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Topology and dynamics of magnet-vortical matter
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Dynamics of vortices, topological invariants and underlying symmetries for ideal fluid dynamics and magnetohydrodynamics

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Part 1

Topological Invariants of Fluid Motion

Vorticity equation

Suppose that the fluid is inviscid and barotropic and that the body force is conservative

$$-(\nabla p)/\rho = -\nabla^{\exists} P$$

$$\boldsymbol{f} = -\nabla^{\exists}\Phi$$

Euler equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \boldsymbol{f} \quad \Leftrightarrow \quad \frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = -\nabla\left(\frac{\boldsymbol{u}^2}{2} + P + \Phi\right)$$

Vorticity $\boldsymbol{\omega} := \mathrm{rot}\,\boldsymbol{u} = \nabla\times\boldsymbol{u}$

rot [Euler equations] becomes, combined with eq, of continuity,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}) \quad \Leftrightarrow \quad \boxed{\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\boldsymbol{u}} \qquad \frac{D}{Dt} := \frac{\partial}{\partial t} + (\boldsymbol{u}\cdot\nabla)$$

Vorticity equations

Helmholtz

Crelle's J. (1858)

Transl. **Tait:** Phil. Mag. (1867)

Helmholtz' law II:
A vortex line remains continually composed of the same fluid particles.
i.e. a vortex line is *frozen into the fluid*.

LXIII. *On Integrals of the Hydrodynamical Equations, which express Vortex-motion.* By H. HELMHOLTZ*.

HITHERTO, in integrating the hydrodynamical equations, the assumption has been made that the components of the velocity of each element of the fluid in three directions at right angles to each other are the differential coefficients, with reference to the coordinates, of a definite function which we shall call the *velocity-potential*. Lagrange† no doubt has shown that this assumption is lawful if the motion of the fluid has been produced by, and continued under, the action of forces which have a potential; and also that the influence of moving solids which are in contact with the fluid does not affect the lawfulness of the assumption. And, since the greater number of natural forces which can be defined with mathematical strictness can be expressed as differential coefficients of a potential, by far the greater number of mathematically investigable cases of fluid-motion belong to the class in which a velocity-potential exists.

Yet Euler‡ has distinctly pointed out that there are cases of fluid-motion in which no velocity-potential exists,—for instance, the rotation of a fluid about an axis when every element has the same angular velocity. Among the forces which can produce such motions may be named magnetic attractions acting upon a fluid conducting electric currents, and particularly friction, whether among the elements of the fluid or against fixed bodies. The effect of fluid friction has not hitherto been mathematically defined; yet it is very great, and, except in the case of indefinitely small oscillations, produces most marked differences between theory and fact. The difficulty of defining this effect, and of finding expressions for its measurement, mainly con-

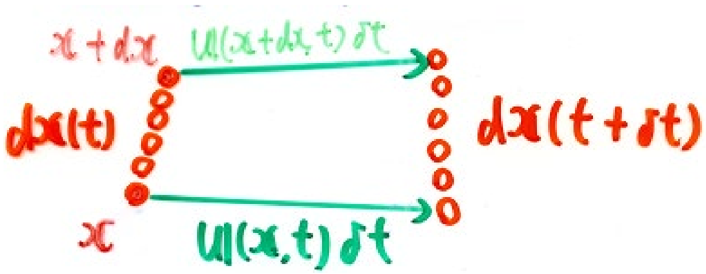
* From Crelle's *Journal*, vol. lv. (1858), kindly communicated by Professor Tait. (1867)
† *Mécanique Analytique* (Paris, 1815), vol. ii. p. 304.
‡ *Histoire de l'Académie des Sciences de Berlin* (1755), p. 292.

Helmholtz's law (1858)

Vorticity equations

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\boldsymbol{u}$$

an infinitesimal material line element: $\mathrm{d}\boldsymbol{x}$



$$d\boldsymbol{x} = \frac{\partial \boldsymbol{\varphi}}{\partial X_A} dX_A = \left(d\boldsymbol{X}\cdot\frac{\partial}{\partial \boldsymbol{X}}\right)\boldsymbol{\varphi}(\boldsymbol{X},t) \quad \Rightarrow$$

$$\frac{D}{Dt}d\boldsymbol{x}(t) = (d\boldsymbol{x}(t)\cdot\nabla)\,\boldsymbol{u}$$

$$\left(\partial/\partial|_{\boldsymbol{X}} = D/Dt\right)$$

Helmholtz's law

Since $$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho}\cdot\nabla\right)\boldsymbol{u} \quad \Leftrightarrow \quad \frac{D}{Dt}d\boldsymbol{x}(t) = (d\boldsymbol{x}(t)\cdot\nabla)\,\boldsymbol{u}$$

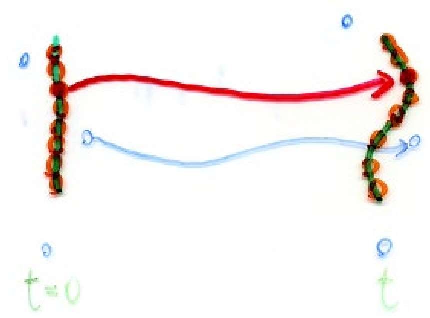
Theorem (Helmholtz' s law II)

A vortex line remains continually composed of the same fluid particles, and therefore moves with the fluid.

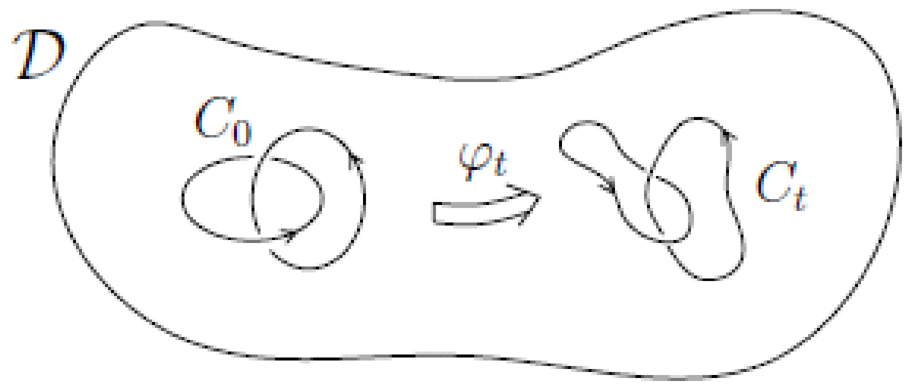
i.e. a vortex line is ***frozen into the fluid***.

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho} - c\,d\boldsymbol{x}\right) = \left(\frac{\boldsymbol{\omega}}{\rho} - c\,d\boldsymbol{x}\right)\cdot\nabla\boldsymbol{u} \quad (c : \text{a constant})$$

$$\frac{\boldsymbol{\omega}}{\rho} - \mathrm{c}d\boldsymbol{x} = \boldsymbol{0} \;\; \text{at} \;\; t = 0 \quad \Leftrightarrow \quad \frac{\boldsymbol{\omega}}{\rho} - \mathrm{c}d\boldsymbol{x} = \boldsymbol{0} \;\; \text{for} \;\; \text{all } t$$



Preservation of vortex-line topology



河内明夫「結び目理論」

"Theory of Knots" by A. Kawauchi (ed.)

Definition A (PL) link $L\ (\in S^3)$ is said to be equivalent to a link $L'\ (\in S^3)$ if there exists a homeomorphic map $h(: S^3 \to S^3)$ *s.t.* $h(L) = L'$.
If further h and its restriction $h|L : L \to L'$ maintains the orientation, L is said to be of the same type with $L'\ (\in S^3)$; $L \cong L'$.

$\Downarrow \qquad \{\varphi_\tau : \mathcal{D} \to \mathcal{D}\,|\,\tau \in [0,t], \varphi_0 = \mathrm{id}\}$ 'ambient isotopy'

From **Helmholtz's law II**,

Corollary Knot and link types of vortex loops are invariant in time.

Helicity and linking number

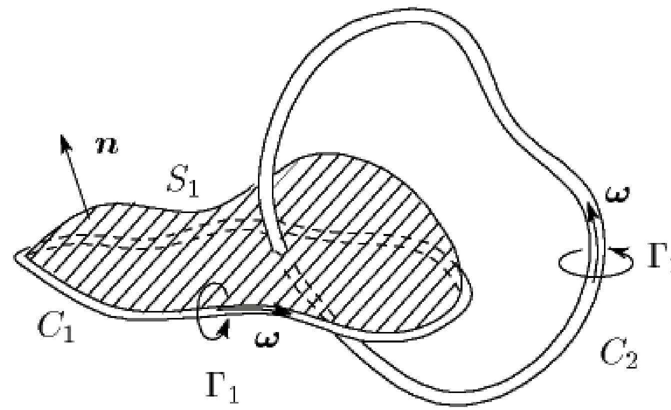
Theorem (Moreau '61, Betchov '61, Moffatt '69)

In the same assumptions, $\int_{\mathcal{D}} \boldsymbol{u} \cdot (\nabla \times \boldsymbol{u}) \, \mathrm{d}V = \int_{\mathcal{D}} \boldsymbol{u} \cdot \boldsymbol{\omega} \, \mathrm{d}V = \text{const.} \quad (\mathcal{D}: \text{ flow domain.})$

linking number (Gauss 1833) Moffatt '69

$$\boldsymbol{\omega}(\boldsymbol{x}) = \oint_C \Gamma \delta\left(\boldsymbol{x} - \boldsymbol{X}(s)\right) \boldsymbol{t}(s) \, \mathrm{d}s$$

$$\begin{aligned} \int \boldsymbol{u} \cdot \boldsymbol{\omega} \, \mathrm{d}V &= \Gamma_1 \oint_{C_1} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} + \Gamma_2 \oint_{C_2} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} \\ &= \Gamma_1 \int_{S_1} \boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}A + \Gamma_2 \int_{S_2} \boldsymbol{\omega} \cdot \boldsymbol{n} \, \mathrm{d}A \\ &= -2\Gamma_1 \Gamma_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \Gamma_i \Gamma_j \end{aligned}$$



α_{ij} is called the *linking number*

linking number (Gauss 1833) Moffatt '69

writhe + twist Călugăreanu '61, Moffatt & Ricca '92

Arnol'd & Khesin '98

Contents

1. Topological Invariants of fluid motion

Motion of an ideal barotropic fluid

Nambu mechanics: Helicity=Casimir invariant

Abundance of topological invariants for baroclinic effect and MHD

generalized enstrophy, Ertel invariant, Cross helicity, generalized helicity

2. Topological Invariants of magnetohydrodynamics by Noether's theorem

Particle relabeling symmetry

Variational symmetry → Cross helicity

Divergence symmetry → Generalized helicity

Nambu Mechanics

Nambu: Phys. Rev. D **7** (1973) 2405

Yoichiro Nambu 1921-2015

Nambu bracket for incompressible flow

$$\frac{dF}{dt} = \{F, H\}$$

Lie-Poisson bracket

$$\{F, H\} = \left\langle \left[\frac{\delta F}{\delta \boldsymbol{v}}, \frac{\delta H}{\delta \boldsymbol{v}} \right], \boldsymbol{v} \right\rangle = \int_{\mathcal{D}} \left[\frac{\delta F}{\delta \boldsymbol{v}}, \frac{\delta H}{\delta \boldsymbol{v}} \right] \cdot \boldsymbol{v} dV$$

$$H[\boldsymbol{v}] = \frac{1}{2} \int_{\mathcal{D}} \boldsymbol{v}^2 \, dV$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = \mathcal{P} \left[\frac{\delta H}{\delta \boldsymbol{v}} \times (\nabla \times \boldsymbol{v}) \right]$$

Helicity $h[\boldsymbol{v}] = \dfrac{1}{2} \displaystyle\int_{\mathcal{D}} \boldsymbol{v} \cdot \nabla \times \boldsymbol{v} dV$

Casimir invariant

$$\{h, K\} = 0 \quad \text{for any } K$$

Nambu bracket

Fukumoto: Nagare **28** (2009) 499

$$\{F, H\} = \int_{\mathcal{D}} \boldsymbol{v} \cdot \nabla \times \left(\frac{\delta F}{\delta \boldsymbol{v}} \times \frac{\delta H}{\delta \boldsymbol{v}} \right) dV = \int_{\mathcal{D}} \nabla \times \boldsymbol{v} \cdot \left(\frac{\delta F}{\delta \boldsymbol{v}} \times \frac{\delta H}{\delta \boldsymbol{v}} \right) dV$$

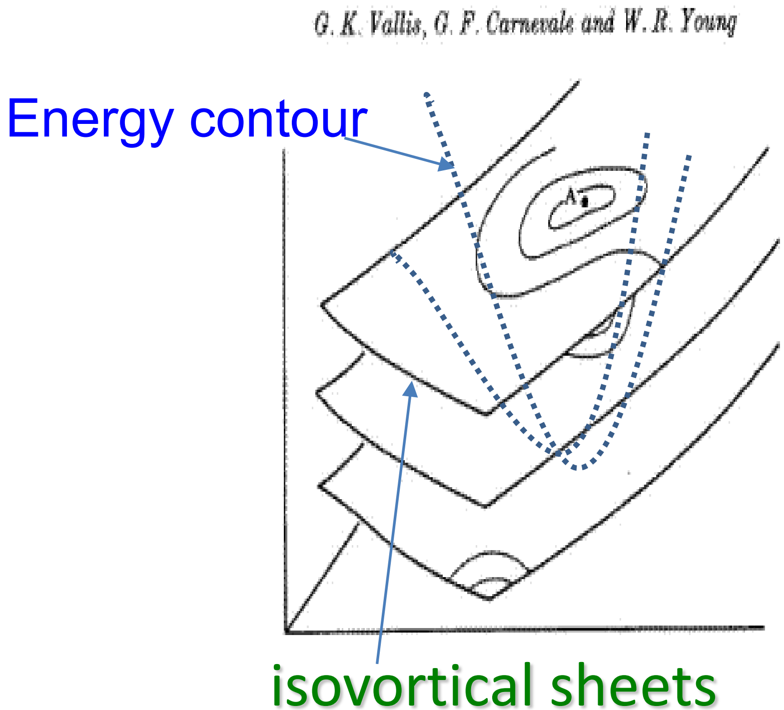
$$= \int_{\mathcal{D}} \frac{\delta h}{\delta \boldsymbol{v}} \cdot \left(\frac{\delta F}{\delta \boldsymbol{v}} \times \frac{\delta H}{\delta \boldsymbol{v}} \right) dV = \int_{\mathcal{D}} \frac{\delta F}{\delta \boldsymbol{v}} \cdot \left(\frac{\delta H}{\delta \boldsymbol{v}} \times \frac{\delta h}{\delta \boldsymbol{v}} \right) dV = \{F, h, H\}_{vvv}.$$

Euler equation

$$\frac{\partial \boldsymbol{v}}{\partial t} = \mathcal{P} \left[\frac{\delta H}{\delta \boldsymbol{v}} \times \frac{\delta h}{\delta \boldsymbol{v}} \right]$$

Arnold's theorem for steady Euler flows

Vallis, Carnevale & Young: J. Fluid Mech.
J. Fluid Mech. **207** (1989) 133



Kinematically accessible variation
(= preservation of circulation)

$$\boldsymbol{x} \to \tilde{\boldsymbol{x}} \quad \Rightarrow \quad \omega = \tilde{\omega}$$

$$\frac{1}{2}\epsilon_{ijk}\omega_k(\boldsymbol{x},t)\mathrm{d}x_i \wedge \mathrm{d}x_j$$
$$= \frac{1}{2}\epsilon_{pqr}\tilde{\omega}_r(\tilde{\boldsymbol{x}},t)\mathrm{d}\tilde{x}_p \wedge \mathrm{d}\tilde{x}_q$$
$$(\tilde{\omega}_r = \omega_r + \delta\omega_r)$$

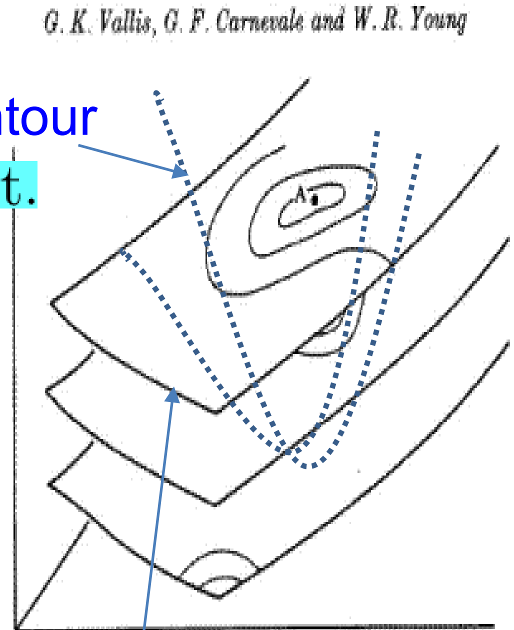
Theorem A steady Euler flow is a coditional extremum of energy H w.r.t. *isovortical perturbations* (kinematically accessible perturbations).

V. I. Arnold: Ann. Inst. Fourier Grenoble **16** (1966) 319

Arnold's theorem

Energy contour

$H = \text{const.}$



isovortical sheets

$h = \text{const.}$

Euler equation

$$\frac{\partial \boldsymbol{v}}{\partial t} = \mathcal{P}\left[\frac{\delta H}{\delta \boldsymbol{v}} \times \frac{\delta h}{\delta \boldsymbol{v}}\right]$$

Isovortical perturbation

$$\delta \boldsymbol{v} = \mathcal{P}\left[\boldsymbol{\xi} \times \frac{\delta h}{\delta \boldsymbol{v}}\right]; \quad \boldsymbol{\xi} = \frac{\delta K}{\delta \boldsymbol{v}}$$

$K:$ arbitrary functional

Arnold's theorem

Kelvin (1887)

$$\delta H = \int_{\mathcal{D}} \frac{\delta H}{\delta \boldsymbol{v}} \cdot \delta \boldsymbol{v} d^3 x = \int_{\mathcal{D}} \boldsymbol{\xi} \cdot \left(\frac{\delta h}{\delta \boldsymbol{v}} \times \frac{\delta H}{\delta \boldsymbol{v}}\right) d^3 x = 0$$

for a steady Euler flow $\mathcal{P}\left[\frac{\delta h}{\delta \boldsymbol{v}} \times \frac{\delta H}{\delta \boldsymbol{v}}\right] = \mathbf{0}$

There are an abundance of topological invariants.

Topological invariants of an ideal barotropic fluid

3D *Helicity (=Hopf invariant, Casimir invariant)*

$$\mathcal{H}[\boldsymbol{\omega}] = \int_{\mathcal{D}} \boldsymbol{v}\cdot\boldsymbol{\omega}\,\mathrm{d}V; \quad \boldsymbol{\omega} := \nabla\times\boldsymbol{v}$$

vorticity

2D Generalized enstrophy (=Casimir invariant)

$$Q[\omega,\rho] = \int_{\mathcal{A}} f\left(\frac{\omega}{\rho}\right)\omega\,\mathrm{d}A$$

$$\frac{D}{Dt}\left(\frac{\omega}{\rho}\right) = 0$$

where f is an arbitrary function, because

Question:

Why the helicity of 3D looks different from 2D Casimirs?
How far is the 2D Casimir invariant from the helicity?

3D non-isentropic Euler flow

Baroclinic effect DESTROYS Helicity

Euler equations for a non-isentropic flow

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = 0,$$

$$\frac{Ds}{Dt} = 0,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p - \nabla \Phi$$

$$= -\nabla w + \boxed{T \nabla s} - \nabla \Phi$$

specific enthalpy

$$w = e + p/\rho$$

$$\mathrm{d}w = T\mathrm{d}s + \frac{1}{\rho}\mathrm{d}p$$

Ertel's potential vorticity

$$Q := \frac{1}{\rho}(\nabla \times \boldsymbol{u}) \cdot \nabla s; \quad \frac{DQ}{Dt} = 0$$

Casimir invariants

$$C = \int \rho F(s, Q) dV$$

Magnetohydrodynamics (MHD) of a compressible adiabatic conducting fluid

Baroclinic effect

Lorentz force

Equations of MHD

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} = 0, \quad \left(\frac{D}{Dt} := \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla w + T\nabla s + \frac{1}{\rho}\boldsymbol{j}\times\boldsymbol{B} - \nabla\Phi,$$

$$(w = w = e + p/\rho, \quad \boldsymbol{j} = \nabla\times\boldsymbol{B})$$

$$\frac{Ds}{Dt} = 0,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B},). \quad \nabla\cdot\boldsymbol{B} = 0.$$

Baroclinic effect and ***Lorentz force*** **DESTROY Helicity**

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}) + \nabla T\times\nabla s + \nabla\times\left(\frac{1}{\rho}\boldsymbol{j}\times\boldsymbol{B}\right)$$

The complete set of Casimir constants of the motion in magnetohydrodynamics

Eliezer Hameiri[a)]

E. Hameiri: Phys. Plasmas **11** (2004) 3423

Cross-helicity

$$\mathcal{M}[\rho] = \int_{\mathcal{D}} \rho dV, \quad \mathcal{S}[\sigma] = \int_{\mathcal{D}} \rho s dV = \int_{\mathcal{D}} \sigma dV,$$

$$h_{\mathrm{c}}[\boldsymbol{v}, \boldsymbol{C}] = \int_{\mathcal{D}} \boldsymbol{v} \cdot \boldsymbol{D} dV, \quad h_{\mathrm{m}}[\boldsymbol{A}] = \frac{1}{2} \int_{\mathcal{D}} \boldsymbol{A} \cdot \boldsymbol{B} dV$$

$$(\boldsymbol{D} = \nabla \times \boldsymbol{C}), \quad (\boldsymbol{B} = \nabla \times \boldsymbol{A})$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{u} \times \boldsymbol{\omega} + \nabla(\pi - \Phi) + T \nabla s + \frac{1}{\rho} \boldsymbol{j} \times \boldsymbol{B},$$

$$\frac{\partial \boldsymbol{D}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{D}),$$

$$\nabla \cdot \boldsymbol{D} = 0, \quad (\boldsymbol{D} \cdot \nabla) s = 0, \quad \nabla \times \left(\frac{\boldsymbol{B}}{\rho} \times \boldsymbol{D} \right) = 0$$

Nambu bracket

Y. Fukumoto & R. Zou: Prog. Theor. Exp. Phys. (PTEP) (2024)

Casimirs

$$h_{\mathrm{c}}[\boldsymbol{M},\boldsymbol{d}] = \int_{\mathcal{D}} \boldsymbol{M}\cdot\boldsymbol{d}\,dV : \quad \boldsymbol{M} := \rho\boldsymbol{v},\ \boldsymbol{d} := \boldsymbol{D}/\rho$$

$$\frac{D\boldsymbol{d}}{Dt} = (\boldsymbol{d}\cdot\nabla)\boldsymbol{u}$$

$$\mathcal{S}[\rho,s] = \int_{\mathcal{D}} \rho s\,dV, \quad h_{\mathrm{m}}[\boldsymbol{A}] = \frac{1}{2}\int_{\mathcal{D}} \boldsymbol{A}\cdot\boldsymbol{B}\,dV \quad (\boldsymbol{B} = \nabla\times\boldsymbol{A})$$

Nambu-bracket representation for MHD equations

$$\frac{d}{dt}F[\boldsymbol{M},\boldsymbol{d},\boldsymbol{B},\rho,s] = \{F,h_{\mathrm{c}},H\}_{MMd} + \{F,\mathcal{S},H\}_{M\rho s} + \{F,h_{\mathrm{m}},H\}_{MBB}$$

$$\{F,h_{\mathrm{c}},H\}_{MMd} := -\int_{\mathcal{D}}\left\{\frac{\delta h_{\mathrm{c}}}{\delta\boldsymbol{d}}\left[\left(\frac{\delta F}{\delta\boldsymbol{M}}\cdot\nabla\right)\frac{\delta H}{\delta\boldsymbol{M}} - \left(\frac{\delta H}{\delta\boldsymbol{M}}\cdot\nabla\right)\frac{\delta F}{\delta\boldsymbol{M}}\right] + \mathrm{cyc}(F,h_{\mathrm{c}},H)\right\}dV$$

$$\{F,\mathcal{S},H\}_{M\rho s} := -\int_{\mathcal{D}}\left\{\left(\frac{\delta F}{\delta s}\frac{\delta H}{\delta\boldsymbol{M}} - \frac{\delta H}{\delta s}\frac{\delta F}{\delta\boldsymbol{M}}\right)\cdot\nabla\frac{\delta\mathcal{S}}{\delta\rho} + \mathrm{cyc}(F,\mathcal{S},H)\right\}dV$$

$$\{F,h_{\mathrm{m}},H\}_{MBB} := -\int_{\mathcal{D}}\left\{\left(\nabla\times\frac{\delta H}{\delta\boldsymbol{B}}\right)\cdot\left[\left(\nabla\times\frac{\delta F}{\delta\boldsymbol{B}}\right)\times\frac{\delta h_{\mathrm{m}}}{\delta\boldsymbol{M}}\right]\right.$$
$$\left. + \left(\nabla\times\frac{\delta F}{\delta\boldsymbol{B}}\right)\cdot\left[\left(\nabla\times\frac{\delta h_{\mathrm{m}}}{\delta\boldsymbol{B}}\right)\times\frac{\delta H}{\delta\boldsymbol{M}}\right] + \mathrm{cyc}(F,h_{\mathrm{m}},H)\right\}dV$$

Generalized helicity *1.* baroclinic flow

Mobbs: J. Fluid Mech. (1981)

Euler-Poincarė equations

$$\frac{\partial v_i}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,v_i + v_j\frac{\partial u_j}{\partial x_i} = \frac{\partial \pi}{\partial x_i} + T\frac{\partial s}{\partial x_i} \qquad \left(v_i := \frac{1}{\rho}\frac{\partial \mathcal{L}}{\partial u_i},\ \ \pi := \frac{\partial \mathcal{L}}{\partial \rho},\ \ T := -\frac{1}{\rho}\frac{\partial \mathcal{L}}{\partial s}\right)$$

$$\frac{\partial}{\partial t}(\boldsymbol{v} + \tau\nabla s) - \boldsymbol{u}\times[\nabla\times(\boldsymbol{v}+\tau\nabla s)] = \nabla\left[\pi - (\boldsymbol{v}+\tau\nabla s)\cdot\boldsymbol{u}\right]$$

$$\frac{D\tau}{Dt} = -T$$

$$\hat{\boldsymbol{\omega}} := \nabla\times(\boldsymbol{v}+\tau\nabla s); \qquad \frac{\partial\hat{\boldsymbol{\omega}}}{\partial t} = \nabla\times(\boldsymbol{u}\times\hat{\boldsymbol{\omega}})$$

Baroclinic helicity $\quad \hat{\boldsymbol{v}} = \boldsymbol{v} + \tau\nabla s$

$$\mathcal{H}[\boldsymbol{v}] = \int_{\mathcal{D}} \hat{\boldsymbol{v}}\cdot(\nabla\times\hat{\boldsymbol{v}})d^3x = \int_{\mathcal{D}} \hat{\boldsymbol{v}}\cdot\hat{\boldsymbol{\omega}}d^3x$$

Generalized helicity **2.** barotropic **MHD**

Calkin: Can. J. Phys. (1963), Bekenstein & Ooron: Phys. Rev. E (2000) *compressible*
Vladimirov & Moffatt: J. Fluid Mech. (1995) *incompressible*

$$\frac{\partial \boldsymbol{m}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{m}) = \boldsymbol{j}$$

$$\hat{\boldsymbol{v}} = \boldsymbol{v} + \frac{1}{\rho} \boldsymbol{B} \times \boldsymbol{m}$$

$$\hat{\boldsymbol{\omega}} = \nabla \times \hat{\boldsymbol{v}} \qquad \frac{\partial \hat{\boldsymbol{\omega}}}{\partial t} = \nabla \times (\boldsymbol{u} \times \hat{\boldsymbol{\omega}})$$

Generalized helicity for MHD

$$\mathcal{H}[\boldsymbol{v}] = \int_{\mathcal{D}} \hat{\boldsymbol{v}} \cdot (\nabla \times \hat{\boldsymbol{v}}) d^3x = \int_{\mathcal{D}} \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{\omega}} d^3x$$

Underlying symmetries?

Part 2

Topological Invariants of Magnetohydrodynamics from Noether's Theorem

Topological Invariants of MHD associated with Relabeling symmetry

Barotropic fluids:

A. Yahalom: J. Math. Phys. **36** (1995) 1324
Y. Fukumoto: Topologica **1** (2008) 003

Baroclinic (non-isentropic) fluids:

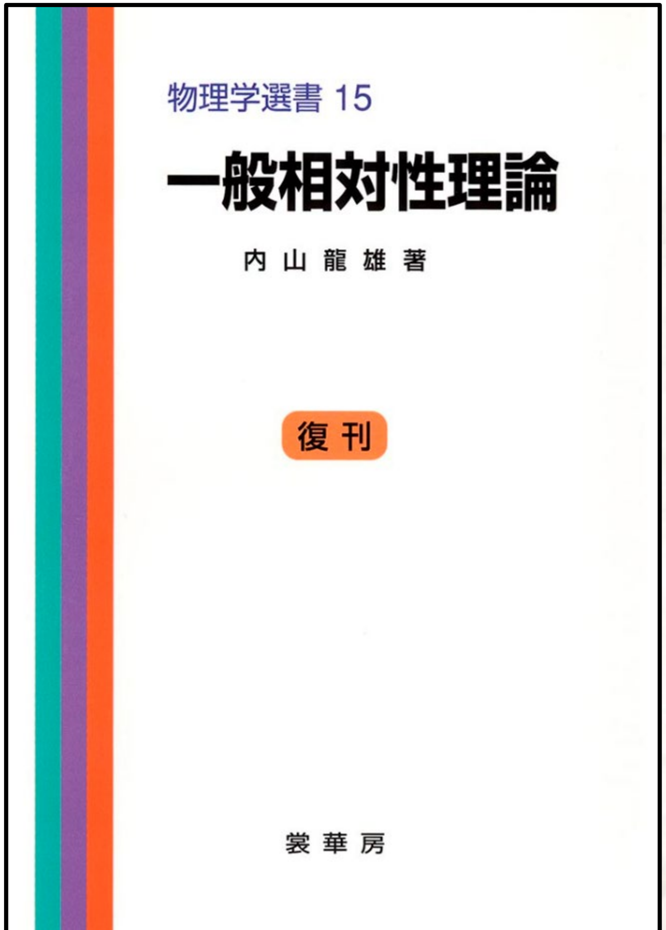
N. Padhye & P. J. Morrison: Phys. Lett. A **219** (1996) 287
N. Padhye & P. J. Morrison: Plasma Phys. Rep. **22** (1996) 960
Y. Fukumoto & H. Sakuma: Procedia IUTAM **7** (2013) 213
C. J. Cotter & D. D. Holm: Found Comput Math.**13** (2013) 457

MHD:

Webb, Dasgupta, McKenzie, Hu & Zank: J. Math. Theor. **47** (2014) 095501
Webb, Dasgupta, McKenzie, Hu & Zank: J. Math. Theor. **47** (2014) 095502

Y. Fukumoto & R. Zou: Prog. Theor. Exp. Phys. (PTEP) (2024)

Ryoyu Utiyama: General Relativity



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第5章　不変変分論

§25.　作用積分を不変にする変分

これからしばらく一般相対論から離れて，まず Noether の定理について述べる．つづいてこの定理の応用として種々の保存則や，§15，あるいは§23 にでてきた恒等式

$$\nabla_\mu \mathbf{G}^{\mu\nu} \equiv 0, \qquad (25.1)$$

$$\mathbf{G}^{\mu\nu} \overset{d}{\equiv} \sqrt{-g}\, G^{\mu\nu} \overset{d}{\equiv} \sqrt{-g}\left\{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right\}$$

を導き出してみよう．

まず，時空内の部分域 Ω のなかの各点には座標 $x^\mu (\mu = 0, 1, 2, 3)$ が与えられているとする．時空とよばれる連続体内でどのような距離の定義が成り立つかはどうでもよい．たとえばこれから扱う問題が特殊相対論のワクのなかに属するものならば，時空を Minkowski 空間とみなせばよい．

さて，Ω 内には N 個の成分をもつ場 $\varphi_A(x)$ が与えられているものとする．$A(=1, 2, \cdots, N)$ はこの成分を識別する添字である．しかし φ_A をテンソル場に限る必要はない．たとえば特殊相対論にでてくるスピノール場でもよい．また，これは必ずしも一種類の場を表わすとは限らない．2種類以上の場（たとえば，電磁ポテンシャルを表わすベクトル場と電子場を表わすスピノール場）を一緒にして通し番号をつけたものが φ_A（A は通し番号）であると考えて差しつかえない．もし時空を一般相対論的に扱うときは，その計量テンソル $g_{\mu\nu}(x)$ も $\varphi_A(x)$ のなかに組み入れるものとする．

さきにも注意したように Ω の幾何学的性質はどんなものでもよいが，この中の点は一意的に，また連続的に座標づけされているものとし，特に x^0 を時間座標，x^k $(k = 1, 2, 3)$ を空間座標とよぶことにする．しかし計量を具体的に与えてない以上，この呼び名は単なる便宜上のものにすぎない．しかし計量

§25.　作用積分を不変にする変分　　147

$g_{\mu\nu}$（あるいは $\eta_{\mu\nu}$）が与えられているときは，x^0 軸（つまり x^k を固定して x^0 だけを変化させることにより描かれる曲線）の接線ベクトルはどこででも時間的とする．

さて，場 $\varphi_A(x)$ の振舞は場の方程式とよばれる N 個の連立偏微分方程式により決められる．一般に場の方程式は Hamilton の原理に従い Lagrange 関数 L から変分法により導かれる．

いま，Lagrange 関数

$$L\{\varphi_A(x),\ \varphi_{A,\mu}(x)\} \qquad (25.2)$$

が与えられているとする．ここで

$$\varphi_{A,\mu} \overset{d}{\equiv} \frac{\partial \varphi_A}{\partial x^\mu}$$

である．(25.2) に対応する場の方程式は Hamilton の原理によれば，作用積分

$$I \overset{d}{\equiv} \int_\Omega L(\varphi_A, \varphi_{A,\mu}) d^4x \qquad (25.3)$$

の φ_A に関する変分を 0 とおくことにより求まる．すなわち，(25.2) の L から導かれる Euler の方程式

$$[L]^A \overset{d}{\equiv} \frac{\partial L}{\partial \varphi_A} - \partial_\mu\left(\frac{\partial L}{\partial \varphi_{A,\mu}}\right) = 0 \qquad (A = 1, 2, \cdots, N) \qquad (25.4)$$

が場の方程式である．

さて，I は $\varphi_A(x)$ の汎関数である．この $\varphi_A(x)$ を他の関数 $\varphi_A'(x)$ に変換したり，あるいは座標変換 $x^\mu \to x^{\mu'}$ を行った際に，I の値が不変な場合がある．そのようなときは，ある種の物理量に対して保存則が成立したり，あるいは恒等式が存在する．このような問題を一般的に扱う数学の一部門が**不変変分論**である．これについて特にくわしい研究を行なったのは Noether* である．

いま，無限小座標変換

$$x^\mu \to x^{\mu'} \overset{d}{\equiv} x^\mu + \delta x^\mu \qquad (25.5)$$

を考えよう．ここで座標が x^μ という点の座標変換後の新座標が $x^{\mu'}$ である．

* E. Noether: Gött. Nachr. (1918) 235.

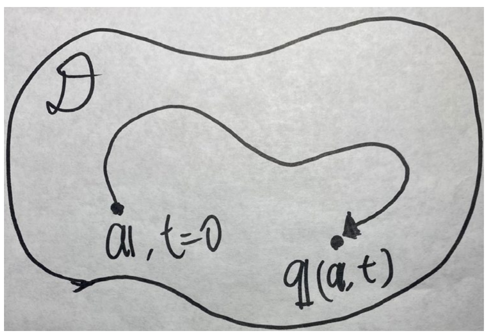
Relabeling transformation and symmetry

Particle relabeling transformation

$$\boldsymbol{a} \to \hat{\boldsymbol{a}} = \boldsymbol{a} + \delta\boldsymbol{a}$$

$$\Longrightarrow \boldsymbol{q}(\boldsymbol{a},t) \to \hat{\boldsymbol{q}}(\hat{\boldsymbol{a}},t) = \boldsymbol{q}(\boldsymbol{a}+\delta\boldsymbol{a},t) + \delta\boldsymbol{q}(\boldsymbol{a},t) = \boldsymbol{q}(\boldsymbol{a},t) + \Delta\boldsymbol{q};$$

$$\Delta q^i(\boldsymbol{a},t) = \delta q^i(\boldsymbol{a},t) + \delta a^k \partial_k q^i \equiv 0$$



Action

$$I[q] = \int_{\mathcal{D}} \mathcal{L}_0 d^4 a = \int dt \int_{\mathcal{D}} \mathcal{L}_0(q, \partial q, a) d^3 a;$$

$$\mathcal{L}_0 = \rho_0 \left[\frac{1}{2}\dot{q}^2 - e(\rho, s) - \Phi(q)\right];$$

$$\rho_0 = \rho(\boldsymbol{a}, 0), \quad \dot{\boldsymbol{q}}(\boldsymbol{a},t) = \boldsymbol{u}(\boldsymbol{q},t)$$

Variational symmetry

$$\int_{\hat{\mathcal{D}}} \mathcal{L}_0(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) d^3\hat{a} = \int_{\mathcal{D}} \mathcal{L}_0(q, \partial q, a) d^3 a \quad \text{for } \boldsymbol{a} \to \boldsymbol{a} + \delta\boldsymbol{a} \text{ s.t. } \Delta\boldsymbol{q} = 0$$

$$\Longleftrightarrow \quad \mathcal{L}_0(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) - \mathcal{L}_0(q, \partial q, a) + (\nabla_a \cdot \delta\boldsymbol{a})\mathcal{L}_0 + \nabla_a \cdot \delta\boldsymbol{\Lambda} = 0.$$

Divergence symmetry

Relabeling symmetry

$$\Delta \boldsymbol{q}(\boldsymbol{a},t) = \hat{\boldsymbol{q}}(\hat{\boldsymbol{a}},t) - \boldsymbol{q}(\boldsymbol{a},t) = \boldsymbol{0}$$

$$\Longleftrightarrow \quad \mathcal{L}_0(\hat{q}, \hat{\partial}\hat{q}, \hat{a}) - \mathcal{L}_0(q, \partial q, a) + (\nabla_a \cdot \delta \boldsymbol{a})\mathcal{L}_0 + \nabla_a \cdot \delta \boldsymbol{\Lambda} = 0.$$

$$\left\{ \frac{\partial \mathcal{L}_0}{\partial q^i} - \partial_j \left(\frac{\partial \mathcal{L}_0}{\partial(\partial_j q^i)} \right) \right\} \delta q^i + \partial_j (\delta J^j) = 0;$$

$$\delta J^j = \mathcal{L}_0 \delta a^j + \frac{\partial \mathcal{L}_0}{\partial(\partial_j q^i)} \delta q^i + \delta \Lambda^j$$

Noether's current in term of $\Delta \mathbf{q}$

$$\delta J^i = \mathcal{L}_0 \delta a^i + \frac{\partial \mathcal{L}_0}{\partial(\partial_j q^i)} \left(\Delta q^i - (\partial_j q^i) \delta a^j \right) + \delta \Lambda^i$$

$$= \frac{\partial \mathcal{L}_0}{\partial(\partial_j q^i)} \Delta q^i - T^i_k \delta a^k + \delta \Lambda^i; \quad T^i_k := \frac{\partial \mathcal{L}_0}{\partial(\partial_i q^j)} \partial_k q^j - \delta^i_k \mathcal{L}_0.$$

Relabeling for non-relativistic fluids

$$\delta \boldsymbol{a} = (0, \delta a^1, \delta a^2, \delta a^3), \quad \text{Assume } \delta \boldsymbol{\Lambda} \equiv \boldsymbol{0}$$

$$T^0_0 = \frac{\partial \mathcal{L}_0}{\partial \dot{q}^j} \dot{q}^j - \mathcal{L}_0 = \rho_0 \left[\frac{1}{2} \dot{q}^2 + e(\rho, s) + \Phi(q) \right]$$

$$T^0_k = \frac{\partial \mathcal{L}_0}{\partial \dot{q}^j} \partial_k q^j = \rho_0 v_j \partial_k q^j =: \rho_0 V_k \quad (k = 1, 2, 3),$$

$$v_j := \frac{1}{\rho_0} \frac{\partial \mathcal{L}_0}{\partial \dot{q}^j}, \quad V_k := v_j \partial_k q^j$$

Energy momentum tensor

Noether's charge

For relabeling transformation $\Delta \boldsymbol{q} = \boldsymbol{0}$,

$$\delta J^0 = -T^0_k \delta a^k = -\rho_0 v_j \left(\delta \boldsymbol{a} \cdot \frac{\partial}{\partial \boldsymbol{a}} \right) q^j$$

$$T^0_k = \rho_0 v_j \partial_k q^j;$$

$$v_j = \frac{1}{\rho_0} \frac{\partial \mathcal{L}_0}{\partial \dot{q}^j}$$

$$Q = \int \delta J^0 d^3 a = -\int \boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{D}(\boldsymbol{x}, t) d^3 x$$

Cross helicity

$$\boldsymbol{D} := -\rho \left(\delta \boldsymbol{a} \cdot \frac{\partial}{\partial \boldsymbol{a}} \right) \boldsymbol{q} = -\rho \delta \boldsymbol{q}$$

$$\frac{\partial \boldsymbol{D}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{D})$$

$$\mathcal{L}_0(q, \partial q, a) = \rho_0 \left[\frac{1}{2} \dot{q}^2 - e(\rho, s) - \Phi(q) \right]$$

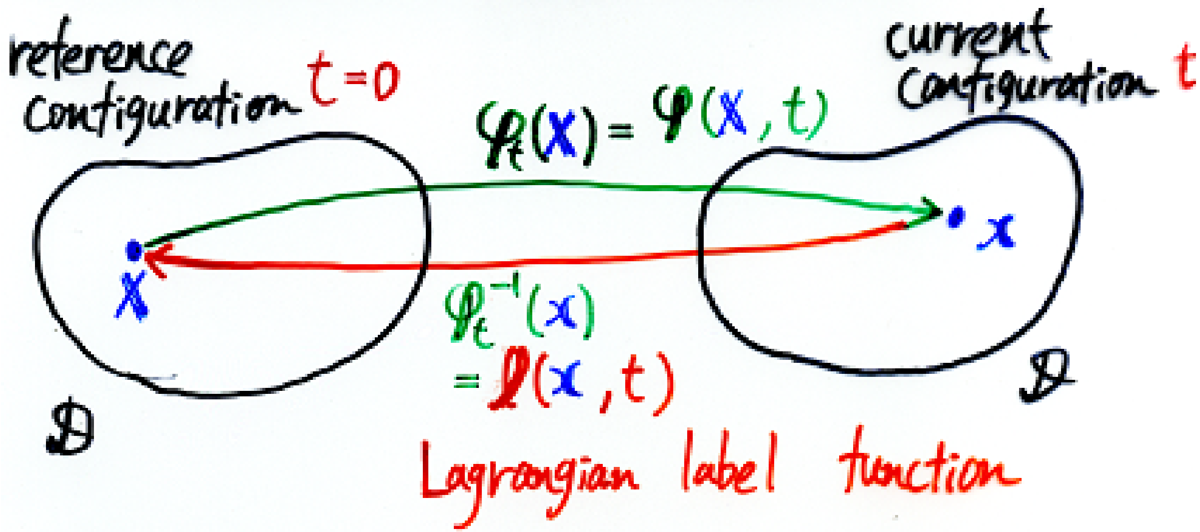
$$\delta \rho = -\nabla \cdot (\rho \delta \boldsymbol{q}) = \nabla \cdot \boldsymbol{D},$$

$$\delta s = -\frac{1}{\rho} \boldsymbol{D} \cdot \nabla s, \quad \delta \boldsymbol{u} = -\left(\frac{\partial \delta \boldsymbol{a}}{\partial t} \cdot \frac{\partial}{\partial \boldsymbol{a}} \right) \boldsymbol{q}$$

Relabeling trnsformaytion

$$\nabla \cdot \boldsymbol{D} = 0, \quad \boldsymbol{D} \cdot \nabla s = 0, \quad \frac{\partial \delta \boldsymbol{a}}{\partial t} = \boldsymbol{0}$$

Hamilton's principle for Euler-Poincaré equation for a non-homentropic flow



Lagrangian label function $l\left(\varphi(\boldsymbol{X}, t), t\right) = \boldsymbol{X}; \quad 0 = \left.\frac{\partial l_A}{\partial t}\right|_X + \frac{\partial l_A}{\partial x_j}\frac{\partial \varphi_j}{\partial t},$

D. D. Holm

$$u_i = -\left(\hat{D}^{-1}\right)_{iA}\frac{\partial l_A}{\partial t}; \quad \left(\hat{D}\right)_{Ai} = \frac{\partial l_A(\boldsymbol{x}, t)}{\partial x_i}$$

Define $D := \det \hat{D}$, then

$$\rho(\boldsymbol{x}, t) = \rho(\boldsymbol{X}, 0) D$$

Variation of action: Baroclinic case

$$l_A \to l_A + \delta l_A \Longrightarrow \begin{cases} \delta u_i = -(\hat{D}^{-1})_{iA}\left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}\right)\delta l_A, \\ \delta\rho = \frac{\partial}{\partial x_i}\left[\rho(\hat{D}^{-1})_{iA}\delta l_A\right], \quad \delta s = (\hat{D}^{-1})_{jA}\frac{\partial s}{\partial x_j}\delta l_A, \\ \delta \boldsymbol{B} = \nabla \times (\boldsymbol{B} \times \hat{D}^{-1}\delta \boldsymbol{l}). \end{cases}$$

$$\delta S = \int_{t_0}^{t_1} \mathrm{d}t \int_{\mathcal{D}} \left(\frac{\partial \mathcal{L}}{\partial u_i}\delta u_i + \frac{\partial \mathcal{L}}{\partial \rho}\delta\rho + \frac{\partial \mathcal{L}}{\partial s}\delta s + \frac{\partial \mathcal{L}}{\partial B_i}\delta B_i\right) \quad \mathcal{L}[\boldsymbol{u}, \rho, s, \boldsymbol{B}; \boldsymbol{x}] = \rho\left(\frac{\boldsymbol{u}^2}{2} - e(\rho, s) - \frac{\boldsymbol{B}^2}{2} - \Phi(\boldsymbol{x})\right)$$

$$\delta S = -\int_{t_0}^{t_1} \mathrm{d}t \int_{\mathcal{D}} \mathrm{d}V \left\{ \frac{\partial}{\partial t}\left[\rho(\hat{D}^{-1})_{iA}\delta l_A \frac{1}{\rho}\frac{\partial \mathcal{L}}{\partial u_i}\right] + \frac{\partial}{\partial x_j}\left[\rho(\hat{D}^{-1})_{iA}\delta l_A\left(\frac{1}{\rho}\frac{\partial \mathcal{L}}{\partial u_i}u_j - \frac{\partial \mathcal{L}}{\partial \rho}\delta_{ij}\right)\right] + \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{B}} \times (\boldsymbol{B} \times \hat{D}^{-1}\delta \boldsymbol{l})\right)\right\}$$
$$+ \int_{t_0}^{t_1} \mathrm{d}t \int_{\mathcal{D}} \mathrm{d}V \left\{ \rho(\hat{D}^{-1})_{iA}\delta l_A \left[\frac{D}{Dt}\left(\frac{1}{\rho}\frac{\partial \mathcal{L}}{\partial u_i}\right) + \frac{1}{\rho}\frac{\partial \mathcal{L}}{\partial u_k}\frac{\partial u_k}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{L}}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial \mathcal{L}}{\partial s}\frac{\partial s}{\partial x_i} + \frac{1}{\rho}\cdot\left(\left(\nabla \times \frac{\partial \mathcal{L}}{\partial \boldsymbol{B}}\right) \times \boldsymbol{B}\right)_i\right]\right\}.$$

$$v_i := \frac{1}{\rho}\frac{\partial \mathcal{L}}{\partial u_i}, \quad \pi := \frac{\partial \mathcal{L}}{\partial \rho}, \quad T := -\frac{1}{\rho}\frac{\partial \mathcal{L}}{\partial s}, \quad \boldsymbol{h} := -\frac{\partial \mathcal{L}}{\partial \boldsymbol{B}}, \quad \boldsymbol{j} = \nabla \times \boldsymbol{h}.$$

$$\frac{Dv_i}{Dt} + v_k \frac{\partial u_k}{\partial x_i} = \frac{\partial \pi}{\partial x_i} + T\frac{\partial s}{\partial x_i} + \frac{1}{\rho}(\boldsymbol{j} \times \boldsymbol{B})_i$$

Euler-Poincaré equations

Hamilton's principle in material coordinates

Define $V_A = \frac{\partial x_i}{\partial X_A} v_i$ $\quad$ $H_A = \frac{\partial x_i}{\partial X_A} h_i$

Rewrite the variation in terms of the Lagrangian variables ***X***

$$\delta \boldsymbol{l} = \delta \boldsymbol{l}(\boldsymbol{X}, t)$$

$$\delta S = \int_{t_0}^{t_1} \mathrm{d}t \int_{\mathcal{D}} \mathrm{d}V_X \left\{ -\frac{\partial}{\partial t}\left(V_A \rho_0\, \delta l_A\right) + \frac{\partial}{\partial X_A}\left(\pi \rho_0\, \delta l_A\right) + \nabla_X \cdot [(\boldsymbol{B}_0 \times \delta \boldsymbol{l}) \times \boldsymbol{H}] \right.$$
$$\left. + \rho_0\, \delta l_A \left(\frac{\partial V_A}{\partial t} - \frac{\partial \pi}{\partial X_A} - T \frac{\partial s}{\partial X_A} - \frac{1}{\rho_0} (\boldsymbol{J} \times \boldsymbol{B}_0)_A \right) \right\}.$$

$$\boldsymbol{J} := \nabla_X \times \boldsymbol{H}, \quad \rho_0 := \rho(\boldsymbol{X}, 0), \quad \boldsymbol{B}_0 := \boldsymbol{B}(\boldsymbol{X}, 0)$$

for arbitrary variation $\delta \boldsymbol{l}$ subject to B.C. and I.C.

⇓

Euler-Poincaré equations in terms of the material coordinates

$$\frac{\partial V_A}{\partial t} - \frac{\partial \pi}{\partial X_A} - T \frac{\partial s}{\partial X_A} - \frac{1}{\rho_0} (\boldsymbol{J} \times \boldsymbol{B}_0)_A = 0$$

Particle relabeling symmetry

Time-independent change of particle labels $\boldsymbol{X} \to \boldsymbol{X}' = \boldsymbol{\eta}(\boldsymbol{X})$;

$$\rho(\boldsymbol{X},0)dX \wedge dY \wedge dZ = \rho(\boldsymbol{X}',0)dX' \wedge dY' \wedge dZ' = \rho(\boldsymbol{x},t)dx \wedge dy \wedge dz$$

$$s(\boldsymbol{X},0) = s(\boldsymbol{X}',0) = s(\boldsymbol{x},t).$$

$$\frac{1}{2}\epsilon_{ABC}B_C(\boldsymbol{X},0)dX_A \wedge dX_B = \frac{1}{2}\epsilon_{ABC}B_C(\boldsymbol{X}',0)dX'_A \wedge dX'_B = \frac{1}{2}\epsilon_{ijk}B_k dx_i \wedge dx_j$$

Infinitesimal particle relabeling

$$\boldsymbol{X} \to \boldsymbol{X}' = \boldsymbol{X} + \delta\boldsymbol{l};$$

$$\frac{\partial}{\partial \boldsymbol{X}} \cdot (\rho_0 \delta\boldsymbol{l}) = 0, \quad (\delta\boldsymbol{l}\cdot\nabla)s_0 = 0, \quad \frac{\partial}{\partial \boldsymbol{X}} \times (\boldsymbol{B}_0 \times \delta\boldsymbol{l}) = \boldsymbol{0}$$

Under particle relabeling, the action is unchanged $\delta S = 0$:

$$\frac{\partial}{\partial t}(V_A\rho_0\,\delta l_A) - \frac{\partial}{\partial X_A}(\pi\rho_0\,\delta l_A) - \nabla_X\cdot[(\boldsymbol{B}_0\times\delta\boldsymbol{l})\times\boldsymbol{H}]$$
$$-\rho_0\,\delta l_A\left(\frac{\partial V_A}{\partial t} - \frac{\partial \pi}{\partial X_A} - T\frac{\partial s}{\partial X_A} - \frac{1}{\rho_0}(\boldsymbol{J}\times\boldsymbol{B}_0)_A\right) = 0$$

$$V_A = \frac{\partial x_i}{\partial X_A}v_i$$

$$H_A = \frac{\partial x_i}{\partial X_A}h_i$$

Assume Euler-Poincaré equations, then

$$\frac{\partial}{\partial t}(V_A\rho_0\,\delta l_A) - \frac{\partial}{\partial X_A}(\pi\rho_0\,\delta l_A) - \nabla_X\cdot[(\boldsymbol{B}_0\times\delta\boldsymbol{l})\times\boldsymbol{H}] = 0$$

Noether's charge

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{D}} V_A\rho_0\,\delta l_A \mathrm{d}V_X = 0$$

Noether's theorem for particle relabeling symmetry

$$\boldsymbol{D}_0 = \rho_0\,\delta\boldsymbol{l};\quad \frac{\partial}{\partial \boldsymbol{X}}\cdot \boldsymbol{D}_0 = 0,\quad \left(\boldsymbol{D}_0\cdot\frac{\partial}{\partial \boldsymbol{X}}\right)s_0 = 0,\quad \frac{\partial}{\partial \boldsymbol{X}}\times\left(\boldsymbol{B}_0\times\frac{\boldsymbol{D}_0}{\rho_0}\right) = \boldsymbol{0}$$

Noether's charge

$$\int_{\mathcal{D}} V_A D_{0A}\,\mathrm{d}^3V = \int_{\mathcal{D}} v_i \frac{\partial x_i}{\partial X_A} D_{0A}\rho_0\,\mathrm{d}V = -\int_{\mathcal{D}} \boldsymbol{v}\cdot\boldsymbol{D}\,\mathrm{d}V.$$

$$V_A = \frac{\partial x_i}{\partial X_A} v_i \qquad D_i := -\frac{\partial x_i}{\partial X_A} D_{0A}$$

Baroclinic case

Utiyama (1959)

$$\varphi'_A(x') := \varphi(x) + \delta\varphi_A(x)$$

$$0 = \delta\varphi_A(x) := \bar{\delta}\varphi_A(x) + \varphi_{A,\mu}\delta x^\mu$$

$$h_c[\boldsymbol{v},\boldsymbol{C}] = \int_{\mathcal{D}} \boldsymbol{v}\cdot\boldsymbol{D}\,dV;\quad (\boldsymbol{D} = \nabla\times\boldsymbol{C})$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{u}\times\boldsymbol{\omega} + \nabla(\pi - \Phi) + T\nabla s + \frac{1}{\rho}\boldsymbol{j}\times\boldsymbol{B},$$

$$\frac{\partial \boldsymbol{D}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{D}),$$

$$\nabla\cdot\boldsymbol{D} = 0,\quad (\boldsymbol{D}\cdot\nabla)s = 0,\quad \nabla\times\left(\frac{\boldsymbol{B}}{\rho}\times\boldsymbol{D}\right) = 0$$

$$\boldsymbol{v} = \frac{1}{\rho}\frac{\partial\mathcal{L}}{\partial \boldsymbol{u}},\quad \pi = \frac{\partial\mathcal{L}}{\partial\rho},\quad T = -\frac{1}{\rho}\frac{\partial\mathcal{L}}{\partial s},\quad \boldsymbol{h} = -\frac{\partial\mathcal{L}}{\partial\boldsymbol{B}},\quad \boldsymbol{j} = \nabla\times\boldsymbol{h}$$

Baroclinic flow: *Ertel's potential vorticity*

$$\mathcal{H}_c[\boldsymbol{v},\boldsymbol{D}] = \int \boldsymbol{v}\cdot\boldsymbol{D}dV$$

$$\nabla\times\boldsymbol{v} = \boldsymbol{\omega} \quad \left[v_i = \frac{1}{\rho}\frac{\partial\mathcal{L}}{\partial u_i}\right]; \quad \frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}) + \nabla T\times\nabla s$$

$$\frac{\partial\boldsymbol{D}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{D}), \quad \nabla\cdot\boldsymbol{D} = 0, \quad (\boldsymbol{D}\cdot\nabla)s = 0$$

Example

$$\boldsymbol{D} = \nabla\times(f(Q,s)\nabla s) = \frac{\partial f}{\partial Q}\nabla Q\times\nabla s$$

$$Q := \frac{1}{\rho}(\nabla\times\boldsymbol{v})\cdot\nabla s$$

$$\frac{\partial\boldsymbol{D}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{D})$$

Casimir invariant

$$\mathcal{H}_C = \int_{\mathcal{D}}\boldsymbol{v}\cdot(\nabla\times f\nabla s)\,\mathrm{d}V = -\int_{\partial\mathcal{D}} f\nabla s\cdot(\boldsymbol{v}\times\boldsymbol{n})\mathrm{d}A + \int_{\mathcal{D}}\overbrace{Qf(Q,s)}^{F(Q,s)}\rho\mathrm{d}V$$

Generalized helicity

Baroclinic effect

$$\frac{D\tau}{Dt} = -T$$

$$\hat{\boldsymbol{v}} = \boldsymbol{v} + \tau \nabla s$$

MHD

$$\frac{\partial \boldsymbol{m}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{m}) = \boldsymbol{j}$$

$$\hat{\boldsymbol{v}} = \boldsymbol{v} + \frac{1}{\rho} \boldsymbol{B} \times \boldsymbol{m}$$

Generalized helicity

$$\mathcal{H}[\boldsymbol{v}] = \int_{\mathcal{D}} \hat{\boldsymbol{v}} \cdot (\nabla \times \hat{\boldsymbol{v}}) d^3 x = \int_{\mathcal{D}} \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{\omega}} d^3 x$$

Variational symmetry supplemented by divergence symmetry

Action

$$S[\boldsymbol{l}] = \int dt \int_{\mathcal{D}} d^3x \mathcal{L}(\boldsymbol{u}, \rho, s, \boldsymbol{B}, \boldsymbol{x}) = \int dt \int_{\mathcal{D}} d^3x \mathcal{L}(l_\alpha, \partial_i l_\alpha, \boldsymbol{x})$$

Variational symmetry

$$S[\hat{\boldsymbol{l}}] = S[\boldsymbol{l}]; \quad \int dt \int_{\mathcal{D}} d^3\hat{x} \mathcal{L}(\hat{l}_\alpha, \hat{\partial}\hat{l}_\alpha, \hat{x}) = \int dt \int_{\mathcal{D}} d^3x \mathcal{L}(l_\alpha, \partial l_\alpha, \boldsymbol{x})$$

$$\Longrightarrow \quad \mathcal{L}(\hat{l}_\alpha, \hat{\partial}\hat{l}_\alpha, \hat{x}) - \mathcal{L}(l_\alpha, \partial l_\alpha, x) + (\nabla \cdot \delta \boldsymbol{x})\mathcal{L} + \frac{\partial \delta \Lambda_0}{\partial t} + \nabla \cdot \delta \boldsymbol{\Lambda} = 0$$

Divergence symmetry

$$\partial_k \delta J_k + S_\alpha \delta l_\alpha = 0 \quad (k = 0, 1, 2, 3; \ \alpha = 1, 2, 3);$$

Euler-Lagrange equation: $S_\alpha = \frac{\partial \mathcal{L}}{\partial l_\alpha} - \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j l_\alpha)} \quad (= 0)$

Current: $\delta J_k = \mathcal{L} \delta x_k + \frac{\partial \mathcal{L}}{\partial(\partial_k l_\alpha)} \delta l_\alpha + \delta \Lambda_k$

Divergence symmetry

$$\mathcal{L}(\hat{l}_\alpha, \hat{\partial}\hat{l}_\alpha, \hat{x}) - \mathcal{L}(l_\alpha, \partial l_\alpha, x) + (\nabla\cdot\delta\boldsymbol{x})\mathcal{L} + \frac{\partial \delta\Lambda_0}{\partial t} + \nabla\cdot\delta\boldsymbol{\Lambda} = 0,$$

Diverrgence symmetry

Define τ and $\boldsymbol{m}$ by

$$\frac{\partial \tau}{\partial t} + (\boldsymbol{u}\cdot\nabla)\tau = -T, \qquad \frac{\partial \boldsymbol{m}}{\partial t} - \nabla\times(\boldsymbol{u}\times\boldsymbol{m}) = \boldsymbol{j}$$

Mobbs '81

Vladimirov & Moffatt '95,
Beckenstein & Ooron '00

$$\hat{\boldsymbol{v}} = \boldsymbol{v} + \tau\nabla s + \frac{1}{\rho}\boldsymbol{B}\times\boldsymbol{m}$$

$$\frac{\partial \hat{\boldsymbol{\omega}}}{\partial t} = \nabla\times(\boldsymbol{u}\times\hat{\boldsymbol{\omega}}); \quad \hat{\boldsymbol{\omega}} := \nabla\times\hat{\boldsymbol{v}}$$

$$\delta\Lambda_0^s = \tau(\boldsymbol{D}\cdot\nabla)s, \quad \delta\Lambda_0^B = \boldsymbol{D}\cdot(\boldsymbol{B}\times\boldsymbol{m})/\rho$$

Divergence symmetry for *baroclinic effect* and *Lorentz force*

$$\delta\Lambda_0^s = \tau(\boldsymbol{D}\cdot\nabla)s, \quad \delta\Lambda^s = \boldsymbol{u}\tau(\boldsymbol{D}\cdot\nabla)s$$

$$\delta\Lambda_0^B = \boldsymbol{D}\cdot(\boldsymbol{B}\times\boldsymbol{m})/\rho, \quad \delta\boldsymbol{\Lambda}^B = \frac{\boldsymbol{u}}{\rho}\left[\boldsymbol{D}\cdot(\boldsymbol{B}\times\boldsymbol{m})\right] + \boldsymbol{h}\times\left(\boldsymbol{B}\times\frac{1}{\rho}\boldsymbol{D}\right)$$

$$\boldsymbol{h} = -\frac{\partial\mathcal{L}}{\partial\boldsymbol{B}}, \quad \boldsymbol{j} = \nabla\times\boldsymbol{h}$$

$$\frac{\partial}{\partial t}\left[\boldsymbol{D}\cdot\left(\boldsymbol{v}+\tau\nabla s+\frac{1}{\rho}\boldsymbol{B}\times\boldsymbol{m}\right)\right] + \nabla\cdot\left\{\boldsymbol{u}\left[\boldsymbol{D}\cdot\left(\boldsymbol{v}+\tau\nabla s+\frac{1}{\rho}\boldsymbol{B}\times\boldsymbol{m}\right)\right] - \pi\boldsymbol{D}\right\} = 0$$

What is the origin of $(\delta\Lambda_0^s, \delta\boldsymbol{\Lambda}^s)$ and $(\delta\Lambda_0^B, \delta\boldsymbol{\Lambda}^B)$???

Constrained Eulerian variational principle

$$S=\int dt\int_{\mathcal{D}} d^3x\left\{\mathcal{L}(\boldsymbol{u},\rho,s,\boldsymbol{B})+\rho\tau\left[\frac{\partial s}{\partial t}+(\boldsymbol{u}\cdot\nabla)s\right]-\boldsymbol{\alpha}\cdot\left[\frac{\partial \boldsymbol{B}}{\partial t}-\nabla\times(\boldsymbol{u}\times\boldsymbol{B})\right]\right\},$$

$\tau,\ \boldsymbol{\alpha}$: Lagrangian multipliers

$(\boldsymbol{d}=\delta\boldsymbol{l}\ \Rightarrow)\boldsymbol{d}(\boldsymbol{x},t)$ Eulerian representation of relabeling

$$\delta\boldsymbol{u}=\frac{\partial \boldsymbol{d}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{d}-(\boldsymbol{d}\cdot\nabla)\boldsymbol{u},\quad \delta\rho=-\nabla\cdot(\rho\boldsymbol{d})$$

$\boldsymbol{d}$, s and $\boldsymbol{B}$ are taken to be independent.

$$\begin{aligned}\delta S=\int dt\int_{\mathcal{D}} d^3x&\left\{\frac{\partial}{\partial t}\left[\rho\boldsymbol{d}\cdot\left(\boldsymbol{v}+\tau\nabla s+\frac{1}{\rho}\boldsymbol{B}\times\boldsymbol{m}\right)\right]\right.\\&+\nabla\cdot\left(\rho\boldsymbol{u}\left[\boldsymbol{d}\cdot\left(\boldsymbol{v}+\tau\nabla s+\frac{1}{\rho}\boldsymbol{B}\times\boldsymbol{m}\right)\right]-\pi\rho\boldsymbol{d}-\boldsymbol{\alpha}\times(\delta\boldsymbol{u}\times\boldsymbol{B})\right)+\cdots\\&-\rho\boldsymbol{d}\cdot\left[\frac{\partial}{\partial t}\left(\boldsymbol{v}+\tau\nabla s+\frac{1}{\rho}\boldsymbol{B}\times\boldsymbol{m}\right)+(\boldsymbol{u}\cdot\nabla)\left(\boldsymbol{v}+\tau\nabla s+\frac{1}{\rho}\boldsymbol{B}\times\boldsymbol{m}\right)\right.\\&\qquad\left.+\left(v_k+\tau\partial_k s+\frac{1}{\rho}(\boldsymbol{B}\times\boldsymbol{m})_k\right)\nabla u_k-\nabla\pi\right]\\&\left.-\rho\delta s\left[\frac{\partial\tau}{\partial t}+(\boldsymbol{u}\cdot\nabla)\tau+\mathcal{T}\right]+\delta\boldsymbol{B}\cdot\left[\frac{\partial\boldsymbol{\alpha}}{\partial t}-\boldsymbol{u}\times(\nabla\times\boldsymbol{\alpha})-\boldsymbol{h}\right]\right\}\end{aligned}$$

$$\frac{\delta S}{\delta s}=0\quad\Rightarrow\quad\frac{\partial\tau}{\partial t}+(\boldsymbol{u}\cdot\nabla)\tau=-\mathcal{T}$$

$$\frac{\delta S}{\delta\boldsymbol{B}}=0\quad\Rightarrow\quad\frac{\partial\boldsymbol{\alpha}}{\partial t}-\boldsymbol{u}\times(\nabla\times\boldsymbol{\alpha})=\boldsymbol{h};\quad\boldsymbol{m}=\nabla\times\boldsymbol{\alpha}$$

$$\nabla\times\qquad\frac{\partial\boldsymbol{m}}{\partial t}-\nabla\times(\boldsymbol{u}\times\boldsymbol{m})=\boldsymbol{j};\quad\boldsymbol{j}=\nabla\times\boldsymbol{h}$$

Generator of divergence symmetry and generalized cross helicity

$$\frac{\delta S}{\delta \boldsymbol{d}} = \boldsymbol{0} \quad \Rightarrow \quad \frac{\partial}{\partial t}\left(\boldsymbol{v} + \tau \nabla s + \frac{1}{\rho}\boldsymbol{B} \times \boldsymbol{m}\right) + (\boldsymbol{u} \cdot \nabla)\left(\boldsymbol{v} + \tau \nabla s + \frac{1}{\rho}\boldsymbol{B} \times \boldsymbol{m}\right)$$

$$+ \left(v_k + \tau \partial_k s + \frac{1}{\rho}(\boldsymbol{B} \times \boldsymbol{m})_k\right) \nabla u_k - \nabla \pi = \boldsymbol{0}$$

$$\frac{\partial \hat{\boldsymbol{v}}}{\partial t} - \boldsymbol{u} \times (\nabla \times \hat{\boldsymbol{v}}) = \nabla(\pi - \boldsymbol{u} \cdot \hat{\boldsymbol{v}}); \quad \hat{\boldsymbol{v}} = \boldsymbol{v} + \tau \nabla s + \frac{1}{\rho}\boldsymbol{B} \times \boldsymbol{m}$$

By Noether's theorem, $\delta S = 0$ for arbitrary $\boldsymbol{d}, \delta s$ and $\boldsymbol{B}$ produces

$$\frac{\partial}{\partial t}\left[\rho \boldsymbol{d} \cdot \left(\boldsymbol{v} + \tau \nabla s + \frac{1}{\rho}\boldsymbol{B} \times \boldsymbol{m}\right)\right] + \nabla \cdot \left(\rho \boldsymbol{u}\left[\boldsymbol{d} \cdot \left(\boldsymbol{v} + \tau \nabla s + \frac{1}{\rho}\boldsymbol{B} \times \boldsymbol{m}\right)\right] - \pi \rho \boldsymbol{d} - \boldsymbol{\alpha} \times (\delta \boldsymbol{u} \times \boldsymbol{B})\right) = 0,$$

$$\Longrightarrow \quad \delta \Lambda_0^s = \tau(\rho \boldsymbol{d} \cdot \nabla)s, \quad \delta \Lambda_0^B = \boldsymbol{d} \cdot (\boldsymbol{B} \times \boldsymbol{m}) \qquad (\rho \boldsymbol{d} = \boldsymbol{D})$$

Generalized cross helicity $\displaystyle \int \rho \boldsymbol{d} \cdot \left(\boldsymbol{v} + \tau \nabla s + \frac{1}{\rho}\boldsymbol{B} \times \boldsymbol{m}\right) d^3x = \int \boldsymbol{D} \cdot \hat{\boldsymbol{v}}\, d^3x$

Relabeling $\delta \boldsymbol{u} = \boldsymbol{0}, \delta \rho = 0$ $(\delta s = 0, \delta \boldsymbol{B} = \boldsymbol{0})$ ensures $\delta S = 0$

$$\frac{\partial \boldsymbol{d}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{d} - (\boldsymbol{d} \cdot \nabla)\boldsymbol{u} = \boldsymbol{0}, \quad \nabla \cdot (\rho \boldsymbol{d}) = 0$$

Summary

A unified view of topological invariants for a non-isentropic MHD from Noether's theorem

Casimirs

$$\mathcal{M}[\rho] = \int_\mathcal{D} \rho dV, \quad \mathcal{S}[\sigma] = \int_\mathcal{D} \rho s dV = \int_\mathcal{D} \sigma dV,$$
$$h_c[\boldsymbol{v}, \boldsymbol{C}] = \int_\mathcal{D} \boldsymbol{v} \cdot \boldsymbol{D} dV, \quad h_m[\boldsymbol{A}] = \frac{1}{2} \int_\mathcal{D} \boldsymbol{A} \cdot \boldsymbol{B} dV$$
$$(\boldsymbol{D} = \nabla \times \boldsymbol{C}), \quad (\boldsymbol{B} = \nabla \times \boldsymbol{A})$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{u} \times \boldsymbol{\omega} + \nabla(\pi - \Phi) + T \nabla s + \frac{1}{\rho} \boldsymbol{j} \times \boldsymbol{B},$$
$$\frac{\partial \boldsymbol{D}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{D}),$$
$$\nabla \cdot \boldsymbol{D} = 0, \quad (\boldsymbol{D} \cdot \nabla) s = 0, \quad \nabla \times \left(\frac{\boldsymbol{B}}{\rho} \times \boldsymbol{D} \right) = 0$$

Nambu-bracket representation for Lie-Poisson equation

$$\frac{d}{dt} F[\boldsymbol{M}, \boldsymbol{d}, \boldsymbol{B}, \rho, s] = \{F, h_c, H\}_{MMd} + \{F, \mathcal{S}, H\}_{M\rho s} + \{F, h_m, H\}_{MBB}$$

$$\boldsymbol{M} = \rho \boldsymbol{v}, \ \ \boldsymbol{d} = \boldsymbol{D}/\rho$$

Variational symmetry $h_c = \int_\mathcal{D} \boldsymbol{D} \cdot \boldsymbol{v} \, dV$

Generalized enstrophy for 2D barotropy $\Omega = \int_\mathcal{A} f\left(\frac{\omega}{\rho}\right) \omega dA$

Ertel's potential vorticity Q $C = \int \rho F(s, Q) dV; \quad Q = \frac{1}{\rho} (\nabla \times \boldsymbol{v}) \cdot \nabla s$

Divergence symmetry

Generalized cross-helicity

$$\int \rho \boldsymbol{d} \cdot \left(\boldsymbol{v} + \tau \nabla s + \frac{1}{\rho} \boldsymbol{B} \times \boldsymbol{m} \right) d^3 x = \int \boldsymbol{D} \cdot \hat{\boldsymbol{v}} \, d^3 x$$