Phase diagram of QCD matter with magnetic field: domain-wall Skyrmion chain in chiral soliton lattice

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Topology and Dynamics of Magneto-Vortical Matter
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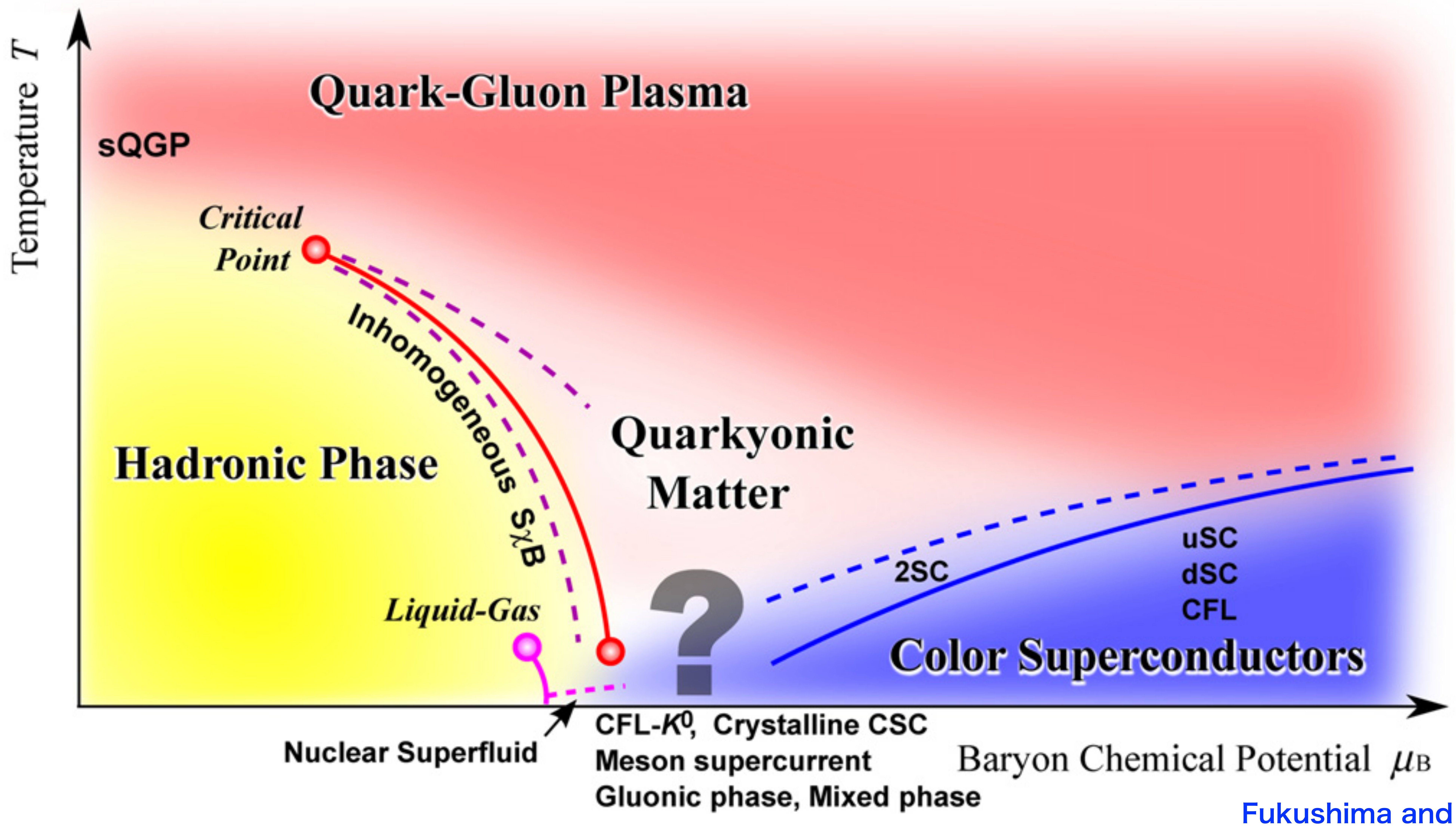
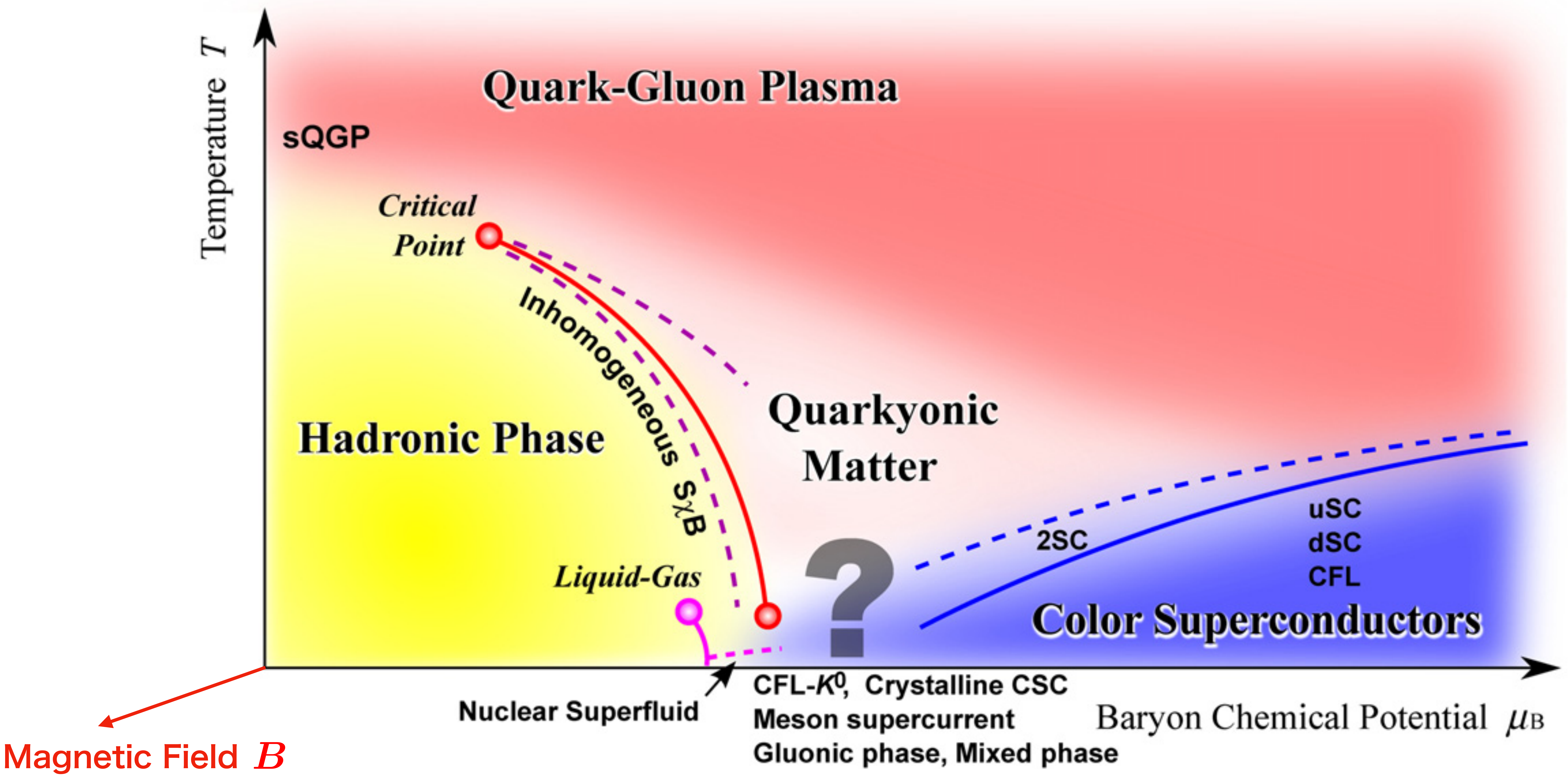
JHEP 12 (2023) 032

QCD phase diagram



Fukushima and Hatsuda (2008)

QCD phase diagram with B



What I want to discuss today

=How phase structure is modifies?

- I will use the chiral perturbation theory.
  - It is useful for making model-independent predictions because it is based on the symmetry of the microscopic QCD Lagrangian.
  - Consider the finite-B modification in a region with a small μ_B.
- Consider zero-temperature and only μ_B.

Since pions do not carry baryon number,
nothing seems to happen even if μ_B is considered.

- Skyrmion plays an important role to determine the phase structure.

Chiral perturbation theory

- **Order parameter is the chiral condensate:** $\langle \bar{q}q \rangle = |\langle \bar{q}q \rangle| \Sigma$

- **Nambu-Goldstone boson:** $\Sigma = \exp(\mathrm{i}\sigma_a \phi_a)\,, \quad \phi_a \equiv \pi_a / f_\pi$

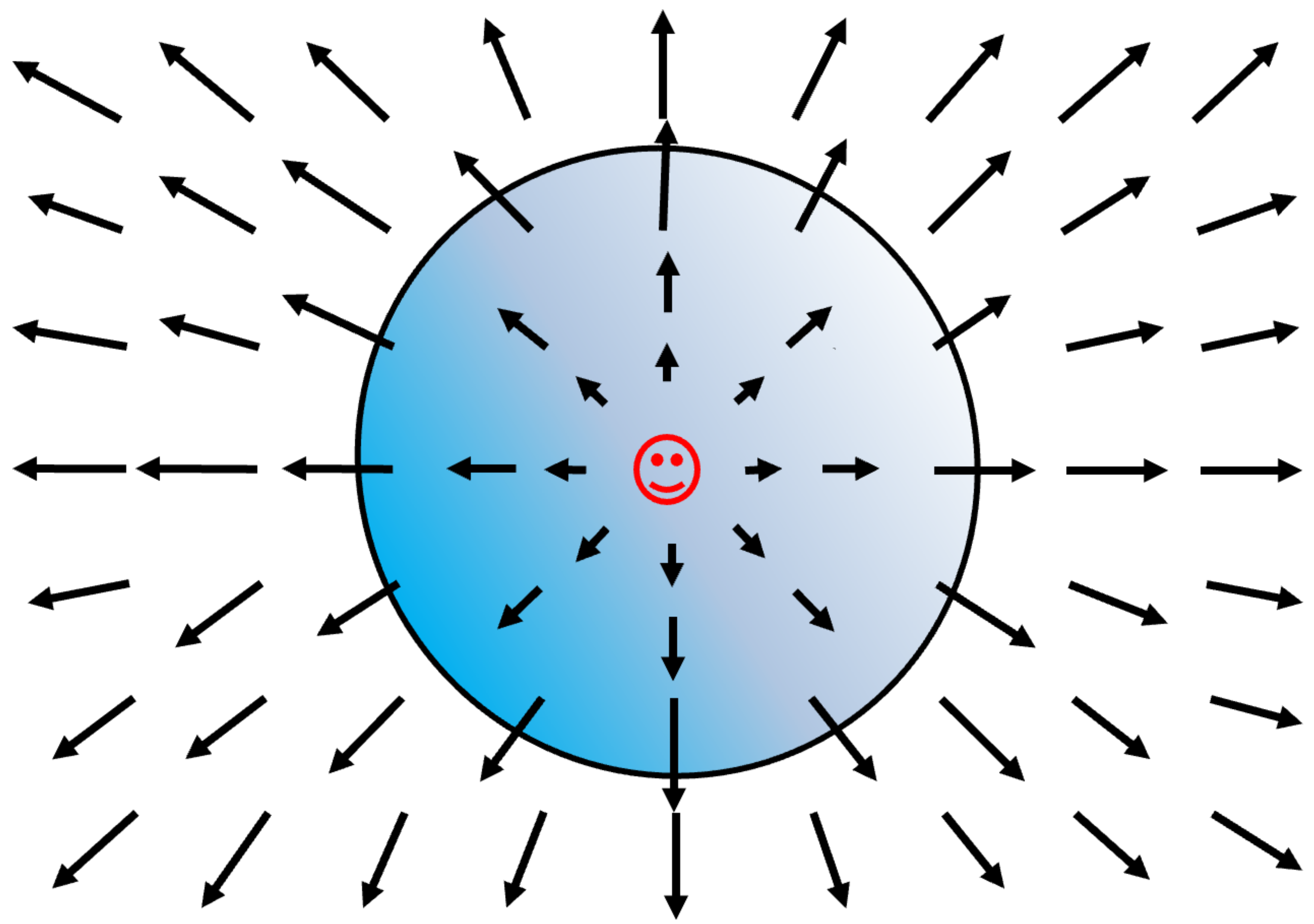
- **Effective Lagrangian:** $$\mathcal{L}_{\mathrm{ChPT}} = \frac{f_\pi^2}{4} \mathrm{tr}\left(D_\mu \Sigma D^\mu \Sigma^\dagger \right) - \frac{f_\pi^2 m_\pi^2}{4} (2 - \Sigma - \Sigma^\dagger)$$

$$D_\mu \Sigma = \partial_\mu \Sigma + \mathrm{i} A_\mu [Q, \Sigma]\,, \quad Q = \mathrm{diag}(2/3, -1/3)$$

Skyrmions

- Can the baryons be made by pions (rather than quarks)?

Baryon as soliton = Skyrmion



Topological number = Baryons

$$N_{\mathrm{B}} = \frac{1}{24\pi^2}\int \mathrm{d}^3x\,\epsilon^{ijk}\mathrm{tr}(\Sigma\partial_i\Sigma^\dagger\Sigma\partial_j\Sigma^\dagger\Sigma\partial_k\Sigma^\dagger)$$

$$J^\mu_{\mathrm{B}} = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(\Sigma\partial_\mu\Sigma^\dagger\Sigma\partial_\nu\Sigma^\dagger\Sigma\partial_\sigma\Sigma^\dagger)$$

- How many times R^3 surrounds the configuration space of the pions S^3.

ChPT w/ topological terms

- Baryon current couples to U(1)B gauge field (minimal coupling): Son and Zhitnitsky (2002) Son and Stephanov (2008)

$$\mathcal{L}_{\mathrm{B}} = -A^{\mu}_{\mathrm{B}} j_{\mathrm{B}\mu}\,, \quad A^{\mu}_{\mathrm{B}} = (\mu_{\mathrm{B}}, \mathbf{0})$$

- The μ_{B} can modify phase structure of ChPT (only pions theory).

$$j^{\mu}_{\mathrm{B}} = \underbrace{-\frac{\epsilon^{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{tr}\{L_\nu L_\alpha L_\beta}_{\text{Skyrmion charge}} - \underbrace{3ie\partial_\nu\left[A_\alpha Q(L_\beta + R_\beta)\right]\}}_{\mathrm{U(1)_{em}\ gauged\ part}} \quad L_\mu \equiv \Sigma\partial_\mu\Sigma^\dagger\,,\ R_\mu \equiv \partial_\mu\Sigma^\dagger\Sigma$$

$$Q = \mathrm{diag}(2/3, -1/3)$$

✓ "trial and error" U(1)em gauging w/ baryon number conservation. Goldstone and Wilczek (1981); Witten (1983)

- Anomalous coupling of pions to baryons via Skyrmion!

- Due to this term, μ_{B} can modify phase diagram, even though the theory considers only pions

sine-Gordon theory with the topological term

- I first ignore $\pi_\pm$: $\Sigma = \mathrm{e}^{\mathrm{i}\phi_3\tau_3}$

- Reduced Hamiltonian (B is oriented in z-direction) :

$$\mathcal{H} = \frac{f_\pi^2}{2}(\partial_z\phi_3)^2 + f_\pi^2 m_\pi^2(1-\cos\phi_3) - \frac{e\mu_\mathrm{B}}{4\pi^2}B\partial_z\phi_3$$

- The last term stems from the 2nd term of the Skyrmion term.

$$\mathcal{L}_\mathrm{B} = -\mu_\mathrm{B}\frac{\epsilon^{0ijk}}{24\pi^2}\mathrm{tr}\{\cancel{L_iL_jL_k} - \underline{3\mathrm{i}e\partial_i\left[A_jQ(L_k+R_k)\right]}\}$$

- B≠0 → Finite 1st derivative term → Favor ϕ inhomogenity

- What is a ground state at finite B?

Brauner and Yamamoto (2017)

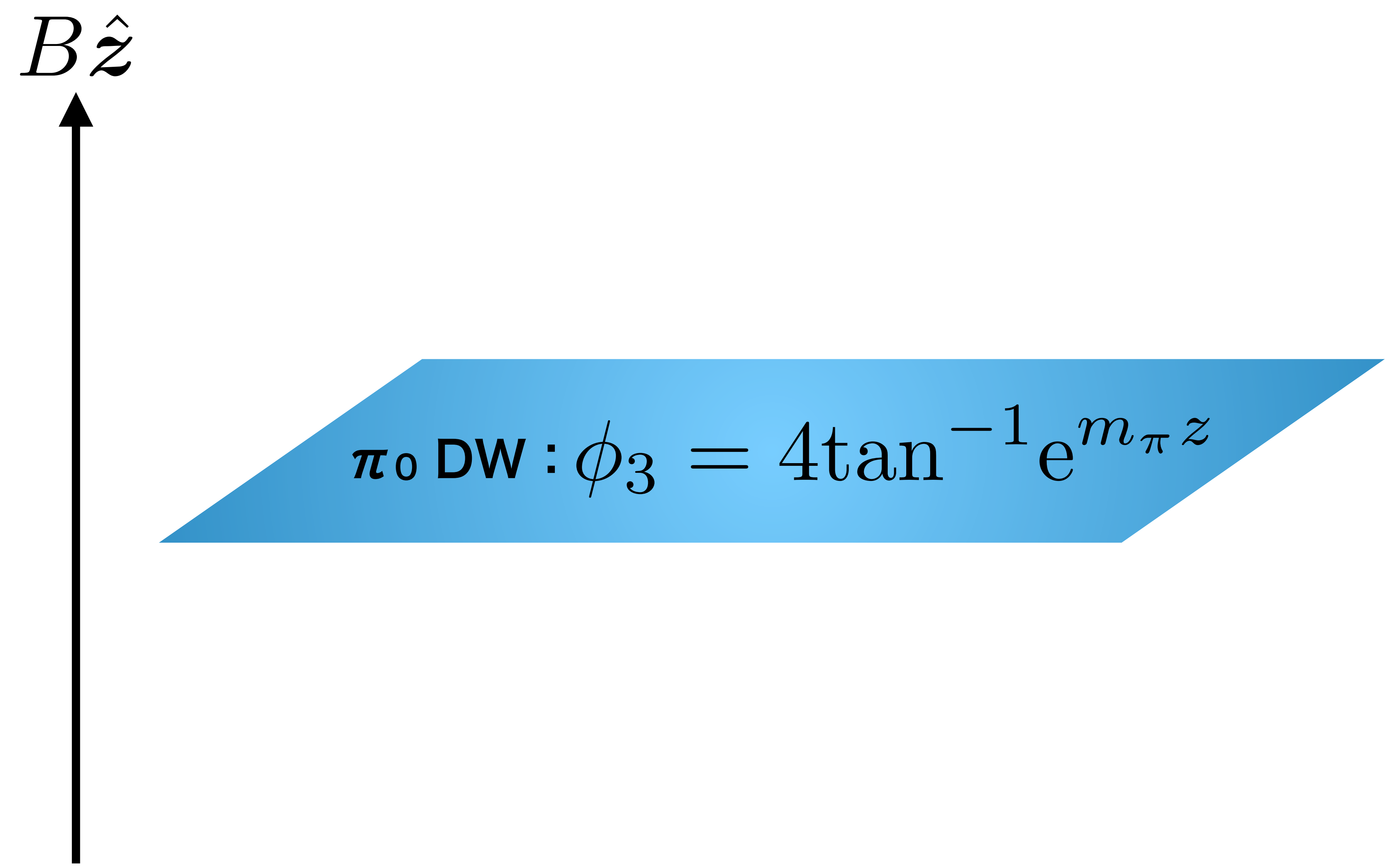
Chiral Soliton Lattice

- **EOM :** $\partial_z^2 \phi_3 = m_\pi^2 \sin \phi_3$

$$\phi_3(-\infty) = 0\,,\ \phi_3(\infty) = 2\pi$$

- **Energy :** $E = \int_{-\infty}^{\infty} \mathrm{d}z\,\mathcal{H} = 8 m_\pi^2 f_\pi - \frac{e\mu_{\mathrm{B}} B}{2\pi}$

- **Critical B :** $B_{\mathrm{CSL}} = \frac{16\pi m_\pi f_\pi^2}{e\mu_{\mathrm{B}}}$



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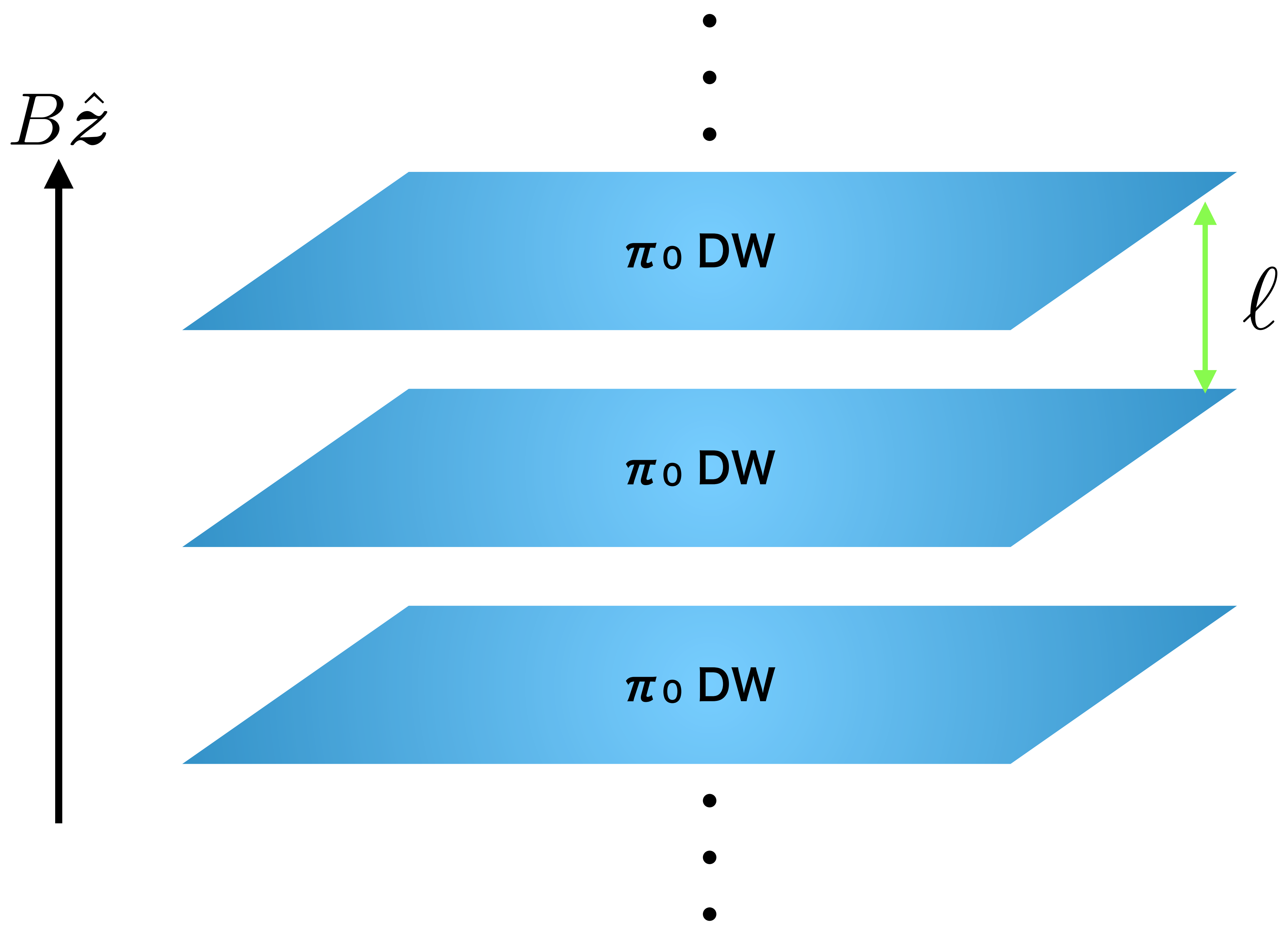


- Pack many DWs in ground state!
- Impossible to pack due to the repulsive force.
- Period depends on B and μ_B.

Son and Stephanov (2008); Brauner and Yamamoto (2017)

Non-Abelian soliton

- **The single soliton:** $\Sigma_0 = \mathrm{e}^{\mathrm{i}\sigma_3\theta}\,,\ \theta = 4\tan^{-1}\mathrm{e}^{m_\pi z}$

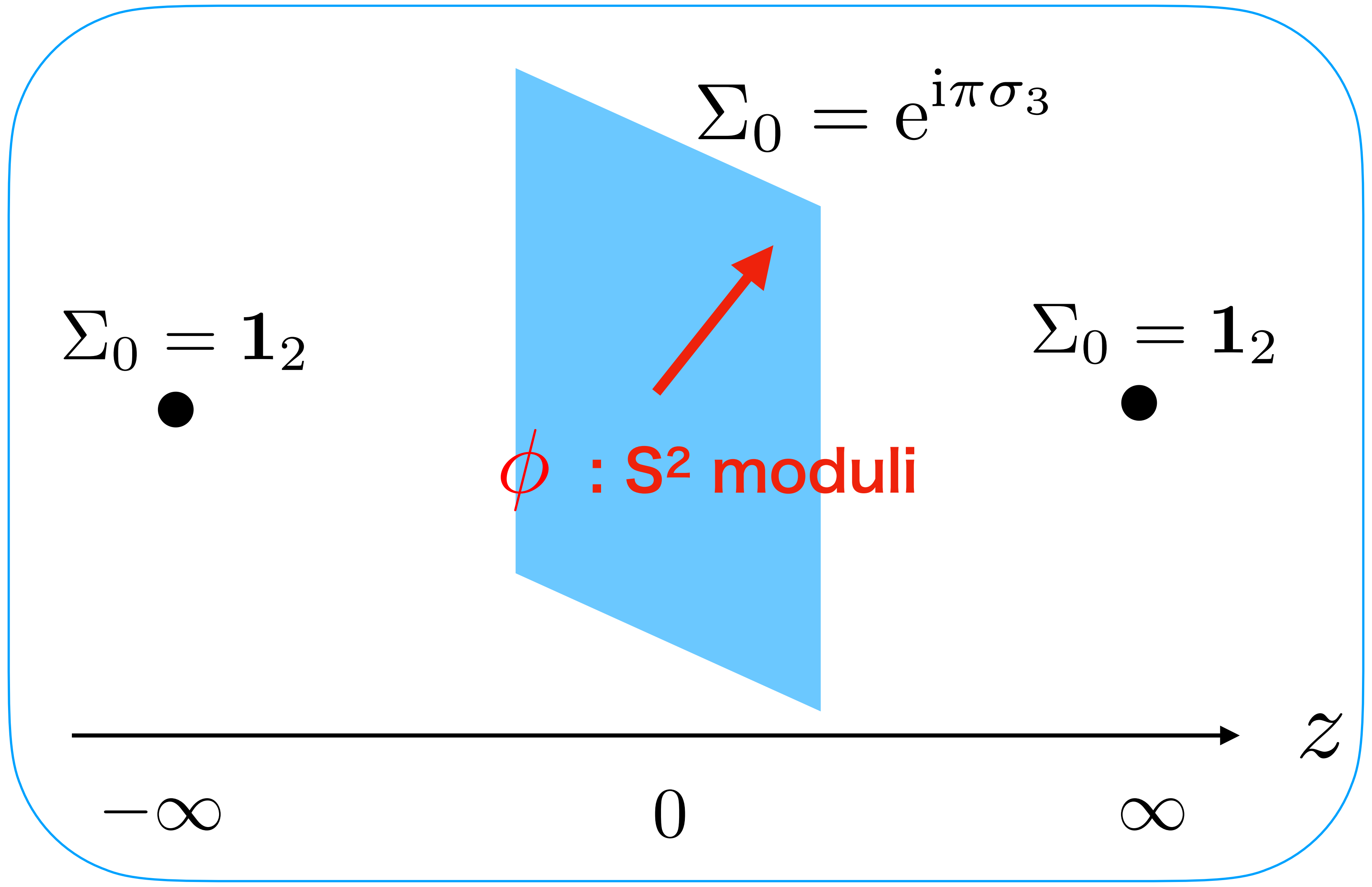
- **More general solution :** $\Sigma = g\Sigma_0 g^\dagger = \exp(\mathrm{i}\theta g\tau_3 g^\dagger)\,, \quad g \in \mathrm{SU}(2)_\mathrm{V}$

- **Σ_0 is invariant under U(1) transformation :** $g = \mathrm{e}^{\mathrm{i}\tau_3\theta}$

- **"SSB" of SU(2)v → U(1)** Moduli $\mathrm{SU}(2)/\mathrm{U}(1) \cong S^2$

- **Coordinate for S²** $\phi = \dfrac{1}{\sqrt{1+|f|^2}}\begin{pmatrix}1\\ f\end{pmatrix}\,, \quad f\in\mathbb{C}$

$$g\sigma_3 g^\dagger = 2\phi\phi^\dagger - 1$$



Nitta (2015); Eto and Nitta (2015); Gaiotto, Komargodski and Seiberg (2018)

CSL with S² moduli

- Each arrow represents S² moduli.

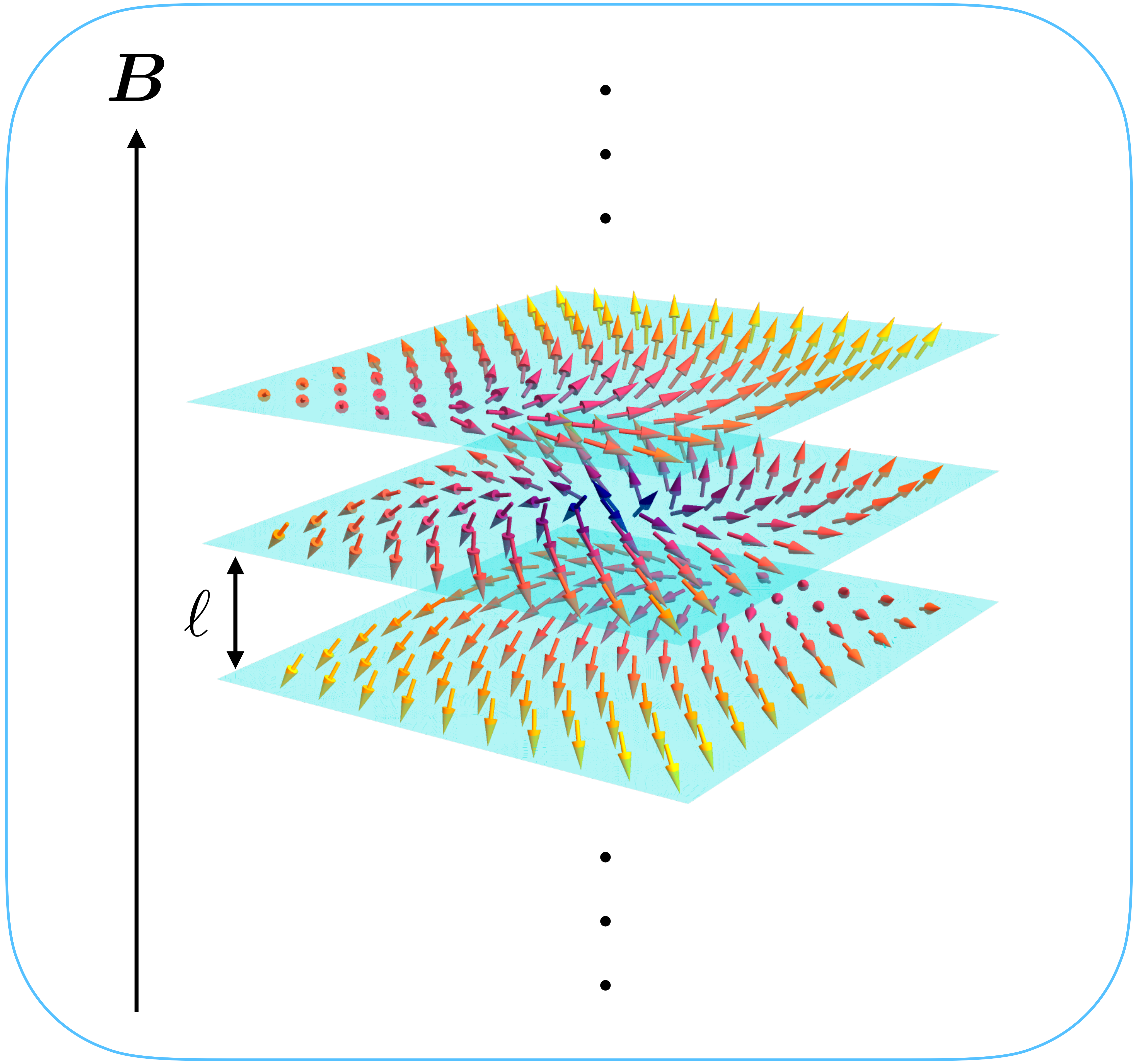
- S² is represented by $\boldsymbol{n} \in \mathbb{R}^3 \, (|\boldsymbol{n}| = 1)$

O(3) nonlinear sigma model

$$n_a \equiv \phi^\dagger \sigma_a \phi \quad |\boldsymbol{n}| = 1$$

- Only $\boldsymbol{\pi}^0$ DW $\rightarrow \quad \boldsymbol{n} = (0, 0, 1)^t$

- Does nontrivial configurations of $\boldsymbol{\phi}$ (or n) occur?



EFT for S^2 moduli

- **Effective Lagrangian :** $\mathcal{L}_{\rm EFT} = \mathcal{L}_{\rm const} + \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm topo}$ Eto, KN and Nitta, JHEP 12 (2023) 032

- **Kinetic term :** $\mathcal{L}_{\rm kin} = \mathcal{C}(\ell)[(\phi^\dagger D_\alpha \phi)^2 + D^\alpha \phi^\dagger D_\alpha \phi]$

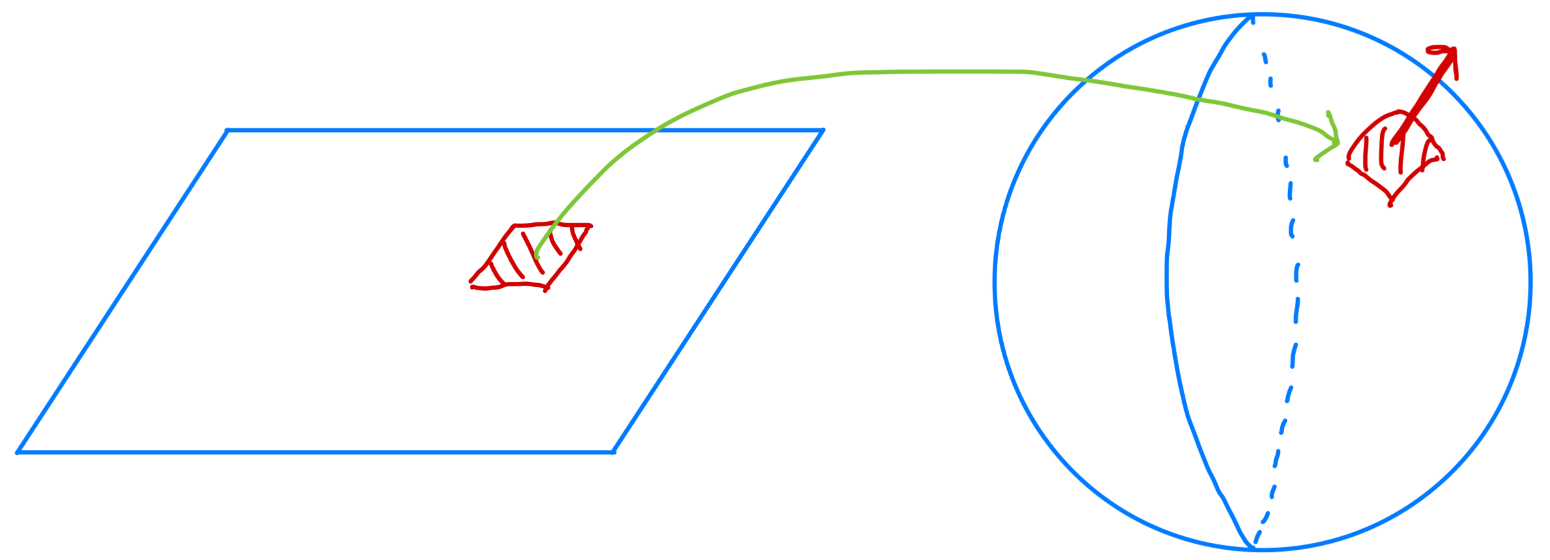
- **Topological terms :** $\mathcal{L}_{\rm topo} = -2\mu_{\rm B} q + \dfrac{e\mu_{\rm B}}{2\pi}\epsilon^{03jk}\partial_j[A_k(1-n_3)]$

O(3) nonlinear sigma model
$n_a \equiv \phi^\dagger \sigma_a \phi \quad |\boldsymbol{n}| = 1$

\- The red term stabilizes the configuration with finite k(<0)!

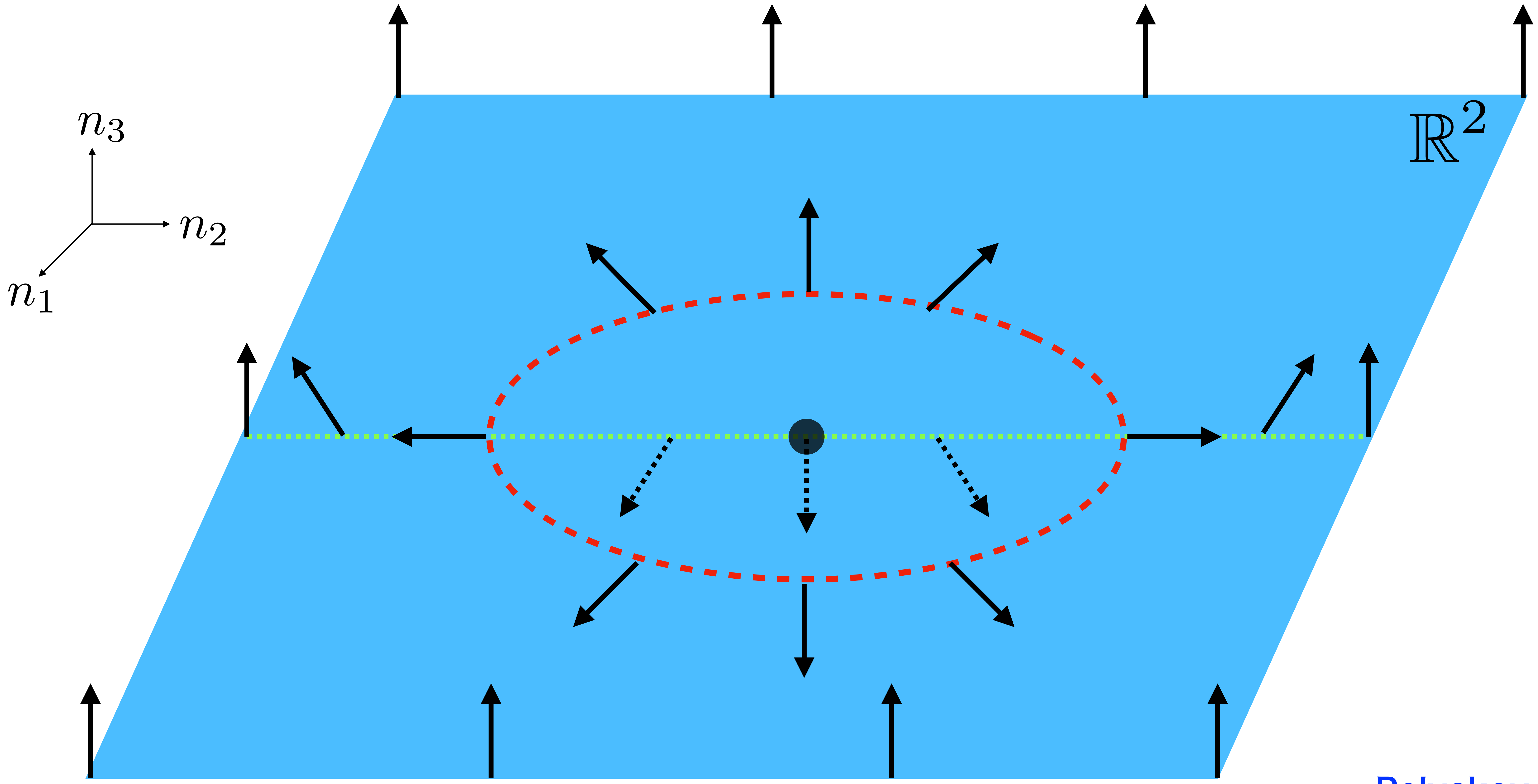
π_2(S^2) topological charge (counting how many times xy plane covers S^2 moduli)

$$
\begin{aligned}
k &= \int \mathrm{d}^2 x\, q \\
&= \frac{1}{4\pi}\int \boldsymbol{n}\cdot\left(\frac{\partial \boldsymbol{n}}{\partial x}\mathrm{d}x \times \frac{\partial \boldsymbol{n}}{\partial y}\mathrm{d}y\right) \\
&\in \mathbb{Z}
\end{aligned}
$$



Baby Skyrmion

- **Configuration on DW surrounding S²:** $\uparrow$ = n_a $\quad \boldsymbol{n^2} = 1$



Polyakov and Belavin (1975)

Energy of baby Skyrmion on DW

- **The effective Hamiltonian:** $\mathcal{H}_{\mathrm{DW}} = \frac{\mathcal{C}(\kappa)}{4}\partial_i \boldsymbol{n}\cdot\partial_i \boldsymbol{n} + 2\mu_{\mathrm{B}} q - \frac{e\mu_{\mathrm{B}}}{2\pi}\epsilon^{03jk}\partial_j[A_k(1-n_3)]$

- **Completing of square of $(\partial_i \boldsymbol{n})^2$:** $(\partial_i \boldsymbol{n})^2 = \frac{1}{2}(\partial_i \boldsymbol{n} \pm \epsilon_{ij}\boldsymbol{n}\times\partial_j \boldsymbol{n})^2 \pm 8\pi q \geq \pm 8\pi q$

- **"Equal" = BPS equation:** Baby Skyrmion! $\boxed{\partial_i \boldsymbol{n} \pm \epsilon_{ij}\boldsymbol{n}\times\partial_j \boldsymbol{n} = 0}$ See Manton and Sutcliff (2004)

- **Total energy:** $E_{\mathrm{DWSk}} \geq \boxed{2\pi\mathcal{C}(\kappa)|k| + 2\mu_{\mathrm{B}} k} - \boxed{\frac{e\mu_{\mathrm{B}}}{2\pi}\int d^2x\,\epsilon^{03jk}\partial_j[A_k(1-n_3)]}$

★The total energy becomes negative at sufficiently large μ_{B}, and baby Skyrmion appears in the ground state!

✓Some constraints on the baby Skyrmion.

Eto, KN and Nitta, JHEP 12 (2023) 032

Constraint on baby Skyrmion

- k anti-Baby Skyrmion solution: $n_3 = \frac{1-|f|^2}{1+|f|^2}, \; f = \frac{b_{k-1}\bar{w}^{k-1} + \cdots + b_0}{\bar{w}^k + a_{k-1}\bar{w}^{k-1} + \cdots + a_0}$

  See Manton and Sutcliff (2004)

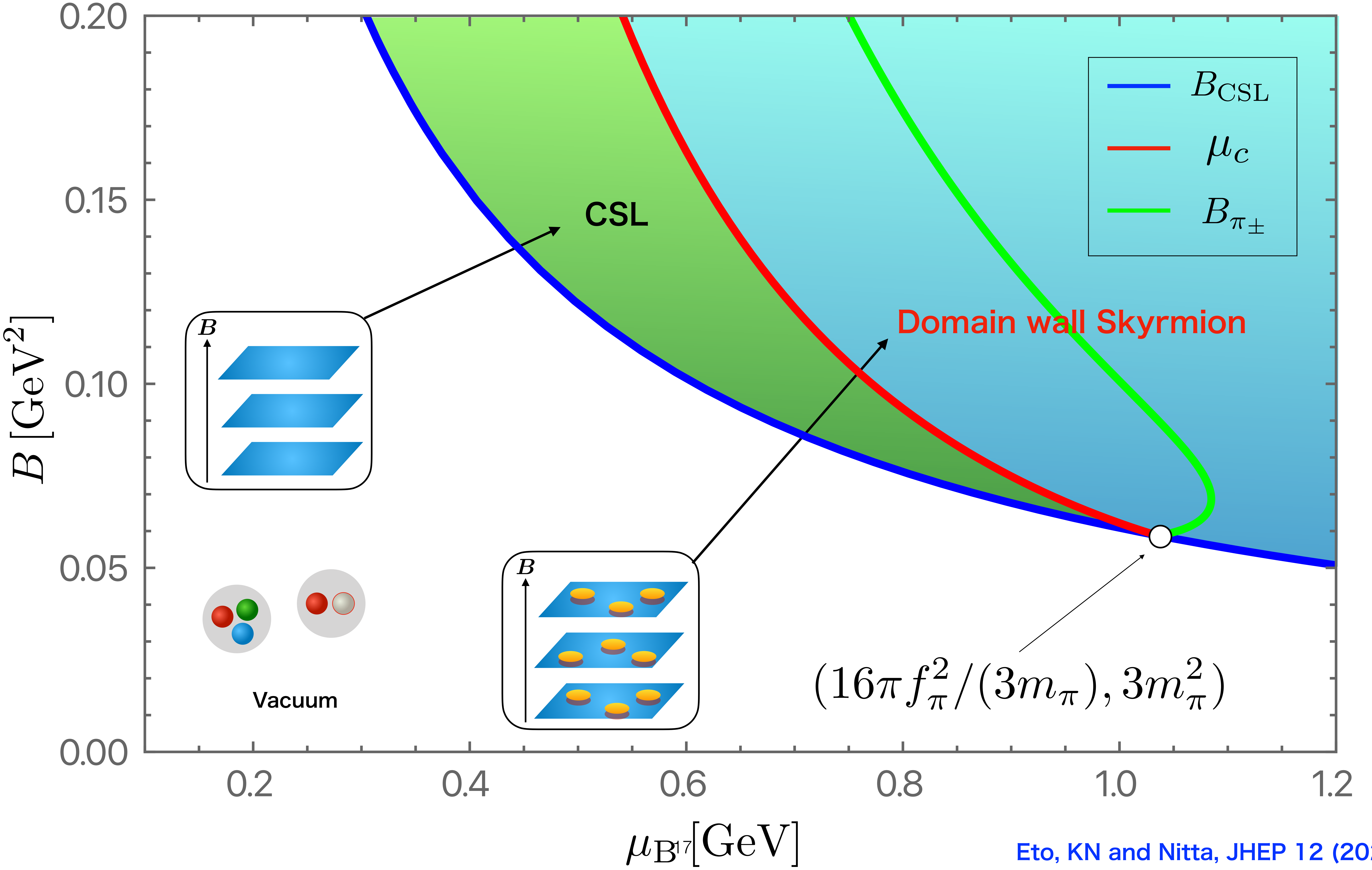
- E_{DWSk} for the k anti-baby Skyrmion: $E_{\mathrm{DWSk}} = \underline{2\pi C(\kappa)|k| - 2\mu_{\mathrm{B}}|k|} + e\mu_{\mathrm{B}}B|b_{k-1}|^2$

  Can it be negative here?

- In order to minimize E_{DWSk}, $b_{k-1}=0$.

- Critical baryon chemical potential: $\mu_{\mathrm{B}} \geq \mu_{\mathrm{c}} = \pi C(\kappa)$ Eto, KN and Nitta, JHEP 12 (2023) 032

  Depending on μ_{B} and B



Eto, KN and Nitta, JHEP 12 (2023) 032

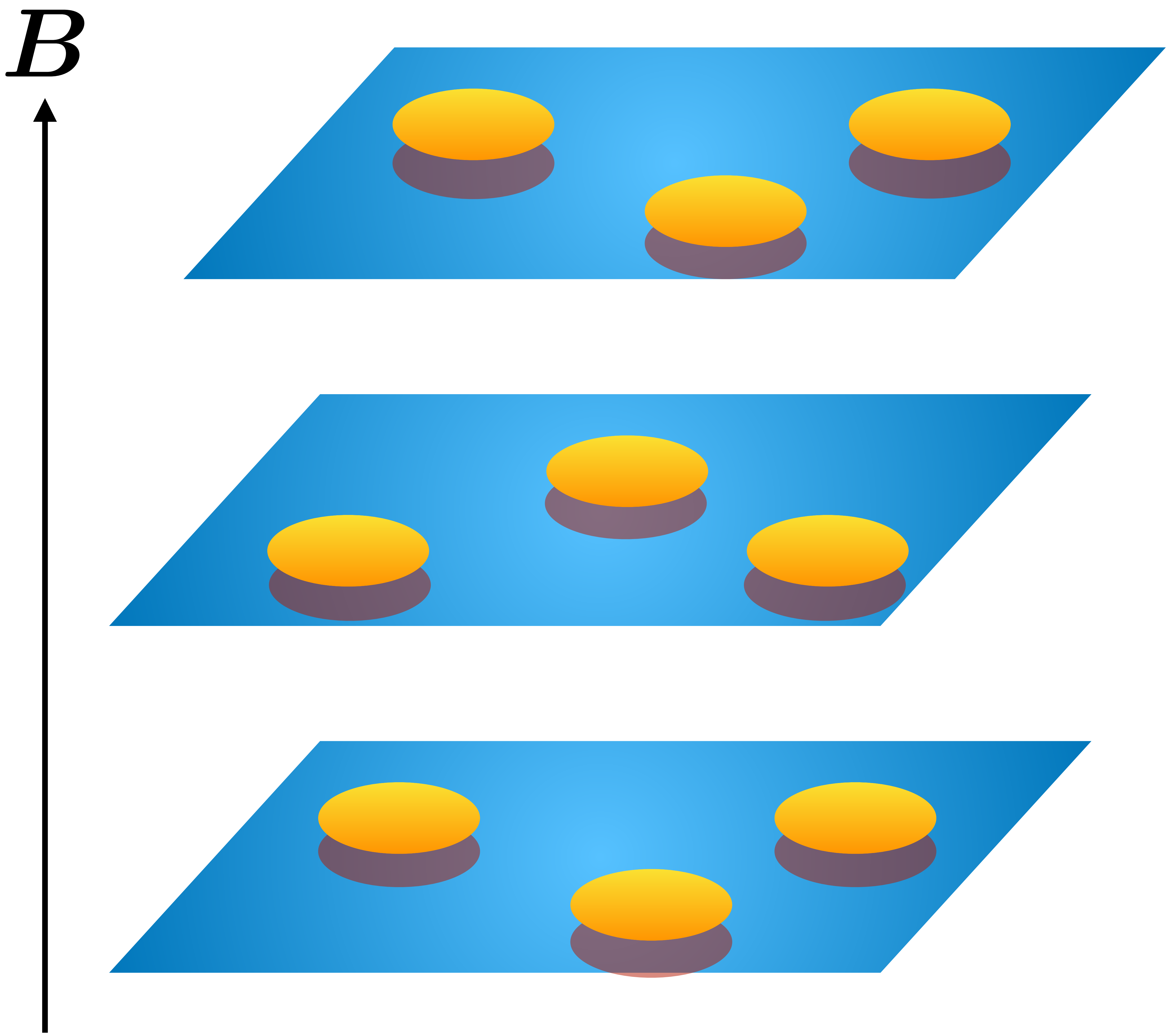
DW Skyrmion

- Baryon and electric charge :

$$N_{\rm B} = 2\,, N_{\rm e} = 1\,, I_3 = 0$$

  - Deuteron? https://en.wikipedia.org/wiki/Deuterium

- What structure is realized?
  - There is no interaction.
  - Beyond BPS approximation



Summary

- We have to include the minimal coupling of Skyrmions to baryons.
- At $B > B_{CSL}$, the parallel stack of π_0 DWs is energetically stable.
- At $\mu > \mu_c$, the baby Skyrmion appears on π_0 DWs.

Future direction

- **DWSk in QCD-like theory (two-color QCD)**

  - Two-color QCD with finite baryon chemical potential and magnetic field has no sign problem.
  - CSL is QCD-like theory has been considered. Brauner, Filions and Kolesova (2019)

- **DWSk in lattice gauge theory**

  - Monte-Carlo simulation
  - Strong-coupling expansion for calculating free energy

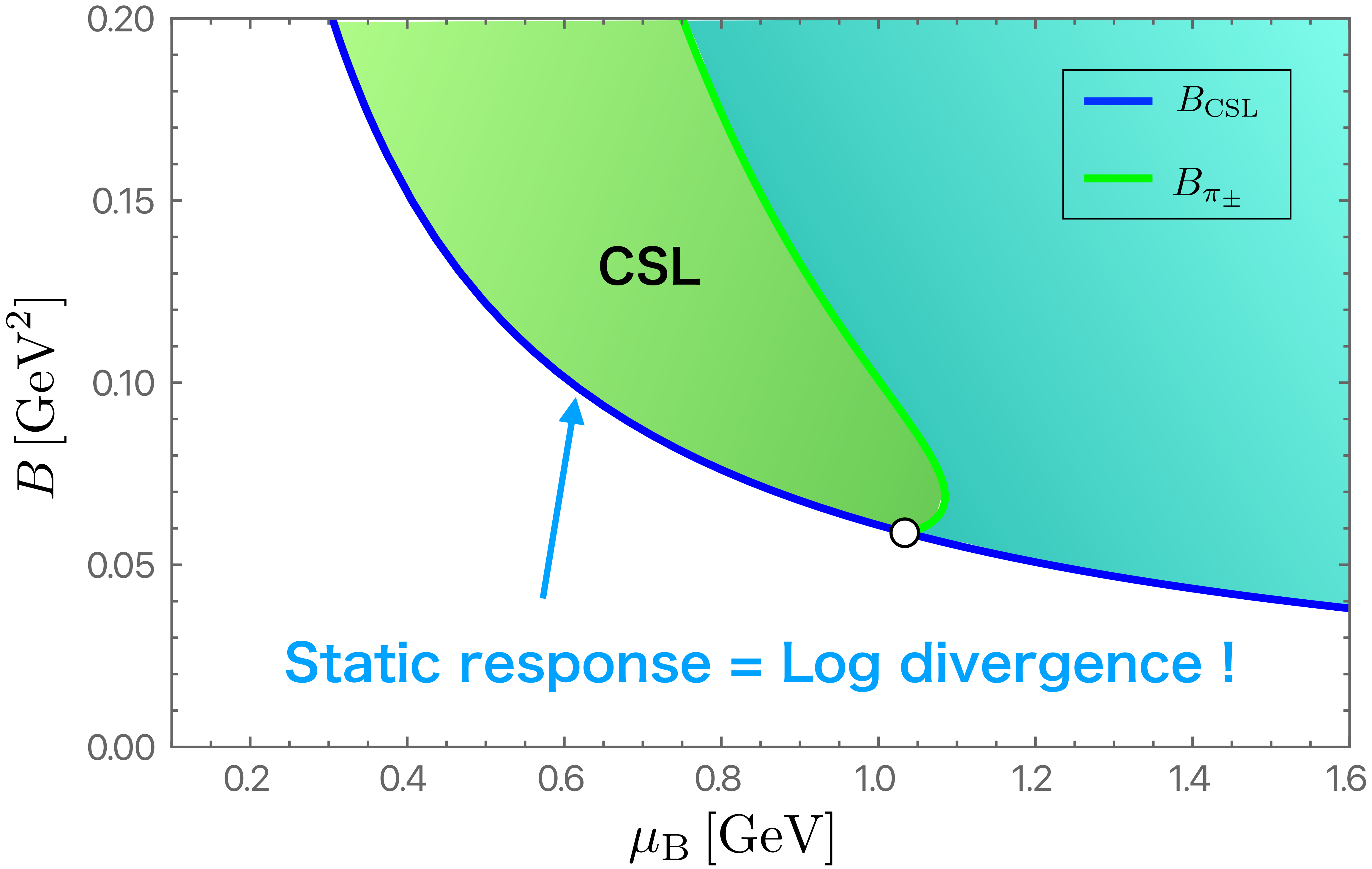
    See also Nishida (2004) and Nishida, Fukushima and Hatsuda (2004)

Other Future Directions

- **Log div (different from ordinary case):**

Solitonic phase

- CSL in QCD with μ_B and B
- CSL in chiral magnet Dzyaloshinsky, JETP (1965)
- Abrikosov lattice in superconductors "Superconductivity," Volume 2, Edited by R. D. Parks (1969)



- Investigation of dynamics near the phase transition to solitonic phases
- Application of the hydro w/ explicit symmetry breaking Hongo, Sogabe, Stephanov and Ho-Ung (2024)
- Previous work may be wrong… Nishimura and Sogabe (2024)

Thank you for your attention!

Back up

Chiral Soliton Lattice

- **EOM = Pendulum:** $\partial_z^2 \phi_3 = m_\pi^2 \sin \phi_3$

  $\phi_3(0) = \pi\,,\ \phi_3(\ell) = 2\pi$

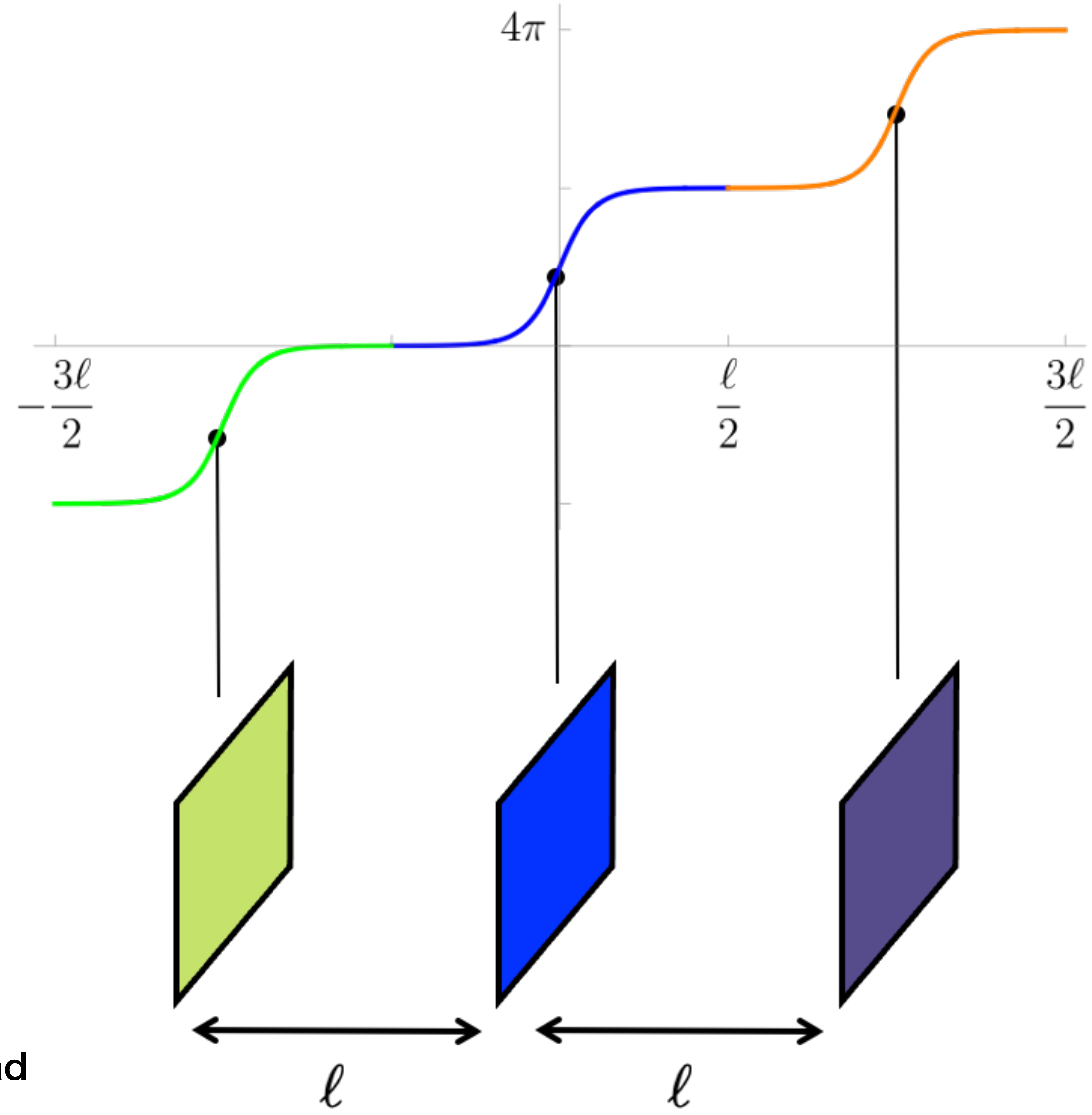
- **Analytic solution :** $\bar{\phi} = 2\mathrm{am}\left(z/\kappa, \kappa\right) + \pi$

  κ : Elliptic modulus

- **Period :** $\phi(z+\ell) = \phi(z) + 2\pi$

  $\ell = 2\kappa K(\kappa)$

  $K(\kappa)$: The complete elliptic integral of the first kind



Brauner and Yamamoto (2017)

Minimization of the total energy

- **Minimizing the total energy gives us the optimal κ.**

$$\mathcal{E}_{\mathrm{tot}} = \int_0^{\ell} \mathrm{d}z \left[\underbrace{\frac{f_\pi^2}{2}(\partial_z\phi)^2 + f_\pi^2 m_\pi^2 (1-\cos\phi)}_{\text{positive}} - \underbrace{\frac{\mu_{\mathrm{B}}}{4\pi^2} B\partial_z\phi}_{\text{negative!}} \right]$$

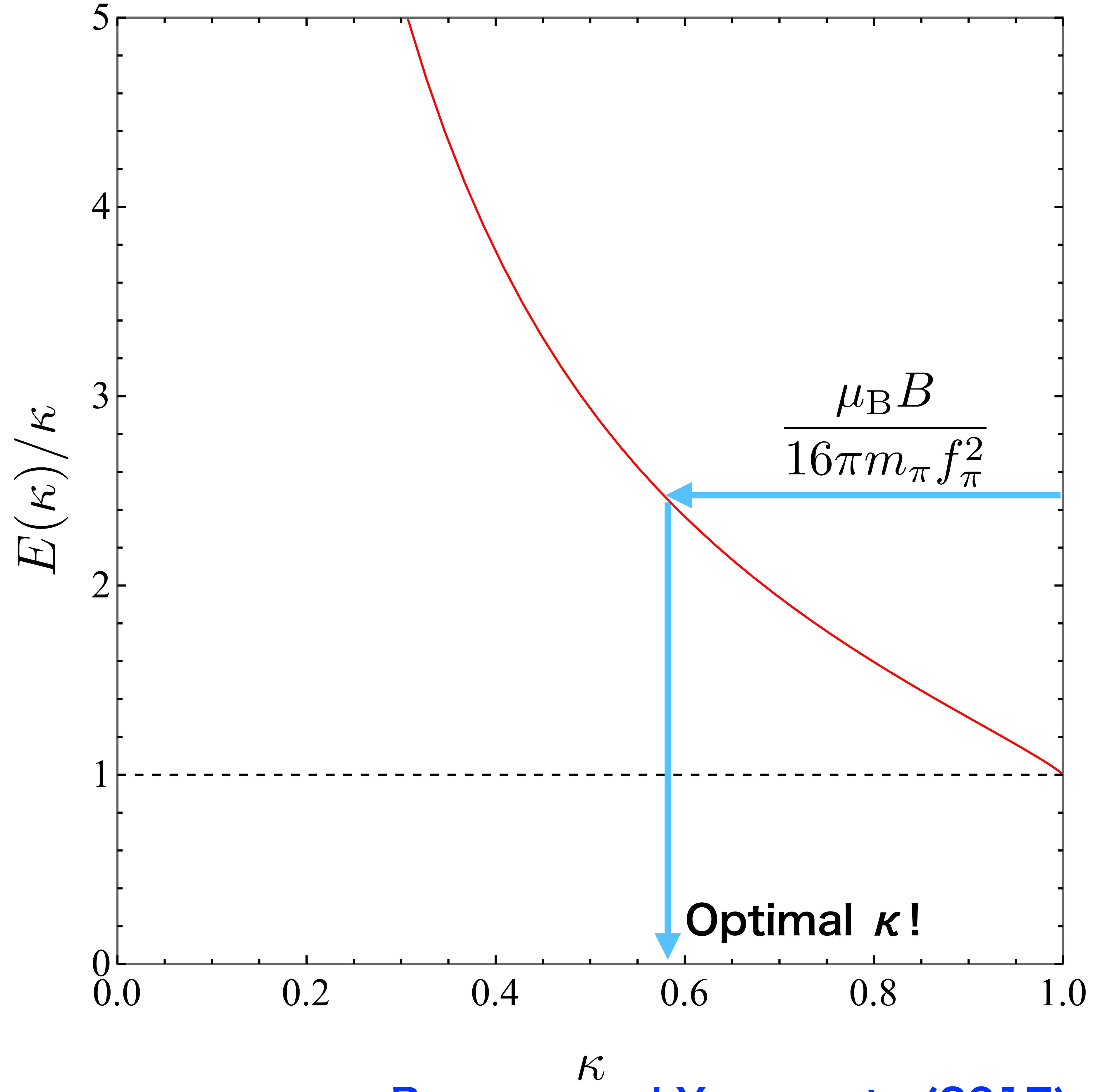
$\phi(\ell) - \phi(0) = 2\pi$

- **Energy minimization condition :**

$$\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{\mathcal{E}_{\mathrm{tot}}}{\ell}\right) \to \frac{E(\kappa)}{\kappa} = \frac{\mu_{\mathrm{B}} B}{16\pi m_\pi f_\pi^2}$$

$E(\kappa)$: The complete elliptic integral of the 2nd kind

- **Critical magnetic field :** $B_{\mathrm{CSL}} = 16\pi f_\pi^2 m_\pi / \mu_{\mathrm{B}}$



Brauner and Yamamoto (2017)

Is the CSL stable?

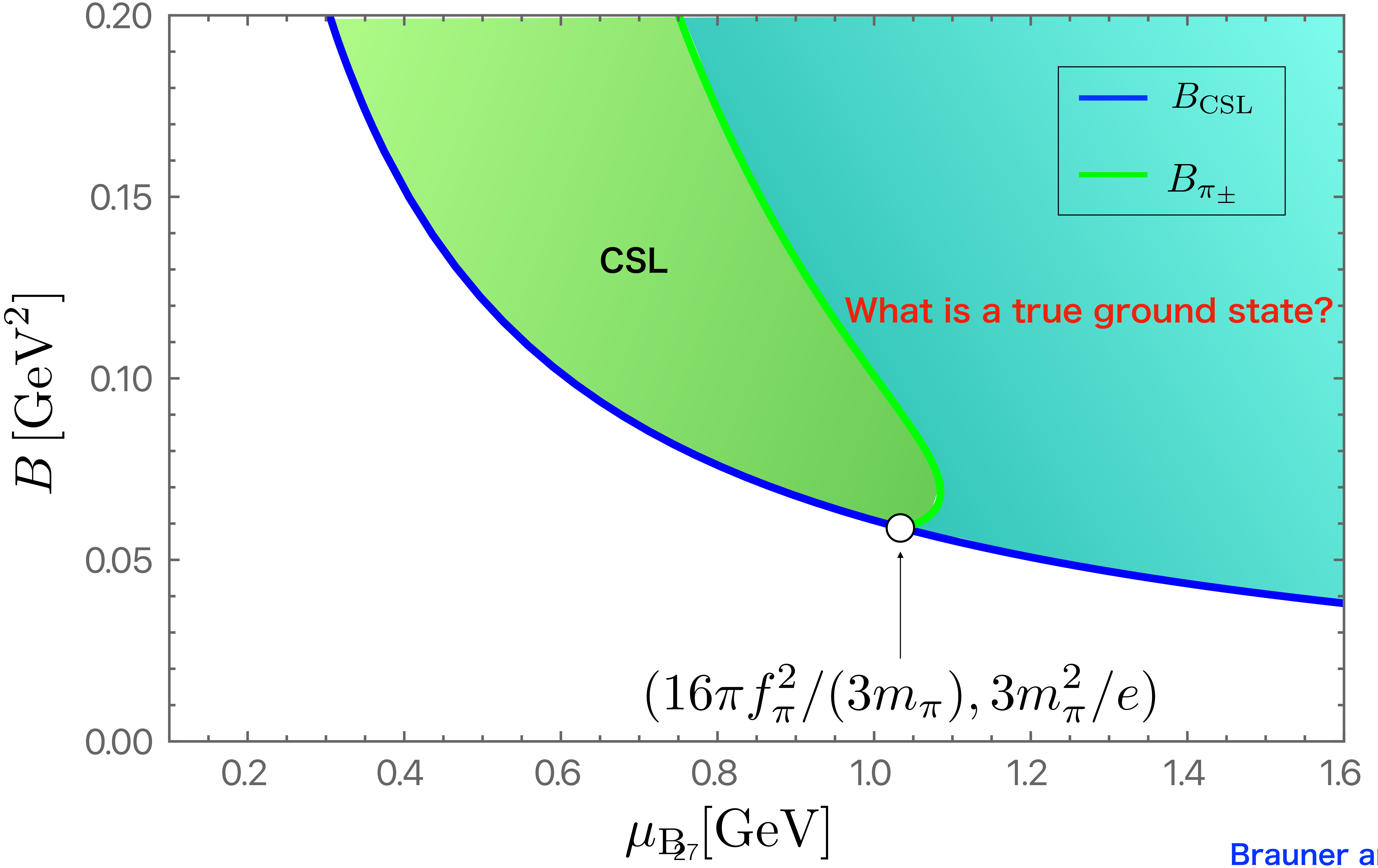
- Fluctuation around the CSL background :

- CSL is unstable against fluctuations of $\pi_\pm$ above $B_{\pi\pm}$
  - Derive the effective action up to the 2nd of the fluctuations from the CSL.
  - Calculate the dispersion relation ω.
  - When $\omega^2<0$, the fluctuation is tachyonic and CSL becomes unstable.

$$\boxed{B_{\pi_\pm} = \frac{m_\pi^2}{\kappa^2}\left(2-\kappa^2+2\sqrt{1-\kappa^2+\kappa^4}\right)} \longleftarrow \kappa = \kappa(B_{\pi_\pm}) \qquad \boxed{\frac{E(\kappa)}{\kappa} = \frac{\mu_\mathrm{B} B_{\pi_\pm}}{16\pi m_\pi f_\pi^2}}$$

Brauner and Yamamoto (2017)

μ_B-B phase diagram



Brauner and Yamamoto (2017)

EFT of the DW

- Construct DW world volume effective theory via the moduli approximation.

- This EFT identifies S^2 moduli as degrees of freedom.

- Promote the moduli to a field on 2+1 dim world volume :

$$\phi \to \phi(x^\alpha)\,, \quad (\alpha = 0, 1, 2)$$

- Integrating over the codimension z :

$$\mathcal{L}_{\mathrm{EFT}} = \int_{-\infty}^{\infty} \mathrm{d}z \, (\mathcal{L}_{\mathrm{ChPT}} + \mathcal{L}_{\mathrm{B}}) \xleftarrow{\text{Substitution}} \Sigma = \exp(2\mathrm{i}\theta\phi\phi^\dagger) u^{-\mathrm{i}\bar{\phi}_3}$$

Elliptic integrals and functions

- **The elliptic integral of the first kind :** $k' = \sqrt{1-k^2}$

$$K(k) = \int_0^{\pi/2} \mathrm{d}\theta \, \frac{1}{\sqrt{1-k^2\sin^2\theta}} \simeq \ln\frac{4}{k'^2} + \frac{k'^2}{4}\left(\ln\frac{4}{k'^2} - 1\right)$$

- **The elliptic integral of the second kind :**

$$E(k) = \int_0^{\pi/2} \mathrm{d}\theta \, \sqrt{1-k^2\sin^2\theta} \simeq 1 + \frac{k'^2}{2}\left(\ln\frac{4}{k'^2} - \frac{1}{2}\right)$$

EOM of the fluctuations

- **Fluctuation around the CSL background :**

$$\omega^2\pi_+ = \underbrace{\left[-\partial_x^2 + B^2\left(x - \frac{p_y}{B}\right)^2\right]}_{\text{Giving the Landau quantization}}\pi_+ + (\partial_z^2 + 2\mathrm{i}\partial_z + m_\pi^2 \mathrm{e}^{\mathrm{i}\phi_3})\pi_+$$

- **Chiral limit :** $\omega^2 = p_z^2 - \underbrace{\frac{\mu_B B p_z}{2\pi^2 f_\pi^2}}_{\text{Deducing the energy!}} + (2n+1)B$

- **ω^2<0 :** $B_{\pi_\pm} = \dfrac{16\pi^4 f_\pi^2}{\mu_B^2}$ Brauner and Yamamoto (2017)