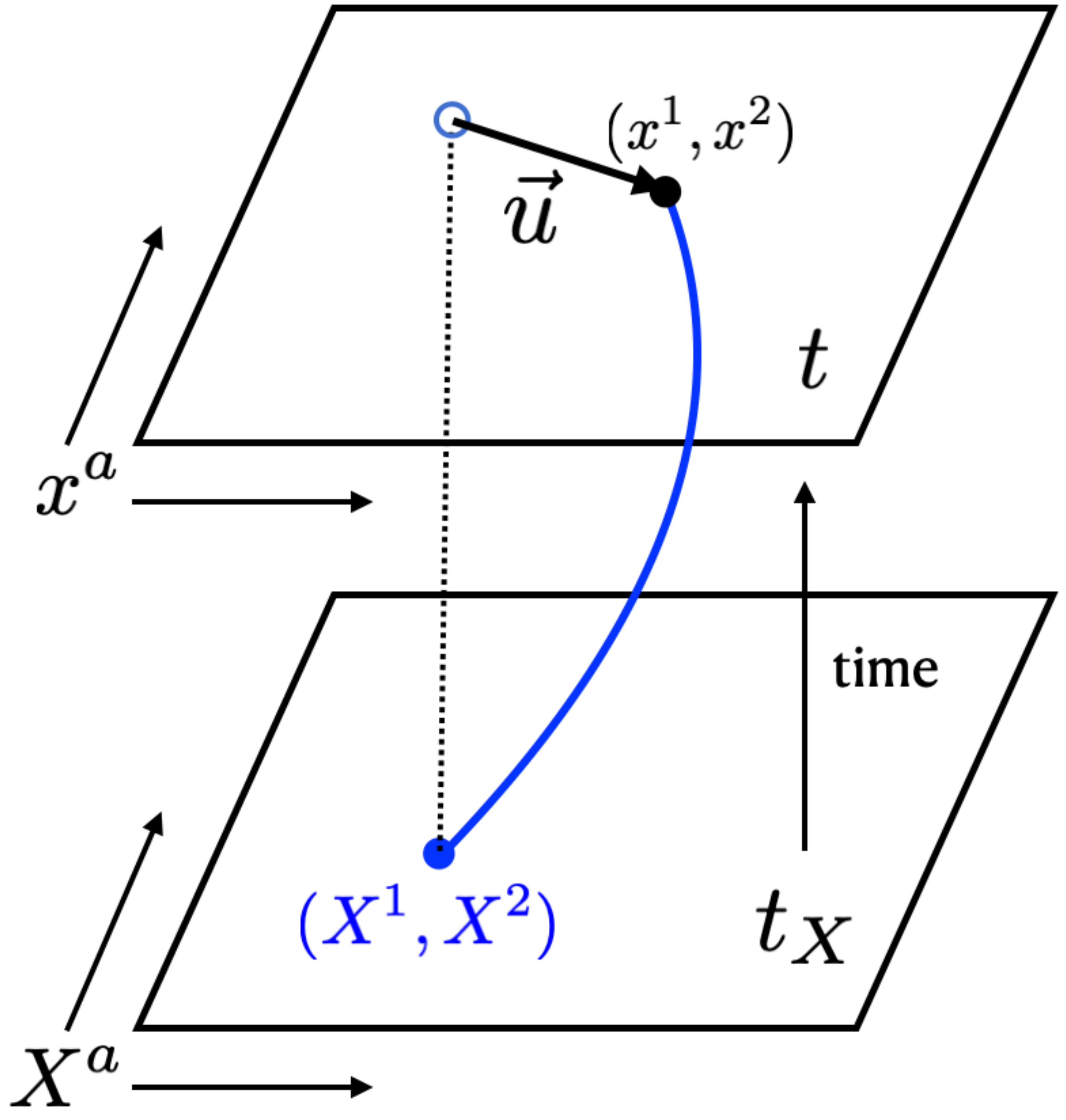
$$x^i \leftrightarrow X^a = \delta^a_i x^i - u^a$$

- Vortex current

$$j^\mu_v = \frac{n_v}{2}\epsilon^{\mu\nu\lambda}\epsilon^{ab}\partial_\nu X^a \partial_\lambda X^b$$

- In the linearized expansion

$$j^i_v = n_v \dot{u}^i$$



Coupling of vortex lattice with dual vector potential

- We reformulate the coupling between vortex current and the dual gauge field

$$a_\mu j_v^\mu = \frac{n_v}{2} a_\mu \epsilon^{\mu\nu\lambda} \epsilon^{ab} \partial_\nu X^a \partial_\lambda X^b$$

- Non-linear effective field theory

$$\mathcal{L} = \frac{n_v}{2} a_\mu \epsilon^{\mu\nu\lambda} \epsilon^{ab} \partial_\nu X^a \partial_\lambda X^b - a_0 n_v + \cdots$$

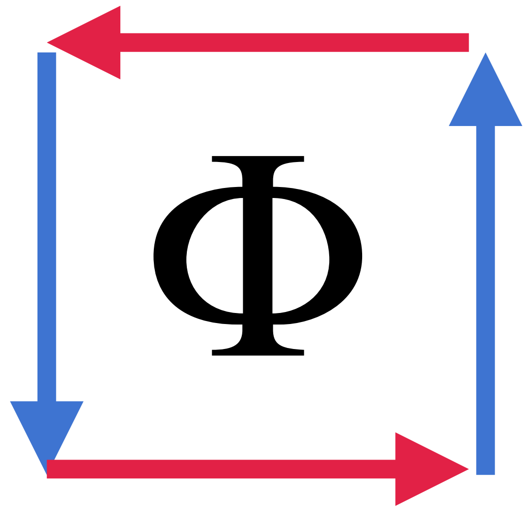
- Non-linear constraint $\dfrac{\delta S}{\delta a_0} = 0 \rightarrow \dfrac{1}{2} \epsilon^{ij} \epsilon^{ab} \partial_i X^a \partial_j X^b = \det \begin{pmatrix} \frac{\partial X^1}{\partial x^1} & \frac{\partial X^2}{\partial x^1} \\ \frac{\partial X^1}{\partial x^2} & \frac{\partial X^2}{\partial x^2} \end{pmatrix} = 1$

- Mapping from x^i to X^a is an **area-preserving diffeomorphism** (incompressible lattice)

Non-commutativity

- Under the effective magnetic field (induced by rotation), the "magnetic" translations do not commute (Aharonov-Bohm effect)

$$[\hat{P}_x, \hat{P}_y] = \frac{i}{\ell^2}\hat{Q} \sim iB\hat{Q}$$



$(\ell^2 = 1/B)$

- In the lowest **Landau level limit**, we treat the coordinates using the non-commutative geometry formalism

$$[\hat{x}, \hat{y}] = -i\ell^2 = i\theta$$

- The non-commutativity becomes important in the construction of the non-linear effective field theory of Tkachenko wave

Non-commutative area preserving

$$[\hat{x}, \hat{y}] = -i\ell^2 = i\theta$$

- The non-commutative version of the area preserving diffeomorphism

$$[\hat{X}, \hat{Y}] = i\theta$$

- The condition can be satisfied by an unitary transformation

$$\hat{X}^a = e^{i\hat{\phi}} \hat{x}^a e^{-i\hat{\phi}}$$

- Tkachenko mode = non-commutative scalar field $\hat{\phi}(\hat{x})$

Non-commutative Dipole symmetry

$$\hat{u}^i = \ell^2 \epsilon^{ij} \partial_j \hat{\phi}$$

- Under the magnetic translation ($\hat{\phi}$ transformed under a dipole symmetry)

$$e^{i\hat{P}_{\vec{c}}}\ \hat{\phi} \to \hat{\phi} + \mathbf{a} \cdot \hat{x} + \frac{1}{2}\theta \vec{c} \cdot \overrightarrow{\nabla} \hat{\phi} + \cdots \qquad e^{iP_{\vec{c}}}\ \hat{u}^i \to \hat{u}^i + c^i \qquad (c^i = \ell^2 \epsilon^{ij} a_j)$$

- Magnetic translations do not commute $[\hat{P}_x, \hat{P}_y] = \frac{i}{\ell^2}\hat{Q}$

- Implies that ϕ transformed under the number operator $\hat{Q}$

$$e^{-i\beta \hat{P}_y} e^{-i\alpha \hat{P}_x} e^{i\beta \hat{P}_y} e^{i\alpha \hat{P}_x} \hat{\phi} = e^{i\frac{\alpha\beta}{\ell^2}\hat{Q}} \hat{\phi} \to \hat{\phi} + \frac{\alpha\beta}{\ell^2}$$

- $\hat{\phi}$ transformed under both magnetic translation and number operator

- $\hat{\phi}$ is the superfluid phase and the shared Nambu-Goldstone boson

Moyal product

- Non-commutative field theory can be mapped to a familiar commutative field theory by replacing usual product by Moyal product
- $\hat{A}\hat{B} \to A \star B = A \exp(\frac{i}{2}\epsilon^{ij}\overleftarrow{\partial}_i \overrightarrow{\partial}_j)B = AB + \frac{i}{2}\theta\epsilon^{ij}\partial_i A \partial_j B + \cdots$
- We trade non-commutativity for non-locality (higher derivative)
- Physics motivation: Since the coordinates do not commute, one can't define the position of operator, operators are fuzzy objects.

Construct non-linear theory

- We formulate the field theory with Moyal product

$$X^a = e^{i\phi} \star x^a \star e^{-i\phi} = x^a + \theta D_i \phi$$

$$D_\mu \phi = -i \left(\partial_\mu e^{i\phi} \right) \star e^{-i\phi}$$

- The symmetry under magnetic translations constraints the action

$$\mathscr{L} = \mathscr{L} \left(D_t \phi, \partial_i D_j \phi \right)$$

- The **symmetries** and the **non-commutative structure** helps us to construct the non-linear effective theory of vortex lattice, which wasn't obtained previously

Decay of Tkachenko wave

- Expanding the non-linear action to cubic order

$$S = \int dt\, d^2\mathbf{x} \left[\dot{\phi}^2 - c^2 \left(\nabla^2 \phi \right)^2 + g_1 \dot{\phi}^3 + g_2 \dot{\phi} \left(\nabla^2 \phi \right)^2 + g_3 \left(\nabla^2 \phi \right)^3 \right]$$

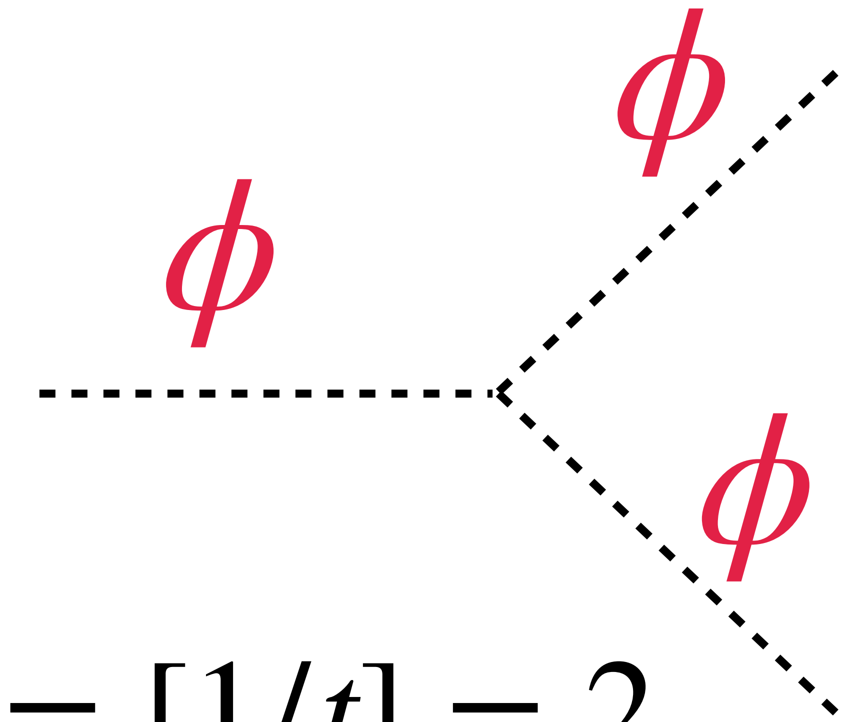
$$[t] = [x^2] = -2$$

$$[E] = [k^2] = 2$$

$$\longrightarrow \quad [g_i] = -2$$

$$(E = ck^2)$$

Fixes the Beliaev decay
of Tkachenko quanta



$$[\Gamma] = [1/t] = 2$$

$$\Gamma \sim g^2 E^3$$

(New result)

Du, Moroz, DXN, Son 2022

Tkachenko is a well defined quasiparticle at low energy

Fracton/elasticity duality

Effective Field Theory

We ignore the coupling with external electromagnetic field and rename some coefficients $B_0 = 2m\Omega$

$$\mathscr{L} = \mathscr{L}_g(a_\mu) - \frac{B_0 n_s}{2}\epsilon_{ij}u^i\partial_t u^j - \frac{1}{2}C_{ij;kl}u_{ij}u_{kl} + B_0 e_i u^i + a_\mu j^\mu_v$$

The las term is the coupling of dual gauge field a_μ with free vortices (on top of the vortex lattice)

Definition of strain

$$u_{ij} = \partial_i u_j - \theta\epsilon_{ij} \qquad \theta = \frac{1}{2}\epsilon_{ij}\partial_i u_j$$

Modulus tensor

$$C_{ij;kl} = 4C_1 P^{(0)}_{ij;kl} + 2C_2 P^{(2)}_{ij;kl},$$

With the definition of projection operators (D=2 is the space dimension)

$$P^{(0)}_{ij;kl} = \frac{1}{D}\delta_{ij}\delta_{kl},$$

$$P^{(2)}_{ij;kl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{1}{D}\delta_{ij}\delta_{kl}.$$

Hubbard-Stratonovich transformation

Pretko, Readzihovsky 2018, Kleinert 1980s, Beekman et al. 2017

We rewrite the Lagrangian by introducing Hubbard-Stratonovich fields $\tilde{\pi}_i$ and $\tilde{\sigma}_{ij}$

$$\mathscr{L} = \mathscr{L}_g(a_\mu) + \frac{1}{2B_0 n_s}\epsilon^{ij}\tilde{\pi}_i \partial_t \tilde{\pi}_j + \tilde{\pi}_i \partial_t u_i + \frac{1}{2}\tilde{C}^{-1}_{ij;kl}\tilde{\sigma}_{ij}\tilde{\sigma}_{kl} - \tilde{\sigma}_{ij}(\partial_i u_j - \theta\epsilon_{ij}) + B_0 e^i u_i + a_\mu j^\mu_v$$

(One can check that integrating out HS fields yields the original Lagrangian)

We then separate the strain fields to smooth (elastic) part and singular (defect induced) part

$$u_i = u_i^s + u_i^e, \qquad \theta = \theta^s + \theta^e$$

Integrating out smooth part gives us the conservation law

$$-\partial_t \tilde{\pi}^i + \partial_j \tilde{\sigma}_{ji} + B_0(\partial_t a_i - \partial_i a_t) = 0$$

And the Ehrenfest constrain (Beekman et al. 2017)

$$\epsilon^{ij}\tilde{\sigma}_{ij} = 0$$

Ehrenfest constrain implies that stress tensor is symmetric

Symmetric tensor gauge theory

We do some field redefinitions

$$B^i = \epsilon^{ij}\pi_j \qquad \pi_i = \tilde{\pi}_i - B_0 a_i$$

$$E^{ij} = \epsilon^{ik}\epsilon^{jl}\sigma_{kl} \qquad \sigma_{ij} = \tilde{\sigma}_{ij} - B_0\delta_{ij}a_t$$

The conservation law and Ehrenfest constraint

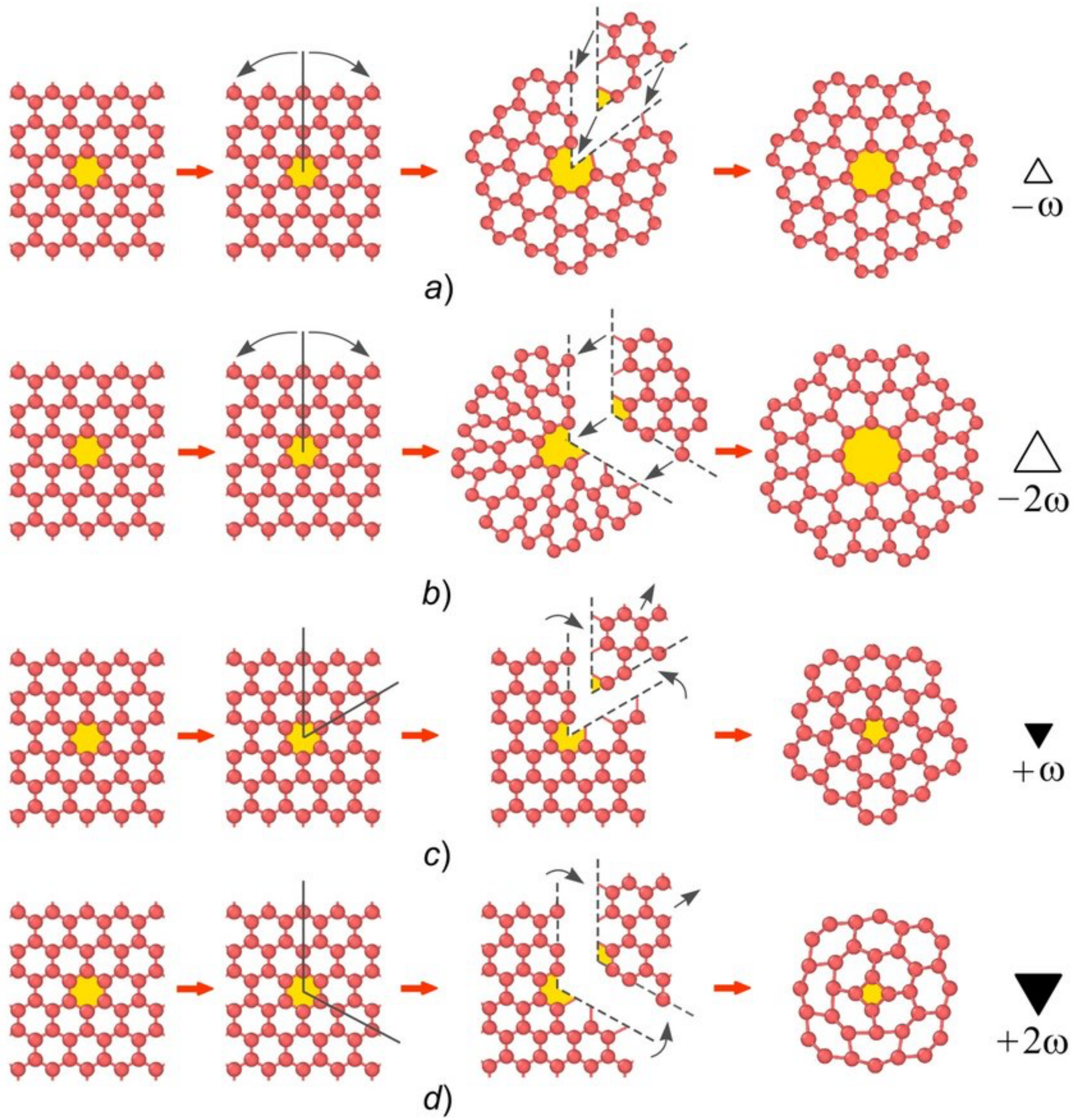
$$\partial_t B^i + \epsilon_{jk}\partial^j E^{ki} = 0$$

$$\epsilon^{ij}E_{ij} = 0$$

Both the conservation law (now is the Bianchi identity) and the Ehrenfest constraint can be solved by introducing symmetric tensor gauge $A_{ij} = A_{ji}$

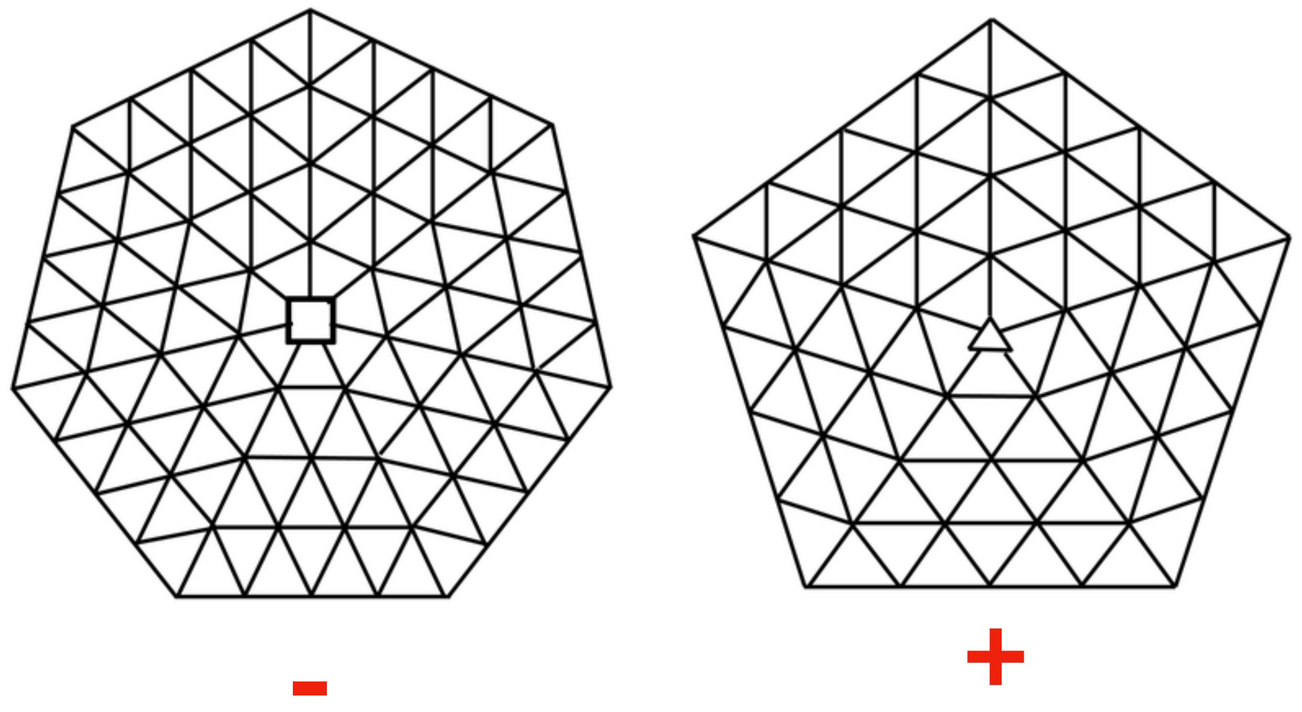
$$B^i = \epsilon_{jk}\partial^j A^{ki}$$

$$E_{ij} = -\partial_t A_{ij} - \partial_i\partial_j\varphi$$

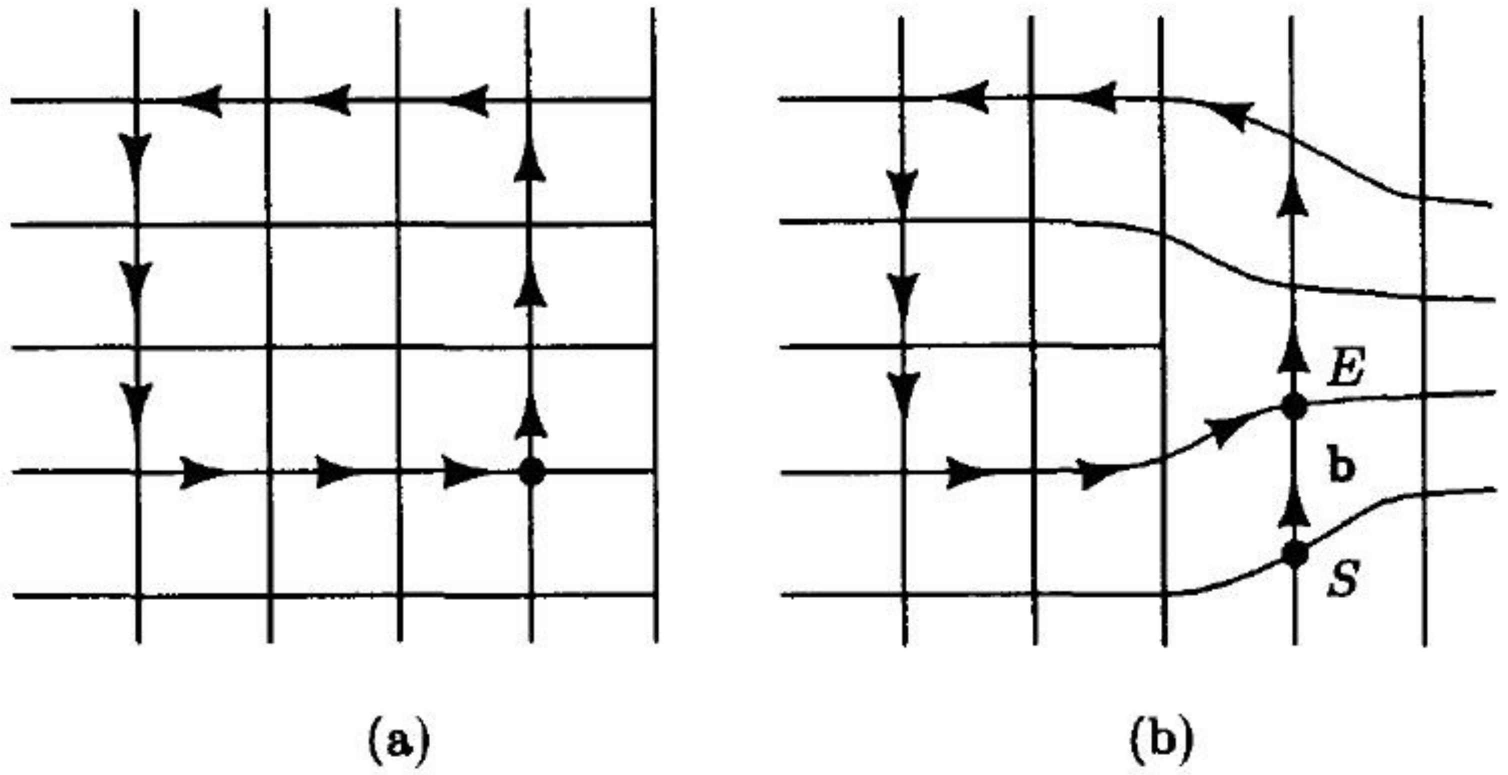


Picture by Mikhail Rozhkov

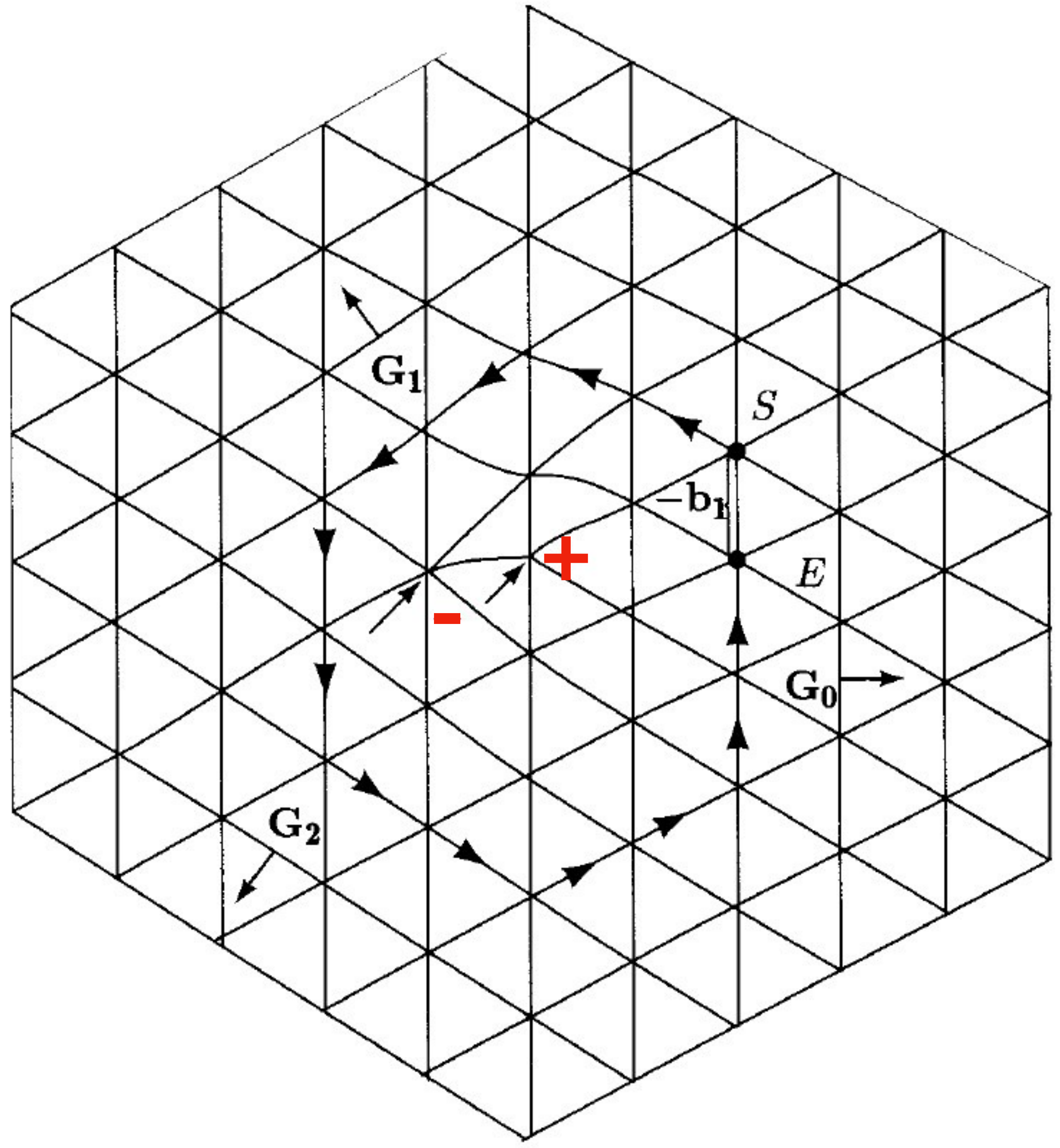
Dislocation and disclination



Disclinations with opposite signs



Dislocation



Dislocation as a disclination dipole $\mathbf{p}_i = -\epsilon_{ij}\mathbf{b}_j$

$$\rho = \mathbf{s} - \partial_i \mathbf{p}_i$$

Free disclinations

Disclinations bounded in dislocations

Dual theory

$$\mathscr{L} = \mathscr{L}_g(a_\mu) + \frac{\epsilon_{ij}}{2B_0 n_s}(B^i + B_0\epsilon^{ik}a_k)\partial_t(B^j + B_0\epsilon^{jl}a_l)$$
$$+\frac{1}{2}\tilde{C}^{-1}_{ij;kl}\left(E^{ij} + B_0\delta^{ij}a_t\right)\left(E^{kl} + B_0\delta^{kl}a_t\right) + A_{ij}J^{ij} + \varphi\rho + a_\mu j^\mu_v$$

Current and density of defects are

$$J^{ij} = \epsilon^{ik}\epsilon^{jl}(\partial_l\partial_t - \partial_t\partial_l)u^s_k$$

$$\rho = \mathbf{s} - \epsilon^{ik}\partial_k\mathbf{b}_i$$

Definition of disclination density and Bugger vector density (dislocation density)

$$\mathbf{s} = \epsilon^{ij}\partial_i\partial_j\theta^s$$
$$\mathbf{b}_i = \epsilon^{lj}\partial_l\partial_j u^s_i$$

Principle of condensed matter physics, Chaikin & Lubensky
(Introduced first by David Nelson)

Gauge symmetries

$$\mathscr{L} = \mathscr{L}_g(a_\mu) + \frac{\epsilon_{ij}}{2B_0 n_s}(B^i + B_0\epsilon^{ik}a_k)\partial_t(B^j + B_0\epsilon^{jl}a_l)$$

$$+\frac{1}{2}\tilde{C}^{-1}_{ij;kl}\left(E^{ij} + B_0\delta^{ij}a_t\right)\left(E^{kl} + B_0\delta^{kl}a_t\right) + A_{ij}J^{ij} + \varphi\rho + a_\mu j^\mu_v$$

The dual Lagrangian satisfies two gauge symmetries

$$A_{ij} \to A_{ij} + \partial_i\partial_j\alpha + B_0\delta_{ij}\chi$$

$$\varphi \to \varphi - \partial_t\alpha$$

$$a_\mu \to a_\mu + \partial_\mu\chi$$

The conservation law of total vortices (both vortex lattice and free vortex)

$$B_0\delta_{ij}J^{ij} - \partial_\mu j^\mu_v = 0$$

It is the modified glide constraint ! We will discuss more about it !

Gauss's law and Fractonic physics

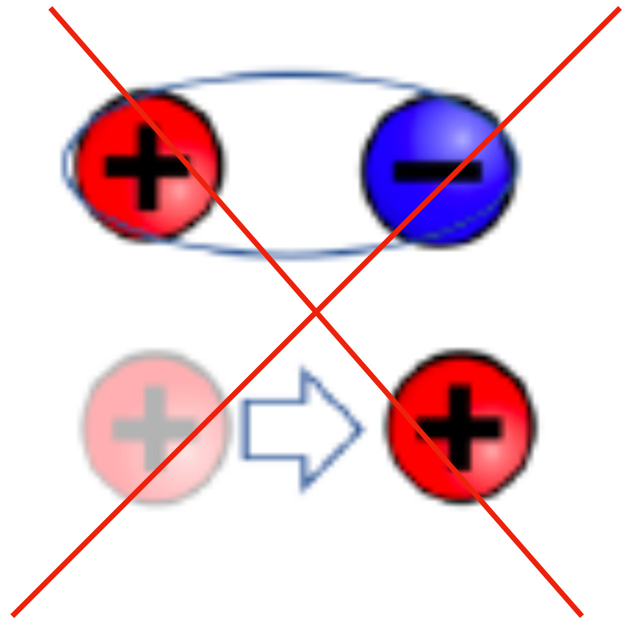
Field equation of φ (after gauging away a_t)

Gauss' law $$\frac{1}{4C_1}\partial_i\partial_j E^{ij} = \rho$$ **Pretko 2016**

- $Q = \int d^2\mathbf{x}\rho =$ **const**
- $P^i = \int d^2\mathbf{x} x^i \rho =$ **const**

→ **Conservation of charge and dipoles**

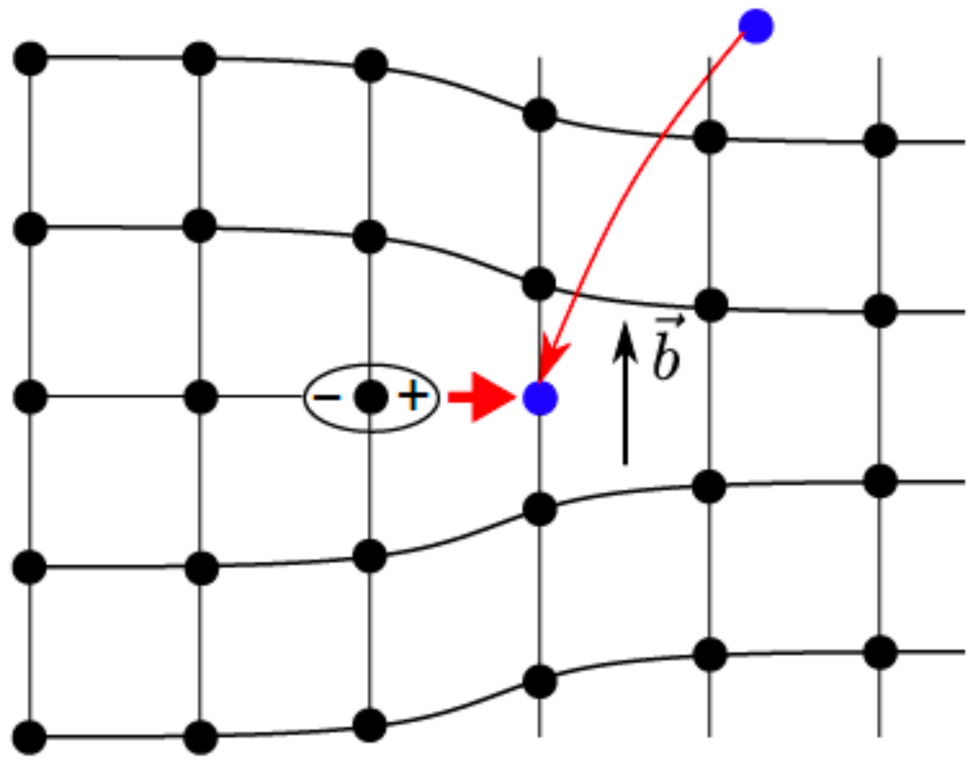
→ **Movement of charge is forbidden by the conservation of dipole moment**

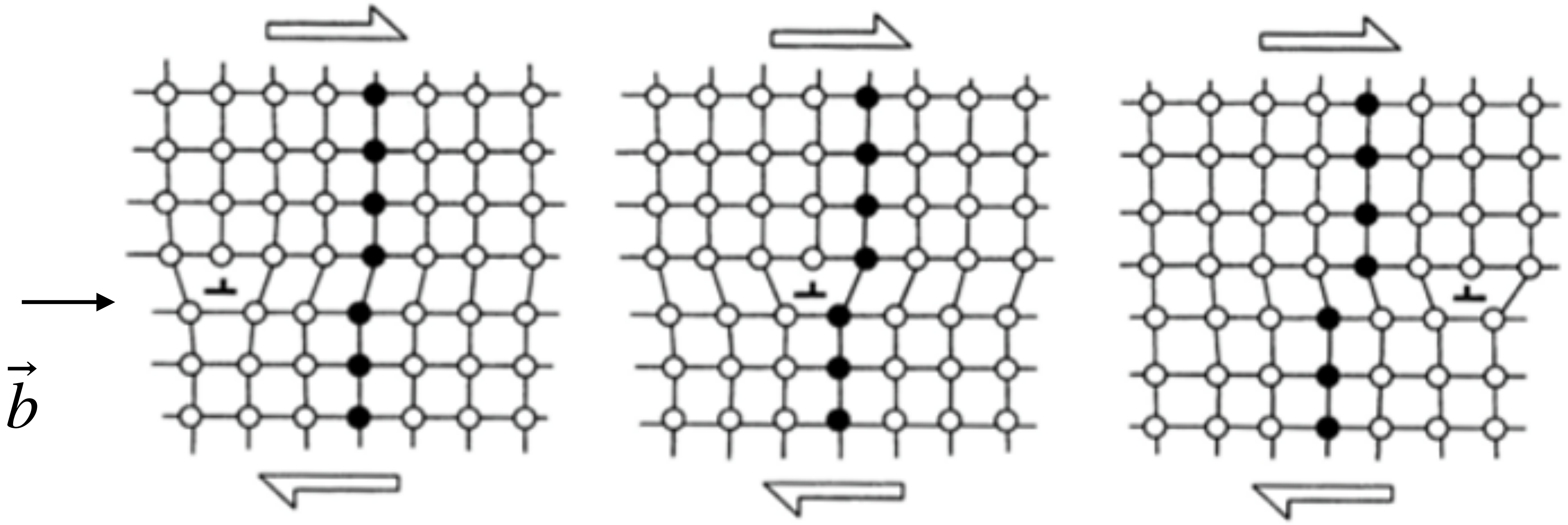


Disclination can't move → **Fractonic physics I**

Modified glide constraint

In normal lattice, dislocation can't climb (glide constraint)





$J^{ij} \rightarrow$ **Movement in $\hat{i}$ direction of dipole parallel to $\hat{j}$ direction**

Glide constraint implies $\delta_{ij}J^{ij} = 0$ ⟶ **Fractonic physics II**

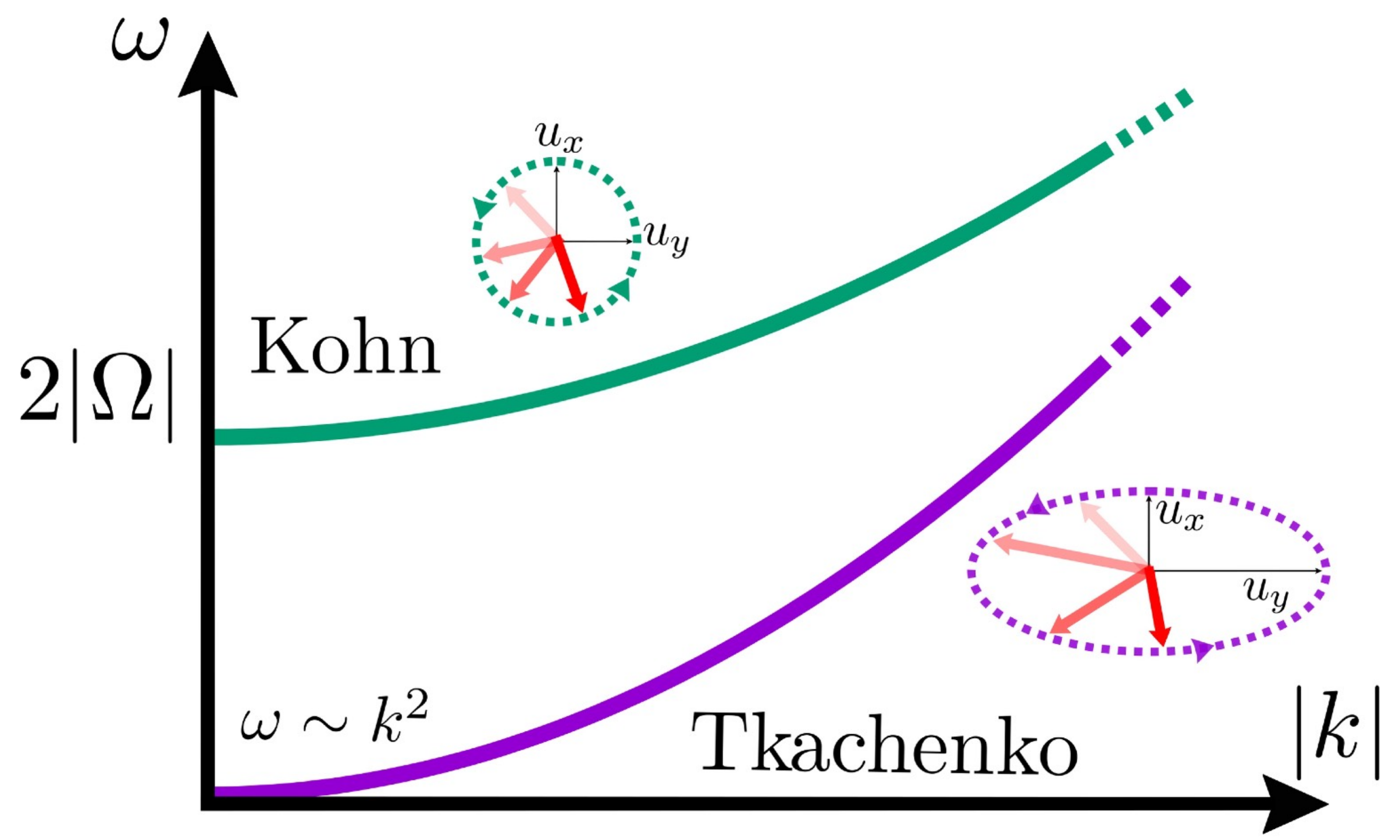
New constraint $B_0\delta_{ij}J^{ij} - \partial_\mu j^\mu_\nu = 0$

Vortices can jump into and out of the lattice

Dispersion relation

We reproduce the dispersion of Tkachenko mode

$$\omega^2 = \frac{2mC_2c_s^2}{B_0^2 n_s}k^4$$



Dispersion relation of the low energy mode doesn't depend on the bulk modulus!

One can gauge away the trace part of A_{ij}

$$A_{ij} \to A_{ij} + B_0\delta_{ij}\chi$$
$$a_\mu \to a_\mu + \partial_\mu\chi$$

(If we ignore the vacancies/intitestials) Compression part decouples from the elastic sector at low energy,
→ we obtained symmetric traceless gauge theory

Static interaction

Vortex-vortex interaction

$$V(\mathbf{q}) = \frac{n_s/m}{q^2 + \lambda_v^{-2}} \qquad \lambda_v = \sqrt{(2C_1 + C_2)m/B_0^2 n_s}$$

- $2C_1 + C_2$ can be either positive or negative Gifford and Baym (2008)
- $2C_1 + C_2 > 0$, inter vortex interaction is repulsive and screened due to the elastic sector.
- At large distance the interaction falls off as $K_0(r/\lambda_v)$ instead of logarithmically in the case of vortices in superfluid.
- $2C_1 + C_2 < 0$, inter vortex interaction is attractive → Type I ↔ Type II ???
- We also reproduce dislocation-dislocation interaction.
  The result agrees with Gifford and Baym (2008).

Quantum melting= Higgs mechanism in symmetric tensor gauge

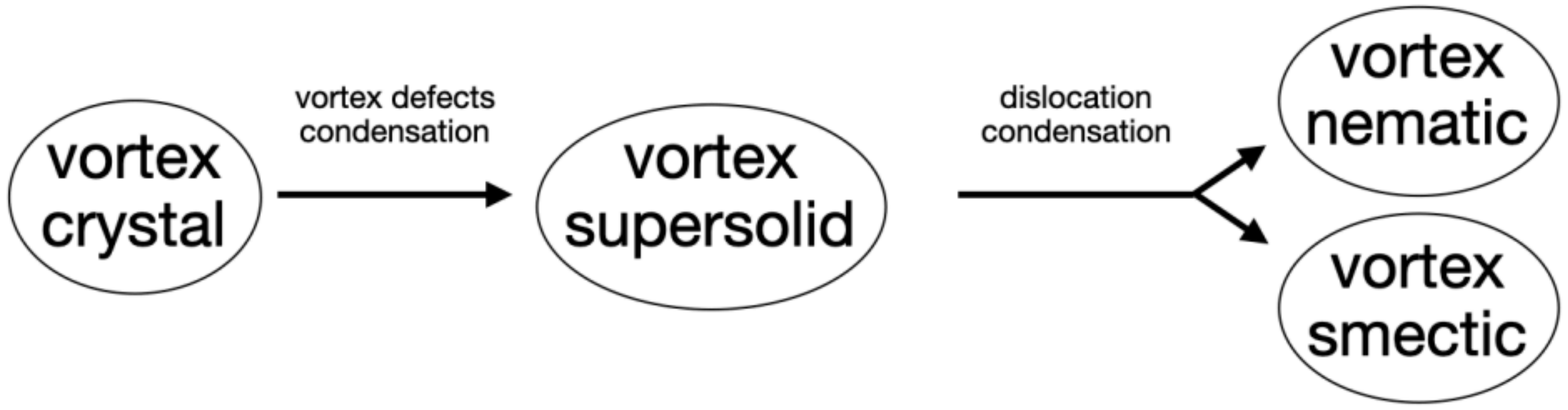
Nguyen, Gromov, Moroz
SciPost 2020

Interplay of superfluidity and elasticity

$$\mathcal{L}(u^i, a_\mu) \to \mathcal{L}(A_{ij}, A_t, a_\mu)$$

with coupling to disclinations, dislocations, vacancies/interstitials

Quantum melting scenario via Higgs condensation of defects



Dual gravity theory

Duality from Lifshitz theory

Radzihovsky 2022
Gorantla et al 2022

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{\lambda}\dot{\phi}^2 - \frac{C_2}{B^2}\left(\partial_i\tilde{\partial}_j\phi + \partial_j\tilde{\partial}_i\phi\right)^2\right]$$

$$\tilde{\partial}_i = \epsilon_{ij}\partial_j$$

Nguyen, Moroz
arXiv:2310.13741

Dualize to symmetric traceless tensor gauge theory

$$\mathcal{L} = \frac{\kappa}{8}e_{ij}e^{ij} - \frac{\lambda}{2}b^2 - \rho a_0 - j^{ij}a_{ij}$$

- $e_{ij}e^{ij}$: symmetric traceless stress tensor
- b^2: superfluid density fluctuations
- ρa_0: disclination density
- $j^{ij}a_{ij}$: dislocation current

Duality from Lifshitz theory

Dualize to symmetric traceless tensor gauge theory

$$\mathcal{L} = \frac{\kappa}{8} e_{ij} e^{ij} - \frac{\lambda}{2} b^2 - \rho a_0 - j^{ij} a_{ij}$$

with Bianchi identity = particle number conservation on LLL

$$\partial_t b - \frac{1}{2B} \left(\partial_i \tilde{\partial}_j + \partial_j \tilde{\partial}_i \right) e^{ij} = 0$$

symmetric tensor gauge theory

$$b = -\frac{1}{2B} \left(\partial_i \tilde{\partial}_j + \partial_j \tilde{\partial}_i \right) a_{ij},$$

$$e_{ij} = \partial_t a_{ij} - \left(\partial_i \partial_j - \frac{1}{2} \delta_{ij} \Delta \right) a_0$$

Duality from Lifshitz theory

Dualize to symmetric traceless tensor gauge theory

$$\mathcal{L} = \frac{\kappa}{8} e_{ij} e^{ij} - \frac{\lambda}{2} b^2 - \rho a_0 - j^{ij} a_{ij}$$

with Bianchi identity = particle number conservation on LLL

$$\partial_t b - \frac{1}{2B} \left(\partial_i \tilde{\partial}_j + \partial_j \tilde{\partial}_i \right) e^{ij} = 0$$

symmetric tensor gauge theory with u(1) gauge redundancy

$$a_0 \to a_0 + \partial_t \beta, \quad a_{ij} \to a_{ij} + \left(\partial_i \partial_j - \frac{1}{2} \delta_{ij} \Delta \right) \beta$$

Lattice defects

$$\mathcal{L} = \frac{\kappa}{8} e_{ij} e^{ij} - \frac{\lambda}{2} b^2 - \rho a_0 - j^{ij} a_{ij}$$

from gauge invariance $\partial_t \rho + \left(\partial_j \partial_i - \frac{1}{2} \delta_{ij} \Delta \right) j^{ij} = 0$

so charge, dipole and quadrupole are conserved

charge → disclination

dipole → dislocations

quadrupole → glide constraint

$$Q = \int d^2 x \rho, \quad Q^i = \int d^2 x \epsilon^{ij} x^j \rho, \quad Q^{tr} = \int d^2 x \mathbf{x}^2 \rho$$

Static interactions between defects are easy to calculate

Vacancies/interstitials couple to trace of the gauge field tensor

Towards dual gravity

inspired by
Du, Mehta, Nguyen, Son
2021

Symmetric traceless metric fluctuation field

$$\mathfrak{h}_{ij} = -l^2 \left(\varepsilon_{ik} a_{jk} + \varepsilon_{jk} a_{ik}\right)$$

Linearized volume-preserving diffeomorphisms $\xi^i = l^2 \tilde{\partial}^i \beta$

$$\partial_i \xi^i = 0$$

$$\mathfrak{h}_{ij} \to \mathfrak{h}_{ij} - \partial_i \xi_j - \partial_j \xi$$

Non-linear generalization: unimodular dynamical metric $\mathfrak{g}_{ij}$

$$\delta_\beta \mathfrak{g}_{ij} = -\xi^k \partial_k \mathfrak{g}_{ij} - \mathfrak{g}_{kj} \partial_i \xi^k - \mathfrak{g}_{ik} \partial_j \xi^k$$

VPDs are non-commutative: $[\delta_\alpha, \delta_\beta] = \delta_{[\alpha,\beta]}$ ← $l^2 \varepsilon^{ij} \partial_i \alpha \partial_j \beta$

Towards dual gravity

Moreover, the temporal gauge potential must transforms as

$$\delta_\beta a_0 = \partial_t \beta - \xi^k \partial_k a_0 = \partial_t \beta - \ell^2 \varepsilon^{kl} \partial_k a_0 \partial_l \beta$$

Gauge-invariant building blocks:

$$v^i = l^2 \varepsilon^{ij} \partial_j a_0$$

Ricci scalar $\mathfrak{R} \to \partial_i \partial_j \mathfrak{h}_{ij} = -2b$

shear strain rate $\mathfrak{s}_{ij} = \partial_t \mathfrak{g}_{ij} + \nabla_i v_j + \nabla_j v_i - \mathfrak{g}_{ij} \nabla_k v^k$

simple non-linear gravity guess

$$\mathcal{L} = \frac{\kappa}{8\ell^4} \mathfrak{s}_{ij} \mathfrak{s}^{ij} - \frac{\lambda}{8} \mathfrak{R}^2$$

Conclusions

- Tkachenko is a **shared NGB** of a non commutative field theory
- **Non-linear effective field theory** of vortex lattice
- Tkachenko is a stable quasi particle $\Gamma(E) \sim E^3$
- Fractonic behaviour of topological defects and quantum melting

DXN, Gromov,Moroz, SciPost Phys 9 (5), 0762020 (2020)

- Symmetric tensor gauge dual theory of the Liftshitz model and new scenarios of quantum melting
- Tkachenko wave as an massless emergent non-relativistic graviton

(DXN, Moroz, arXiv:2310.13741(2023)