Anisotropic linear waves in spin magnetohydrodynamics

Zhe Fang (Zhejiang), KH, Jin Hu (Fuzhou), 2402.18601; 2409.07096.
Cf. KH, X.-G. Huang, M. Hongo, 2207.12794 for a review.

Koichi Hattori
Zhejiang University

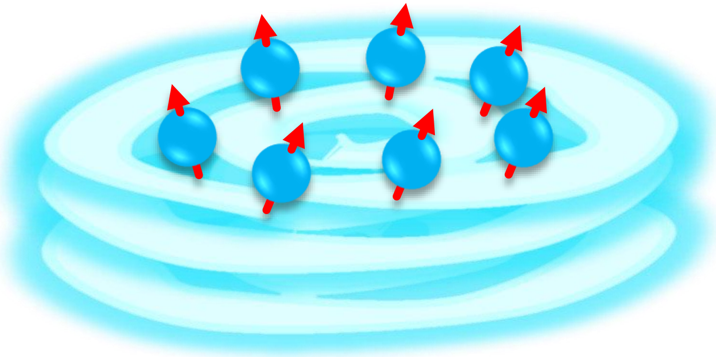
Topology and Dynamics of Magneto-Vortical Matter Jan. 13 - 24, 2025

Plan of this talk

1. Generality I: Quark-gluon plasma and relativistic heavy-ion collisions

2. Generality II: Relativistic hydrodynamics

3-1. Magnetohydrodynamics (MHD) and the linear waves
3-2 Chiral MHD and instability
3-3 Spin MHD

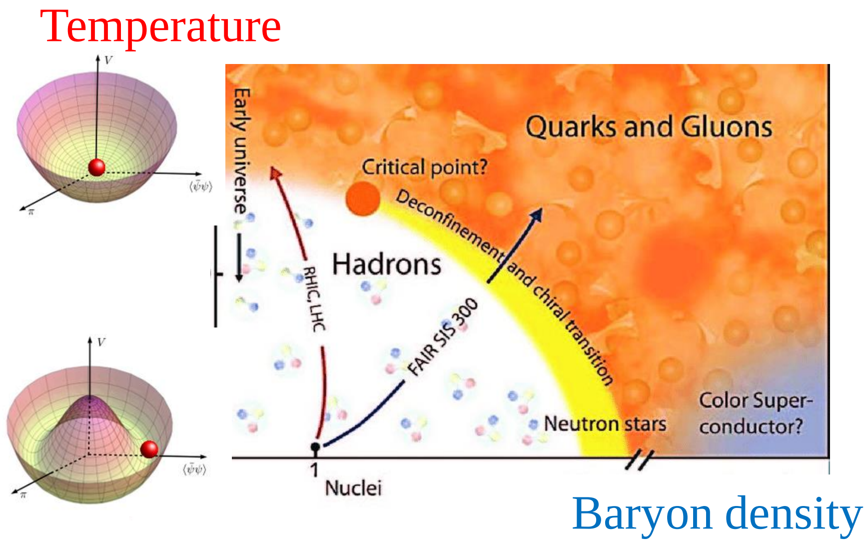


Generality I: Quark-gluon plasma and relativistic heavy-ion collisions

- *QCD phase diagram*
- *Relativistic heavy-ion collisions*
- *Magnetic and vortical fields*
- *Hadron polarization measurement*

"QCD phase diagram"

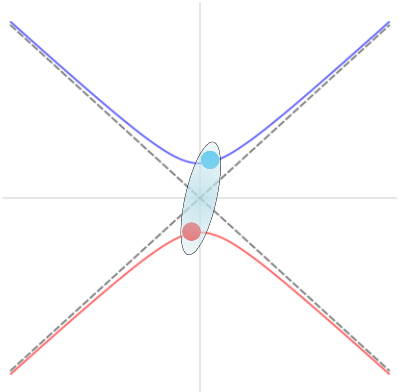
$$\mathcal{L}_{\rm QCD} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}G^{a\mu\nu}G^{a}_{\mu\nu}$$



Chiral symmetry

Paring induces a quark mass gap like a superconducting gap.

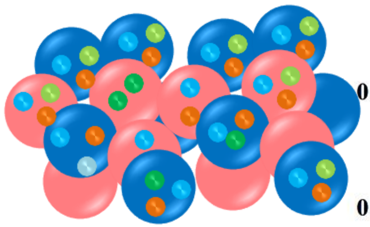
Order parameter: Chiral condensate $\langle \bar{q}q \rangle$

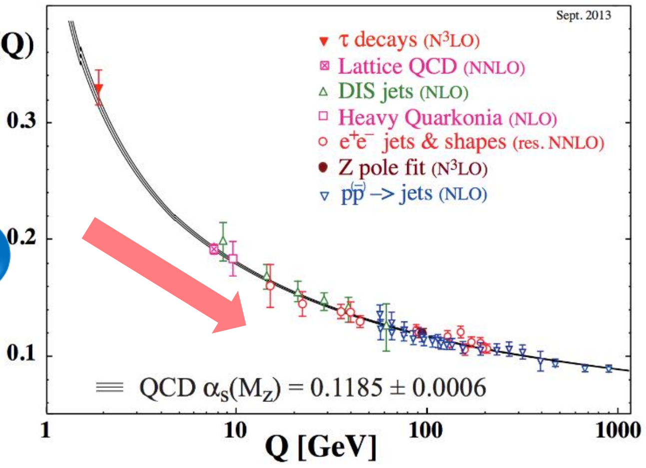


Confinement-deconfinement

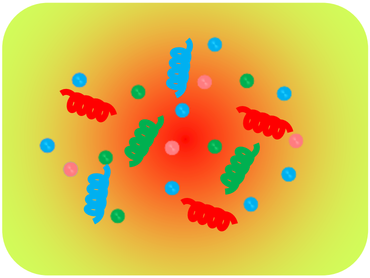
- Asymptotic freedom in QCD

Confinement





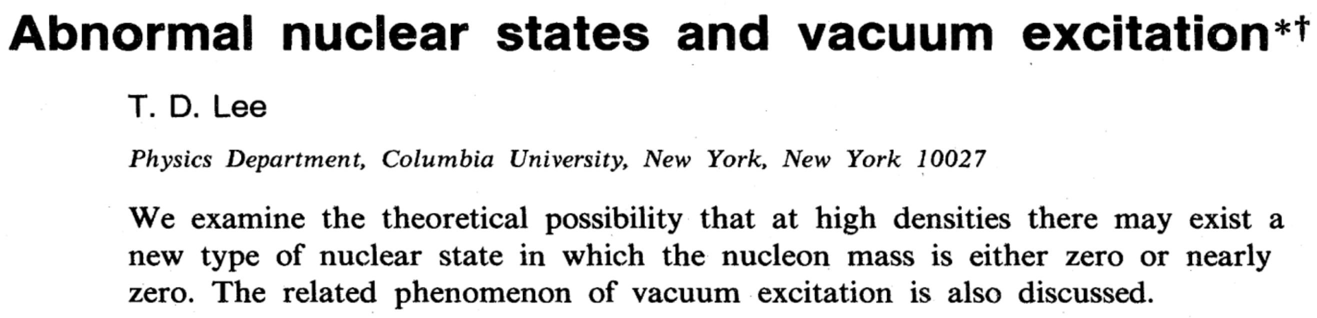
Deconfinement



Strong coupling
(Non-perturbative)

Weak coupling
(Perturbative)

Early ideas of relativistic heavy-ion collisions



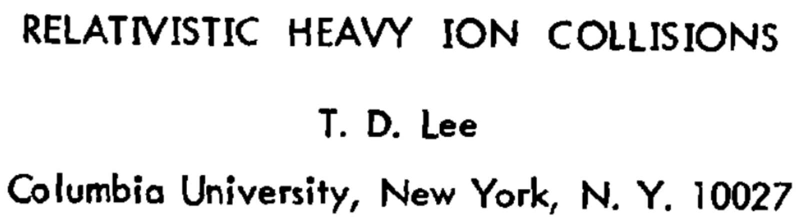
Abnormal nuclear states and vacuum excitation*†

T. D. Lee

Physics Department, Columbia University, New York, New York 10027

We examine the theoretical possibility that at high densities there may exist a new type of nuclear state in which the nucleon mass is either zero or nearly zero. The related phenomenon of vacuum excitation is also discussed.

T. D. Lee (1975)

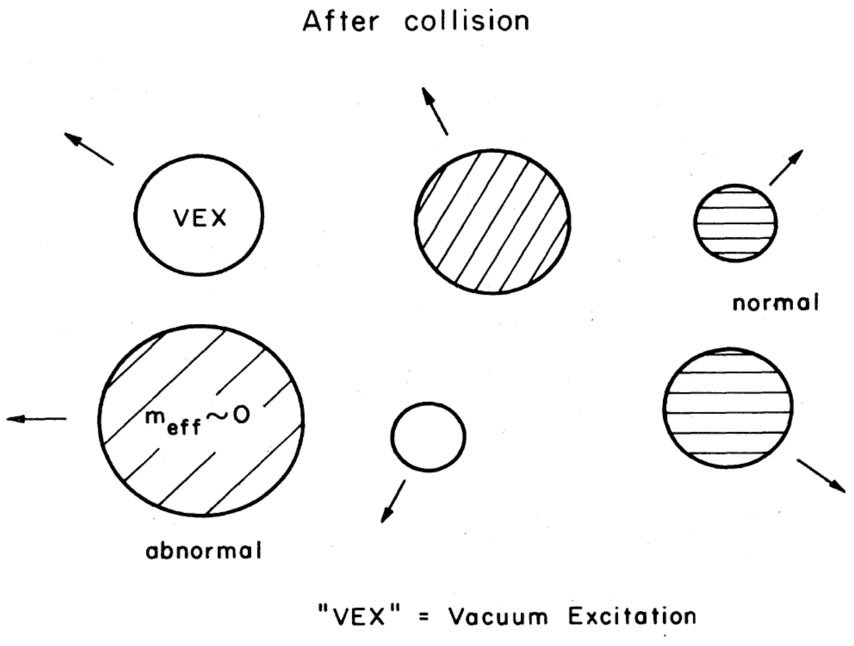


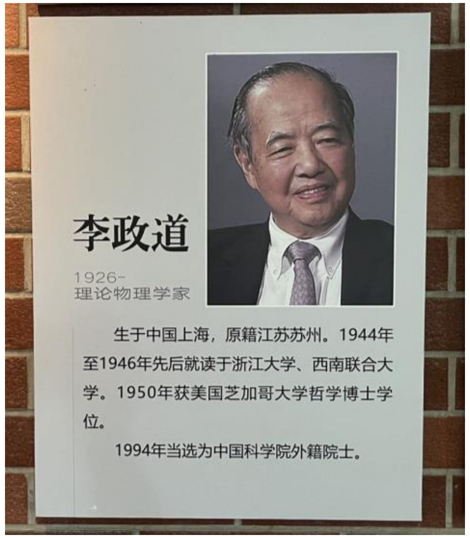
RELATIVISTIC HEAVY ION COLLISIONS

T. D. Lee

Columbia University, New York, N. Y. 10027

T. D. Lee (1982)





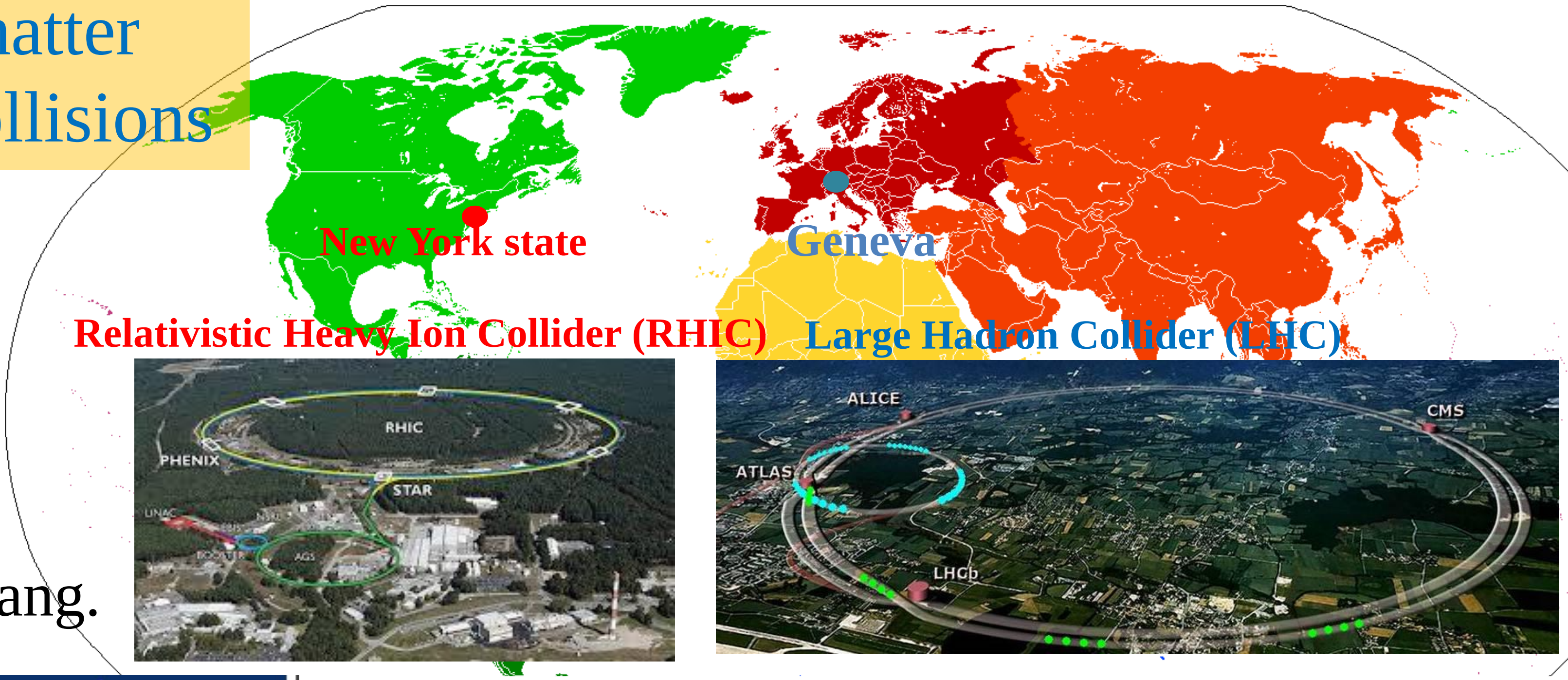
李政道

1926-
理论物理学家

生于中国上海，原籍江苏苏州。1944年至1946年先后就读于浙江大学、西南联合大学。1950年获美国芝加哥大学哲学博士学位。

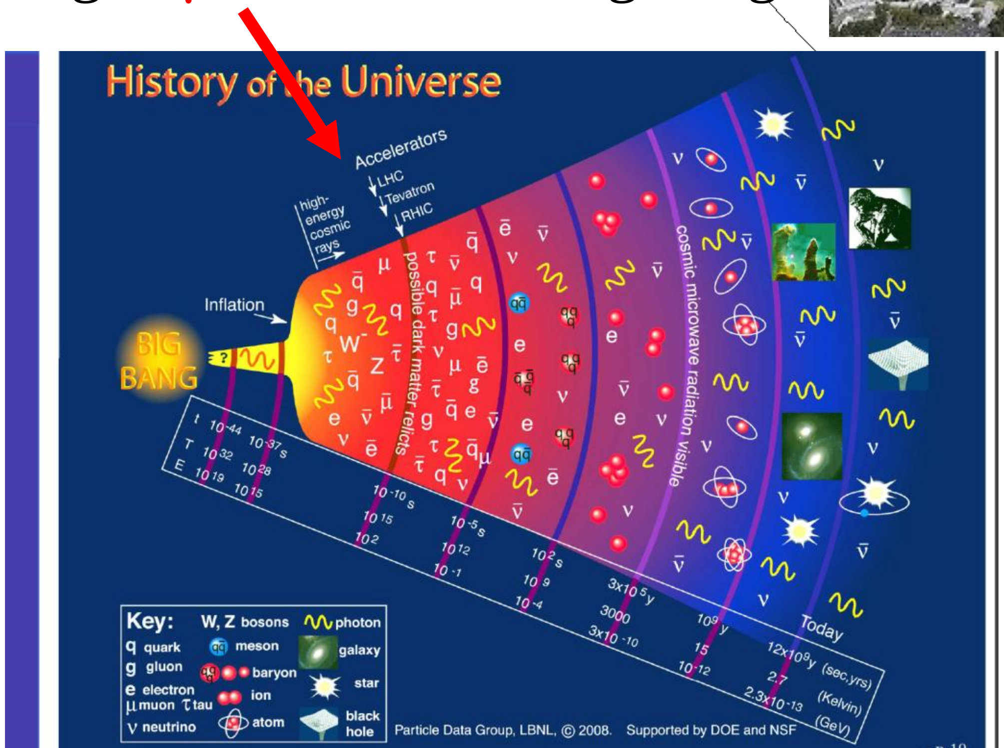
1994年当选为中国科学院外籍院士。

Quest to the extreme state of matter by the relativistic heavy-ion collisions

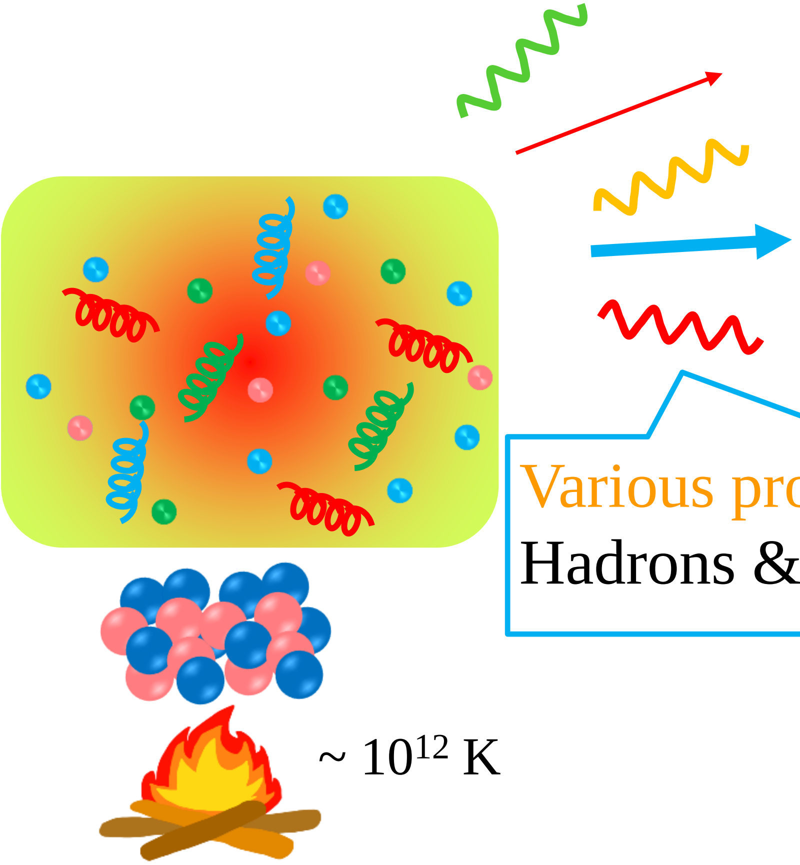


“Little bang”

Reproducing 10μ sec after the big bang.



Observables in heavy-ion collisions



γ

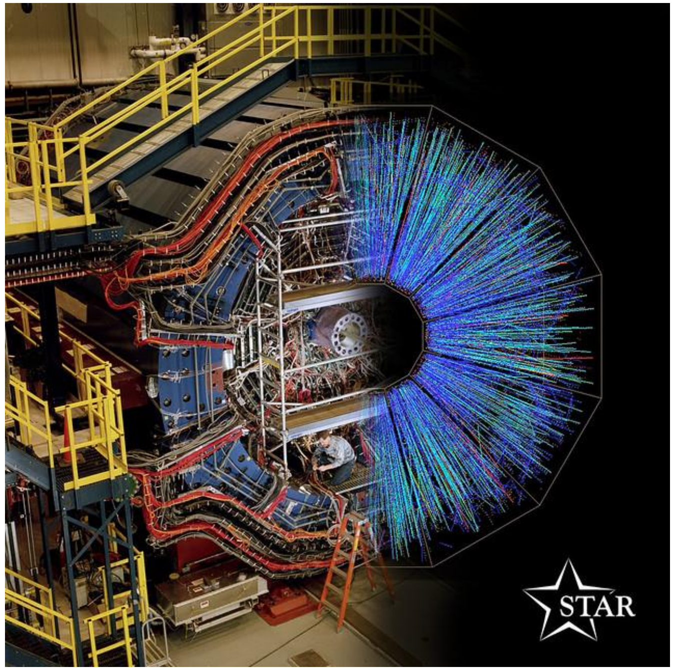
e, μ

π, p, n...
Flavors up to b quarks

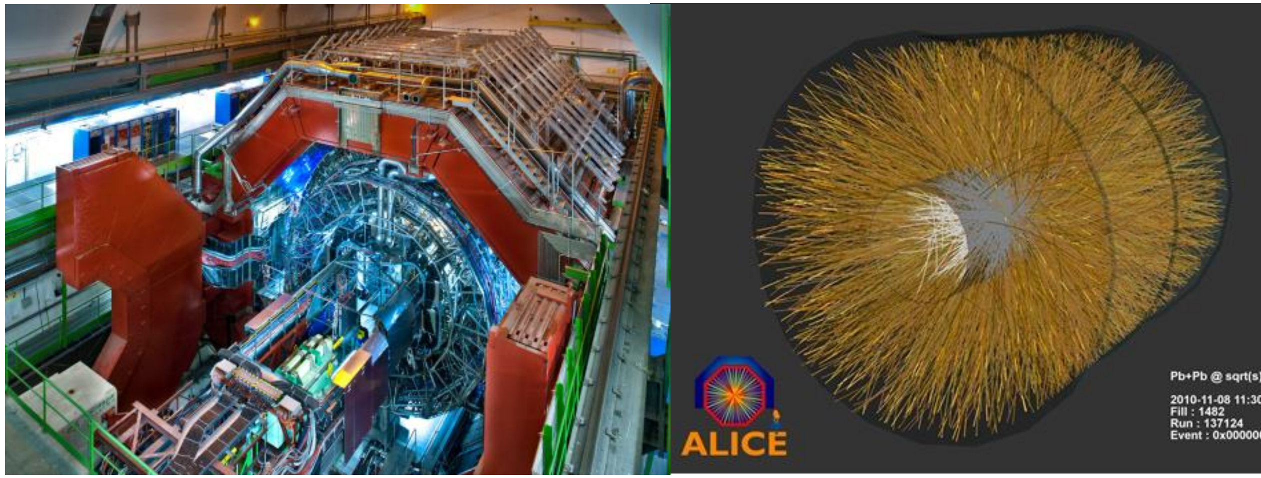
Various probes
Hadrons & photons/leptons

~ 10^{12} K

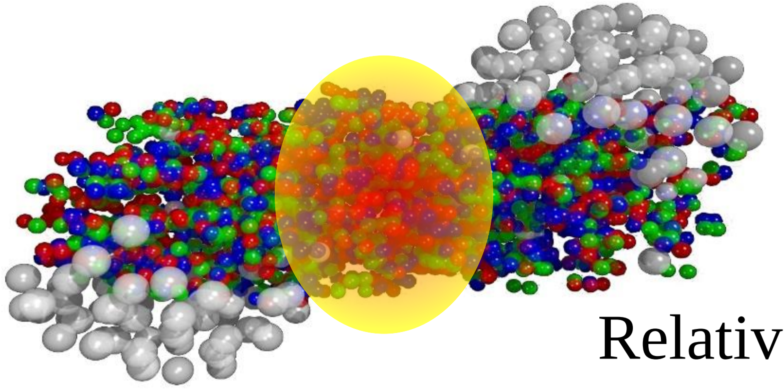
RHIC STAR collaboration



LHC ALICE collaboration



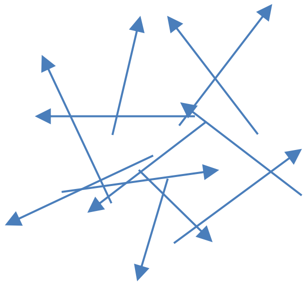
Hydrodynamics behavior of the quark-gluon plasma



Relativistic heavy-ion collisions at RHIC and LHC

✗ Free-streaming limit?
(Asymptotic-free limit)

✗ Random momentum distribution



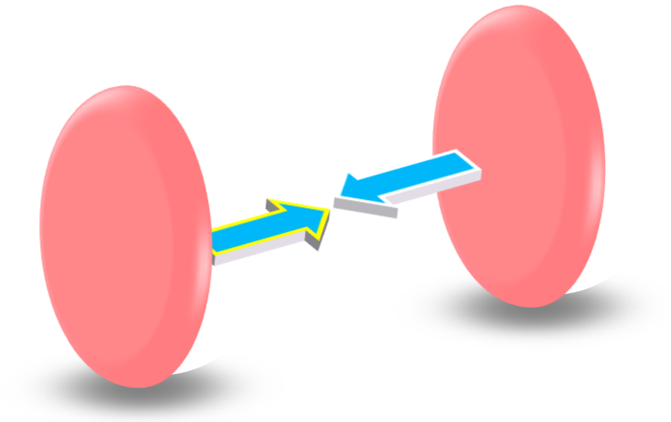
✓ Strongly correlated matter
→ Collective flow at RHIC and LHC

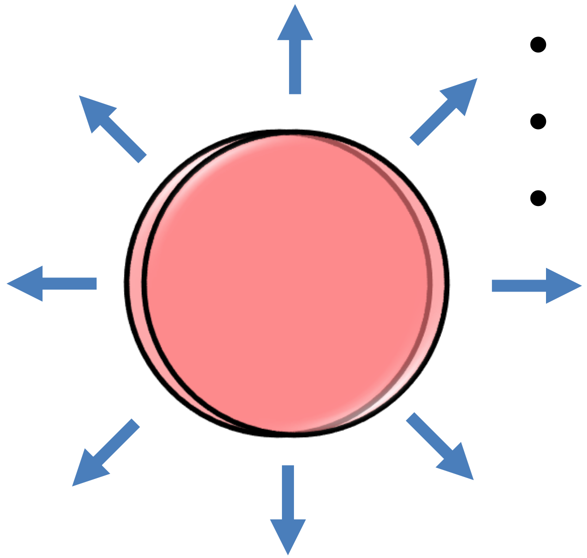
✓Correlated momentum angle distribution

Frequent interactions induce strong correlations (even if the coupling constant is perturbatively small.)

Hydrodynamic expansion

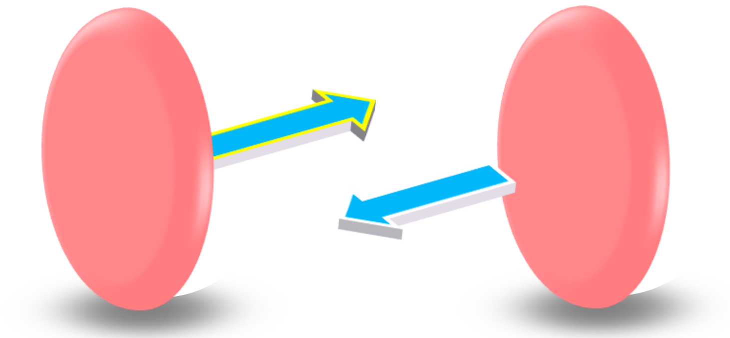
Central collisions (Head-on collisions)

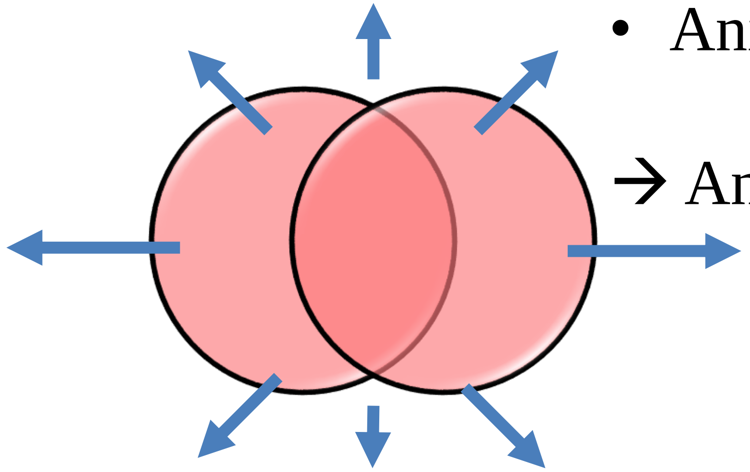




- Isotropic overlap region
- Isotropic pressure gradient
- Isotropic expansion

Non-central collisions (Peripheral collisions)

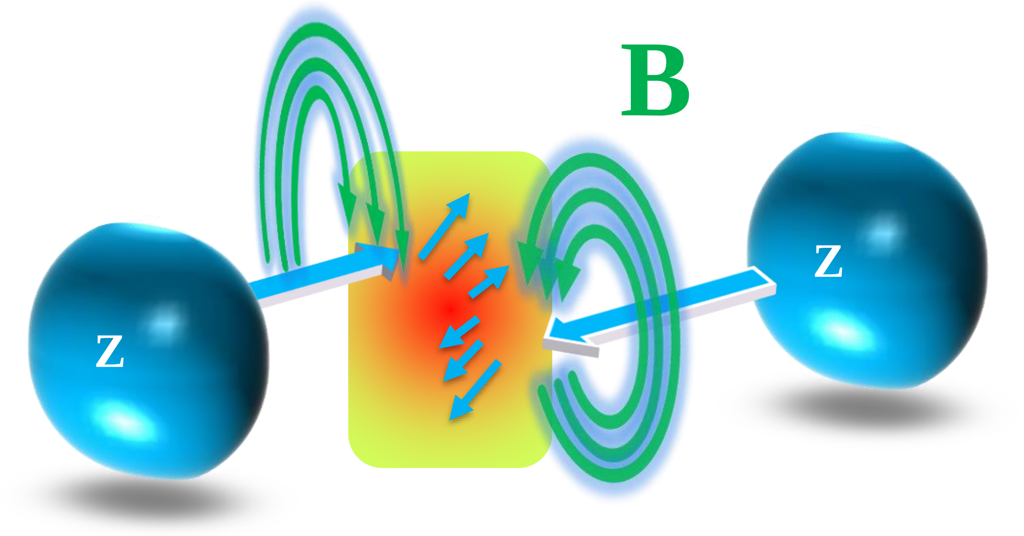




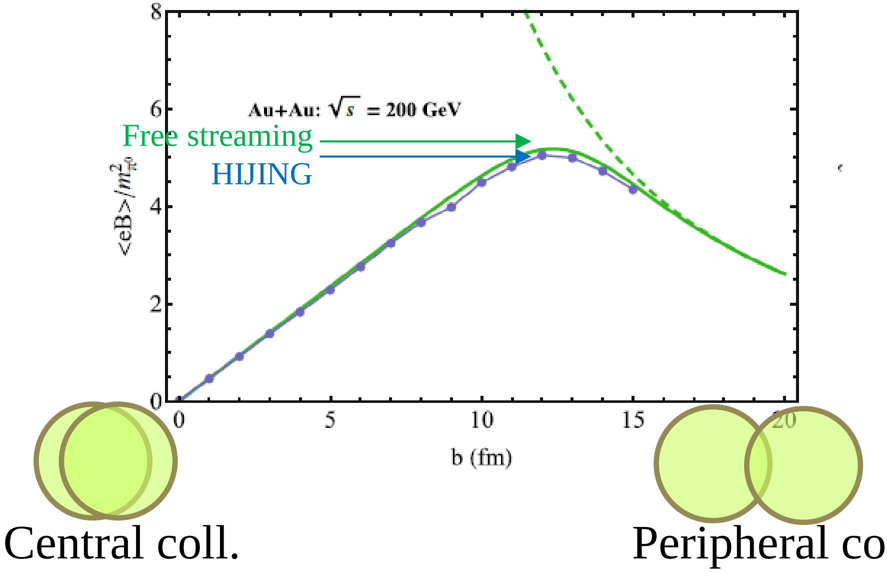
- Anisotropic overlap region
- Anisotropic pressure gradient
- Anisotropic expansion

→ Anisotropic hadron spectrum

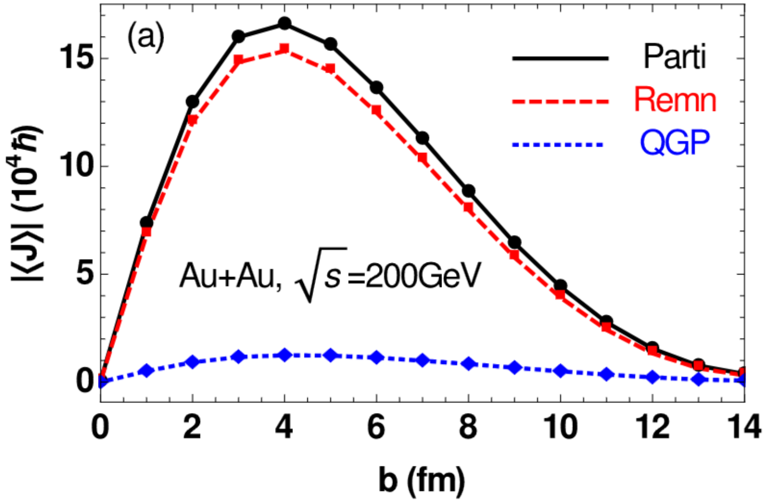
Strong magnetic and vortical fields



$\omega = \nabla \times \mathbf{v}$

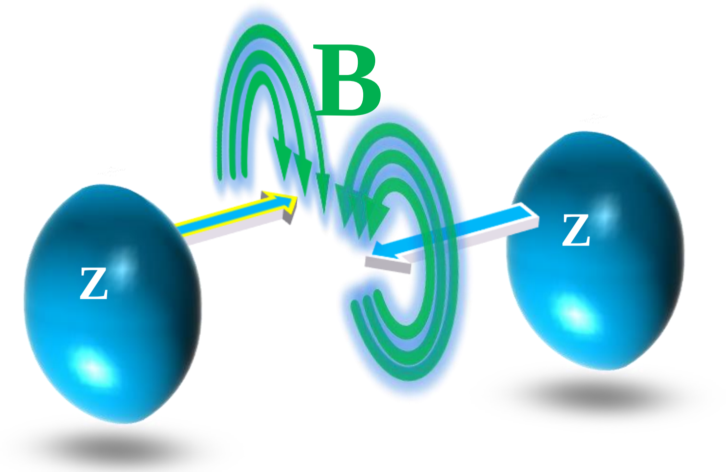


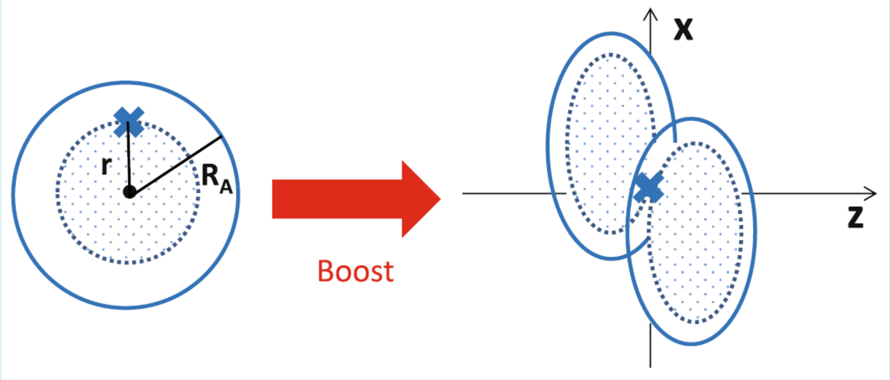
Deng, Huang 1201.5108; KH, Huang 1609.00747



Deng, Huang, 1603.06117

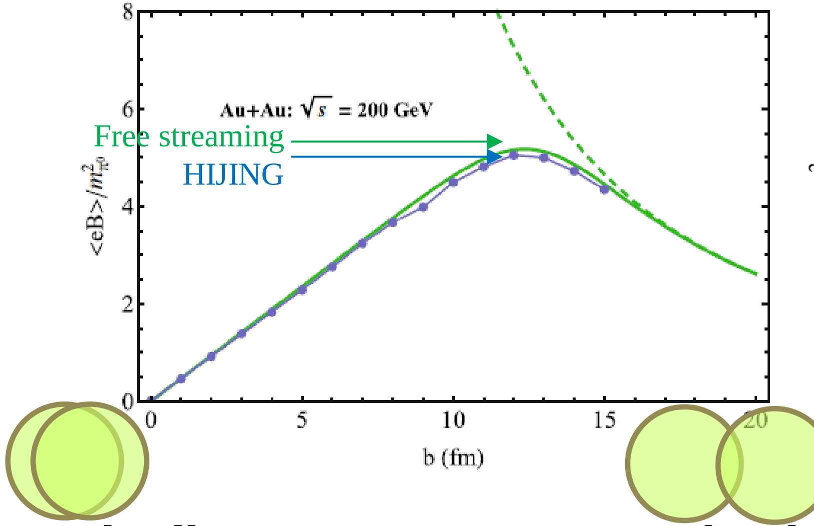
Strong magnetic fields induced by relativistic heavy-ion collisions





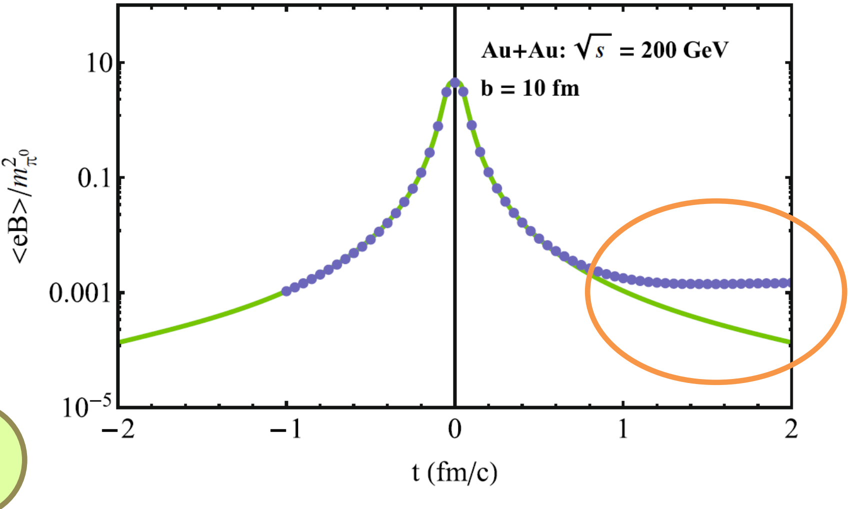
Static Coulomb field

Boost → Lienard-Wiechert potential



Central coll.

Peripheral coll.



Deng & Huang; KH & Huang [1609.00747]

Chiral magnetic effect

Vilenkin (1982)
Nielsen, Ninomiya (1983)
Kharzeev, McLerran, Warringa (2007)
Kharzeev, Fukushima, Warringa (2008)

Chiral magnetic effect

$$j_V^\mu = \sigma_{\mathrm{CME}} B^\mu$$

Ohmic current

$$j_V^\mu = \sigma_{\mathrm{Ohm}} E^\mu$$

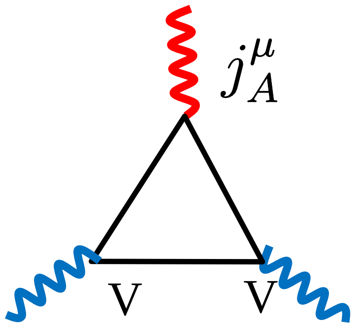
Parity	Odd	Odd	Even	Odd	Even	Odd

Parity-odd environment is provided by the chirality imbalance:

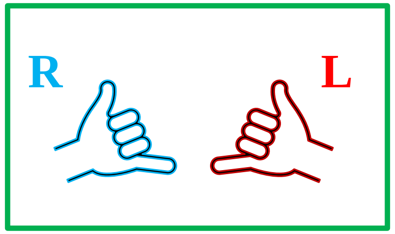
$$\mu_A = (\mu_R - \mu_L)/2$$

Anomalous transport coefficients ∝ Anomaly coefficient C_A

$$\sigma_{\mathrm{CME}} \propto \mu_A C_A \qquad \partial_\mu j_A^\mu = -e^2 C_A E^\mu B_\mu$$



Chirality-momentum locking



Dirac equation in the chiral representation
→ Chirality-spin-momentum locked in the massless limit

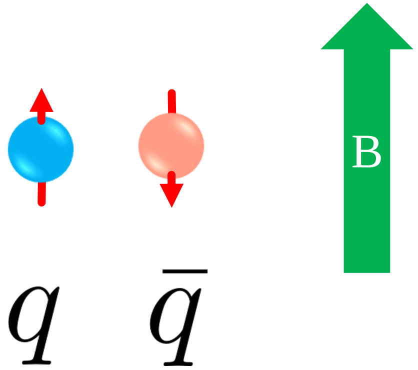
{Chirality, Spin, Momentum}

$$\begin{pmatrix} -m & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

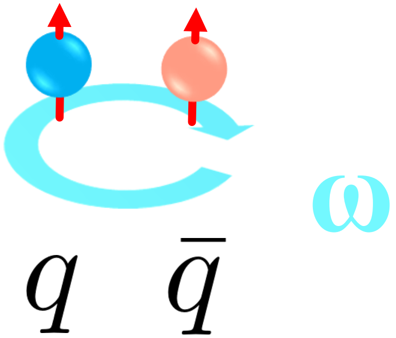
Spin polarization in B or ω

Chirality-momentum locking

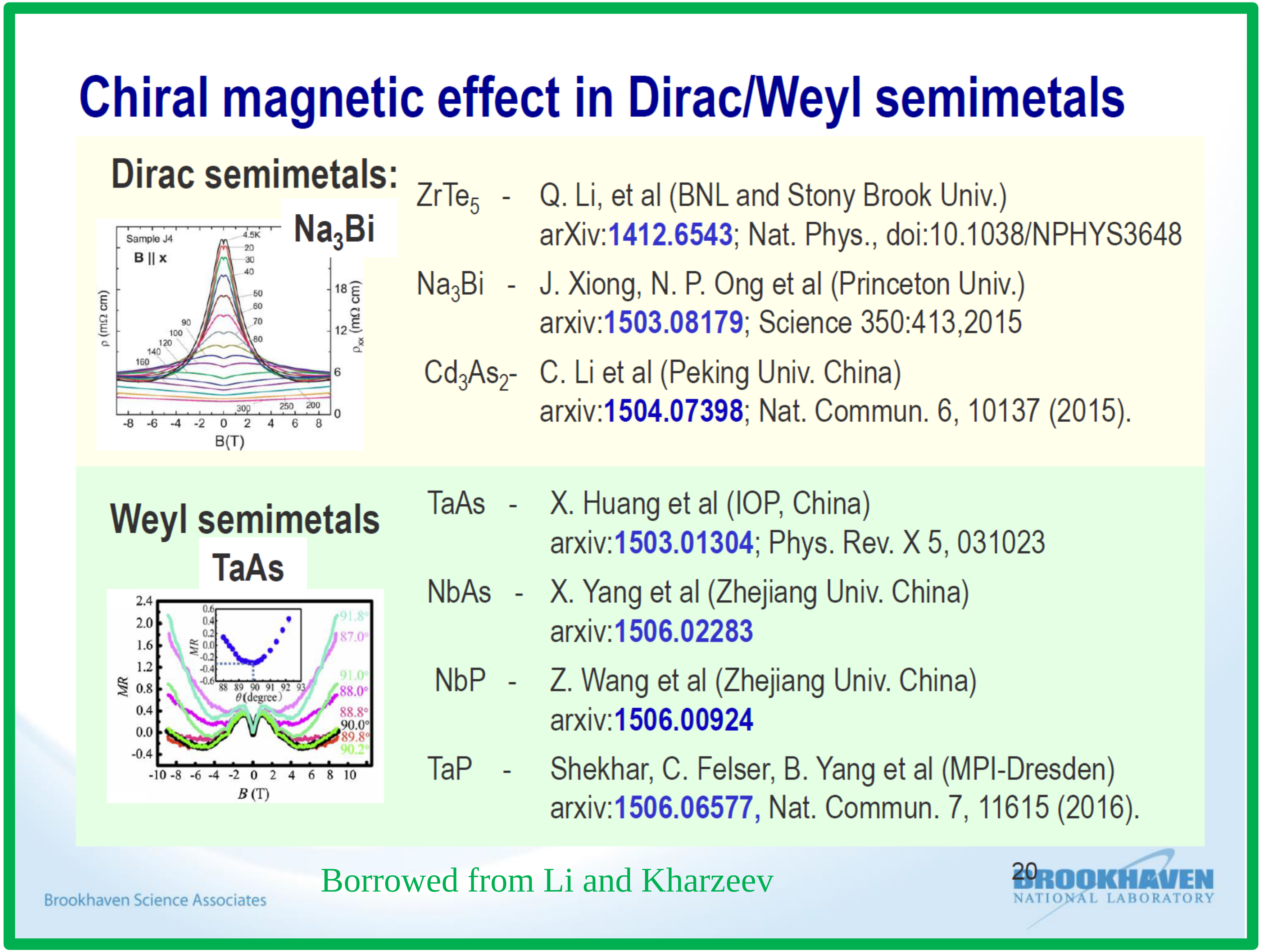
→ Finite currents when one of chirality is favored (via anomaly).



$$\boldsymbol{j}_{\mathrm{CME}} \propto \mu_A \boldsymbol{B}$$



$$\boldsymbol{j}_{\mathrm{CVE}} \propto \mu \mu_A \boldsymbol{\omega}$$



Chiral magnetic effect in Dirac/Weyl semimetals

Dirac semimetals:

Na_3Bi

$ZrTe_5$ - Q. Li, et al (BNL and Stony Brook Univ.)
arXiv:**1412.6543**; Nat. Phys., doi:10.1038/NPHYS3648

Na_3Bi - J. Xiong, N. P. Ong et al (Princeton Univ.)
arxiv:**1503.08179**; Science 350:413,2015

Cd_3As_2- C. Li et al (Peking Univ. China)
arxiv:**1504.07398**; Nat. Commun. 6, 10137 (2015).

Weyl semimetals

TaAs

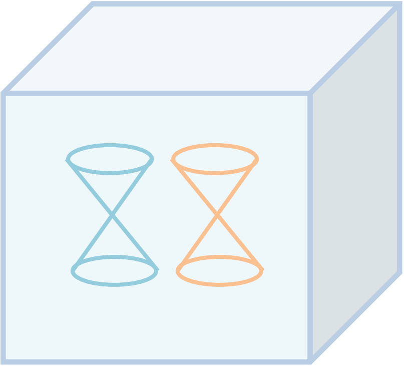
TaAs - X. Huang et al (IOP, China)
arxiv:**1503.01304**; Phys. Rev. X 5, 031023

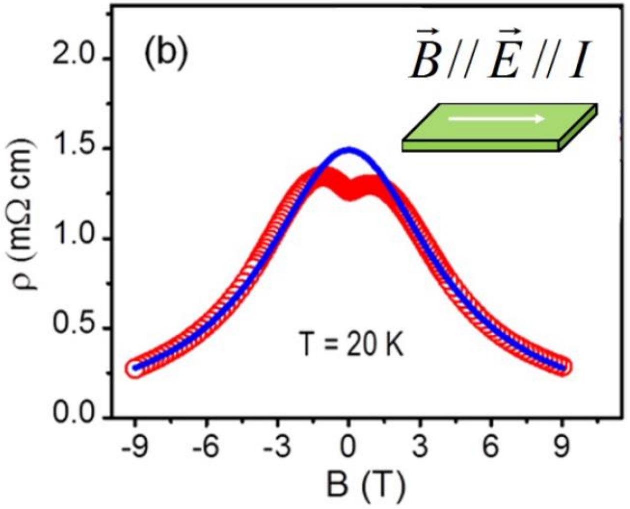
NbAs - X. Yang et al (Zhejiang Univ. China)
arxiv:**1506.02283**

NbP - Z. Wang et al (Zhejiang Univ. China)
arxiv:**1506.00924**

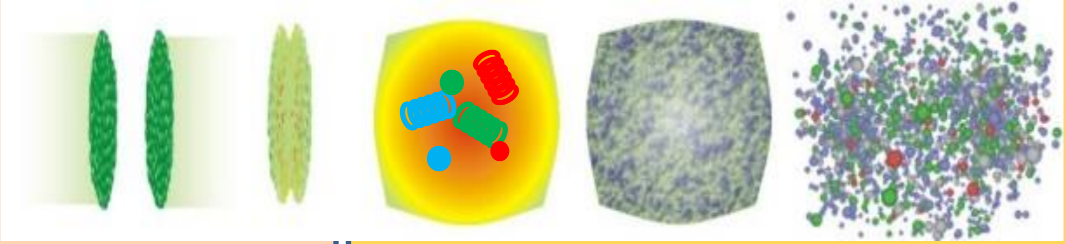
TaP - Shekhar, C. Felser, B. Yang et al (MPI-Dresden)
arxiv:**1506.06577**, Nat. Commun. 7, 11615 (2016).

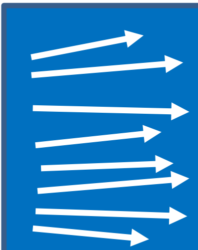
Borrowed from Li and Kharzeev



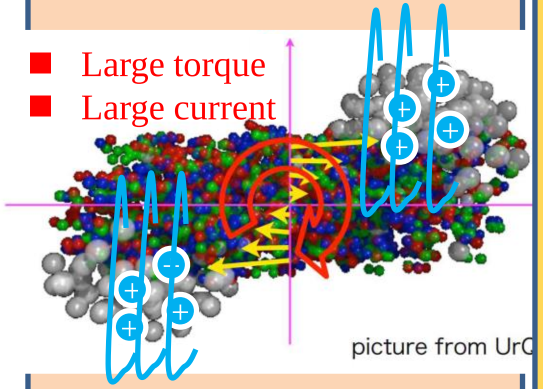


Hadron spin measurements





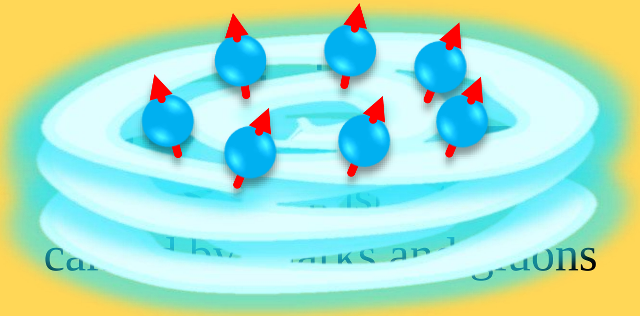
Collisions → Thermal QGP → Hardonization → Measurements



- Large torque
- Large current

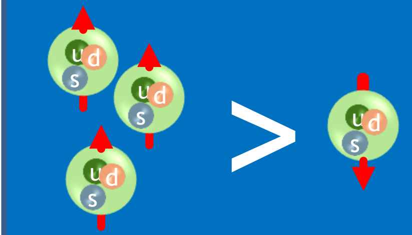
picture from Ur

Strong B and ω
in the initial state



"Relativistic spintronics" in QGP

Spin-polarized hadrons



STAR (2007 – 2017, 2023)

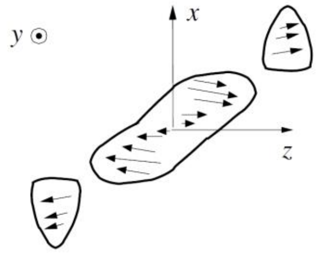
$\Lambda_{\uparrow} > \Lambda_{\downarrow}$

PRL 94, 102301 (2005) PHYSICAL REVIEW LETTERS week ending 18 MARCH 2005

Globally Polarized Quark-Gluon Plasma in Noncentral $A + A$ Collisions

Zuo-Tang Liang[1] and Xin-Nian Wang[2,1]
[1]*Department of Physics, Shandong University, Jinan, Shandong 250100, China*
[2]*Nuclear Science Division, MS 70R0319, Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*

(Received 25 October 2004; published 14 March 2005)



Spin transport theories for QGP

Spin hydrodynamics

KH, Hongo, Huang, Matsuo, Taya, 1901.06615
Fukushima, Pu, 2010.01608
Li, Stephanov, Yee, 2011.12318
She, Huang, Hou, Liao, 2105.04060
Hongo, Huang, Kaminski, Stephanov, Yee, 2107.14231
Biswas, Daher, Das, Florkowski, Ryblewski, 2304.01009
etc

Quantum kinetic theory

Weickgenannt, Sheng, Speranza, Wang, Rischke, 1902.06513
Gao, Liang, 1902.06510
KH, Hidaka, Yang, 1903.01653; 2002.02612
Liu, Mameda, Huang, 2002.03753
etc.

Generality II: Relativistic hydrodynamics

\+

Magnetic field

Chiral anomaly

Spin dof

⇒

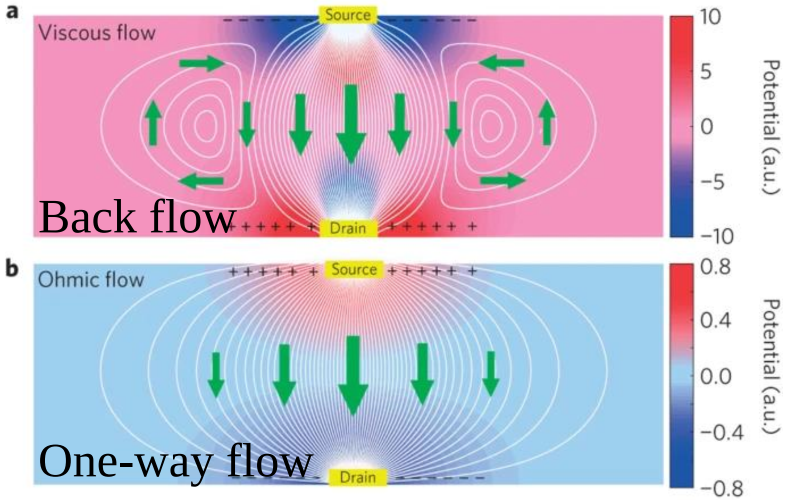
Magneto-hydrodynamics

Chiral hydrodynamics

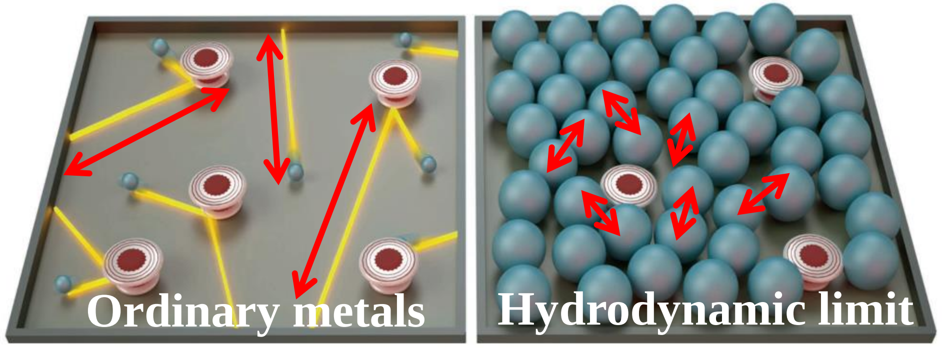
Spin hydrodynamics

Collective "hydrodynamic" motion in various systems

Condensed matter systems

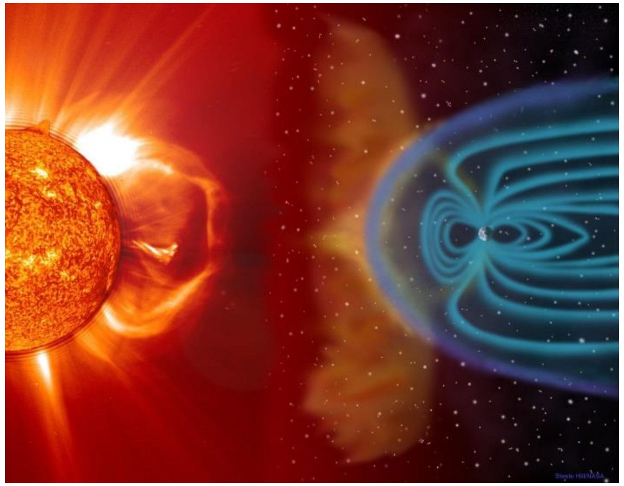


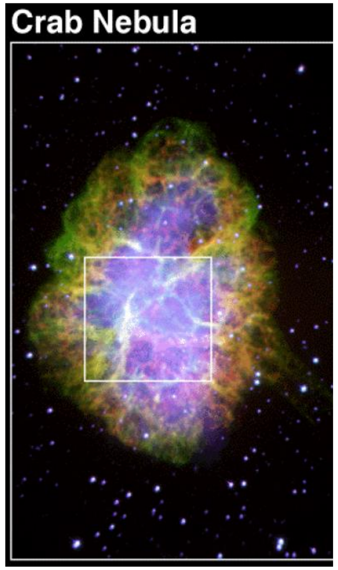
Graphene
Nature Phys., Levitov & Falkovich

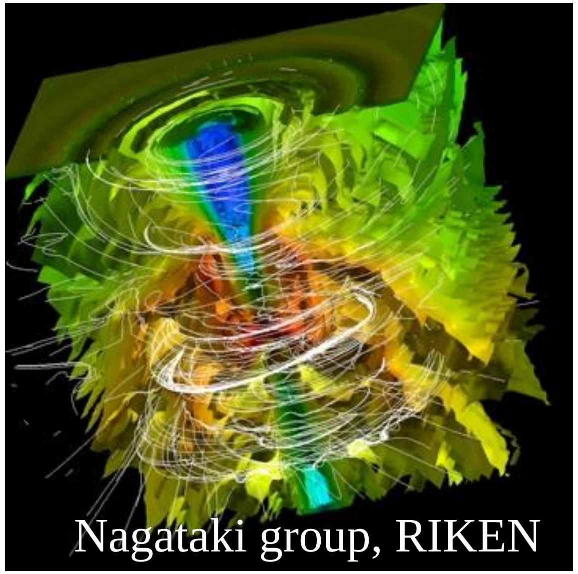


Science, Zaanen

Astrophysics







Nagataki group, RIKEN

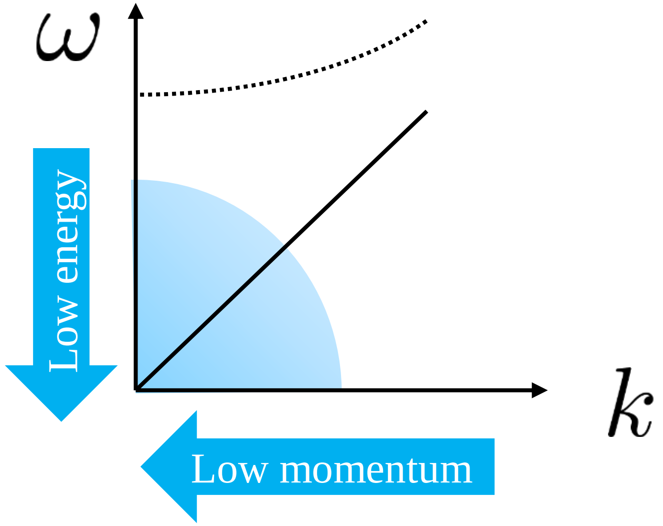
Hydrodynamics as universal low-energy EFT

Relevant dof. in low energy regime = Gapless modes
= Conserved charges surviving in a long spacetime scale.

A set of conservation laws
= Equations of motion

E.g., Translational symmetry

$$\partial_\mu T^{\mu\nu} = 0$$



Hydrodynamics = Universal low-energy EFT based on symmetries

Closing the system of equations

E.g., the simplest case

$$\partial_\mu T^{\mu\nu} = 0$$

Spacetime translational symmetry

# of conserved charges = # of equations

To get a closed system of eqs., one needs constitutive eqs.

$$T^{ij}[T^{0\mu}] = \sum_{n=0}^{\infty} \partial^{(n)} T^{0\mu}$$

Derivative expansion works
in the long spacetime scale.

Derivation of constitutive equations

See a review, KH, Huang, Hongo (2022)

Statistical description of underlying theory

Weak coupling

Strong coupling

Kinetic theory

Operator method (QFT)

Grad, Israel-Stewart,
Frankfurt group

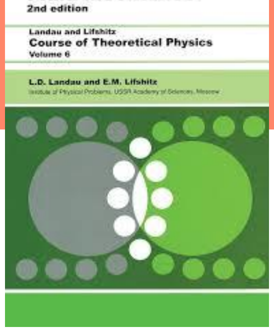
Zubarev, Becattini, Hayata-Hidaka-Noumi-Hongo, Hongo-KH (for MHD), Hongo-Huang-Stephanov-Yee (for spin)

Dynamics of averaged charges
(Course graining)

Thermodynamics

Constraints on constitutive eqs. from the 2nd law
aka the "entropy-current analysis"

Hydrodynamics



Fluid Mechanics
2nd edition
Landau and Lifshitz
Course of Theoretical Physics
Volume 6
L.D. Landau and E.M. Lifshitz
Institute of Physical Problems, USSR Academy of Sciences, Moscow

Simplest case

Translational symmetries

$$\partial_\mu \Theta^{\mu\nu} = 0$$

Hydrodynamic variables

$$\{\, \epsilon, v^i \,\}$$

Thermodynamics

- 1st law $\quad Tds = d\epsilon \qquad \beta \equiv \dfrac{\partial s}{\partial \epsilon}$

Inverse temperature
- Thermodynamic conjugates
  (Lagrange multipliers)

- 2nd law $\quad \partial_\mu s^\mu \geq 0$

Fluid velocity: $u^\mu = (1, v^i)/\sqrt{1 - |\boldsymbol{v}|^2}$

Co-moving derivative: $D \equiv u^\mu \partial_\mu$

$$\begin{aligned} \partial_\mu s^\mu &= \partial_\mu (s u^\mu + \delta s^\mu) \\ &= s(\partial_\mu u^\mu) + \beta D\epsilon + \partial_\mu \delta s^\mu \geq 0 \end{aligned}$$

Derivatives of conserved charges
→ Connection to equations of motion

“Entropy-current analysis”

$$D \equiv u^\mu \partial_\mu$$

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$$

$$\text{Cf.} \quad \Theta^{\mu\nu} = \begin{pmatrix} \epsilon & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

in the rest frame

$$\partial_\mu s^\mu \quad = \quad s(\partial_\mu u^\mu) + \beta D\epsilon + \partial_\mu \delta s^\mu \geq 0$$

⬇

$$\Theta^{\mu\nu} = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \Theta^{\mu\nu}_{(1)}$$

$$\partial_\mu s^\mu \quad = \quad \boxed{\beta(Ts - \epsilon - p)\partial_\mu u^\mu}$$
$$\boxed{-\Theta^{\mu\nu}_{(1)}\partial_\mu\beta_\nu} + \partial_\mu(s^\mu_{(1)} + \beta u_\nu \Theta^{\mu\nu}_{(1)})$$

$$\beta^\mu = \beta u^\mu$$

Leading order (ideal fluid)

- Should be vanishing

NLO in derivative

- Should be semi-positive

Constraints from the second law

LO $$p = Ts - \epsilon$$

NLO $$\Theta^{\mu\nu}_{(1)} = [\zeta \Delta^{\mu\nu}\Delta^{\alpha\beta} + \eta(\Delta^{\mu\langle\alpha}\Delta^{\beta\rangle\nu})]\partial_\alpha\beta_\beta$$

Trace part
→ Bulk viscosity

Traceless part
→ Shear viscosity

$$-\Theta^{\mu\nu}_{(1)}\partial_\mu\beta_\nu \geq 0 \text{ if } \zeta,\ \eta \geq 0$$

Formulation of magnetohydrodynamics with the entropy-current analysis

KH, Hirono, Yee, Yin (2017)
Hongo, KH (2020) for a QFT approach
KH, Hongo, Huang (2022) for a review

Fang, KH, Hu, 2402.18601 for solutions

Conserved charges in EM fields

Grozdanov et al. (2017)
KH, Hirono, Yee, Yin (2017)
Hongo, KH (2020)

Hydrodynamic variables

Conservation laws

{Total energy density, fluid flow velocity}

Translational symmetries

$$\{\epsilon,\ u^\mu\} \quad \Longleftrightarrow \quad \partial_\mu T^{\mu\nu}_{\mathrm{total}} = 0$$

Electromagnetism

$$\boldsymbol{B} \quad \Longleftrightarrow \quad \partial_\mu \tilde{F}^{\mu\nu} = 0$$

A static charge distribution does not screen the B-field.
(No "magnetic Coulomb field" without a magnetic monopole).

$$B \not\to 0$$

- Conservation of magnetic flux.

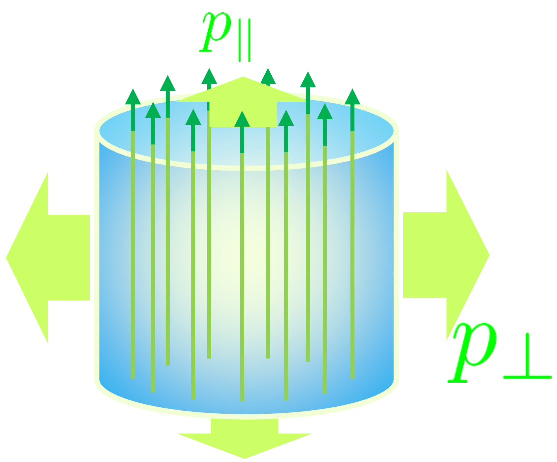
E-field is screened by a static charge distribution, i.e., the Debye screening effect.

$$E \to 0$$

E(u, B) and j(u, B) are induced at off-equilibrium

Anisotropies in constitutive equations

$$\Xi^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu - b^\mu b^\nu$$



Anisotropic pressure

$$\Theta^{\mu\nu}_{(0)} \quad = \quad \epsilon u^\mu u^\nu - p_\perp \Xi^{\mu\nu} + p_\parallel b^\mu b^\nu$$

$p_\perp$ and $p_\parallel$ are different by the Maxwell stress.

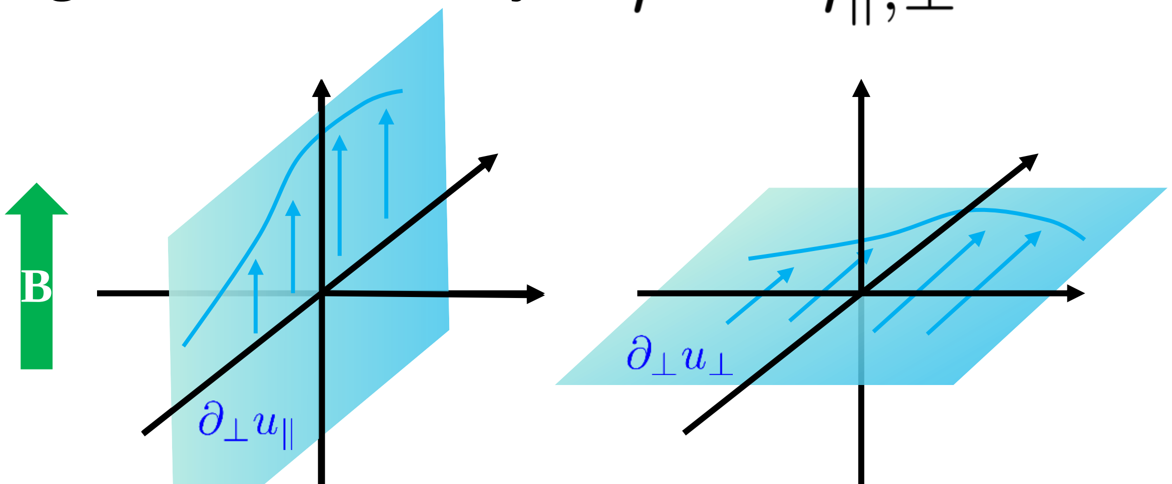
Anisotropic viscosities

$$\delta\Theta^{(\mu\nu)} = -T\eta^{\mu\nu\rho\sigma}\partial_{(\rho}\beta_{\sigma)}$$

Splitting of shear viscosity $\eta \to \eta_{\parallel,\perp}$

2 shear viscosities

3 bulk viscosities

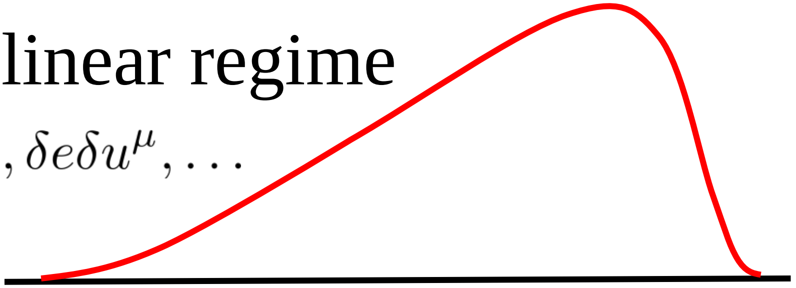


See a review, KH, Huang, Hongo (2022)

Solving spin MHD within linear-mode analysis

Linear regime

Nonlinear regime



$\delta e^2, \delta e \delta u^\mu, \ldots$

For numerical methods beyond linear regime, see talk by Benoit

$$
\begin{aligned}
e &\to e + \delta e(x), \\
u^\mu &\to u^\mu + \delta u^\mu(x), \\
B^\mu &\to B^\mu + \delta B^\mu(x)
\end{aligned}
$$

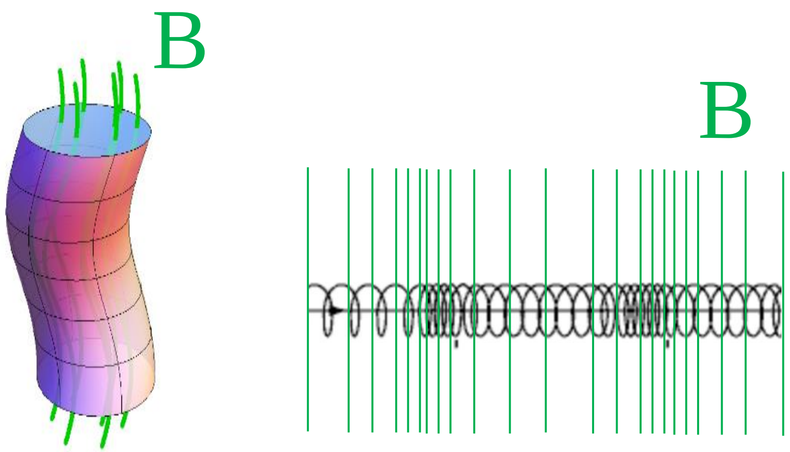
- Energy density (1)
- Flow velocity (3)
- Magnetic field (2)

$$
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
\begin{pmatrix}
\delta u_y \\
\delta B_y \\
\delta \epsilon \\
\cdot \\
\cdot \\
\cdot
\end{pmatrix}
= 0
$$

Exactly diagonalized with an analytic algorithm.

Fang, KH, Hu, 2402.18601; 2409.07096

An issue in preceding works



Magnetic tension and pressure

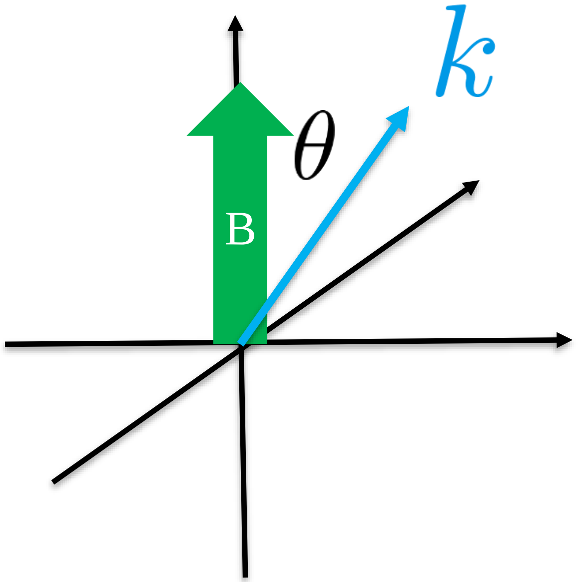
- Ideal MHD (well-known)

  Alfven waves and Magneto-sonic waves

- First-order MHD (Not known due to strong anisotropy)

  Solutions had been known only at θ = 0 or π/2.

  Moreover, there was disagreement between those known solutions at θ = π/2.



Grozdanov et al. (2017)
Armas and Camilloni (2022) ↔ Hernandez and Kovtun (2017)

“Critical angle”

Fang, KH, Hu, 2402.18601

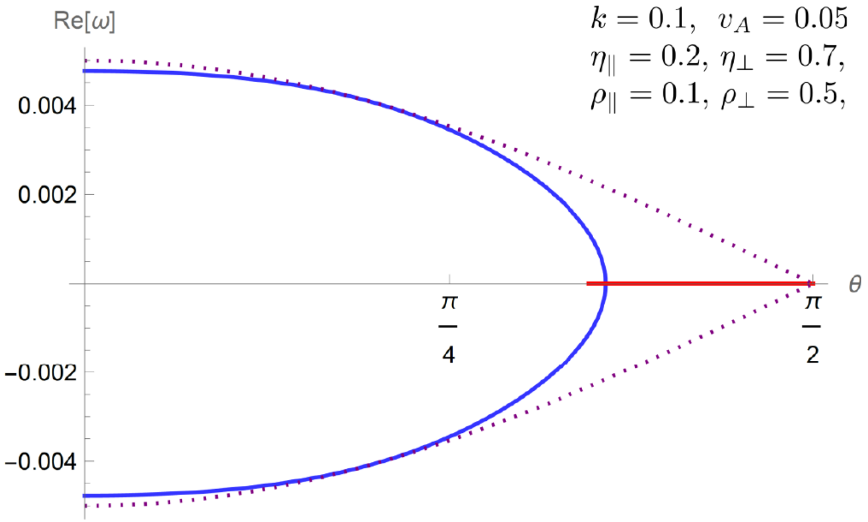
Solutions at θ = 0 and π/2 are not smoothly connected.

$$\omega = \pm v k - i\Gamma k^2 + O\left(\frac{k^n}{\cos^n\theta}\right)$$

Small k expansion breaks down due to the anisotropy.
→ Small k and cos θ limit do not commute.

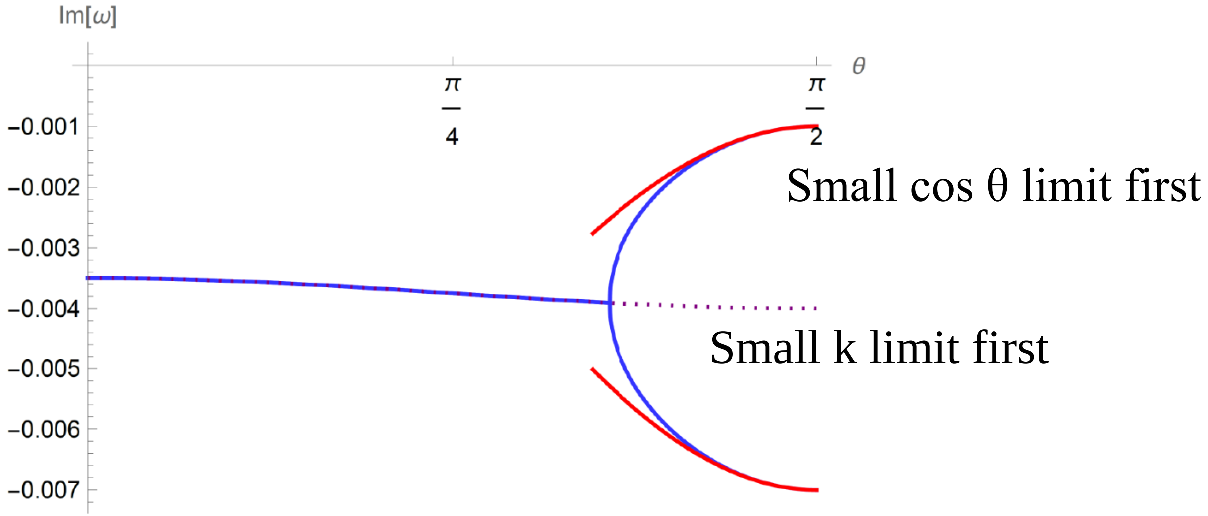
Small cosine expansion $\neq$ Small k expansion (Dotted lines)
Without any expansion

Phase velocities



Purely diffusive near the transverse direction.

Damping rates



Small cos θ limit first

Small k limit first

Split of the diffusion rates

Chiral magnetohydrodynamics

Chirality in dynamical magnetic fields

KH, Hirono, Yee, Yin, arXiv:1711.08450

Conservation laws in chiral MHD

Energy-momentum conservation
[Translational symmetry]

$$\partial_\mu \Theta^{\mu\nu} = 0$$

Magnetic flux conservation
[Magnetic one-form symmetry]

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

Absence of a magnetic monopole

$$\partial_\mu \tilde{F}^{\mu 0} = \nabla \cdot \boldsymbol{B} = 0$$

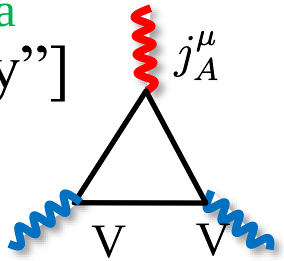
Grozdanov, Hofman, Iqbal
KH, Hirono, Yee, Yin

Maxwell equation (Electric lines terminating at charges)

$$\partial_\mu F^{\mu\nu} = j^\mu \qquad \partial_\mu F^{\mu 0} = \nabla \cdot \boldsymbol{E} = j^0$$

Chiral charge (non)-conservation
[Chiral symmetry and its breaking by "quantum anomaly"]

Son, Surowka

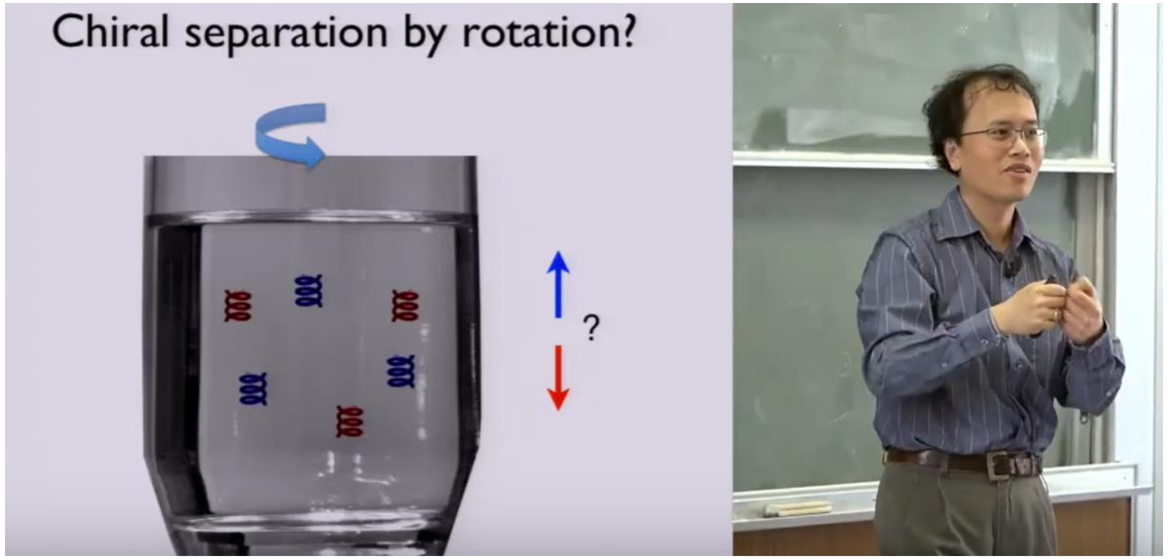


$$\partial_\mu j_A^\mu = C_A E_\mu B^\mu$$

$C_A = \frac{1}{2\pi^2}$: Chiral anomaly coefficient

Chiral hydrodynamics

Son, Surowka (2009)



Chiral separation by rotation?

D. T. Son - Hydrodynamics and quantum anomalies from youtube

- The answer is positive.
- Thermodynamic stability requires induced currents called the chiral magnetic/vortical effect.
- Manifestation of a quantum effect in fluid

$$\begin{aligned} \boldsymbol{\omega} &= \nabla \times \boldsymbol{v} \\ \boldsymbol{B} &= \nabla \times \boldsymbol{A} \end{aligned}$$

Chiral magnetohydrodynamics
in STRONG & DYNAMICAL magnetic fields

-- Chiral hydrodynamics $n_A \neq 0,\ B^\mu \sim \mathcal{O}(\partial^1)$ and external

Son & Surowka

-- Chiral magnetohydrodynamics (MHD) $n_A \neq 0,\ B^\mu \sim \mathcal{O}(\partial^0)$

KH, Hirono, Yee, Yin

and dynamical

Variables in chiral MHD: $\{\epsilon, u^\mu, B^\mu, \text{and } n_A\}$

n_A: # density of axial charge

Axial chemical potential: Lagrange multiplier to n_A

$$\mu_A = -T\frac{\partial s}{\partial n_A}$$

Constitutive eqs. and the entropy production in the first order

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} \\ \tilde{F}^{\mu\nu} &= \tilde{F}^{\mu\nu}_{(0)} - \epsilon^{\mu\nu\alpha\beta} u_{\alpha} E_{(1)\beta} \\ j^{\mu}_{A} &= j^{\mu}_{A(0)} + j^{\mu}_{A(1)} \end{aligned}$$

$$T^{\mu\nu}_{(1)},\ E^{\mu}_{(1)},\ j^{\mu}_{A(1)} \sim \mathcal{O}(\partial^1)$$

NB) $\partial_{\mu} j^{\mu}_{A} = \boxed{-C_A E^{\mu}_{(1)} B_{\mu}}.$

Chiral anomaly

Computing the entropy current,

$$\begin{aligned} \partial_{\mu}\left(s u^{\mu} + \mathcal{O}(\partial^1)\right) &= T^{\mu\nu}_{(1)} \partial_{\mu}(\beta u_{\nu}) - j^{\mu}_{A(1)} \partial_{\mu}(\beta \mu_A) \\ &\quad + E^{\mu}_{(1)} \{ \boxed{\mu_A C_A B_{\mu}} - \epsilon_{\mu\nu\alpha\beta} u^{\nu} \partial^{\alpha} (\beta H^{\beta}) \} \end{aligned}$$

Conductivities: CME and dissipative terms

From the constitutive eq. of $E^\mu_{(1)}$ and the Maxwell eq., $j^\nu = \partial_\mu F^{\mu\nu}$

$$J^\mu_V = \boxed{C_A \mu_A B^\mu} + \left[\sigma_\parallel E^\mu_\parallel + \sigma_\perp E^\mu_\perp + \sigma_{\mathrm{Hall}} \epsilon^{\mu\nu\alpha\beta} u_\nu b_\alpha E_\beta \right] + \cdots$$

CME

Ohmic currents
(∥ and ⊥ to B-field)

Hall current

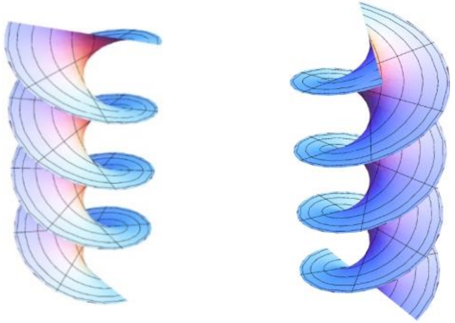
- The CME current is completely fixed by C_A.
- Without the CME, the thermodynamic stability is not insured.

Effects of chirality imbalance

How does the CME modify the waves in the chiral fluid?

$$j^{\mu} = \sigma_{\text{CME}} B^{\mu}$$

→ Helical waves

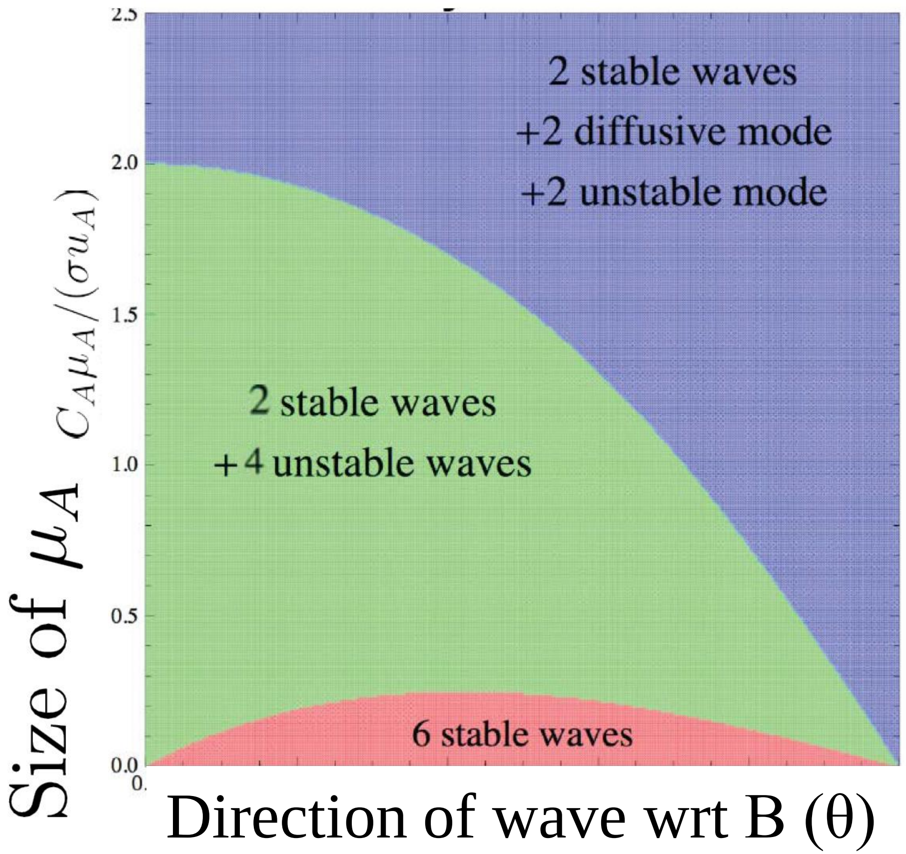


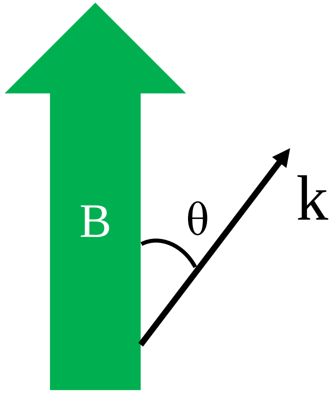
“Phase diagram” of the eigenmodes

Three modes propagating in the opposite directions (6 solutions in total)

$$(\omega^2 - x_1)(\omega^4 + b\omega^2 + c) = 0 \qquad x_1\text{: Real solution}$$

Stability of the waves from classification of solutions





1 real and 2 pure imag. sols.

1 real and 2 complex sols.

3 real solutions

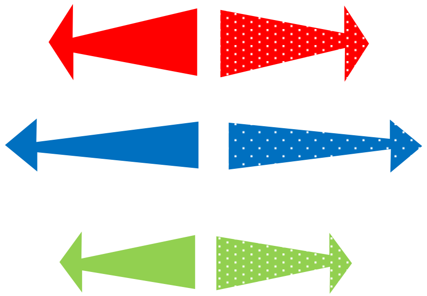
Alfven and magneto-sonic waves

Helical unstable modes in Chiral MHD

Magnetohydrodynamics

6 stable modes
- Alfven waves
- Magneto-sonic waves

Damping due to dissipation
Linear polarizations

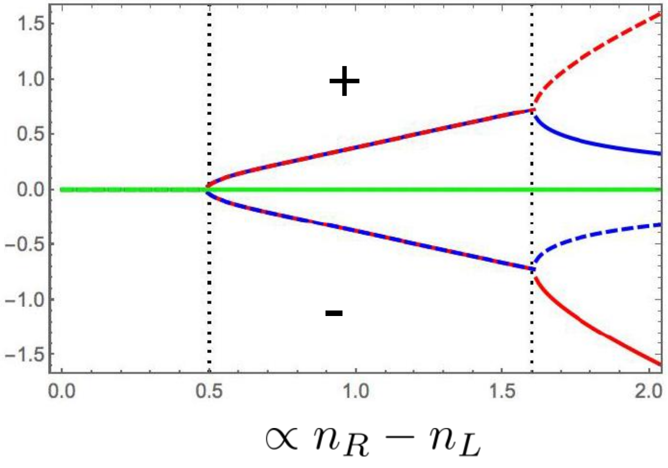


+

Chirality imbalance ($n_R \neq n_L$)
→ "Chiral magnetic effect"

=

Imaginary part of the solutions

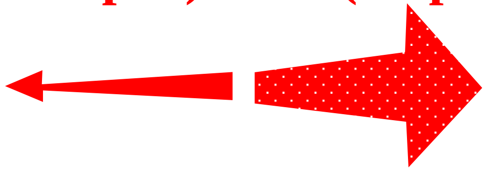


Chiral MHD

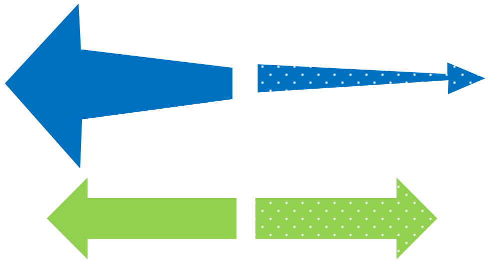
- **4 unstable modes**
- 2 stable modes

Helicity eigenstates

L (damped) **R (amplified)**

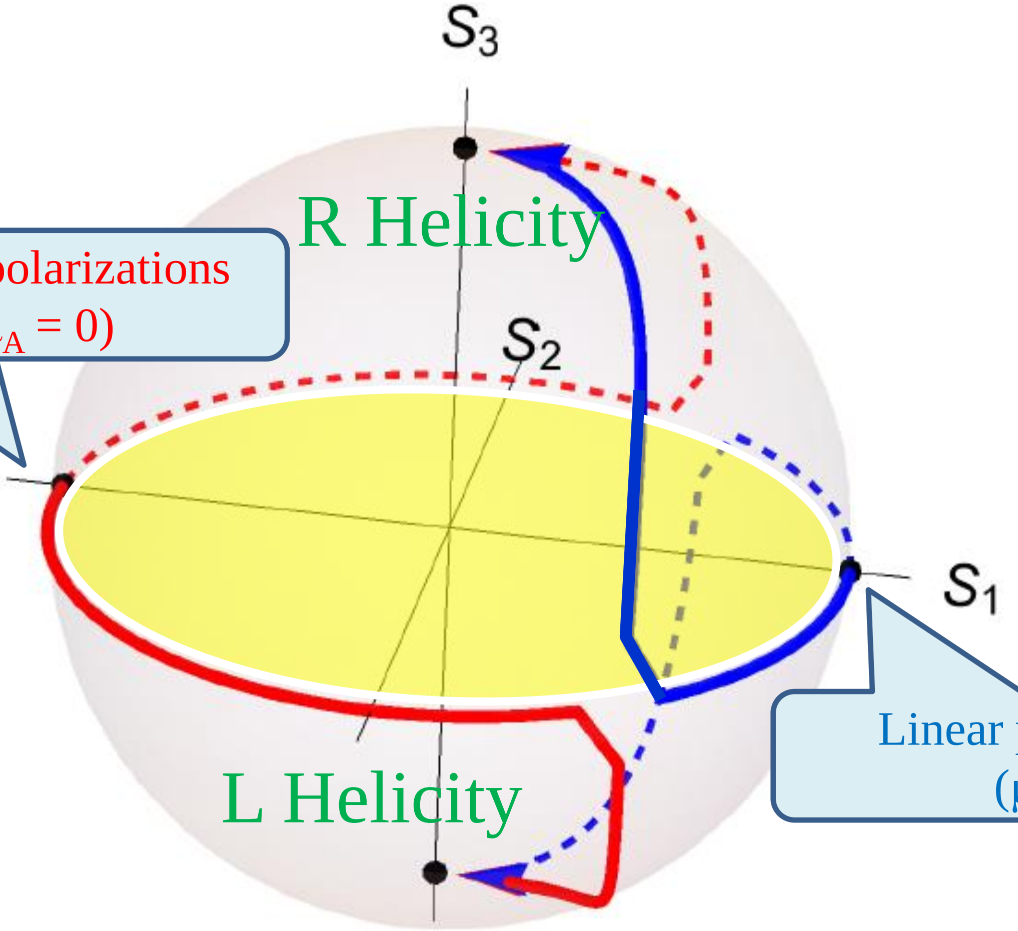


R (amplified) **L (damped)**



Hattori, Hirono, Yee, Yin (PRD, 2019)

Polarizations on the Poincare sphere with a varying μ_A



Linear polarizations
($\mu_A = 0$)

Linear polarizations
($\mu_A = 0$)

$$\frac{v_1}{v_2} = \frac{b_1}{b_2} = \frac{\epsilon_A V}{u_A^2 \cos\theta - V^2}$$

Stokes vector

$$s = \frac{\left(b_1^2 - b_2^2, 2\mathrm{Re}\left[b_1\, b_2^*\right], 2\,\mathrm{Im}\left[b_1\, b_2^*\right]\right)}{b_1^2 + b_2^2}$$

Equator: Linear polarizations

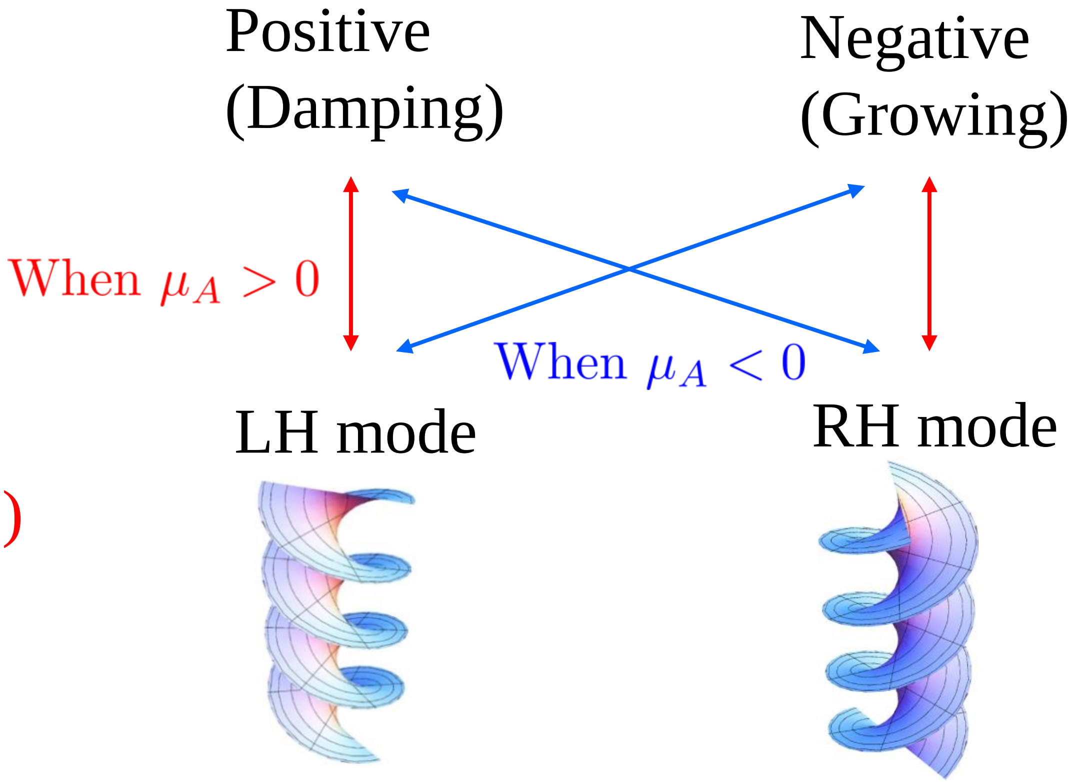
Upper and lower hemispheres: R and L polarizations

(Poles: R and L circular polarizations)

The unstable modes are helical in nature.

New hydrodynamic instability in a chiral fluid

Signs of the imaginary parts
(Damping/growing modes in the hydrodynamic time evolution)



Positive
(Damping)

Negative
(Growing)

When $\mu_A > 0$

When $\mu_A < 0$

LH mode

RH mode

Helicity decomposition
(Circular R/L polarizations)

$$\nabla \times \boldsymbol{e}_{R/L} = \pm \boldsymbol{e}_{R/L}$$

A helicity selection, depending on the sign of μ_A.

Helicity conversions as a topological origin of the instability

Total helicity conservation: $$\frac{d}{dt}\int_V d^3\boldsymbol{x}\left[n_A + \boldsymbol{B}\cdot\boldsymbol{A} + \boldsymbol{\omega}\cdot\boldsymbol{v}\right] = 0$$

Fermion

Chiral anomaly

Fluid helicity

New hydrodynamic instability
→ Helicity conversion mechanism

"Chiral imbalance"
btw the # of R and L fermions:

$$n_A = n_R - n_L$$

CPI

Helical fluid instability

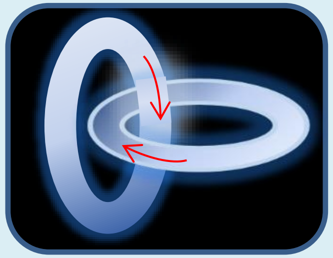
Magnetic helicity $\int_V d^3\boldsymbol{x}\,\boldsymbol{B}\cdot\boldsymbol{A}$

Hirono

?

Fluid helicity
(vortex strings)

$$\int_V d^3\boldsymbol{x}\,\boldsymbol{\omega}\cdot\boldsymbol{v}_{\text{fluid}}$$



Magnetic fields are amplified!!!

Spin hydrodynamics

KH, Hongo, Huang, Matsuo, Taya, 1901.06615
Fang, KH, Hu, 2409.07096

Spin as a quasi-hydro variable

- Total AM conservation from rotational symmetry : $$\partial_\mu J^{\mu\alpha\beta} = 0$$

- Spin-orbit coupling for relativistic constituents

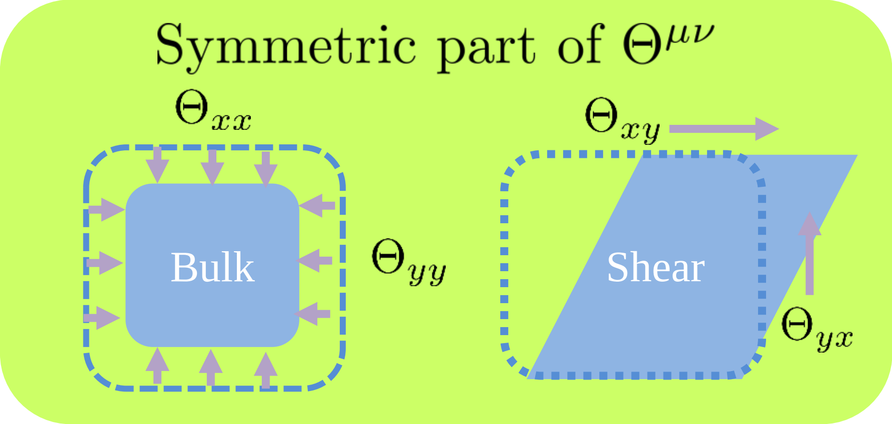
$$\partial_\mu \big(\underbrace{L^{\mu\alpha\beta}}_{\text{Orbital AM}} + \underbrace{\Sigma^{\mu\alpha\beta}}_{\text{Spin}} \big) = 0$$

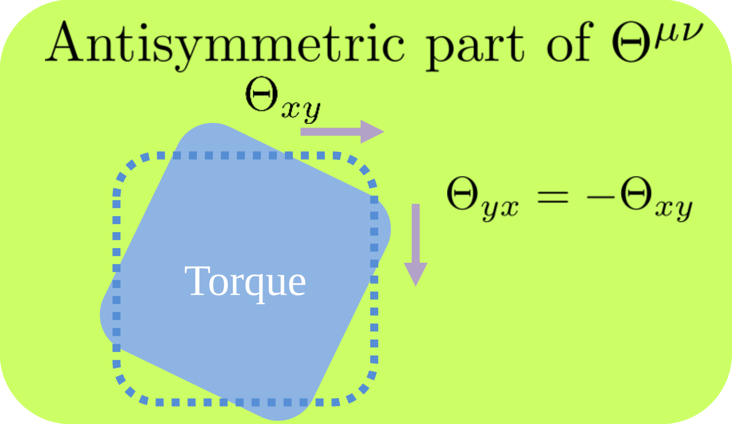
$$L^{\mu\alpha\beta} = x^\alpha \Theta^{\mu\beta} - x^\beta \Theta^{\mu\alpha}$$

→ Spin "non-conservation" equation

No symmetry protecting spin conservation.

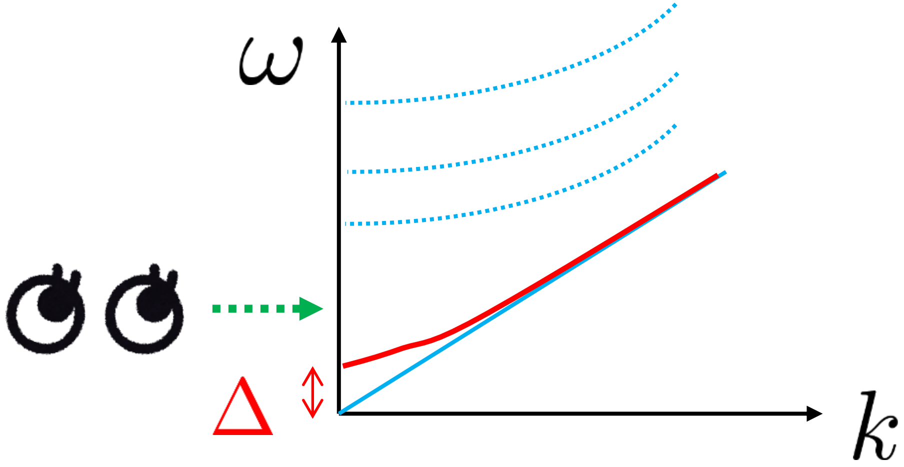
$$\begin{aligned} \partial_\mu \Sigma^{\mu\alpha\beta} &= -\partial_\mu L^{\mu\alpha\beta} \\ &= -\left(\Theta^{\alpha\beta} - \Theta^{\beta\alpha}\right) \end{aligned}$$





Explicit, but weak, symmetry breaking
→ Small energy gap

Quasi-hydrodynamic modes



Cf. Hydro+ (Stephanov-Yin) for a long-lived mode due to the critical slowing down.

(kinetic relaxation time) $\lesssim$ 1/gap << (spacetime scale of our interest)

Momentum transfer rate
(randomization of momentum)

Transfer rate of quasi-conserved charge

Rotational viscosities (Anti-symmetric part)

$$\delta\Theta^{[\mu\nu]} \sim -T\gamma^{\mu\nu\rho\sigma}(\partial_{[\rho}\beta_{\sigma]} - \beta\mu_{\rho\sigma})$$

Fang, KH, Hu, 2409.07096

Thermal vorticity　Spin potential

- Extended first law

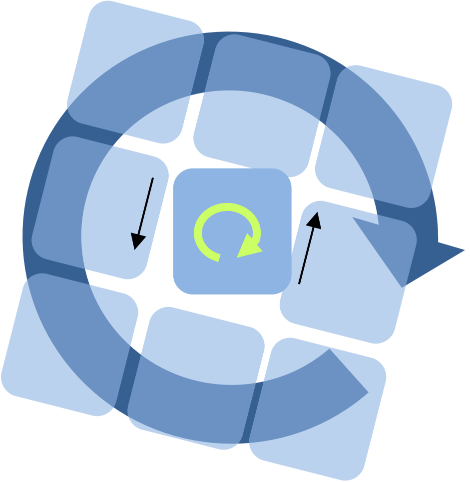
$$Tds = de - \frac{1}{2}\mu^{\nu\rho}d\sigma_{\nu\rho} - H_\mu dB^\mu$$

Energy density　Spin density　Magnetic flux

$$\beta\mu^{\nu\rho} \equiv -2\frac{\partial s}{\partial\sigma_{\nu\rho}}$$

Spin potential

"No-slip condition" with the rotational fluid motion



Isotropic limit
KH, Hongo, Huang, Matsuo, Taya, 1901.06615

Spin hydrodynamics

KH, Hongo, Huang, Matsuo, Taya, 1901.06615
Fukushima, Pu, 2010.01608
Li, Stephanov, Yee, 2011.12318
She, Huang, Hou, Liao, 2105.04060
Hongo, Huang, Kaminski, Stephanov, Yee, 2107.14231
Biswas, Daher, Das, Florkowski, Ryblewski, 2304.01009

Solving spin MHD within linear-mode analysis

$$e \to e + \delta e(x), \quad u^\mu \to u^\mu + \delta u^\mu(x),$$
$$B^\mu \to B^\mu + \delta B^\mu(x), \quad S^{\mu\nu} \to S^{\mu\nu} + \delta S^{\mu\nu}(x)$$

- Energy density (1)
- Flow velocity (3)
- Magnetic field (2)
- Spin (3)

9 × 9 matrix !!

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \delta u_y \\ \delta B_y \\ \delta S_x \\ \delta S_z \\ \cdot \\ \cdot \end{pmatrix} = 0$$

Still exactly diagonalized with an analytic algorithm.

Fang, KH, Hu, 2402.18601 2409.07096

Alfven modes with spin mixing

From the 4×4 matrix,

$$\begin{aligned}\omega &= \pm v_A k_\parallel - \frac{i}{2}\Big(\rho'_\perp k_\parallel^2 + \rho'_\parallel k_\perp^2 + \frac{(\eta_\parallel\gamma_\parallel - \xi^2)k_\parallel^2 + \eta_\perp\gamma_\parallel k_\perp^2}{h\gamma_\parallel}\Big),\\ \omega &= -i\Gamma_\perp - i\frac{\gamma_\perp}{h}k_\perp^2\,,\\ \omega &= -i\Gamma_\parallel - i\frac{(\gamma_\parallel-\xi)^2}{h\gamma_\parallel}k_\parallel^2\end{aligned}$$

$\Gamma_{\parallel,\perp} = \frac{8}{\chi}\gamma_{\parallel,\perp}$

Spin susceptibility: $S^{\mu\nu} = \chi\mu^{\mu\nu}$

Rotational viscosity:

$\delta\Theta^{[\mu\nu]} = -T\gamma^{\mu\nu\rho\sigma}(\partial_{[\rho}\beta_{\sigma]} - \beta\mu_{\rho\sigma})$

Spins are not conserved → Gapped modes at k = 0.

$$\partial_\mu\Sigma^{\mu\alpha\beta} = -2\Theta^{[\mu\nu]}$$

$$\begin{aligned}\omega &= -i\Gamma_\perp\\ \omega &= -i\Gamma_\parallel\end{aligned}$$

All solutions are always damping (stable around the equilibrium)

Magneto-sonic modes with spin mixing

From the 5×5 matrix,

$$
\begin{aligned}
\omega &= \pm v_1 k - i w_1 k^2 \\
\omega &= \pm v_2 k - i w_2 k^2 \\
\omega &= -i\Gamma_{||} - i w_3 k^2
\end{aligned}
$$

$$
\begin{aligned}
w_1 &= -\frac{W_1 - v_1^2 W_2}{2(v_1^2 - v_2^2)} \\
w_2 &= +\frac{W_1 - v_2^2 W_2}{2\gamma_{||}(v_1^2 - v_2^2)} \\
w_3 &= \frac{(\gamma'_{||} - \xi')^2}{\gamma'_{||}} \frac{1 - v_A^2 \cos^2\theta}{1 - v_A^2} + 4\xi' \frac{\sin^2\theta}{1 - v_A^2}
\end{aligned}
$$

Always damping (stable around the equilibrium)

$$
\begin{aligned}
W_1 &= \frac{\gamma'_{||}\eta'_{||} - \xi'^{\,2}}{\gamma_{||}} \left(c_s^2 \cos^2(2\theta) + \frac{v_A^2}{1 - v_A^2} \sin^2\theta \right) + \rho'_\perp c_s^2 \left(1 + \frac{v_A^2}{1 - v_A^2} \sin^2\theta \right) \\
&\quad + \cos^2\theta \left[\zeta'_{||} \left(\frac{v_A^2}{1 - v_A^2} + c_s^2 \sin^2\theta \right) + c_s^2 \left(\zeta'_\perp + \eta'_\perp - 2\zeta'_\times \right) \sin^2\theta \right] \\
W_2 &= \frac{\gamma'_{||}\eta'_{||} - \xi'^{\,2}}{\gamma_{||}} \left(1 + \frac{v_A^2}{1 - v_A^2} \sin^2\theta \right) + \rho'_\perp \left(1 + \frac{c_s^2 v_A^2}{1 - v_A^2} \sin^2\theta \right) \\
&\quad + \frac{\zeta'_{||}}{1 - v_A^2} \cos^2\theta + (\zeta'_\perp + \eta'_\perp) \sin^2\theta
\end{aligned}
$$

Split of damping spin modes

$$\begin{aligned} \omega &= -i\Gamma_\perp - i\frac{\gamma_\perp}{h}k_\perp^2 \\ \omega &= -i\Gamma_{||} - i\frac{(\gamma_{||}-\xi)^2}{h\gamma_{||}}k_{||}^2 \\ \omega &= -i\Gamma_{||} - iw_3 k^2 \end{aligned}$$

$\Gamma_{||,\perp} = \frac{8}{\chi}\gamma_{||,\perp}$

Spin susceptibility: $S^{\mu\nu} = \chi\mu^{\mu\nu}$

$$w_3 = \frac{(\gamma'_{||}-\xi')^2}{\gamma'_{||}}\frac{1-v_A^2\cos^2\theta}{1-v_A^2} + 4\xi'\frac{\sin^2\theta}{1-v_A^2}$$

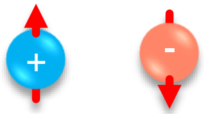
Spins are not conserved → Gapped modes at k = 0.
All solutions are always damping.

Summary

1. We went though a brief overview of relativistic hydro. for QGP.
2. We discussed formulation of MHD with chirality and spin.
3. We developed an analytic algorithm for solutions search on an order-by-order basis in derivative.
4. We pointed out the breakdown of a small k expansion in anisotropic systems and an instability in chiral systems.

Outlook

Extension to a finite (baryon) charge density for low-energy collisions.
-- Charge density and E field may be included as quasi-hydro variables.
-- Spin polarization induced by a magnetic field.

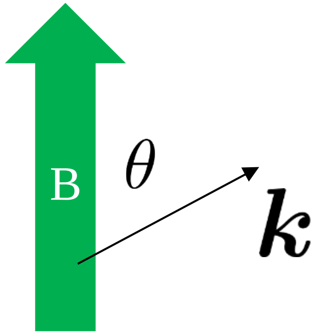


Cancelled in neutral fluid

Back-up slides

Breakdown of a small k expansion in anisotropic systems

Trig. functions serve as another small parameter.



Two limits, $\theta \to \pi/2$ and $\boldsymbol{k} \to 0$, do not commute.

Demonstration with the simplest case: The Alfven mode (w/o spin)

$$\begin{aligned} \omega &= \pm\sqrt{v_A^2 k_\parallel^2 - \frac{1}{4}(\tilde{\rho}-\tilde{\eta})^2 k^4} - \frac{i}{2}(\tilde{\rho}+\tilde{\eta})k^2 \\ &= \pm v_A k_\parallel - \frac{i}{2}(\tilde{\rho}+\tilde{\eta})k^2 + \mathcal{O}(k^3) \end{aligned}$$

$$\boxed{\begin{aligned} \omega(\theta \to \frac{\pi}{2}) &= \pm\frac{i}{2}|\tilde{\rho}-\tilde{\eta}|k_\perp^2 - \frac{i}{2}(\tilde{\rho}+\tilde{\eta})k_\perp^2 \\ &= -i\eta'_\perp k_\perp^2, \quad -i\rho'_\parallel k_\perp^2 \end{aligned}}$$

$$\boxed{\omega(\theta \to \frac{\pi}{2}) = -\frac{i}{2}(\rho'_\parallel + \eta'_\perp)k_\perp^2 \pm \lim_{\theta\to\pi/2} \sum_{n=1}^{\infty} \frac{c_n(\rho'_\parallel - \eta'_\perp)^{2n}}{(v_A \cos\theta)^{2n-1}} k_\perp^{2n+1}}$$

Split of diffusive modes
from an alternative expansion

For $\quad |\cos\theta| < \hat{k}$:

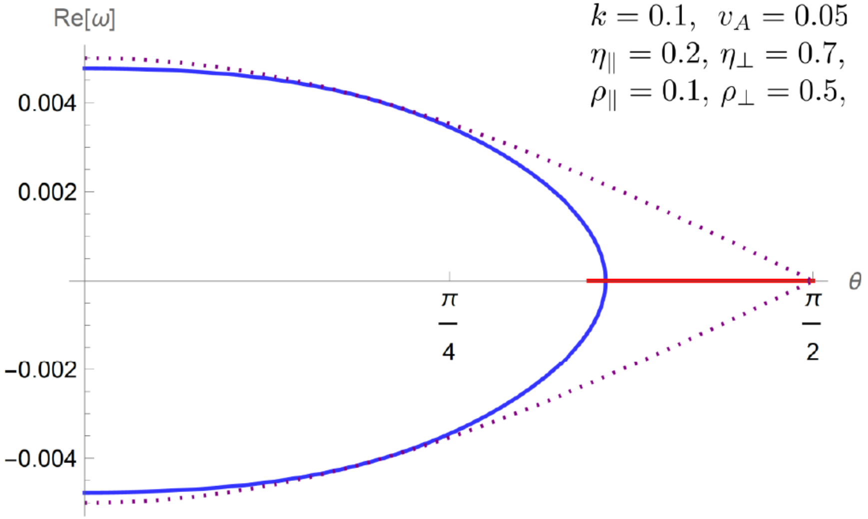
$$\omega_+ \;=\; -ik^2\rho'_{||} - i\left((\rho'_\perp - \rho'_{||})k^2 + \frac{v_A^2}{\eta'_\perp - \rho'_{||}}\right)\cos^2\theta + \mathcal{O}(\cos^4\theta),$$

$$\omega_- \;=\; -ik^2\eta'_\perp - i\left((\eta'_{||} - \eta'_\perp)k^2 - \frac{v_A^2}{\eta'_\perp - \rho'_{||}}\right)\cos^2\theta + \mathcal{O}(\cos^4\theta)$$

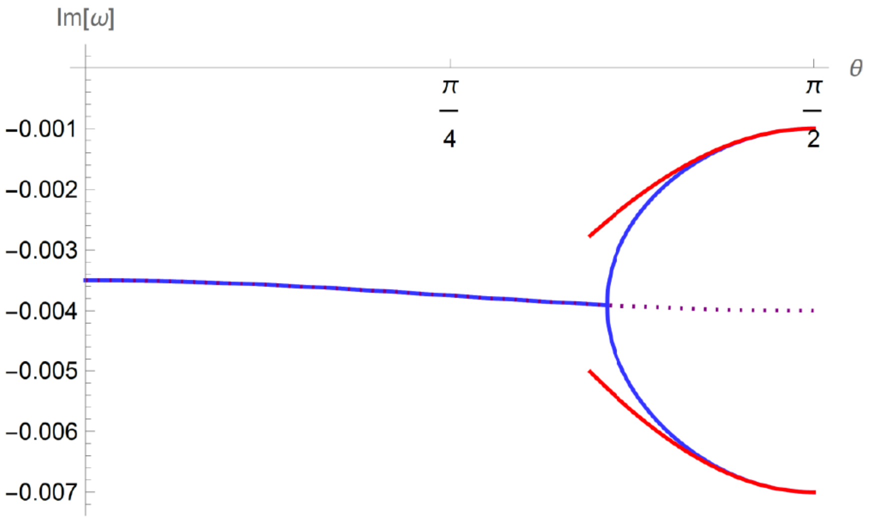
Small cosine expansion $\quad \not\leftrightarrow \quad$ Dotted lines: Small k expansion

Without any expansion

Phase velocity



Damping rate

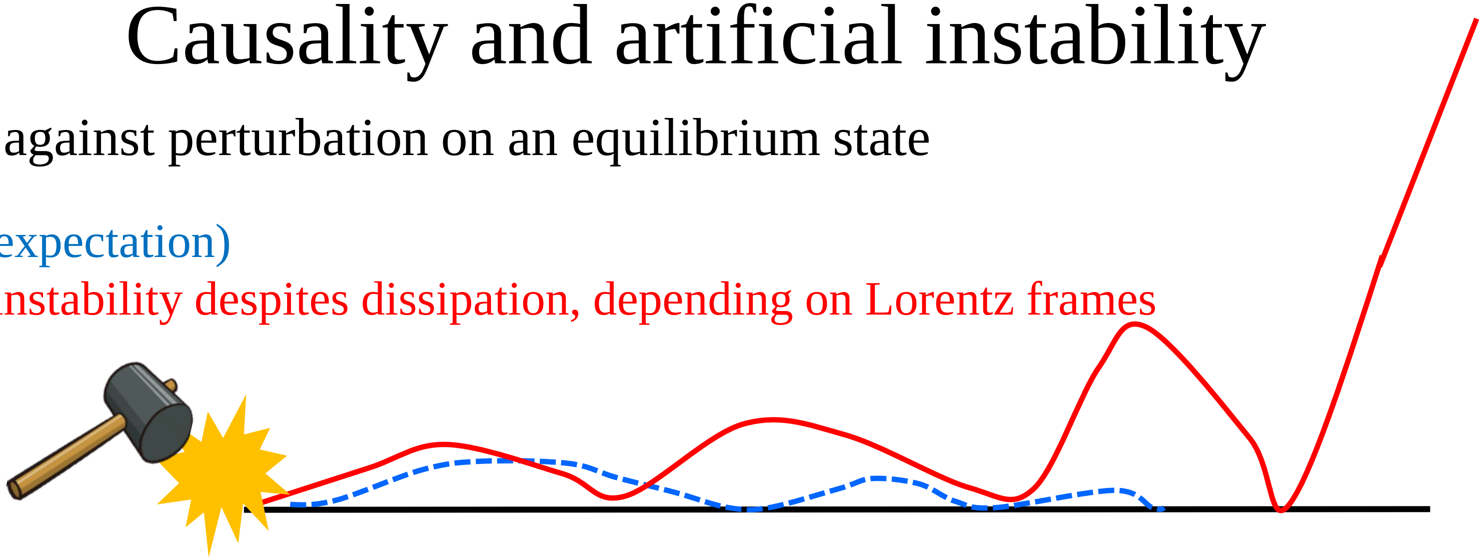


Purely diffusive near the transverse direction.

Splitting of the damping rates

Causality and artificial instability

Stability against perturbation on an equilibrium state



Stability (expectation)

Artificial instability despites dissipation, depending on Lorentz frames

Hiscock & Lindblom (1983-1987)

- Dissipative currents are acausal.

  Systems need finite time to develop dissipative currents.

$$\boldsymbol{j} = -D\nabla \cdot n \qquad \partial_t n = -\nabla \cdot \boldsymbol{j}$$

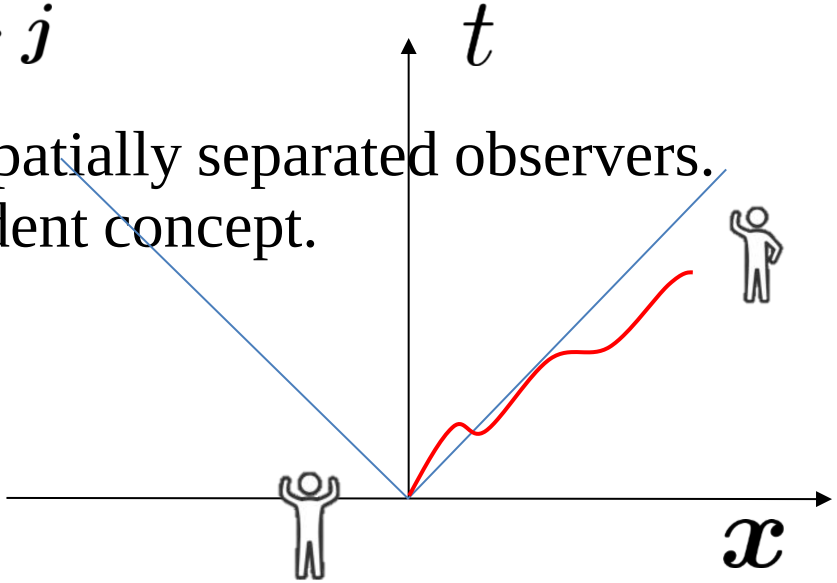
- Chronicle order can be different for spatially separated observers.

✗ Stability becomes an observer-dependent concept.

- Causality is necessary for stability.

  → Israel-Stewart theory, etc.

Gavassino (2022)



MHD from the global symmetries of QED

We argued that E is screened in the equilibrium, but B is not.
→ MHD = fluid dynamics + magnetic flux

Q1: Can we understand MHD from any symmetry
of the underlying microscopic theory?

Grozdanov et al.
Hongo, KH

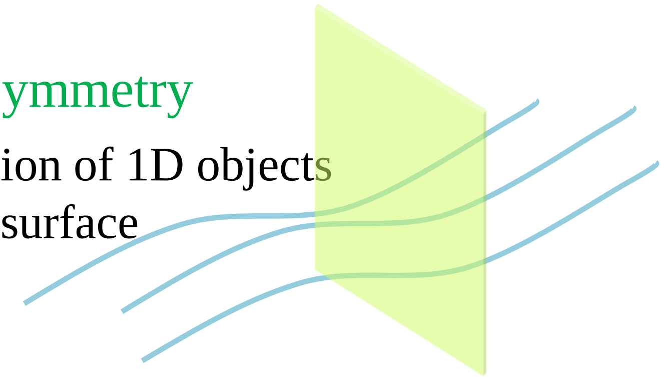
Q2: How is MHD derived from a microscopic theory?

Fluxes as conserved currents

Gaiotto et al.

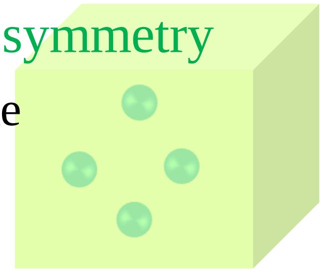
1-form symmetry

Conservation of 1D objects
through a surface



Conventional (0-form) symmetry

Conservation of point-like
charges in a box



[“MHD from QED” a talk at “Spin and Hydrodynamics in Relativistic Nuclear Collisions”](#)

Constitutive equations to the first order
(in a parity-even environment)

Available tensors

$$\boxed{\begin{matrix} g^{\mu\nu} \\ \epsilon^{\mu\nu\alpha\beta} \end{matrix}} \quad \boxed{u^\mu} \quad \boxed{b^\mu = B^\mu/\sqrt{B^\nu B_\nu}} \quad \boxed{\begin{matrix} \Xi^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu - b^\mu b^\nu \\ u_\mu \Xi^{\mu\nu} = 0 = b_\mu \Xi^{\mu\nu} \end{matrix}}$$

$$\begin{aligned} \Theta^{\mu\nu} &= e u^\mu u^\nu + p_\perp \Xi^{\mu\nu} + p_\parallel b^\mu b^\nu + \delta\Theta^{\mu\nu} \\ \Sigma^{\mu\nu\rho} &= \epsilon^{\mu\nu\rho\lambda}(\sigma_\lambda + u_\lambda \delta\sigma) \\ \tilde{F}^{\mu\nu} &= (B^\mu u^\nu - B^\nu u^\mu) + \delta\tilde{F}^{\mu\nu} \end{aligned}$$

$$\sigma^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu \sigma_{\rho\sigma}$$

- Totally anti-symmetric spin tensor $\Sigma^{\mu\nu\rho}$ (motivated by Dirac fermions)

$$\partial_\mu \Sigma^{\mu\alpha\beta} = -\partial_\mu L^{\mu\alpha\beta} = -2\Theta^{[\mu\nu]}$$

$$L^{\mu\alpha\beta} = x^\alpha \Theta^{\mu\beta} - x^\beta \Theta^{\mu\alpha}$$

Thermodynamics

- 1st law $$Tds = de - \frac{1}{2}\mu^{\nu\rho} d\sigma_{\nu\rho} - H_\mu dB^\mu$$

Energy density Spin density Magnetic flux

Thermodynamic conjugates (Lagrange multipliers) $B^\mu = \tilde{F}^{\mu\nu} u_\nu$

$$\beta \equiv \frac{\partial s}{\partial e}, \quad \beta\mu^{\nu\rho} \equiv -2\frac{\partial s}{\partial \sigma_{\nu\rho}}, \quad \beta H_\mu \equiv -\frac{\partial s}{\partial B^\mu}$$

Temperature Spin potential Magnetic field

$$Ts = e + p - \mu_{\alpha\beta}\sigma^{\alpha\beta} - H_\mu B^\mu$$

- 2nd law $\partial_\mu s^\mu \geq 0$

$$\begin{aligned} \partial_\mu s^\mu &= \partial_\mu (s u^\mu + \delta s^\mu) \\ &= s\theta + \beta De - \frac{1}{2}\beta\mu^{\nu\rho} D\sigma_{\nu\rho} - \beta H_\mu DB^\mu + \partial_\mu \delta s^\mu \geq 0 \end{aligned}$$

Time derivative: $D \equiv u^\mu \partial_\mu$

Time derivatives of conserved charges

→ Connection to equations of motion

Number of dof

e, n, and three components for $\sigma_{\nu\rho}$

For a totally anti-symmetric $\Sigma^{\mu\nu\rho}$,
$\sigma_{\nu\rho} = -\sigma_{\rho\nu}$ and $0 = u_\mu u^\nu \Sigma^\mu{}_{\nu\rho} = -u^\nu \sigma_{\nu\rho}$

Constraint equation (no temporal derivative)

$$u^\rho \left(\partial_\mu \Sigma^\mu{}_{\nu\rho} + 2\Theta_{[\nu\rho]} \right) = 0$$

Both # of dof and EoM are reduced by 3.

Throwing away the redundancy

$$\sigma^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_\nu \sigma_{\rho\sigma} \qquad \sigma^\mu u_\mu = 0$$

Divergence of the entropy current

$$\partial_\mu s^\mu \quad = \quad s\theta + \beta De - \frac{1}{2}\beta\mu^{\nu\rho} D\sigma_{\nu\rho} - \beta H_\mu DB^\mu + \partial_\mu \delta s^\mu$$

$$D \equiv u^\mu \partial_\mu$$

Equations of motion

Leading order (ideal fluid)

- Should be vanishing

NLO in derivative

- Should be semi-positive

$$\begin{aligned}
\partial_\mu s^\mu \quad &= \quad s\partial_\mu u^\mu + Ds + \partial_\mu s^\mu_{(1)} \\
&= \boxed{\begin{aligned} &\beta(Ts - \epsilon - p_\perp + \omega_{\alpha\beta}S^{\alpha\beta} + H_\mu B^\mu)\partial_\mu u^\mu \\ &-\beta\big[\,(p_\parallel - p_\perp)b^\mu b^\nu + Bb^\mu H^\nu + 2S^{\mu\alpha}\omega^\nu{}_\alpha\,\big]\partial_\mu u_\nu \end{aligned}} \\
&\ \boxed{\begin{aligned} &-\Theta^{\mu\nu}_{(1)}(\partial_\mu\beta_\nu - 2\beta\omega_{\mu\nu}) \\ &-\Sigma^{\mu\alpha\beta}_{(1)}\partial_\mu(\beta\omega_{\alpha\beta}) + \tilde{F}^{\mu\nu}_{(1)}\partial_\mu(\beta H_\nu) \end{aligned}} \\
&\ +\partial_\mu(s^\mu_{(1)} + \beta u_\nu \Theta^{\mu\nu}_{(1)} + \beta\omega_{\alpha\beta}\Sigma^{\mu\alpha\beta}_{(1)} - \beta H_\nu \tilde{F}^{\mu\nu}_{(1)})
\end{aligned}$$

First-order corrections

$$\begin{aligned}\partial_\mu s^\mu &= -\delta\Theta^{(\mu\nu)}\partial_{(\mu}\beta_{\nu)} - \delta\Theta^{[\mu\nu]}(\partial_{[\mu}\beta_{\nu]} - \beta\mu_{\mu\nu}) + \delta\tilde{F}^{\mu\nu}\partial_\mu(\beta H_\nu) \\ &\geq 0\end{aligned}$$

Symmetric part
(Shear)

Anti-symmetric part
(Torque, vorticity)

Induced EM field

The entropy production stops when the spin potential equals the thermal vorticity.

$$\beta\mu_{\mu\nu} = \partial_{[\mu}\beta_{\nu]}$$

Becattini

Semi-positivity is ensured by bilinear forms

$$\begin{aligned}\delta\Theta^{\mu\nu} &= -T\begin{pmatrix}\eta^{\mu\nu\rho\sigma} & \xi^{\mu\nu\rho\sigma} \\ \xi'^{\,\mu\nu\rho\sigma} & \gamma^{\mu\nu\rho\sigma}\end{pmatrix}\begin{pmatrix}\partial_{(\rho}\beta_{\sigma)} \\ \partial_{[\rho}\beta_{\sigma]} - \beta\mu_{\rho\sigma}\end{pmatrix} \\ \delta\tilde{F}^{\mu\nu} &= -T\rho^{\mu\nu\rho\sigma}\partial_{[\rho}(\beta H_{\sigma]})\end{aligned}$$

Viscous tensors

Resistivity tensor

The off-diagonal component $\xi^{\mu\nu\alpha\beta}$ converts symmetric shear to a torque and antisymmetric vorticity to a shear deformation.

Cross viscosities Shear-Torque conversion

$$\delta\Theta^{\mu\nu} = -T\begin{pmatrix}\eta^{\mu\nu\rho\sigma} & \xi^{\mu\nu\rho\sigma} \\ \xi'^{\,\mu\nu\rho\sigma} & \gamma^{\mu\nu\rho\sigma}\end{pmatrix}\begin{pmatrix}\partial_{(\rho}\beta_{\sigma)} \\ \partial_{[\rho}\beta_{\sigma]} - \beta\mu_{\rho\sigma}\end{pmatrix}$$

$$\xi^{\mu\nu\rho\sigma} = 2\xi\left(b^{\mu}\Xi^{\nu[\rho}b^{\sigma]} + b^{\nu}\Xi^{\mu[\rho}b^{\sigma]} + b^{\mu}\Xi^{\nu(\rho}b^{\sigma)} - b^{\nu}\Xi^{\mu(\rho}b^{\sigma)}\right)$$

Onsager's reciprocal relations from time-reversal symmetry

$$\eta^{\mu\nu\rho\sigma}(b) = \eta^{\rho\sigma\mu\nu}(-b),\ \gamma^{\mu\nu\rho\sigma}(b) = \gamma^{\rho\sigma\mu\nu}(-b),$$

$$\xi^{\mu\nu\rho\sigma}(b) = \xi^{\rho\sigma\mu\nu}(-b),\ \xi'^{\,\mu\nu\rho\sigma}(b) = \xi^{\rho\sigma\mu\nu}(-b)$$

Short summary

Spin MHD is formulated based on

- Energy-momentum conservation
- Spin (non-)conservation
- Magnetic flux conservation

$$\partial_\mu \Theta^{\mu\nu} = 0$$
$$\partial_\mu \Sigma^{\mu\nu\rho} = -2\Theta^{[\mu\nu]}$$
$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

5 shear and bulk viscosities
2 rotational viscosities
1 cross viscosity
2 resistivity

$$\begin{aligned}
\eta^{\mu\nu\rho\sigma} &= \begin{pmatrix} b^\mu b^\nu & \Xi^{\mu\nu} \end{pmatrix} \begin{pmatrix} \zeta_{\parallel} & \zeta_{\times} \\ \zeta'_{\times} & \zeta_{\perp} \end{pmatrix} \begin{pmatrix} b^\rho b^\sigma \\ \Xi^{\rho\sigma} \end{pmatrix} \\
&\quad +2\eta_{\parallel}(b^\mu \Xi^{\nu(\rho} b^{\sigma)} + b^\nu \Xi^{\mu(\rho} b^{\sigma)}) \\
&\quad +\eta_{\perp}(\Xi^{\mu\rho}\Xi^{\nu\sigma} + \Xi^{\mu\sigma}\Xi^{\nu\rho} + \Xi^{\mu\nu}\Xi^{\rho\sigma}), \\
\gamma^{\mu\nu\rho\sigma} &= \gamma_{\perp}(\Xi^{\mu\rho}\Xi^{\nu\sigma} - \Xi^{\mu\sigma}\Xi^{\nu\rho}) - 2\gamma_{\parallel}(b^\mu \Xi^{\nu[\rho} b^{\sigma]} - b^\nu \Xi^{\mu[\rho} b^{\sigma]}), \\
\xi^{\mu\nu\rho\sigma} &= 2\xi(b^\mu \Xi^{\nu[\rho} b^{\sigma]} + b^\nu \Xi^{\mu[\rho} b^{\sigma]} + b^\mu \Xi^{\nu(\rho} b^{\sigma)} - b^\nu \Xi^{\mu(\rho} b^{\sigma)}), \\
\rho^{\mu\nu\rho\sigma} &= -2\rho_{\perp}(b^\mu \Xi^{\nu[\rho} b^{\sigma]} - b^\nu \Xi^{\mu[\rho} b^{\sigma]}) + 2\rho_{\parallel}\Xi^{\mu[\rho}\Xi^{\sigma]\nu}
\end{aligned}$$

Inequalities for semi-positivity

$$\zeta_{\perp,\parallel} \geq 0, \quad \eta_{\perp,\parallel} \geq 0, \quad \gamma_{\perp,\parallel} \geq 0, \quad \rho_{\perp,\parallel} \geq 0,$$
$$\zeta_{\perp}\zeta_{\parallel} - \zeta_{\times}^2 \geq 0, \quad \eta_{\parallel}\gamma_{\parallel} - \xi^2 \geq 0 \quad (\text{Matrix} \geq 0)$$

Solving the hydrodynamic equations

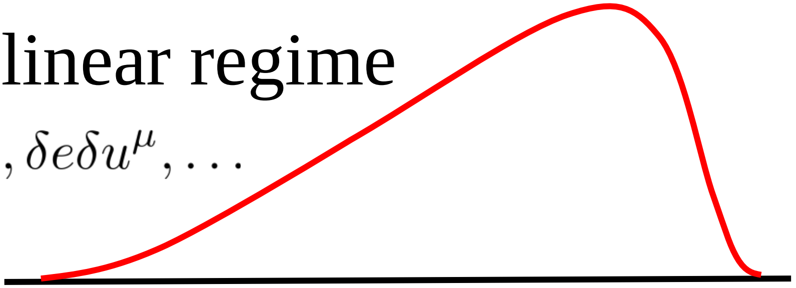
Linear-mode analysis

$$e \to e + \delta e(x), \quad u^\mu \to u^\mu + \delta u^\mu(x),$$
$$B^\mu \to B^\mu + \delta B^\mu(x), \quad S^{\mu\nu} \to S^{\mu\nu} + \delta S^{\mu\nu}(x)$$

Linear regime

Nonlinear regime

$\delta e^2, \delta e \delta u^\mu, \ldots$



Linearized equation in a matrix form

→ Secular equation (det = 0)

→ Dispersion relations

9 × 9 matrix !!

- Energy density (1)
- Flow velocity (3)
- Magnetic field (2)
- Spin (3)

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \delta u_y \\ \delta B_y \\ \delta S_x \\ \delta S_z \\ \cdot \\ \cdot \end{pmatrix} = 0$$

We found an analytic algorithm for the solution search.

Spinless case (usual MHD)

$$\rho'_{\|,\perp} = \rho_{\|,\perp}/\mu_m$$

$$\begin{pmatrix} \omega + i(\rho'_\perp k_\|^2 + \rho'_\| k_\perp^2)/\mu_m & B k_\| \\ h\frac{v_A^2}{B} k_\| & h\omega + i(\eta_\| k_\|^2 + \eta_\perp k_\perp^2) \end{pmatrix} \begin{pmatrix} \delta u_y \\ \delta B_y \end{pmatrix} = 0$$

$$\begin{pmatrix} -i\frac{B/h}{1-v_A^2}\rho'_\perp c_s^2 k_\perp^2 & -k_\perp B & 0 & -i\rho'_\perp k_\| k_\perp & \omega + i\rho'_\perp k_\perp^2 \\ i\frac{B/h}{1-v_A^2}\rho'_\perp c_s^2 k_\perp k_\| & k_\| B & 0 & \omega + i\rho'_\perp k_\|^2 & -i\rho'_\perp k_\perp k_\| \\ \omega & -hk_\perp & -h(1-v_A^2)k_\| & 0 & \frac{h}{B}v_A^2\omega \\ -c_s^2 k_\perp & h\omega + i[(\zeta_\perp + \eta_\perp)k_\perp^2 + \eta_\| k_\|^2] & i(\zeta_\times + \eta_\|)k_\perp k_\| & \frac{h}{B}v_A^2 k_\| & -\frac{h}{B}v_A^2 k_\perp \\ -c_s^2 k_\| & +i(\zeta_\times + \eta_\|)k_\perp k_\| & \bar{h}(1-v_A^2)\omega + i(\zeta_\| k_\|^2 + \eta_\| k_\perp^2) & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta e \\ \delta u_x \\ \delta u_z \\ \delta B_x \\ \delta B_z \end{pmatrix} = 0$$

$$(\nabla \cdot \boldsymbol{B} = 0)$$

Alfven waves (Transverse waves)

$$\tilde{\rho} = (\rho_\perp \cos^2\theta + \rho_\| \sin^2\theta)/\mu_m$$
$$\tilde{\eta} = (\eta_\| \cos^2\theta + \eta_\perp \sin^2\theta)/h.$$

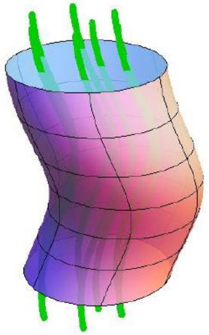
Secular equation for the 2*2 matrix

$$v_A^2 k_\|^2 - (\omega + i\tilde{\rho}k^2)(\omega + i\tilde{\eta}k^2) = 0$$

Dispersion relation for the Alfven waves

$$\omega \;=\; \pm v_A k_\| - \frac{i}{2}(\tilde{\rho} + \tilde{\eta})k^2 + \mathcal{O}(k^3)$$

Alfven velocity: $v_A^2 = \frac{B^2}{\mu_m h}$



Spinless case continued (usual MHD)

Magneto-sonic waves (Longitudinal waves)

Secular equation w/o dissipative corrections (the lowest order in k)

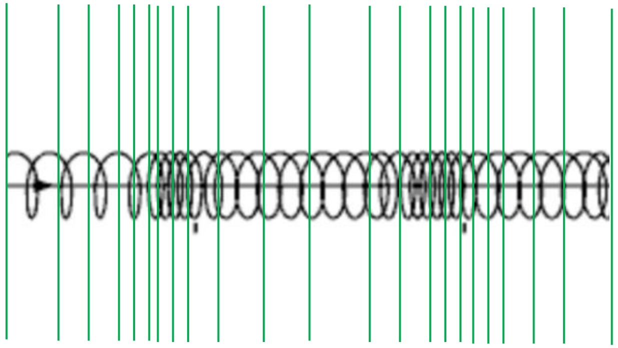
$$(\omega^2)^2 - \mathcal{V}^2 k^2 \omega^2 + v_A^2 c_s^2 k^2 k_\parallel^2 = 0$$

Modifications of the sound velocity due to the magnetic pressure.

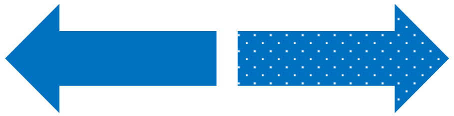
$$\omega = \pm v_1 k, \quad \omega = \pm v_2 k$$

$$v_{1,2} = \frac{1}{\sqrt{2}} \sqrt{\mathcal{V}^2 \pm \sqrt{\mathcal{V}^4 - 4 v_A^2 c_s^2 \cos^2 \theta}}$$

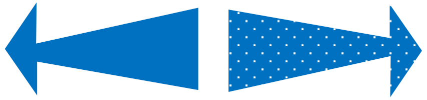
Fast and slow magneto-sonic waves



Magneto-sonic waves with dissipative corrections
- Solutions have not been known in the literature



- 0th order solutions:
  Pairs of waves propagating in opposite directions



- 1st order dissipative corrections:
  Pairs of waves damping at the same rate
  (w/o parity breaking effects)

The same magnitudes of
velocities and damping rates

Ansatz for the four solutions

$$\omega = \pm v_1 k - i w_1 k^2, \quad \omega = \pm v_2 k - i w_2 k^2$$

$w_{1,2}$: Unkown damping rates to be determined
$v_{1,2}$: Velocities already determined at the LO.

Analytic solutions: Damping rates

$$w_1 = -\frac{W_1 - v_1^2 W_2}{2(v_1^2 - v_2^2)}, \quad w_2 = +\frac{W_1 - v_2^2 W_2}{2(v_1^2 - v_2^2)}$$

$$\begin{aligned}
W_1 &= \eta'_{\|}\left(c_s^2 \cos^2(2\theta) + \frac{v_A^2 \sin^2\theta}{1-v_A^2}\right) + \rho'_{\perp} c_s^2 \frac{1 - v_A^2 \cos^2\theta}{1-v_A^2} \\
&\quad + \cos^2\theta\left[\zeta'_{\|}\frac{v_A^2}{1-v_A^2} + (\zeta'_{\|} + \zeta'_{\perp} + \eta'_{\perp} - 2\zeta'_{\times}) c_s^2 \sin^2\theta\right], \\
W_2 &= \eta'_{\|}\left(1 + \frac{v_A^2}{1-v_A^2}\sin^2\theta\right) + \rho'_{\perp}\left(1 + \frac{v_A^2 c_s^2}{1-v_A^2}\sin^2\theta\right) \\
&\quad + \frac{\zeta'_{\|}}{1-v_A^2}\cos^2\theta + (\zeta'_{\perp} + \eta'_{\perp})\sin^2\theta
\end{aligned}$$

One can show that $W_{1,2} \geq 0$.

All solutions are always damping (stable around the equilibrium)

Linearized spin MHD

4×4

$$\left(A_{(0)} + iA_{(1)}\right) \begin{pmatrix} \delta u_y \\ \delta B_y \\ \delta S_x \\ \delta S_z \end{pmatrix} = 0$$

$$A_{(0)} = \begin{pmatrix} Bk_{\|} & \omega & 0 & 0 \\ h\omega & \frac{1}{\mu_m} Bk_{\|} & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

$$A_{(1)} = \begin{pmatrix} 0 & \rho'_{\perp} k_{\|}^2 + \rho'_{\|} k_{\perp}^2 & 0 & 0 \\ (\eta_{\|} - 2\xi + \gamma_{\|})k_{\|}^2 + (\eta_{\perp} + \gamma_{\perp})k_{\perp}^2 & 0 & -\frac{4i}{\chi}(\xi - \gamma_{\|})k_{\|} & -\frac{4i}{\chi}\gamma_{\perp} k_x \\ -i2(\gamma_{\|} - \xi)k_{\|} & 0 & \Gamma_{\|} & 0 \\ i2\gamma_{\perp} k_x & 0 & 0 & \Gamma_{\perp} \end{pmatrix}$$

5×5

$(\nabla \cdot \boldsymbol{B} = 0)$

$$\left(M_{(0)} + iM_{(1)}\right) \begin{pmatrix} \delta e \\ \delta u_x \\ \delta u_z \\ \delta B_x \\ \delta B_z \\ \delta S_y \end{pmatrix} = 0$$

$$M_{(0)} = \begin{pmatrix} 0 & -k_{\perp} B & 0 & 0 & \omega & 0 \\ 0 & k_{\|} B & 0 & \omega & 0 & 0 \\ \omega & -hk_{\perp} & h(v_A^2 - 1)k_{\|} & 0 & \frac{h}{B} v_A^2 \omega & 0 \\ -c_s^2 k_{\perp} & h\omega & 0 & \frac{h}{B} v_A^2 k_{\|} & -\frac{h}{B} v_A^2 k_{\perp} & 0 \\ -c_s^2 k_{\|} & 0 & -h(v_A^2 - 1)\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \end{pmatrix}$$

$$v_A^2 = B^2/(\mu_m h)$$

$$M_{(1)} = \begin{pmatrix} -\frac{v_A^2}{1-v_A^2} \frac{\rho_{\perp}}{B} c_s^2 k_{\perp}^2 & 0 & 0 & -\frac{\rho_{\perp}}{\mu_m} k_{\|} k_{\perp} & \frac{\rho_{\perp}}{\mu_m} k_{\perp}^2 & 0 \\ \frac{v_A^2}{1-v_A^2} \frac{\rho_{\perp}}{B} c_s^2 k_{\perp} k_{\|} & 0 & 0 & \rho_{\perp} \mu_m^{-1} k_{\|}^2 & -\rho_{\perp} \mu_m^{-1} k_{\perp} k_{\|} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\zeta_{\perp} + \eta_{\perp})k_{\perp}^2 + (\eta_{\|} - 2\xi + \gamma_{\|})k_{\|}^2 & (\zeta_{\times} + \eta_{\|} - \gamma_{\|})k_{\perp} k_{\|} & 0 & 0 & 4\frac{i}{\chi}(\xi - \gamma_{\|})k_{\|} \\ 0 & (\zeta_{\times} + \eta_{\|} - \gamma_{\|})k_{\perp} k_{\|} & \zeta_{\|} k_{\|}^2 + (\eta_{\|} + 2\xi + \gamma_{\|})k_{\perp}^2 & 0 & 0 & +4\frac{i}{\chi}(\xi + \gamma_{\|})k_{\perp} \\ 0 & 2i(\gamma_{\|} - \xi)k_{\|} & -2i(\gamma_{\|} + \xi)k_x & 0 & 0 & 8\Gamma_{\|} \end{pmatrix}$$

Sweeping trashes away

Our goal: Solutions at $\mathcal{O}(k^2)$.
(Higher orders do not mean anything for the first-order hydro.)

Determinant should be factorized with ansatz

$$\begin{aligned} \det M_{4\times 4} &= \{\omega - (v_1 k - i w_1 k^2)\}\{\omega - (-v_1 k - i w_1 k^2)\} \\ &\quad \times \{\omega - (v_2 k - i w_2 k^2)\}\{\omega - (-v_2 k - i w_2 k^2)\} \\ &\quad + \mathcal{O}(\omega^3 k^3) + \mathcal{O}(\omega^2 k^4) + \mathcal{O}(\omega^1 k^5) + \mathcal{O}(\omega^0 k^6) \end{aligned}$$

However, actual determinant contains higher orders in k.

Greatly simplified if unnecessary higher orders are identified and thrown away appropriately at this stage.

Alfven modes with spin mixing

From the 4×4 matrix,

$$\begin{aligned}
\omega &= \pm v_A k_\parallel - \frac{i}{2}\left(\rho'_\perp k_\parallel^2 + \rho'_\parallel k_\perp^2 + \frac{(\eta_\parallel \gamma_\parallel - \xi^2)k_\parallel^2 + \eta_\perp \gamma_\parallel k_\perp^2}{h\gamma_\parallel}\right), \\
\omega &= -i\Gamma_\perp - i\frac{\gamma_\perp}{h}k_\perp^2 , \\
\omega &= -i\Gamma_\parallel - i\frac{(\gamma_\parallel - \xi)^2}{h\gamma_\parallel}k_\parallel^2
\end{aligned}$$

$$\Gamma_{\parallel,\perp} = \frac{8}{\chi}\gamma_{\parallel,\perp}$$

Spin susceptibility: $S^{\mu\nu} = \chi\mu^{\mu\nu}$

Rotational viscosity:

$$\delta\Theta^{[\mu\nu]} = -T\gamma^{\mu\nu\rho\sigma}(\partial_{[\rho}\beta_{\sigma]} - \beta\mu_{\rho\sigma})$$

Spins are not conserved → Gapped modes at k = 0.

$$\partial_\mu \Sigma^{\mu\alpha\beta} = -2\Theta^{[\mu\nu]}$$

$$\begin{aligned}
\omega &= -i\Gamma_\perp \\
\omega &= -i\Gamma_\parallel
\end{aligned}$$

All solutions are always damping (stable around the equilibrium)

Magneto-sonic modes with spin mixing

From the 5×5 matrix,

$$
\begin{aligned}
\omega &= \pm v_1 k - i w_1 k^2 \\
\omega &= \pm v_2 k - i w_2 k^2 \\
\omega &= -i\Gamma_{\|} - i w_3 k^2
\end{aligned}
\qquad
\begin{aligned}
w_1 &= -\frac{W_1 - v_1^2 W_2}{2(v_1^2 - v_2^2)} \\
w_2 &= +\frac{W_1 - v_2^2 W_2}{2\gamma_{\|}(v_1^2 - v_2^2)} \\
w_3 &= \frac{(\gamma'_{\|} - \xi')^2}{\gamma'_{\|}} \frac{1 - v_A^2 \cos^2\theta}{1 - v_A^2} + 4\xi' \frac{\sin^2\theta}{1 - v_A^2}
\end{aligned}
$$

Always damping (stable around the equilibrium)

$$
\begin{aligned}
W_1 &= \frac{\gamma'_{\|}\eta'_{\|} - \xi'^{\,2}}{\gamma_{\|}} \left(c_s^2 \cos^2(2\theta) + \frac{v_A^2}{1 - v_A^2} \sin^2\theta \right) + \rho'_{\perp} c_s^2 \left(1 + \frac{v_A^2}{1 - v_A^2} \sin^2\theta \right) \\
&\quad + \cos^2\theta \left[\zeta'_{\|} \left(\frac{v_A^2}{1 - v_A^2} + c_s^2 \sin^2\theta \right) + c_s^2 \left(\zeta'_{\perp} + \eta'_{\perp} - 2\zeta'_{\times} \right) \sin^2\theta \right] \\
W_2 &= \frac{\gamma'_{\|}\eta'_{\|} - \xi'^{\,2}}{\gamma_{\|}} \left(1 + \frac{v_A^2}{1 - v_A^2} \sin^2\theta \right) + \rho'_{\perp} \left(1 + \frac{c_s^2 v_A^2}{1 - v_A^2} \sin^2\theta \right) \\
&\quad + \frac{\zeta'_{\|}}{1 - v_A^2} \cos^2\theta + (\zeta'_{\perp} + \eta'_{\perp}) \sin^2\theta
\end{aligned}
$$

The analytic solutions tell us that

1. All the phase velocities are subluminal.
2. All the imaginary parts act as damping effects
   in the fluid rest frame.

This is, however, not sufficient to insure causality
and stability in arbitrary Lorentz frame (see a back-up slide).