Vortex and Topology:
Recent topics in Topological Fluid Mechanics

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21 Jan. 2025, Topology and Dynamics of Magneto-Vortical Matter

Vortex Dynamics on Curved Torus
(a manifold with a handle)

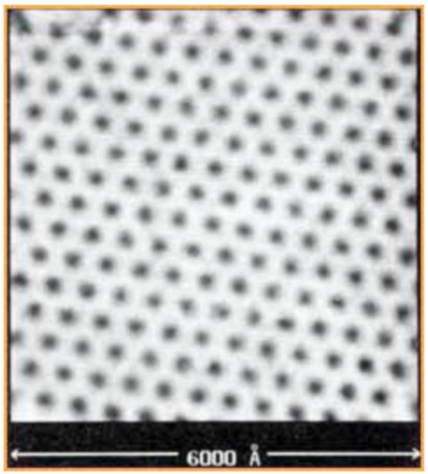
Vortex Crystals

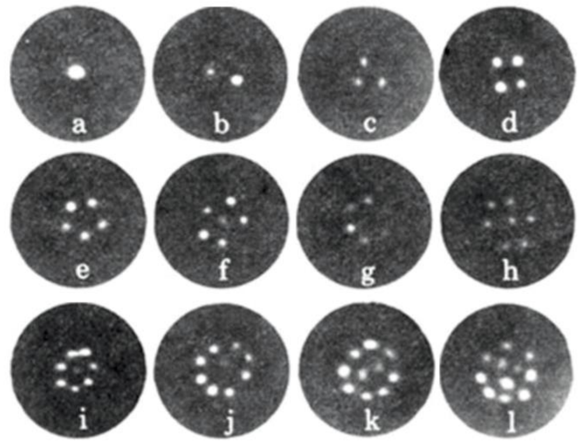
"Localized" patterns are observed in many physical systems

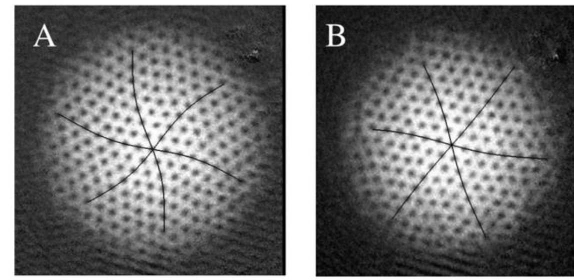
Super Conductors (Hess et. Al PRL 62 1989)

Super fluid helium II (Yarmuchuck et. Al PRL62 1979)

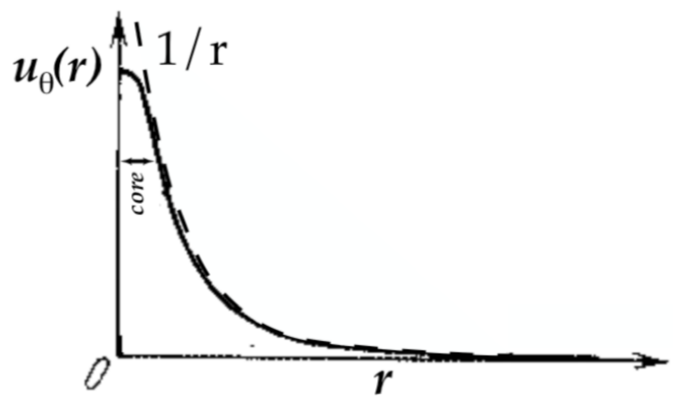
Bose-Einstein Condensates (Coddington et.al 2003)







A. A. Abrikosov (2003, Nobel Laureates, Rev. Mod. Phys. '04) Those particle systems can be described by a phenomenological model with particle interactions under potentials with a logarithmic singularity.



Vortex crystals = Equilibrium states of points with log-singularities

Mathematical Models: Vortex Dynamics

» Physical quantities: velocity field $\boldsymbol{u}(t,\boldsymbol{x}) = (u(t,x,y), v(t,x,y))$

pressure $p(t,x,y)$ density (unity) $\rho = 1$

» The incompressible 2D Euler equations

$$\operatorname{div}\boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}, \qquad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}.$$

» This is a mathematical model for less viscous (e.g. water and air) or non-viscous flows.

» vorticity: $\omega = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

$$\frac{d}{dt}\omega(\boldsymbol{x}(t),t) = 0$$

In 2D, vorticity is conserved quantity along the path of fluid particle.

» Vorticity-streamfunction formulation = vortex dynamics

$$u = \psi_y \quad v = -\psi_x \qquad \Delta\psi = -\omega \qquad \frac{d\boldsymbol{x}}{dt}(t) = \boldsymbol{u}(\boldsymbol{x}(t),t)$$

Vortex Dynamics on 2D Riemannian manifolds

(M,g) M orientable 2D Riemannian manifold g Riemannian metric

Hydrodynamic Green's function (Lin 1941)

$G_H(\zeta,\zeta_0)$ $\zeta,\zeta_0\in M$

$G_H(\zeta,\zeta_0)=G_H(\zeta_0,\zeta)$ Reciprocal condition

$$\triangle G_H(\zeta,\zeta_0)=\begin{cases}\delta_{\zeta_0}-\frac{1}{\mathrm{Area}(M)} & \text{if } M \text{ is compact,}\\ \delta_{\zeta_0} & \text{if } M \text{ is non-compact,}\end{cases}$$

$\triangle$ Laplace Beltrami operator δ_{ζ_0} Dirac delta function on $\zeta_0\in M$

The Green's function has a logarithmic singularity

Streamfunction-vorticity formulation

ω vorticity ψ Streamfunction

Poisson equation $-\triangle\psi=\omega$

$$\Longrightarrow\ \psi(\zeta)=-\int_M G_H(\zeta,\zeta_0)\omega(\zeta_0)d\mu(\zeta_0).$$

μ Riemannian volume form

The velocity field is recovered from the stream function.
Construction of the Green's function is critical.

Mathematical Models: Point Vortex Dynacmis

N Point vortices

$\zeta_j \in M$ Locations of N point vortices $\Gamma_j \in \mathbb{R}$ Strengths

$$\omega = \sum_{j=1}^{N} \Gamma_j \delta_{\zeta_j} \implies \psi(\zeta) = -\sum_{j=1}^{N} \Gamma_j G_H(\zeta, \zeta_j)$$

Modified stream function (Helmholtz principle)

Robin function

$$R(\zeta_m) = \lim_{\zeta \to \zeta_m} \left[G_H(\zeta, \zeta_m) - \frac{1}{2\pi} \log \underbrace{d(\zeta, \zeta_m)}_{\text{distance}} \right]$$

Modified stream function of the *m*th point vortex

$$\psi_m(\zeta_m) = -\sum_{j \neq m}^{N} \Gamma_j G_H(\zeta_m, \zeta_j) - \frac{1}{2} \Gamma_m R(\zeta_m)$$

Equations of motion for *N* point vortices (in complex form)

$$\frac{\mathrm{d}\zeta_m}{\mathrm{d}t} = -2i\lambda^{-2}(\zeta_m, \bar{\zeta}_m) \frac{\partial \psi_m}{\partial \bar{\zeta}_m} \implies \frac{\partial \psi_m}{\partial \bar{\zeta}_m} = 0 \quad \text{Vortex crystal}$$

$\lambda(\zeta_m, \bar{\zeta}_m)$ conformal factor

Vortex flows on a curver torus

A toroidal surface $\mathbb{T}_{R,r}$ $\quad R$ major radius $\quad r$ minor radius

$\alpha = R/r > 1$ aspect ratio

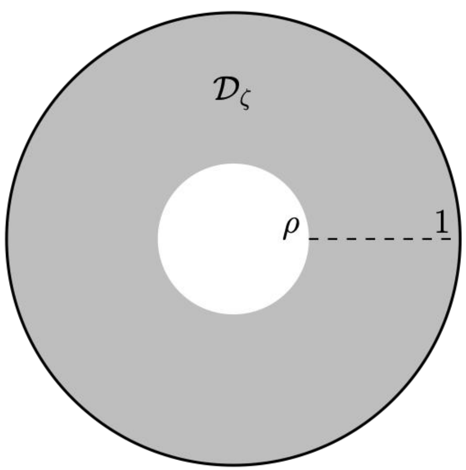
Embed the torus in the 3D Euclidean space

$$(\theta, \phi) \in \mathbb{T}_{R,r} \mapsto ((R - r\cos\theta)\cos\phi, (R - r\cos\theta)\sin\phi, r\sin\phi) \in \mathbb{E}^3$$

Complex representation Stereographic projection)

$$\zeta : (\theta, \phi) \longmapsto \mathrm{e}^{i\phi} \exp\left(- \int_0^\theta \frac{du}{\alpha - \cos u} \right) \equiv \mathrm{e}^{i\phi} \exp(r_c(\theta))$$

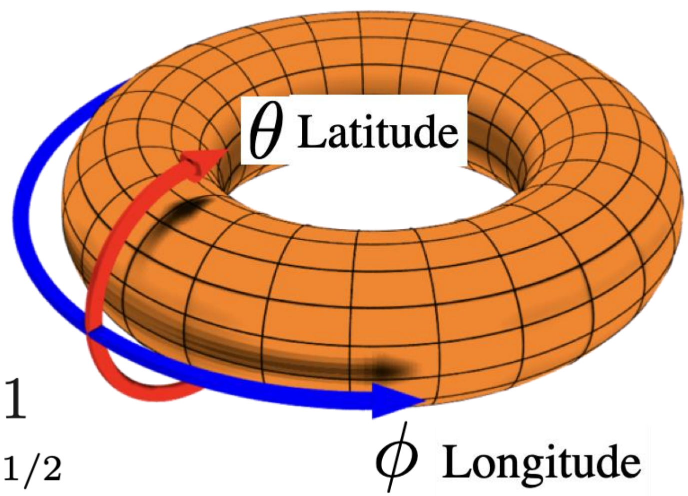
ζ-plane



$$\{\zeta \in \mathbb{C} \,|\, \rho < |\zeta| < 1\}$$

$$\rho = \mathrm{e}^{-2\pi\mathcal{A}} < 1$$

$$\mathcal{A} = \left(\alpha^2 - 1\right)^{-1/2}$$



Hydrodynamics Green's function

Conformal factor

$$\lambda(\zeta,\overline{\zeta}) = \frac{R - r\cos\theta}{|\zeta|}$$

Distance

$$d(\zeta,\zeta_0) = \lambda(\zeta,\zeta_0)|\zeta - \zeta_0|$$

Hydrodynamic Green's function (Green-Mashall '12, S-, Shimizu '16)

$$\boxed{G_H(z_1,z_2) = \frac{1}{2\pi}\log\left|z_2 P\left(\frac{z_1}{z_2}\right)\right| + F(\theta_1) + F(\theta_2) + \frac{1}{4\pi^2\mathcal{A}}G(\theta_1)G(\theta_2) - \frac{1}{4\pi}G(\theta_1) - \frac{1}{4\pi}G(\theta_2)}$$

Schottky Klein Prime function associated with the annulus

$$P(\zeta,\sqrt{\rho}) = (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^k\zeta)(1-\rho^k/\zeta).$$

Indefinite integrals

$$F(\theta) = -\frac{1}{4\pi^2\alpha}\int_0^\theta \frac{\alpha u - \sin u}{\alpha - \cos u}du, \qquad G(\theta) = -\int_0^\theta \frac{du}{\alpha - \cos u} = \log|z|.$$

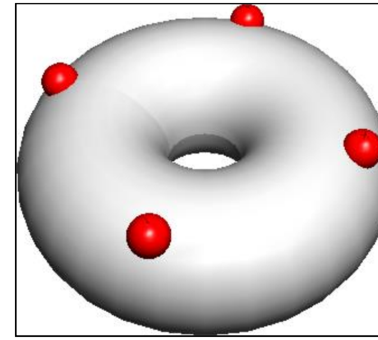
Robin function

$$\boxed{R(\zeta_m) = \frac{1}{2\pi}\log\prod_{n\geq 1}(1-\rho^n)^2 + 2F(\theta_m) + \frac{1}{4\pi^2\mathcal{A}}G^2(\theta_m) - \frac{1}{2\pi}\log(R - r\cos\theta_m)}$$

$$\mathcal{A} = (\alpha^2 - 1)^{-1/2}$$

Evolution equation for point vortices

Point vortices = Vortex distribution consists of Dirac's delta functions



$$\omega = \sum_{j=1}^{N} \Gamma_j \delta_{\zeta_j}$$

$\zeta_j \in M$ Locations of N point vortices

$\Gamma_j \in \mathbb{R}$ Strengths

Evolution equations for *N* point vortices

$$r^2(\alpha - \cos\theta_m)\frac{\mathrm{d}\theta_m}{\mathrm{d}t} = i\sum_{j\neq m}^{N}\Gamma_j\left[\frac{K(\zeta_m/\zeta_j) - \overline{K(\zeta_m/\zeta_j)}}{4\pi}\right],$$

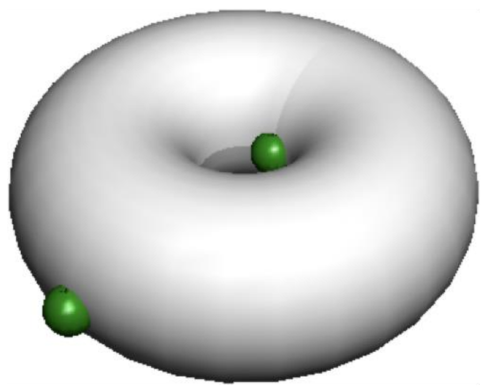
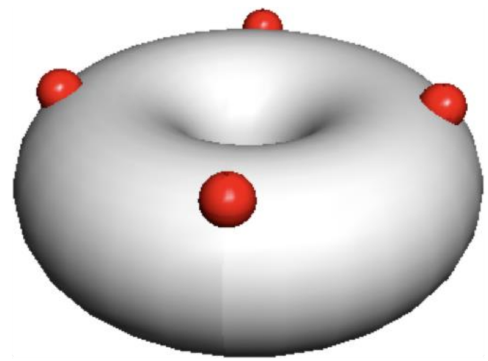
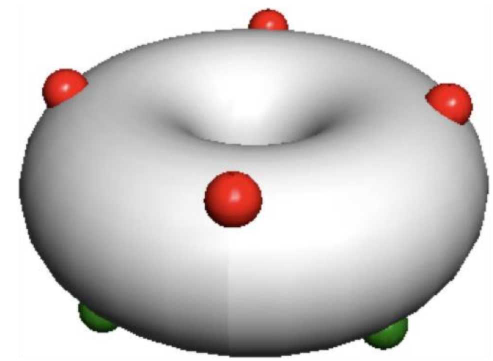
$$r^2(\alpha - \cos\theta_m)^2\frac{\mathrm{d}\phi_m}{\mathrm{d}t} = \sum_{j\neq m}^{N}\Gamma_j\left[\frac{K(\zeta_m/\zeta_j) + \overline{K(\zeta_m/\zeta_j)}}{4\pi} + \frac{\alpha\theta_m - \sin\theta_m}{4\pi^2\alpha} + \frac{r_c(\theta_j)}{4\pi^2\mathcal{A}} - \frac{1}{4\pi}\right]$$
$$+\Gamma_m\left[\frac{\alpha\theta_m - \sin\theta_m}{4\pi^2\alpha} + \frac{r_c(\theta_m)}{4\pi^2\mathcal{A}} + \frac{1}{4\pi}\sin\theta_m\right].$$

$$K(\zeta,\sqrt{\rho}) = \frac{\zeta P'(\zeta,\sqrt{\rho})}{P(\zeta,\sqrt{\rho})}. \quad P(\zeta,\sqrt{\rho}) = (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^k\zeta)(1-\rho^k/\zeta).$$

$$\frac{\mathrm{d}\theta_m}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\phi_m}{\mathrm{d}t} = 0 \quad \Longrightarrow$$ **Vortex crystals on the curved torus**

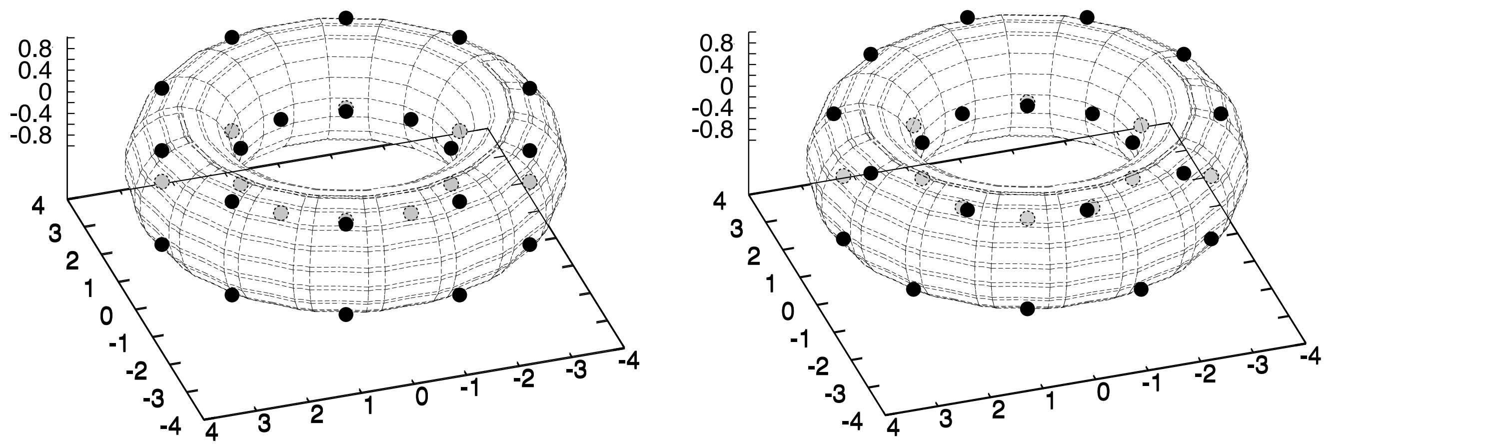
T. S-. and Y. Shimizu Proc. Roy. Soc. A (2016)

Vortex Crystals

Antipodal locations	N-ring configuration	Pairs of M-ring
		
$(\theta, \phi) = (0,0) \quad (0,\pi)$ $(\pi,\pi) \quad (\pi,0)$	$(\theta_m, \phi_m) = (\Theta_0, \frac{2\pi m}{N})$	$(\theta_{2m-1}, \phi_{2m-1}) = (\Theta_1, \frac{2\pi m}{M})$ $(\theta_{2m}, \phi_{2m}) = (\Theta_2, \frac{2\pi m}{M} + \frac{\pi}{M})$ $(\theta_{2m-1}, \phi_{2m-1}) = (\Theta_1, \frac{2\pi m}{M})$ $(\theta_{2m}, \phi_{2m}) = (\Theta_2, \frac{2\pi m}{M})$ $\Theta_1 + \Theta_2 = 2\pi$
Any strength	$\Gamma_m = \Gamma$	$\Gamma_{2m} = \Gamma \quad \Gamma_{2m-1} = -\Gamma$
Fixed equilibrium	Relative equilibrium	Relative equilibrium

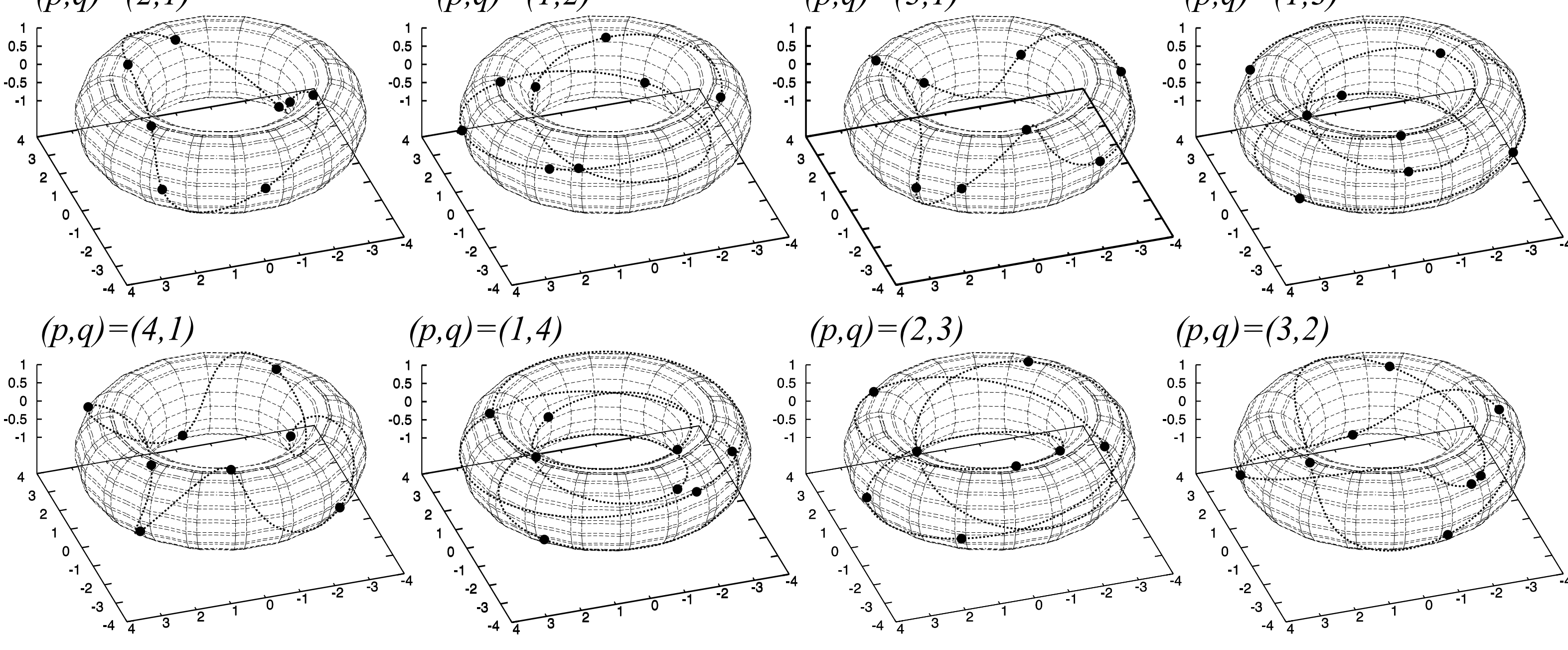
More Vortex Crystals

(a) K=3, M=10, aligned　　(b) K=3, M=10, staggered



(p,q)=(2,1)　(p,q)=(1,2)　(p,q)=(3,1)　(p,q)=(1,3)

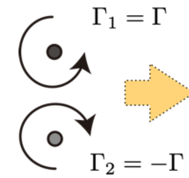
(p,q)=(4,1)　(p,q)=(1,4)　(p,q)=(2,3)　(p,q)=(3,2)

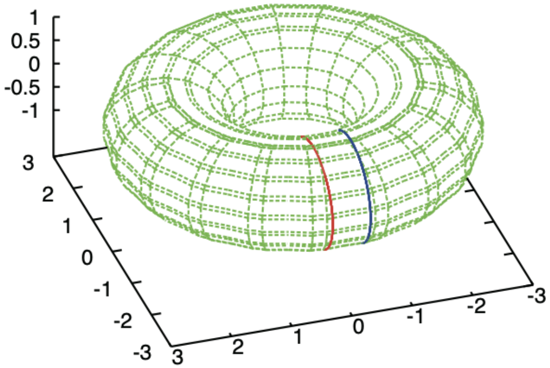


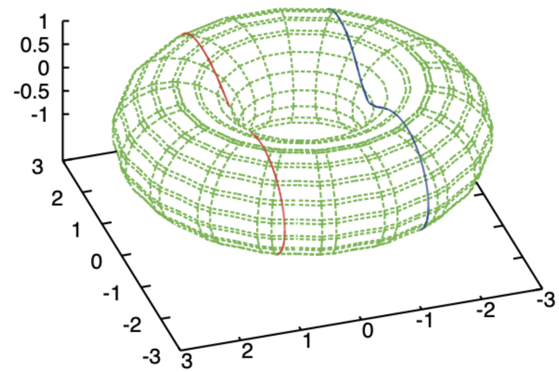
T. S.-, Phil. Trans. Roy Soc. A (2019)

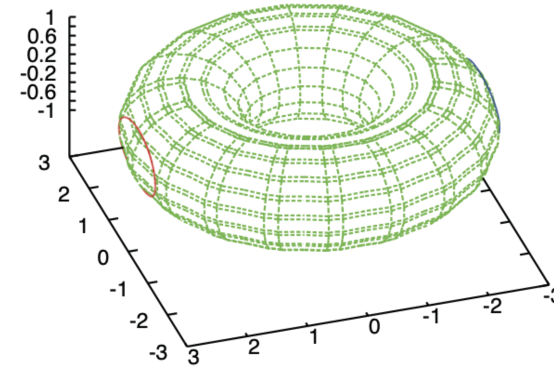
Two Point Vortex Dynamics (A catalogue)

Vortex Dipole (A pair of point vortices with the opposite signs)

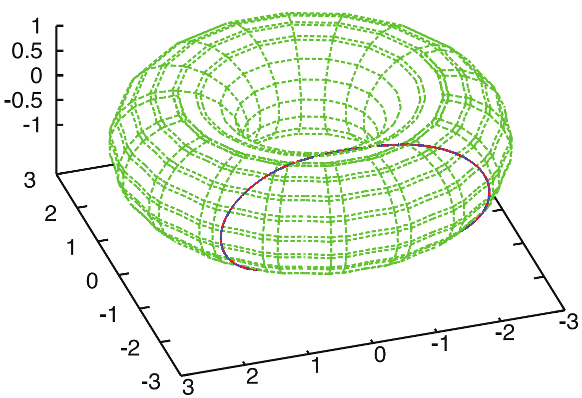


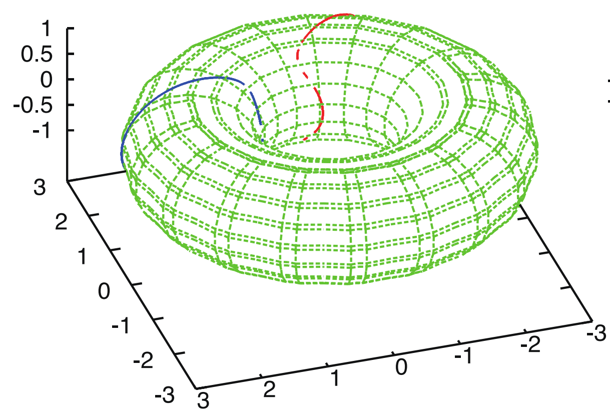


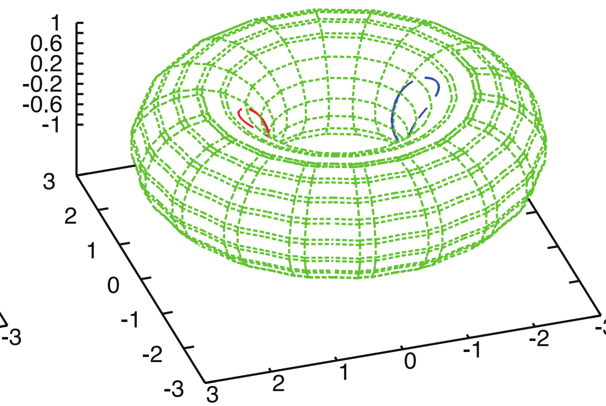




Identical vortices (A pair of point vortices with the same sign)







The handle structure affects the dynamics of point vortices.

T. S-. and Y. Shimizu Proc. Roy. Soc. A (2016)

Vortex Crystals with background vortcitity

Liouville-type equation

$$\nabla^2_{\mathbb{T}_{R,r}}\psi = c\mathrm{e}^{d\psi} + \frac{2}{d}\kappa(\mathbb{T}_{R,r}) \equiv -\omega$$

Curvature of the torus

$$\kappa(\mathbb{T}_{R,r}) \equiv -\frac{\cos\theta}{r(R - r\cos\theta)}$$

Analytic solution (S-. PRSA '19, '21)

$$\psi(\zeta,\overline{\zeta}) = \frac{1}{d}\log\left[-\frac{2|f'(\zeta)|^2}{cd(1+|f(\zeta)|^2)^2}\right] - \frac{1}{d}\log\left[\frac{(R-r\cos\theta)^2}{4|\zeta|^2}\right]$$

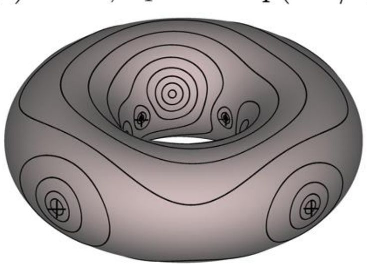
(a) $f(\zeta)$ is analytic on the annular domain D_ζ.

(b) $f(\zeta)$ is loxodromic, i.e., $f(\rho\zeta) = f(\zeta)$.

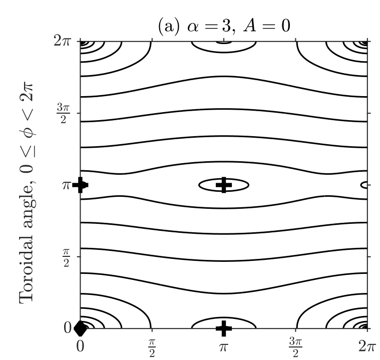
(c) $f'(\zeta) \neq 0$ on D_ζ.

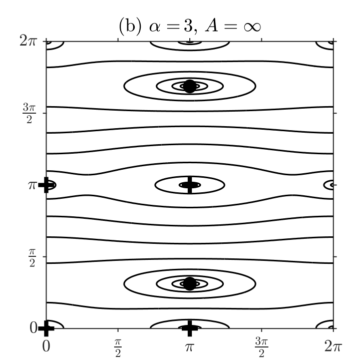
Examples of vortex crystals

(a) $\alpha = 2,\ a_1 = 0.9\exp(\mathrm{i}\,2\pi/3)$



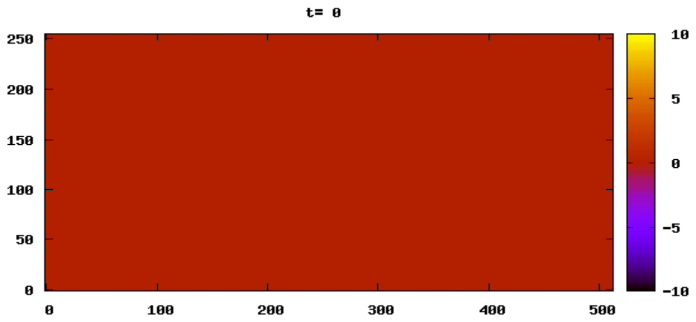
(b) $\alpha = 3,\ a_1 = 0.9\exp(\mathrm{i}\,2\pi/3)$



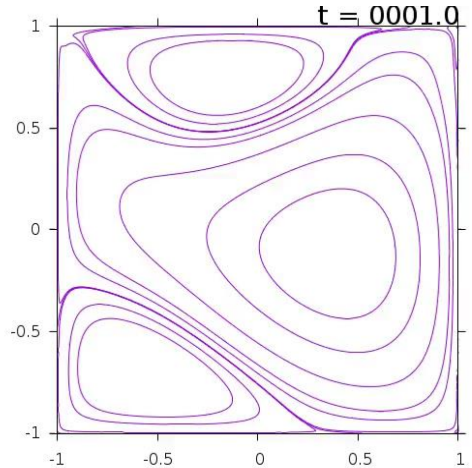


Topological Flow Data Analysis
Tracking “Shape of Flows”

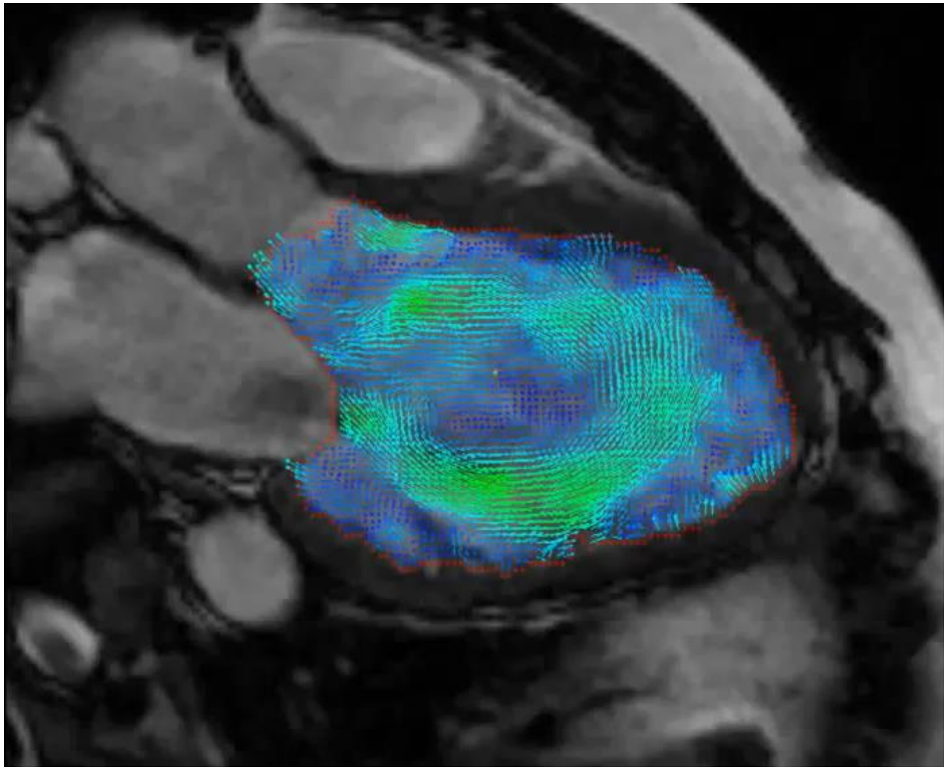
Motivation – complex flow patterns



A flow around a wing (Sawamura 2012)



A flow model of rotating flow in a box.
(Matsumoto 2015)



4D-MRI flow visualization in the left ventricle of the heart

We develop flow pattern characterizations to promote interdisciplinary research using flow simulations and experiments.

What is Topological Flow Data Analysis (TFDA)?

- It is a method of data analysis extracting topological features of flow orbits generated by 2D vector fields.

- **(A reduced-order modeling with discrete graphs)**
  The continuous time evolution of flows is reduced to a discrete dynamical system of graphs (partially Cyclically Ordered rooted Tree = COT) and its symbolic representation (COT representation).

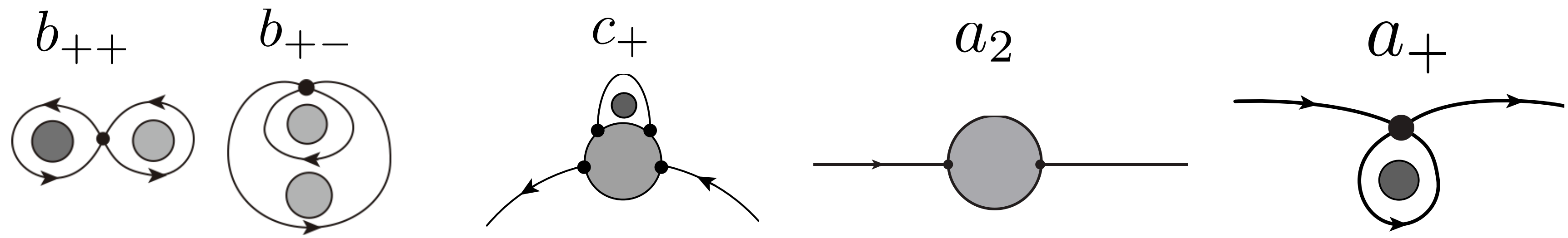
- **(Quantitative estimation of flow topology)**
  It extracts quantitative information associated with topological flow patterns in datasets.

- We explain the basic features of TFDA using 2D incompressible flows, and we then extend the theory suitable for blood flows in the heart.

Reference: Incompressible flows: Proc. Roy. Soc. A ‘13, Physica D ‘15, IMA J. Appl. Math. ’18; Compressible flows: Disc. Math. Algo. Appl. ‘22

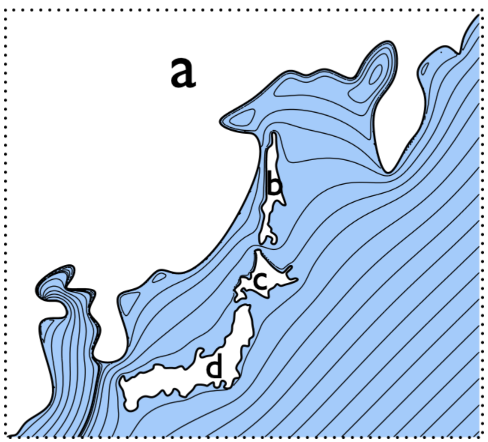
Theoretical Overview of TFDA for 2D incompressible flows

COT symbols = Describing local topological orbit structures (streamlines)

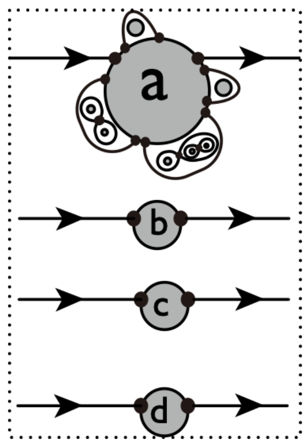


COT (= tree) & COT representation (= symbolic expression)

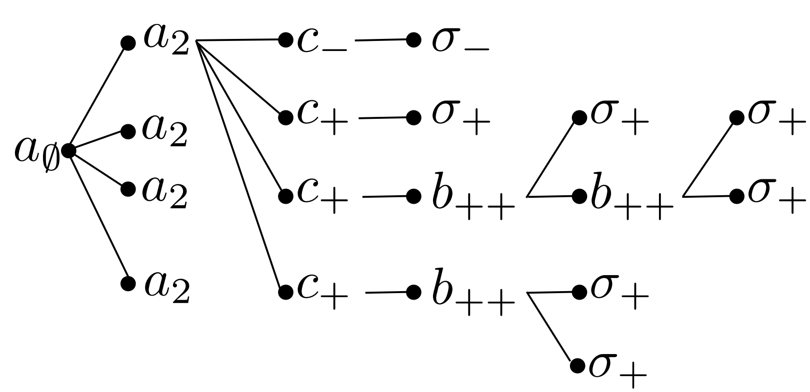
Instantaneous Streamlines



Topological structures



COT(=graph) & COT representation (=symbols)

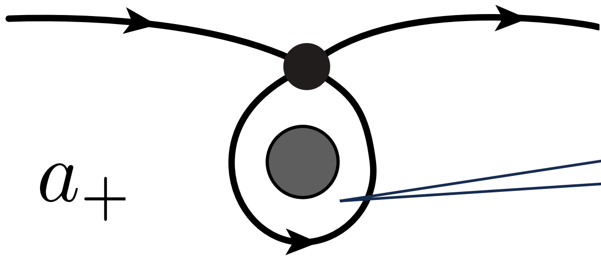


$$a_{\emptyset}(a_2, a_2, a_2, a_2(c_+(b_{++}\{\sigma_+, \sigma_+\}), c_+(b_{++}\{b_{++}\{\sigma_+, \sigma_+\}, \sigma_+\}), c_+(\sigma_+), c_-(\sigma_-))$$

The correspondence between the topological structures of orbits and the COT is one-to-one.

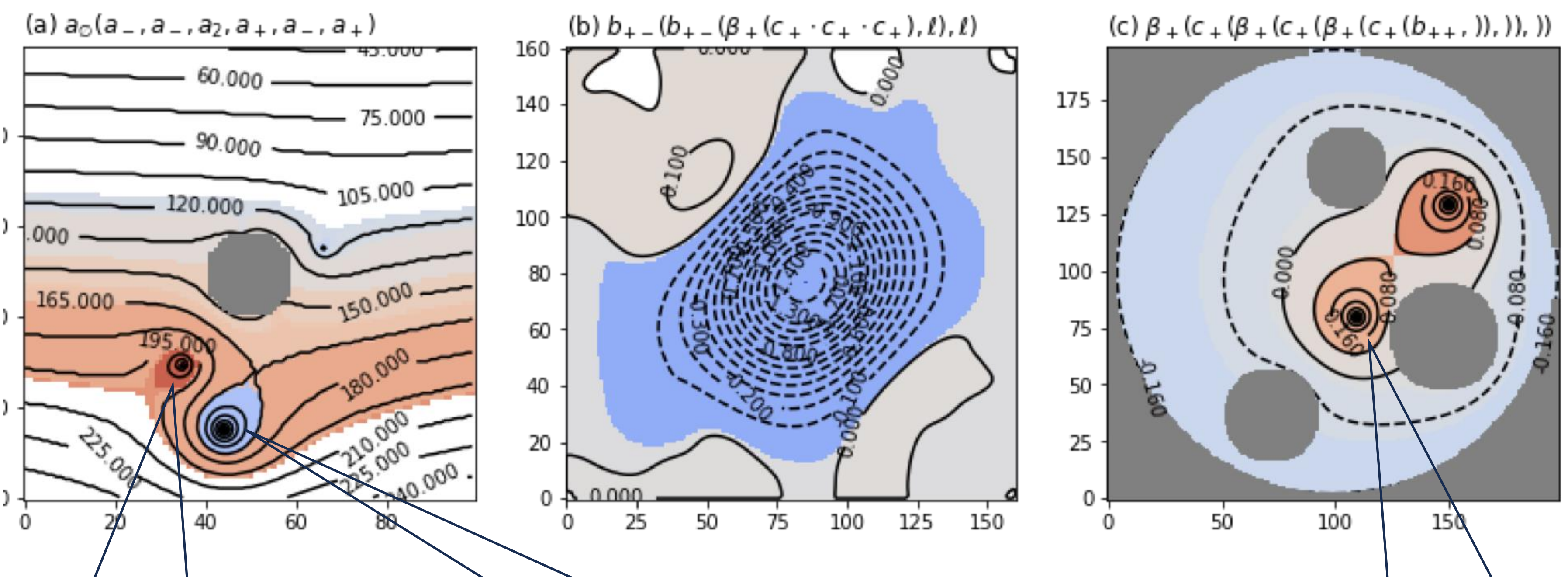
Topological Decomposition of Flow Domain

Each COT symbol is associated with a bounded region = **The region of influence**



The region of influence of a_+= a flow domain enclosed by saddle separatrix

Using COT representation, the flow domain is decomposed into the regions of influence



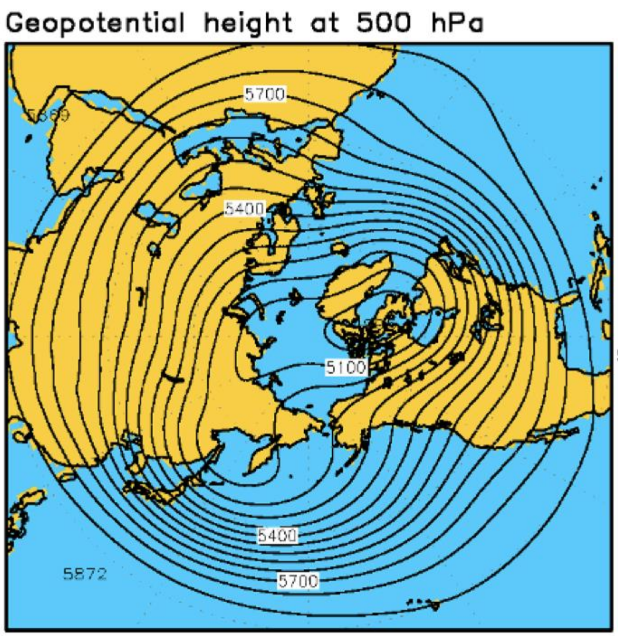
The region of influence of a_+

The region of influence of a_-

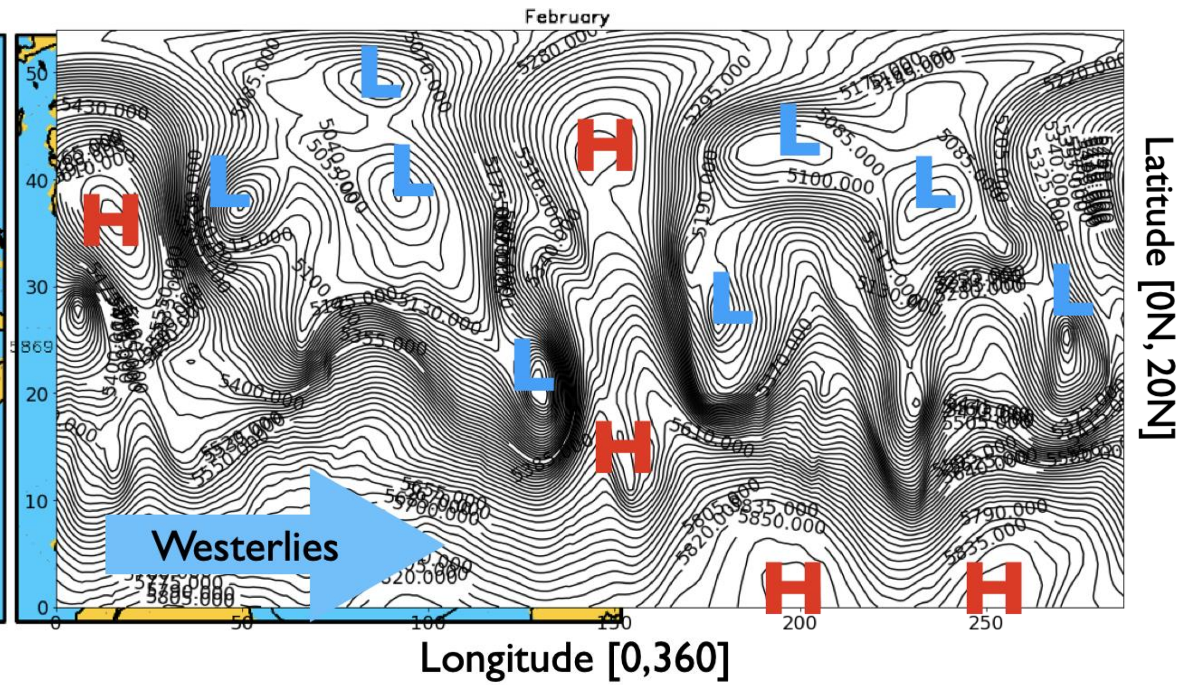
The region of influence of b_{++}

Application 1: Atmospheric Flows

Geopotential height at 500hPa (Data from Japan's Meteorological Agency)



Average Flow (Wikipedia)



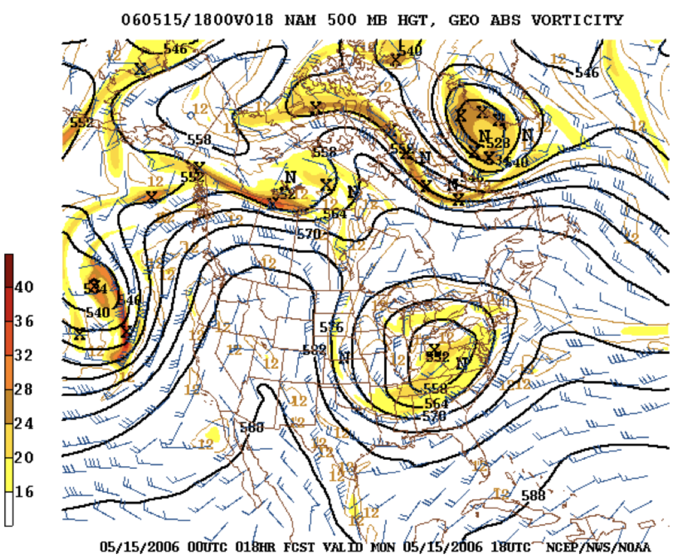
- ✓ The height of the atmosphere where the pressure becomes 500hPa.
- ✓ Under the quasi-geostrophic assumption, level curves of the geo-potential height are equivalent to streamline, yielding Hamiltonian vector fields
- ✓ Existence of westerly jet flow around the north pole.
- ✓ High/low-pressure regions play a significant role in weather/climate.

Blocking phenomena

Atmospheric Blocks (by Wikipedia):

A large-scale pattern in the atmospheric pressure field that is nearly stationary, effectively “blocking” or redirecting migratory cyclones. These blocks can remain in place for several days or even weeks, causing the areas affected by them to have the same kind of weather for an extended period of time – a cause of abnormal weather heatwaves, draught, heavy rain etc…

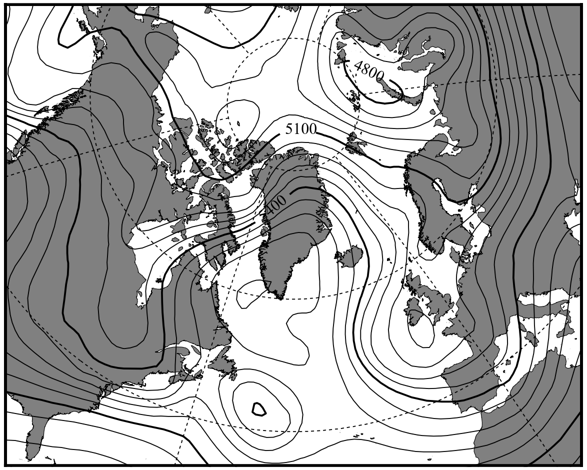
A “block” over western North America in 2006 (Wikipedia)



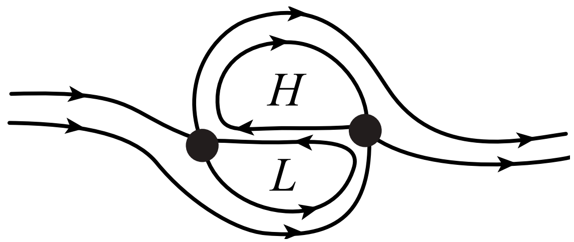
- ✓ It is difficult to identify whether or not the current weather is in the state of blocks.
- ✓ They are usually identified by experts in meteorology authority.
- ✓ We need an objective method to identify the blocking state.

Blocking morphology and COT symbols

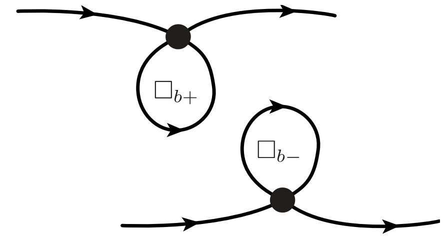
Dipole type



Local Morphology

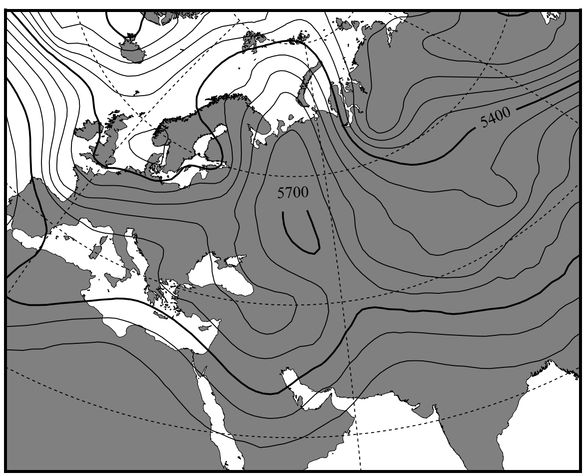


COT $a_{+} \cdot a_{-}$

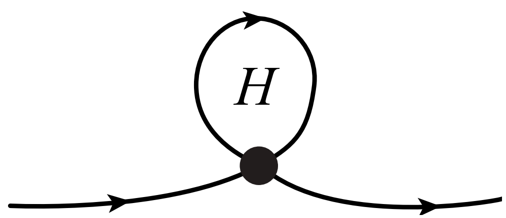


Dipole type blocking = a pair of high and low-pressure regions, which is represented by a sequence of COT symbols “$a_{+} \cdot a_{-}$”

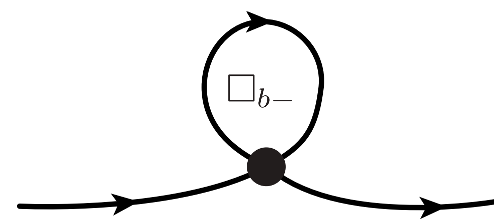
Ω type



Local Morphology



COT a_{+}



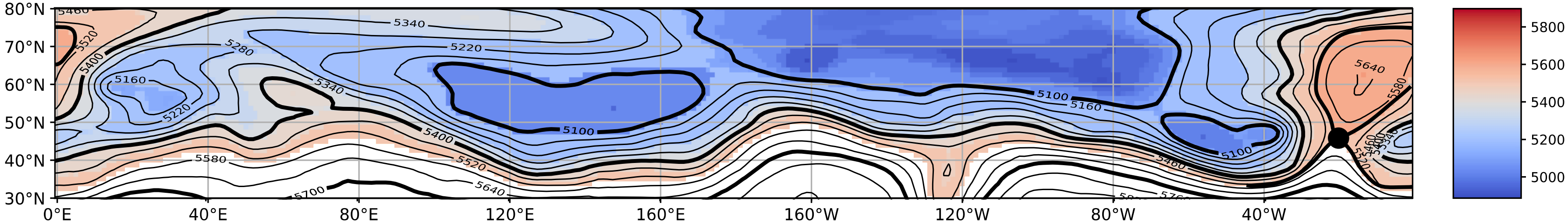
Omega type blocking = an isolated high pressure region, which is represented by a sequence of COT symbols “a_{+}”

TFDA results and a new feature: morphology identification

Idea: We perform TFDA for snapshots of 500hPa geopotential height data, tracking the evolution of regions of influence with a_+ (high-pressure regions) and a_- (low-pressure regions) in the temporal direction.

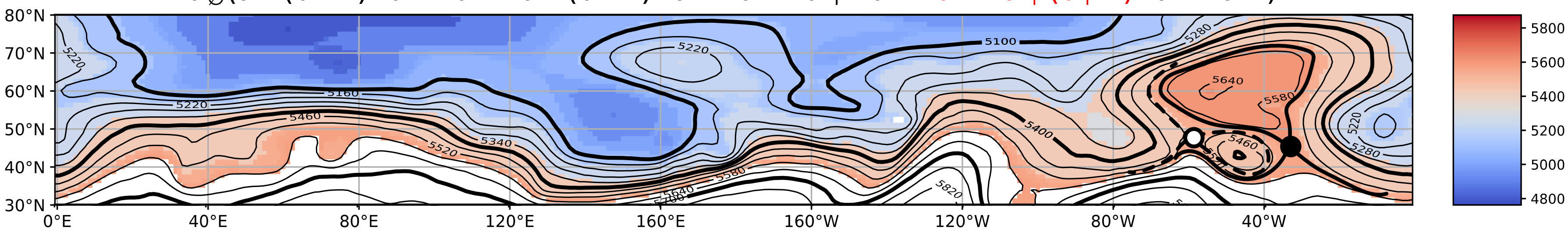
(a) blocking event #01 (Omega type)

$$a_\emptyset(a_-(b_{-+}(b_{--}\{\sigma_-, b_{--}\{\sigma_-, b_{--}\}\}, \sigma_+)) \cdot a_-(b_{--}) \cdot a_- \cdot a_-(b_{--}) \cdot a_- \cdot a_+ \cdot a_+ \cdot a_- \cdot a_+)$$



(b) blocking event #03 (Dipole type)

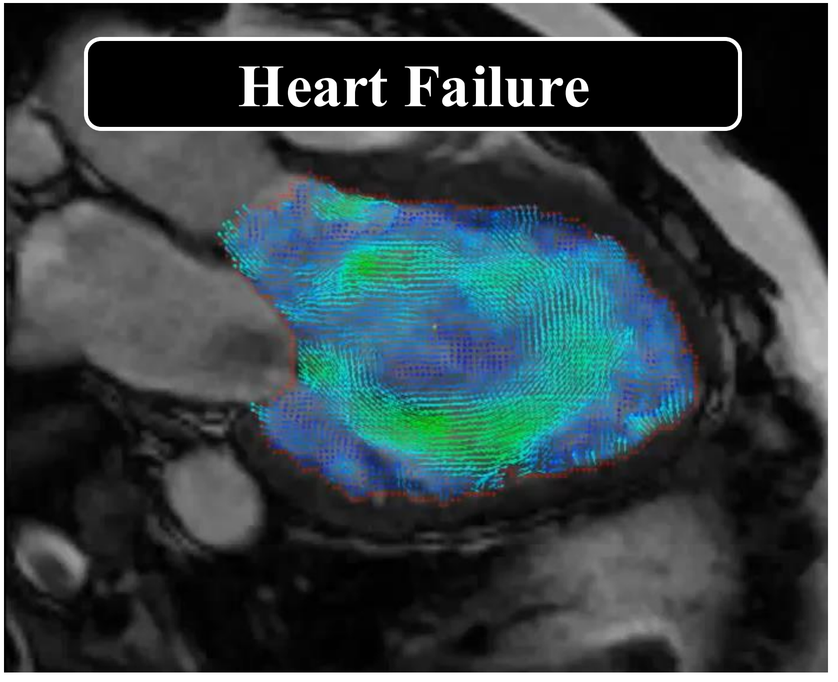
$$a_\emptyset(a_-(b_{--}) \cdot a_- \cdot a_- \cdot a_-(b_{--}) \cdot a_- \cdot a_- \cdot a_+ \cdot a_- \cdot a_- \cdot a_+(b_{+-}) \cdot a_- \cdot a_-)$$

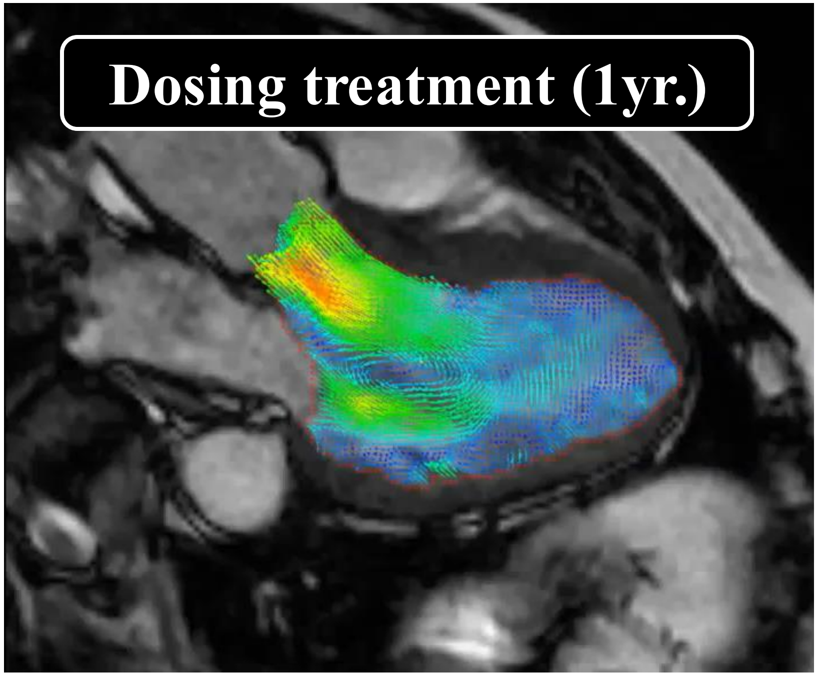


- TFDA successfully detects blocking events at a similar level to conventional methods. In addition, it can identify the morphological type of the blocking by observing the COT representation.

Application 2 : Vortex Structure Identification in the Heart Flow

MRI·Echocardiography = Visualization of blood flows inside the heart



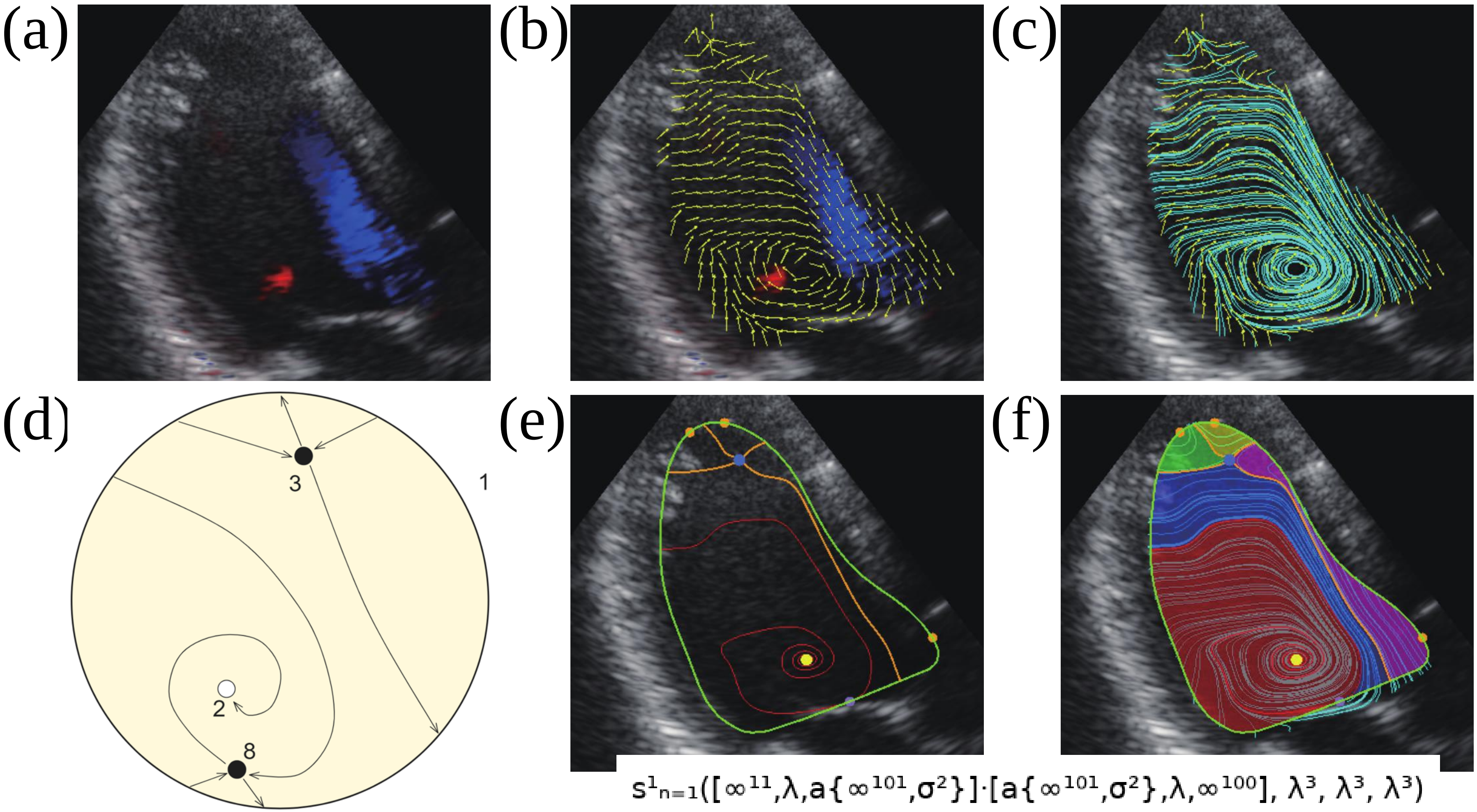


Nabeta T, Itatani K et al. 2015;14;36(11):637

- Disturbances in the vortical patterns cause inefficient flow and worsen heart diseases.
- 2D vortex flow structures in the left ventricle (LV) play an essential role in describing the efficiency of the blood flow in heart diseases
- A few objective definition of vortex structures is present for the blood flow stream -- difficulties describing the vortices comprehensively in clinical imaging examination.

Need to "identify" vortex structures from flow data inside the LV in clinical medicine

Echocardiography Vetor Flow Mapping (VFM)



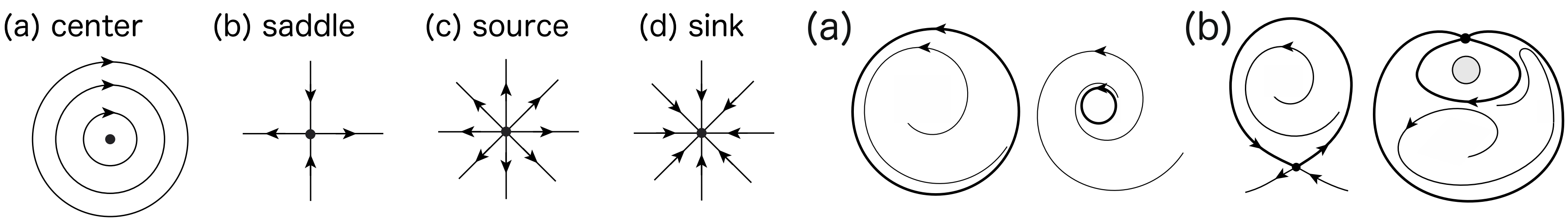
- (a) Original image of echocardiography in the LV symmetrical long-axis plane.
- (b) Instantaneous vector field flow mapping (VFM) in the plane. *Itatani et al., Japan J. Appl. Phys. vol. **52**, 2013*
- (c) Particle orbits are computed from the velocity field on this domain.
- (d) COT (e) saddle connection diagram
- (f) Partition plot

TFDA for the flow in LV

Theoretical difficulties and theoretical extensions:

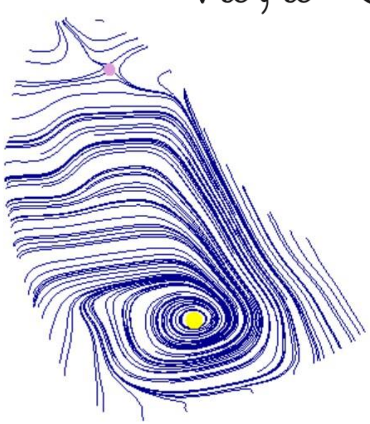
(1) The flow in the 2D section is NOT incompressible ⇒ Compressible flow components
(2) The boundary is constantly MOVING ⇒ transversal boundary condition

“The flow of finite type” : Additional compressible orbit structures



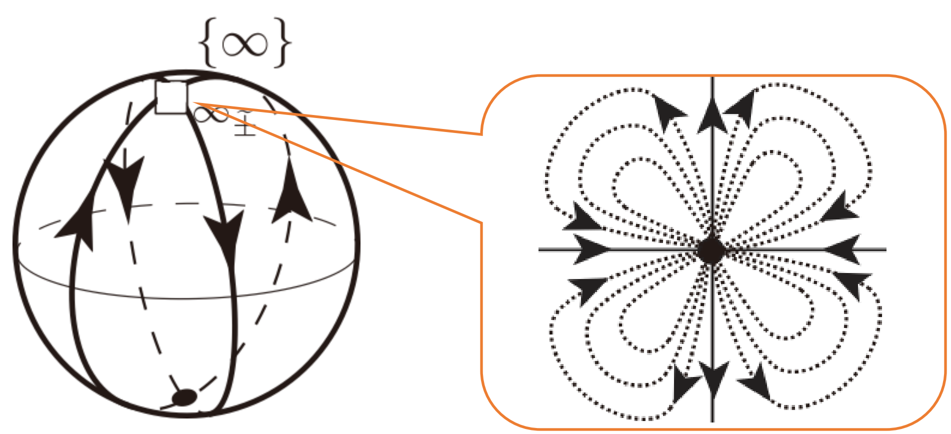
“n-bundled ss-saddle” : Collapse to a degenerate point via equivalence relation

$$\forall x, x' \in \overline{\Omega}, \quad x \sim x' \Longleftrightarrow x, x' \in \partial\Omega$$



collapse by the equivalence relation

$\Longrightarrow$



Flow in the disk with the moving boundary = transversal boundary condition

Flow on the sphere with a degenerate fixed point = “an n-bundled ss-saddle”.

Topological Vortex Structure : A Mathematical Notion

Output of TFDA

COT and COT representation

$$s_{\emptyset 1}\{[\infty_{\tilde{+}}, \lambda_{\sim}, a_{\simeq}\{\sigma_{\simeq -}, \infty_{\simeq}\}], \lambda_{\sim}, \lambda_{\sim}, \lambda_{\sim}\}$$

$[\infty_{\tilde{+}}, \lambda_{\sim}, a_{\simeq}\{\sigma_{\simeq -}, \infty_{\simeq}\}]$ $\quad \lambda_{\sim} \quad \lambda_{\sim} \quad \lambda_{\sim}$

$\infty_{\tilde{+}} \quad \lambda_{\sim} \quad a_{\simeq}\{\sigma_{\simeq -}, \infty_{\simeq}\}$

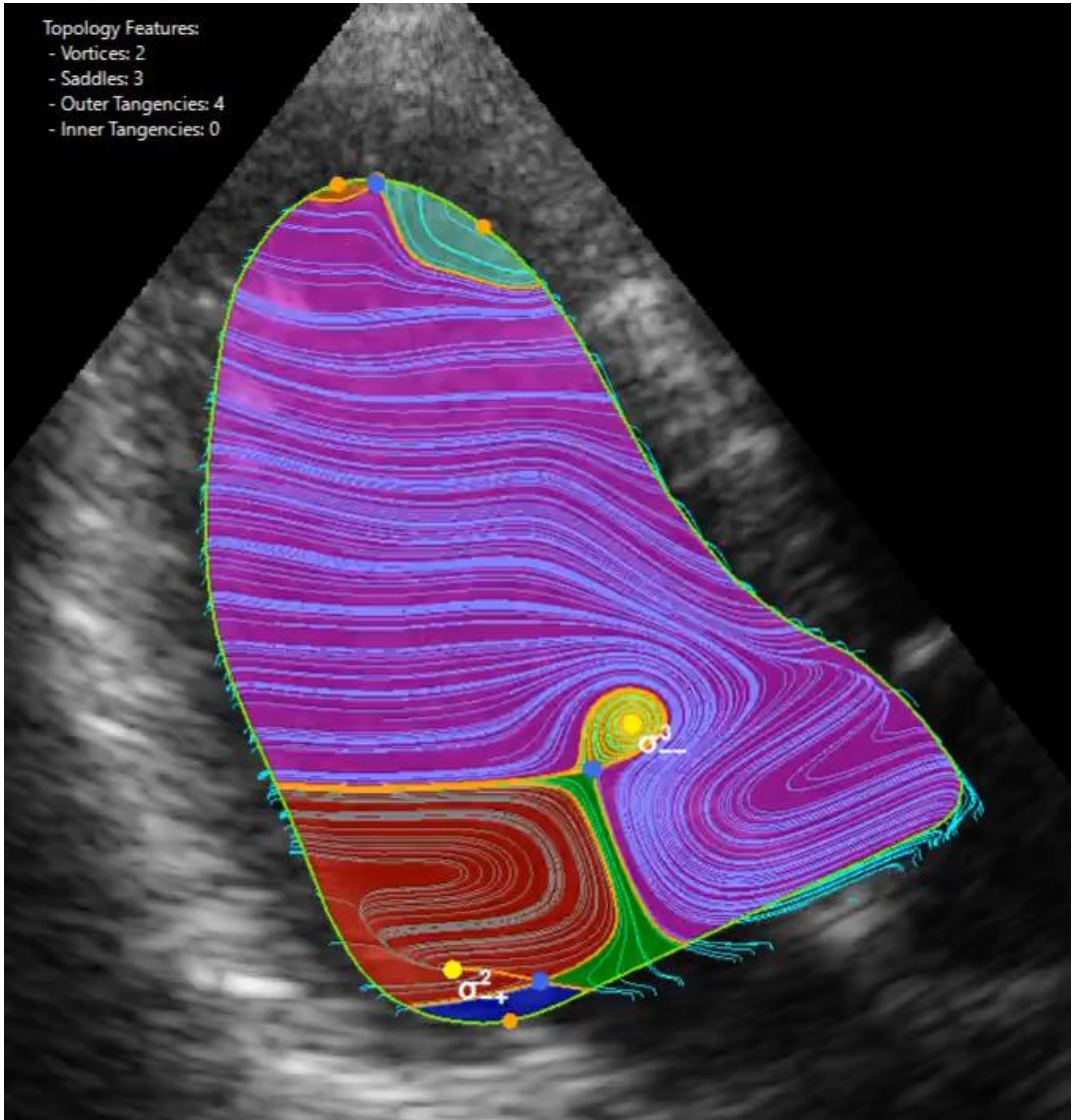
$\sigma_{\simeq -} \quad \infty_{\simeq}$

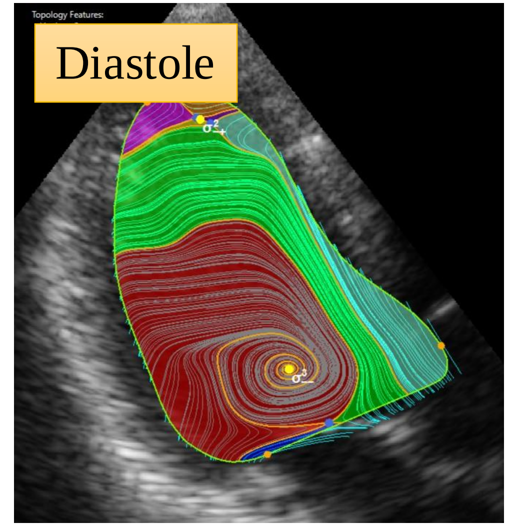
Topological Vortex Structure

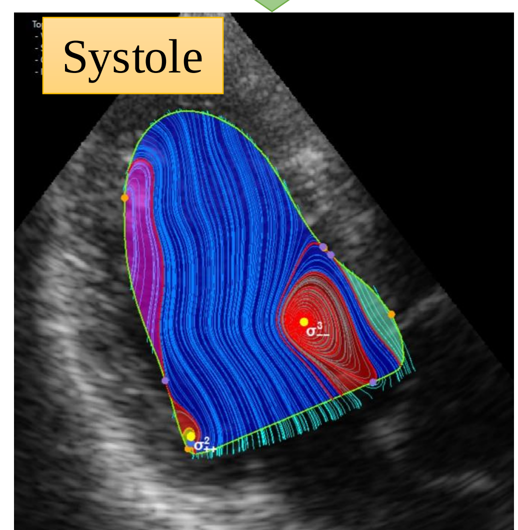
- ✓ It is the region of influence of the triplet notation in the COT representation.
- ✓ It is the domain enclosed by the saddle separatrices.
- ✓ We can quantify the shape, the area, and the geometric center of this domain.
- ✓ We can evaluate the heart function as a pump.

Topological Change during One Cardiac Cycle for Healthy Hearts

$s_{\emptyset 1}\left\{[\infty_{\tilde{\mp}}, \lambda_{\sim}, a_{\simeq}\{\sigma_{\simeq -}, \infty_{\simeq}\}], \lambda_{\sim}, \lambda_{\sim}, \lambda_{\sim}\right\}.$



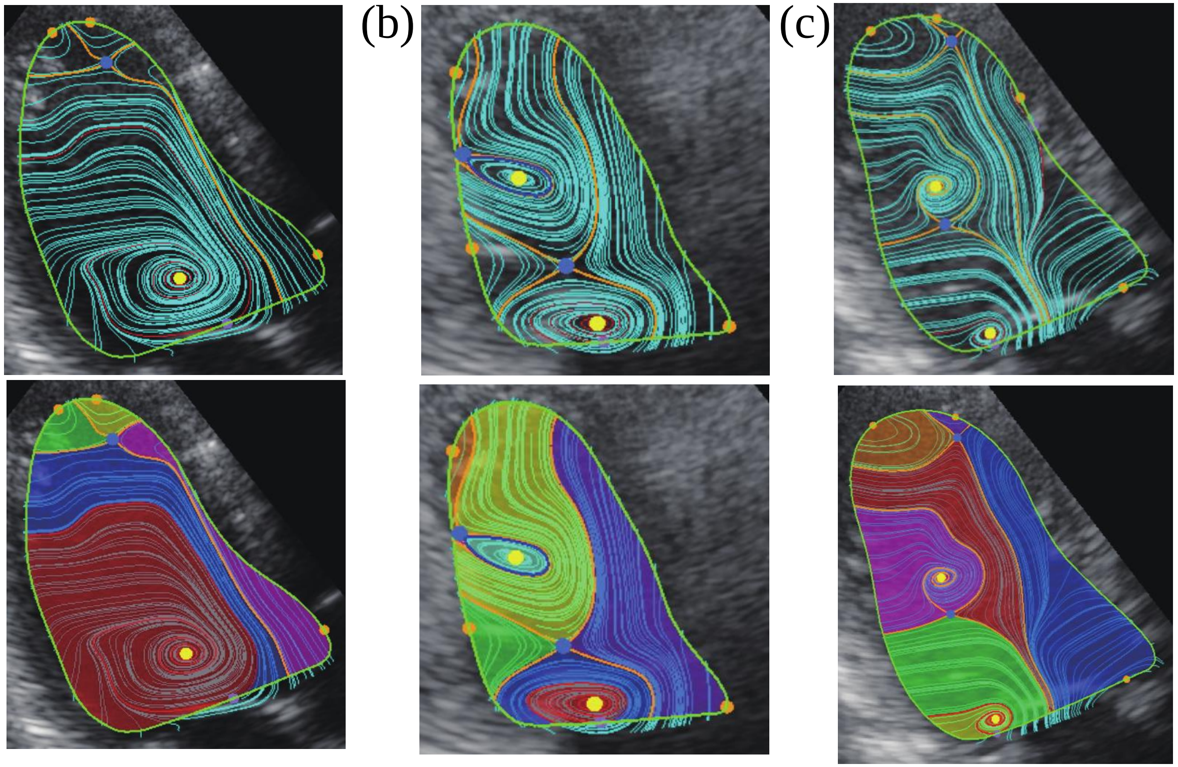




$s_{\emptyset 0}([a_{\tilde{\mp}}\{\sigma_{\tilde{\mp}+}, \infty_{\tilde{\mp}}\}, \lambda_{\sim}, \infty_{\simeq}] \cdot [a_{\tilde{\mp}}\{\sigma_{\tilde{\mp}-}, \infty_{\tilde{\mp}}\}, \lambda_{\sim}, \infty_{\simeq}]).$

Example of Heart Failures in systole

(a) (b) (c)



(a) $s_{\emptyset 1}\left\{[\infty_{\overset{\sim}{+}}, \lambda_{\sim}, a_{\simeq}\{\sigma_{\simeq -}, \infty_{\simeq}\}], \lambda_{\sim}, \lambda_{\sim}, \lambda_{\sim}\right\}.$

(c) $s_{\emptyset 1}\left\{[\infty_{\overset{\sim}{+}}, \lambda_{\sim}, a_{\simeq}\{\sigma_{\simeq -}, \infty_{\simeq}\}] \cdot [\infty_{\overset{\sim}{+}}, \lambda_{\sim}, a_{\simeq}\{\sigma_{\simeq -}, \infty_{\simeq}\}], \lambda_{\sim}, \lambda_{\sim}, \lambda_{\sim}\right\}$

(b) $s_{\emptyset 1}\left\{[\infty_{\overset{\sim}{+}}, \lambda_{\sim}, a_{\simeq}\{\sigma_{\simeq -}, \infty_{\simeq}\}], \lambda_{\sim}, [\infty_{\overset{\sim}{+}}, a_{-}(b_{\overset{\sim}{+}}(\sigma_{\overset{\sim}{+}-}, \lambda_{\sim})), \infty_{\simeq}], \lambda_{\sim}\right\}$

(a) Healthy heart

A good outflow of blood from the valve is supported by the big topological vortex structure.

(b) Heart failure/Case 1

Weak out-flow from the valve and a small topological vortex structure at the basal position.

(c) Heart failure/Case 2

The topological vortex structure is divided into two small ones due to weak ejection of blood flow.

Our ongoing projects with MDs:

- Tracking the time evolution of the topological vortex structures.
- Clinical studies based on the topological vortex structures for heart diseases.

References

Mathematical Theory of topological classification of streamlines:

- T. Yokoyama and T. Sakajo, Word representation of streamline topologies for structurally stable vortex flows in multiply connected domains, Proc. Roy. Soc. A, vol. 469 (2013) (doi:10.1098/rspa.2012.0558)
- T. Sakajo and T. Yokoyama, Transitions between streamline topologies of structurally stable Hamiltonian flows in multiply connected domains, Physica D, vol. 307 (2015) pp. 22--41 (doi:10.1016/j.physd.2015.05.013)
- T. Sakajo and T. Yokoyama, Tree representation of topological streamline patterns of structurally stable 2D Hamiltonian vector fields in multiply connected domains, The IMA Journal of Applied Mathematics, vol. 83 pp. 380--411 (2018) (doi: 10.1093/imamat/hxy005)
- T. Uda, T. Yokoyama and T. Sakajo, Algorithms converting streamline topologies for 2D Hamiltonian vector fields using Reeb graphs and persistent homology, 日本応用数理学会論文誌, vol. 29 No. 1 pp. 187--224 (2019) (in Japanese).
- T. Sakajo and T. Yokoyama, Discrete representations of orbit structures of flows for topological data analysis, Discrere Mathematics, Algorithms and Applications accepted (2022) [arXiv:2010.13434]

Applications to the atmospheric flows, oceanography, and medical imaging:

- T. Uda, T. Sakajo, M. Inatsu and K. Koga, Morphological identification of atmospheric blockings by topological flow data analysis, J. Meteo. Soc. Japan vol. 99 No.5 pp. 1169--1183 (2021) (doi:10.2151/jmsj.2021-057)
- T. Sakajo, S. Ohishi and T. Uda, Identification of Kuroshio meanderings south of Japan via a topological data analysis for sea surface height, Journal of Oceanography (2022) (doi:10.1007/s10872-022-00656-3)
- T. Sakajo and K. Itatani, Topological identification of vortical flow structures in the left ventricle of the heart, SIAM Journal on Imaging Science vol. 16 No.3 pp. 1491--1519 (2023) (doi:10.1137/22M1536923)