Nonlocal(izable) Field Theory. I.

1948 ~ 1949

Princeton ~ Columbia

NATIONAL SUBJECT INDEX

CLASS SCHEDULE

TIME	10.10~11	11.10~12	1.10~2	2.10~3	3.10~4	4.10~5	5.10~6	6.10~7	7.10~8
MON.	1016P	1016P	1016P	~~214P~~	1016P	1016P			
TUE.	1016P	1016P	~~329P~~	1016P	1016P	1016P			
WED.	1016P	1016P	1016P	~~213P~~	1016P	1016P			
THU.	1016P	1016P	~~329P~~	1016P	1016P	1016P			
FRI.	1016P	1016P	1016P	1016P	1016P	1016P			~~Col.~~
SAT.	1016P	1016P							

FINAL EXAMINATIONS

DAY	DATE	TIME	PLACE	COURSE

NAME __________ TELEPHONE University 4-3200 Ext. 506

ADDRESS __________

SCHOOL __________ CLASS __________

NATIONAL No. 3980SI Made in U. S. A.

Columbia 1949~1950 Winter session

Physics 269 Selected topics Seminar

Tuesday 1.10~2 329 Pupin
Thursday 1.10~2 329 Pupin

1949~1950 Spring session

Seminar Rm. 831

Tuesday 3.10~4 pm 831 Pupin

Selected Topics

Selected Topics Seminar
Winter session 1949~1950

~~Topics~~ Lecture I. Sept. 29, 1949, Thursday

I. Nonlocal Field Theory
Introduction
Failure of Mixed Field Theory
(Divergence ~~Difficulty~~ or Negative Energy)
Types of Mixed Field Theories
(a) Bopp, Podolsky, Dirac, Kikuchi, Green
neg. energ. neutral vector meson, neg. energy photons (el. self-en.)
(b) Pais, Sakata, Hara, Tatsuoka,
(el. self. energy. photon self-energy)
~~neg.~~ pos. energy. neutral scalar mesons, charged scalar mesons
(a)′ Pauli-Villars
formalistic (vacuum ~~self-en~~ polarization)
(b)′ Umezawa, Yukawa and Yamada
Feldman
Jost and Rayski, Helv. Phys. Acta
realistic (vacuum polarization)

Electron Self-Energy
Photon-self energy
Vacuum-self energy

Inclusion of Higher Spin Particles
Bhabha
S-Matrix Scheme
Heisenberg, Møller
Feynman
Action at distance → arbitrary function
Nonlocal Field

In this seminar course, as I mentioned yesterday, I would like to discuss ~~a~~ those problems which must be solved in order to arrive at a consistent theory of elementary particles. Apparently there are two separate~~ions~~ problems:

(i) Problem of Convergence of Field Theory

(ii) Problem of Systematics of Elementary Particle. (Spectrum of Masses, Spins, Charges etc) what kinds of elementary particles can and do exist in nature?*

~~We begin with~~ I am not sure whether I can give any definite answer to these problems, but I want to begin with the discussions of the first problem.

~~In this course~~ As to the solution of this problem, it has been generally believed . . .

(continued to Introduction of Part I of "Q. T. of Nonlocal Fields)

"Quantum Theory of Nonlocal Fields"
Part I. Free Fields (Phys. Rev. 77, Jan. 1, 1950)
up to eq. (5).

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* This question is intimately connected with ~~the~~ still deeper question: What is the elementary particle. The def. of el. part. is so intimately connected with the present q.f.t. that once we discard the latter, everything may change.

Lecture II. Oct. 4, Tuesday, 1949
Part I from Equation (6) on.

Remarks on Part I.

R(1) Eqs. (23), (24): the Lorentz transformation ~~in~~ with the velocity v_x, v_y, v_z can be decomposed into products of three transf.

(i) Rotation of space-axes, by which the direction $v_x : v_y : v_z$ is transformed into z-axis.

(ii) Lorentz transformation in z-direction with the velocity $v = \sqrt{v_x^2+v_y^2+v_z^2}$

(iii) Inverse rotation, by which the ~~directions of~~ space-axes ~~return~~ ~~take~~ return to the original directions.

(i)

Lecture III. Oct. 6, Thursday

Part I. From (27) on

R.(2) Eqs. (40); ~~(43)~~ (41):

$$U = \sum_{\mathbf{k}} \iint \left(\frac{2\pi}{L}\right)^3 \frac{\lambda \sin\Theta \, d\Theta \, d\Phi}{4\varkappa\sqrt{k^2+\varkappa^2}} \{ u(\mathbf{k},\Theta,\Phi) U(\mathbf{k},\Theta,\Phi) + v^*(\mathbf{k},\Theta,\Phi) U^*(\mathbf{k},\Theta,\Phi) \}$$

$$U = \int\cdots\int (dk)^4 (dl^4)\, u(k_\mu, l_\mu)\, \delta(k_\mu k^\mu + \varkappa^2)\, \delta(k_\mu l^\mu) \times \delta(l_\mu l^\mu - \lambda^2) \exp(i k_\mu X^\mu) \prod_\mu \delta(x_\mu + l_\mu)$$

$$= \int\cdots\int (dk)^4 (dl^4)\, u(k_\mu, l_\mu)\, \underline{\delta(k_\mu k^\mu + \varkappa^2)}\, \underline{\delta(k_\mu l^\mu)} \times \underline{\delta(l_\mu l^\mu - \lambda^2)} \exp(i k_\mu X^\mu) \exp(i l^\mu p_\mu \ldots)$$

$$= \sum_{\mathbf{k}} \left(\frac{2\pi}{L}\right)^3 \frac{1}{2\sqrt{k^2+\varkappa^2}} \cdot \frac{1}{\varkappa} \cdot \frac{1}{2\lambda} \cdot l'^2 \sin\Theta \, d\Theta \, d\Phi$$

$|k_4| = \sqrt{k^2+\varkappa^2}$

$$\times \{ u(\mathbf{k},\Theta,\Phi)\, U(\mathbf{k},\Theta,\Phi) + v^*(\mathbf{k},\Theta,\Phi)\, U^*(\mathbf{k},\Theta,\Phi) \}$$

R(3) Eqs. (42), (43), (44):

$\tilde{P}_{l'}^{m'}(\Theta,\Phi)$

$$u(\mathbf{k},\Theta,\Phi) = \sum_{l,m} u(\mathbf{k},l,m)\, P_l^m(\Theta,\Phi)$$

$$\iint U(\mathbf{k},\Theta,\Phi)\, P_l^m(\Theta,\Phi) \sin\Theta\, d\Theta\, d\Phi = U(\mathbf{k},l,m)$$

$$\iint u(\mathbf{k},\Theta,\Phi)\, \tilde{P}_{l'}^{m'}(\Theta,\Phi) \sin\Theta\, d\Theta\, d\Phi = \sum_{l,m} \delta_{ll'}\, \delta_{mm'}\, u(\mathbf{k},l,m) = u(\mathbf{k},l',m')$$

Lecture IV. Oct. 11, Tuesday

Part I. From (40) on

R(4) Eqs. (48), (49):

$$[\bar{u}(k_\mu, l_\mu), \bar{u}^*(k'_\mu, l'_\mu)] = \delta(k_\mu k^\mu + \kappa^2)$$
$$\times \delta(l_\mu l^\mu - \lambda^2)\delta(k_\mu l^\mu) \prod_\mu \delta(k_\mu - k'_\mu)\delta(l_\mu - l'_\mu)$$

$\frac{-k_4}{|k_4|}$

$$\int\cdots\int f(k_\mu l_\mu; k'_\mu l'_\mu) \, [u(\underline{k}, \Theta, \Phi), u^*(\underline{k}', \Theta', \Phi')] \, \delta(\,)\delta(\,)\delta(\,) \, \delta(k_\mu k^\mu + \kappa^2)$$
$$\times \delta(l_\mu l^\mu - \lambda^2)\delta(k_\mu l^\mu) = \prod_\mu \delta(k_\mu - k'_\mu)\delta(l_\mu - l'_\mu)$$

$$\iint (\ ) \, \delta(k_\mu k^\mu + \kappa^2)\,\delta(l_\mu l^\mu - \lambda^2)\,\delta(k_\mu l^\mu)(dk)^4 (dl)^4$$

$$\to \sum_{\underline{k}} \iint_{\Theta\,\Phi} \left(\frac{2\pi}{L}\right)^3 \frac{\lambda \sin\Theta \, d\Theta \, d\Phi}{4\kappa\sqrt{k^2+\kappa^2}}$$

$$\to \sum_{\underline{k}} \sum_{l,m} \left(\frac{2\pi}{L}\right)^3 \frac{\lambda}{4\kappa\sqrt{k^2+\kappa^2}}$$

$$\int\cdots\int f(k_\mu l_\mu; k'_\mu l'_\mu)\,\delta \times \delta\,\delta$$

$$\sum_{\substack{klm\\k'l'm'}} [u(\underline{k}, l, m), u^*(\underline{k}', l', m')] \times \left(\frac{2\pi}{L}\right)^3 \frac{\lambda}{4\kappa\sqrt{k^2+\kappa^2}}$$
$$\times \left(\frac{2\pi}{L}\right)^3 \frac{\lambda}{4\kappa\sqrt{k'^2+\kappa^2}} \quad f(\underline{k}, l, m; \underline{k}' l' m')$$
$$= \sum_{\substack{klm\\k'l'm'}} \delta(\underline{k}, \underline{k}')\,\delta(l, l')\,\delta(m, m') \times \cancel{(\ ?\ )}$$
$$\times \left(\frac{2\pi}{L}\right)^3 \frac{\lambda}{4\kappa\sqrt{k^2+\kappa^2}} f(k, l, m; k', l', m')$$

$$\int\cdots\int (dk)^4 (dl)^4 (dl')^4 (dk')^4 f(k_\mu, l_\mu; k'_\mu, l'_\mu)\,[\bar{u}(k_\mu, l_\mu), \bar{u}^*(k'_\mu, l'_\mu)]$$
$$= \int\cdots\int (dk)^4 \cdots (dl')^4 \, f(k_\mu l_\mu; k'_\mu, l'_\mu)\, \delta(k_\mu k^\mu + \kappa^2)\,\delta(l_\mu l^\mu - \lambda^2)\,\delta(k_\mu l^\mu)$$
$$\prod_\mu \delta(k_\mu - k'_\mu)\,\delta(l_\mu - l'_\mu)$$

$$\to \sum \left\{\left(\frac{2\pi}{L}\right)^3 \frac{\lambda}{4\kappa\sqrt{k^2+\kappa^2}}\right\}^2 [u(k, l, m), u^*(k', l' m')] \, f(k, l, m; k', l' m')$$
$$= \int\cdots\int (dk)^4 (dl)^4 \delta(k_\mu k^\mu + \kappa^2)\,\delta(ll)\,\delta(kl) f(k_\mu l_\mu; k_\mu l_\mu)$$

$$\int\cdots\int f(k', l') [\bar{u}(k_\mu, l_\mu), \bar{u}^*(k'_\mu, l'_\mu)] (dk')^4 (dl')^4$$

$$= \int\cdots\int \delta(k_\mu k^\mu + \varkappa^2)\delta(\ )\delta(\ ) \prod_\mu \delta(k_\mu - k'_\mu)\delta(l_\mu - l'_\mu)$$
$$\times f(k', l')(dk')^4(dl')^4$$

$$\sum_{k' l' m'} \int\int f(k'_\mu, l'_\mu) (N_{k'l'm'}) \bar{u}(k_\mu, l_\mu), u(k', \omega', \Phi') \frac{\sin\theta'}{d\omega' d\Phi'}$$
$$u(k, l, m)\, \delta(\ )\delta(\ )\delta(\ )$$

$$= f(k_\mu, l_\mu)\ \delta(kk)\delta(ll)\delta(kl)$$

$$\sum_{k'l'm'} N_{k'l'm'}\ f(k', l', m') [u(k, l, m), u^*(k', l', m')]$$

$$= \cancel{f(k_\mu}\ f(k^*, l, m)$$

Commutation Relations for nonlocal ~~spin~~ scalar field

by Dec. 2, 1949; D. R. Yennie

Local field:

$$\left[\frac{\partial U^*(x',t)}{\partial t}, U(x'',t)\right] = \left[\frac{\partial U(x',t)}{\partial t}, U^*(x'',t)\right]$$

$$= -i\hbar\,\delta(\underline{x}'-\underline{x}'')$$

etc.

$$U(x_\mu) = \int\cdots\int (dk)^4\, u(k_\mu)\exp(ik_\mu x^\mu)\,\delta(k_\mu k^\mu+\varkappa^2)$$

$$U^*(x_\mu) = \int\cdots\int (dk)^4\, u^*(k_\mu)\exp(-ik_\mu x^\mu)\,\delta(k_\mu k^\mu+\varkappa^2)$$

If we take

$$\delta(k_\mu k^\mu+\varkappa^2)\,[u^*(k_\mu), u(k'_\mu)]$$

$$= \frac{k_4}{|k_4|}\frac{\hbar}{c(2\pi)^3}\prod_{\mu=1}^{4}\delta(k_\mu-k'_\mu)$$

Then

$$[U^*(x',t), U^*(x',t)] = \int (dk)^4 \frac{k_4}{|k_4|}\frac{\hbar}{c(2\pi)^3}\delta(k_\mu k^\mu+\varkappa^2)\,\exp\cdot i\underline{k}(\underline{x}''-\underline{x}')$$

$$= 0$$

$$\left[\frac{\partial U^*(x',t)}{\partial t}, U(x'',t)\right] = -\int (dk)^4 \frac{k_4^2}{|k_4|}\frac{i\hbar c}{c(2\pi)^3}$$

$$\times\,\delta(k_\mu k^\mu+\varkappa^2)\exp i\underline{k}(\underline{x}''-\underline{x}')$$

$$= -\int (dk)^3 \frac{i\hbar}{2\cdot(2\pi)^3}\, 2\exp(i\underline{k}(\underline{x}''-\underline{x}'))$$

$$= -i\hbar$$

$$[u^*(x'_\mu), u(x''_\mu)] = \int (dk)^4 \frac{k_4}{|k_4|} \frac{\hbar}{c(2\pi)^3}$$

$$\times \exp(ik_\mu(x''_\mu - x'_\mu))\,\delta(k_\mu k^\mu + \varkappa^2)$$

$$= \frac{\hbar}{ic} \Delta(x''_\mu - x'_\mu)$$

where

$$\Delta(x_\mu) = \frac{1}{(2\pi)^3} \frac{1}{i} \int (dk)^4 \,\delta(k_\mu k^\mu + \varkappa^2)$$

$$\times \frac{k_4}{|k_4|} \exp(ik_\mu x^\mu)$$

Evaluation of $[U(X_\mu, r_\mu), U^*(X'_\mu, r'_\mu)]$

$$U(X_\mu, r_\mu) = \int\cdots\int (dk)^4 (dl)^4 \,\bar{u}(k_\mu, l_\mu) \prod_\lambda \delta(r_\mu + l_\mu) \exp(i k_\mu X^\mu)$$

$$U^*(X'_\mu, r'_\mu) = \int\cdots\int (dk')^4 (dl')^4 \,\bar{u}^*(k'_\mu, l'_\mu) \prod_\mu \delta(r'_\mu + l'_\mu) \exp(-i k'_\mu X'^\mu)$$

$$[\bar{u}(k_\mu, l_\mu), \bar{u}^*(k'_\mu, l'_\mu)] = \varepsilon(k)\,\delta(k_\mu k^\mu + \varkappa^2) \times \delta(k_\mu l^\mu)\,\delta(l_\mu l^\mu - \lambda^2)$$

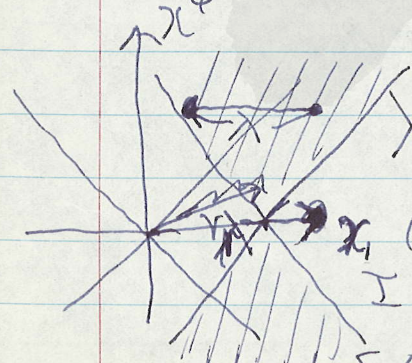
$$\varepsilon(k) \equiv \frac{\varepsilon_\mu k^\mu}{|\varepsilon_\mu k^\mu|} \quad \left(\equiv \frac{k^4}{|k^4|}\right)$$

ε_μ : time-like vector $\varepsilon^4 > 0$,

$$[U(X_\mu, r_\mu), U^*(X'_\mu, r'_\mu)]$$

$$= \int\cdots\int (dk)^4 \,\delta(k_\mu k^\mu + \varkappa^2)\,\delta(k_\mu r^\mu) \times \exp(i k_\mu (X^\mu - X'^\mu))\,\varepsilon(k)\,\delta(r_\mu r^\mu - \lambda^2) \prod_\mu \delta(r_\mu - r'_\mu)$$

$$I(X_\mu, r_\mu) = \int\cdots\int (dk)^4 \,\delta(k_\mu k^\mu + \varkappa^2)\,\delta(k_\mu r^\mu) \exp(i k_\mu X^\mu)\,\varepsilon(k)$$



$$Y_\mu = X_\mu - \frac{(r_\mu X^\mu)\, r_\mu}{\lambda^2}$$

$(Y_\mu r^\mu = 0)$ $\qquad Y_\mu Y^\mu = X_\mu X^\mu - \frac{(r_\mu X^\mu)^2}{\lambda^2}$

$$I(X_\mu, r_\mu) = I(Y_\mu, r_\mu)$$

$$\delta(k_\mu r^\mu) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(i\alpha k_\mu r^\mu)\, d\alpha$$

$I(y_\mu, \gamma_\mu) = 0$ if $y_\mu y^\mu + \alpha^2\lambda^2 > 0$
or $y_\mu y^\mu > 0$

$y_\mu y^\mu < 0$:
$$I(y_\mu, \gamma_\mu) = -2\pi i \int_0^{\pi/2} \frac{\kappa}{\lambda} J_1(\kappa\rho \cancel{\sin\theta}\cos\theta)\, d\theta$$

$$\begin{cases} \alpha = \beta/\lambda \cos\theta \\ \rho = \beta\sin\theta \end{cases} \qquad \{ \rho = \sqrt{c^2t^2 - y^2}$$

Whit., Wat. p.380. Prob. 16.
$$\int_0^{\frac{\pi}{2}} J_{2n}(2z\cos\theta)\, d\theta = \frac{1}{2}\pi \{J_n(z)\}^2$$

$$I = -\pi^2 i \frac{\kappa}{\lambda} \{J_{1/2}(\tfrac{\kappa\beta}{2})\}^2$$

$$= -\pi^2 i \frac{\kappa}{\lambda} \left(\frac{4}{\kappa\beta\pi}\right) \sin^2\left(\frac{\kappa\beta}{2}\right)$$

$$= -\frac{4\pi i}{\beta\lambda} \sin^2\left(\frac{\kappa\beta}{2}\right)$$

$$\beta^2 = -y_\mu y^\mu$$
$$= \frac{(\gamma_\mu x^\mu)^2}{\lambda^2} - x_\mu x^\mu > 0$$

$(I = 0$ for $\beta^2 < 0)$

~~γ^4~~. $\gamma_2 = \gamma_3 = \gamma_4 = 0$ $\gamma_1 = \lambda$:
$y_1 = x_1 - x_1 = 0$ $y_2 = x_2$, $y_3 = x_3$
$y_4 = x_4$.

Lecture V Oct. 13. ~ Lecture VI Oct. 18.

Part I. Nonlocal Spinor Field

Remark to IV.

Instead of introducing extra-degrees of freedom represented by $(\omega_1, \omega_2, \omega_3)$, it might be possible to adopt the following scheme:

$$\gamma^\mu [p_\mu, \psi] + m c \psi = 0$$

$$\beta^1 = \beta_1 = \rho_3 \sigma_1 = i \gamma^2 \gamma^3 \gamma^4 = \cancel{\beta_{234}}\, i\beta^{234}$$

$$\beta^2 = \beta_2 = \rho_3 \sigma_2 = i \gamma^3 \gamma^1 \gamma^4 = -i \gamma^1 \gamma^3 \gamma^4 = i \beta^{314}$$

$$\beta^3 = \beta_3 = \rho_3 \sigma_3 = i \gamma^1 \gamma^2 \gamma^4 = i \beta^{124}$$

$$-\beta^4 = \beta_4 = -i \rho_2 = -i \gamma^1 \gamma^2 \gamma^3 = i \beta^{132}$$

$$\gamma_x^1 \gamma_y^2 \gamma_z^3 \gamma_t^4 = \rho_1$$

$$\beta_\mu [x^\mu, \psi] + \lambda \psi \cancel{= 0}$$

or $$\sum i \beta^{\alpha\beta\gamma} \cancel{{}_{\alpha\beta\gamma}} [x^\delta, \psi] - \lambda \cancel{\eta}\varepsilon^{1234} \psi = 0$$

$\alpha, \beta, \gamma, \delta$: different
and the order is even permutation of $(1, 2, 3, 4)$

or $$\sum \beta^{\alpha\beta\gamma} [x^\delta, \psi] + i \lambda \cancel{\eta}\varepsilon^{1234} \psi = 0$$

where $(\varepsilon^{\alpha\beta\gamma\delta})$ is an antisymmetric tensor of the fourth rank, i.e.

$$\varepsilon^{\alpha\beta\gamma\delta} = \cancel{\pm 1} + 1 \quad \text{or} \quad -1$$

according as $(\alpha\beta\gamma\delta)$ is even or odd permutation of $(1\,2\,3\,4)$, whereas $\beta^{\alpha\beta\gamma}$ can be regarded as the antisymmetric tensor of the third rank.

~~4. By proper Lorentz transformation, the order of ~~fourth axes~~ (x_1, x_2, x_3, x_4) is retained in (x_1', x_2', x_3', x_4'), or~~ (or more generally, may differ from $(x_1' \cdots x_4')$ by an even permutation)
~~If we perform improper Lorentz transformation, the order of (x_1, x_2, x_3, x_4)~~

By ~~proper~~ Lorentz transformation,

$$\varepsilon'^{\alpha\beta\gamma\delta} = \cancel{a_{\alpha\beta\gamma\delta}}\, a_{\alpha\varkappa} a_{\beta\lambda} a_{\gamma\mu} a_{\delta\nu} \varepsilon^{\varkappa\lambda\mu\nu}$$

$$a_{11} a_{22} a_{33} a_{44} - a_{12} a_{21} a_{33} a_{44}$$

$$+ \qquad\qquad -$$

$$\mathrm{Det}(a_{\alpha\varkappa})$$

$$\varepsilon'^{\alpha\beta\gamma\delta} = \mathrm{Det}(a_{\mu\nu})\, \varepsilon^{\alpha\beta\gamma\delta}$$

Commutation Relations

between ~~part II~~. $\exp ik_\mu x^\mu$
and $\exp il^\mu p_\mu/\hbar$

$$[x^\mu, p_\nu] = i\hbar\,\delta_{\mu\nu}$$

$$\cancel{[x^\mu, (p_\nu)^n] = i\hbar}$$

$$[k_\mu x^\mu, -il^\nu p_\nu/\hbar] = \cancel{i\hbar}\, k_\mu l^\mu$$

$$[k_\mu x^\mu, (-il^\nu p_\nu/\hbar)^2] = \cancel{i\hbar}\,[k_\mu x^\mu, -il^\nu p_\nu/\hbar](-il^\nu p_\nu/\hbar)$$
$$+(-il^\nu p_\nu/\hbar)[k_\mu x^\mu, -il^\nu p_\nu/\hbar]$$
$$= 2(k_\mu l^\mu)(-il^\nu p_\nu/\hbar)$$

$$[k_\mu x^\mu, (-il^\nu p_\nu/\hbar)^n]$$
$$= n(k_\mu l^\mu)\,(-il^\nu p_\nu/\hbar)^{n-1}$$

$$[k_\mu x^\mu, \exp(-il^\nu p_\nu/\hbar)]$$
$$= (k_\mu l^\mu)\cdot\exp(-il^\nu p_\nu/\hbar)$$

$$\cancel{ex}\; k_\mu x^\mu \cdot \exp(-il^\nu p_\nu/\hbar)$$
$$= \exp(-il^\nu p_\nu/\hbar)\cdot k_\mu(x^\mu + l^\mu)$$

$$(k_\mu x^\mu)^2\cdot\exp(-il^\nu p_\nu/\hbar)$$
$$= \exp(-il^\nu p_\nu/\hbar)(k_\mu(x^\mu + l^\mu))^2$$

$$\exp(ik_\mu x^\mu)\cdot\exp(-il^\nu p_\nu/\hbar)$$
$$= \exp(-il^\nu p_\nu/\hbar)\cdot \exp ik_\mu x^\mu \cdot \exp ik_\mu l^\mu$$

$$k_\mu x'^\mu (x'|\exp(-il^\nu p_\nu/\hbar)|x'')$$
$$= (x'|\exp(-il^\nu p_\nu/\hbar)|x'')(k_\mu(x''^\mu + l^\mu))$$

$$\{k_\mu(x'^\mu - x''^\mu - l^\mu)\}(x' | \quad | x'') = 0$$

$$(x'|\exp(-il^\nu p_\nu/\hbar)|x'') = \prod_\mu \delta(x'^\mu - x''^\mu - l^\mu)$$

Not Finished

Other Authors

Concept of the Canonical Invariants
Interaction of two fields.

Limiting case of $\lambda \to 0$.

~~If we consider~~ Let us consider invariant operators such as U^*U, VV and

$$F = [p_\mu U^*][p^\mu U], \quad [x_\mu U^*][x^\mu U]$$

$$[p_\mu V][p^\mu V], \quad [x_\mu V][x^\mu V]$$

which are all Hermitian operators. So are the operators

$$[p_\mu U^*][x^\mu U] + [x^\mu U^*][p_\mu U]$$

$$i[p_\mu U^*][x^\mu \cancel{etc}] - i[x^\mu U^*][p_\mu U]$$

$$\because \quad \left([p_\mu U^*][x^\mu U]\right)^* = [x^\mu U^*][p_\mu U]$$

They are all functions of x_μ, p_μ, $b(k,\omega)$, $b^*(k,\mu)$ or of X_μ, P_μ, $b(k\omega)$, $b^*(k\omega)$. If we express each of them as a matrix in the representation, in which x_μ are diagonal, the trace, for example, of F:

$$\text{Trace } F = \int\cdots\int (x'|[p_\mu U^*][p^\mu U]|x'')(dx')^4(dx'')^4$$

is invariant with respect to any canonical transformation of the type

$$\left.\begin{aligned} x'^\mu &= S x^\mu S^{-1} \\ p'_\mu &= S p_\mu S^{-1}, \end{aligned}\right\}$$

where S is any unitary operator, which can be expressed as a function of x^μ, p_μ. Let us call such a quantity as Trace F as a "canonical invariant" in order to discriminate are not ~~canonical~~ invariant, the commut. relations are still canonically invariant.)

ordinary
it from "relativistic invariant".
(Although the field ~~eq. them~~ quantities themselves

Now we postulate that all "canonical invariants", ~~which are reciprocal with respect to respect to x^μ and p_μ, which are Hermitian~~ which are constructed from bilinear products of U and U^*, should be a constant multiple of U^*U (or UU^*).

This condition is satisfied, if the operator U satisfies the relations

$$\left.\begin{array}{l} [x_\mu[x^\mu, U]] - \lambda^2 U = 0 \\ [p_\mu[p^\mu, U]] + m^2c^2 U = 0 \\ [p_\mu[x^\mu, U]] (= [x^\mu[p_\mu, U]]) = 0 \end{array}\right\}$$

~~Whether the latter relations are~~
~~Conversely~~ Now, ~~It~~ can easily be shown that any operator U can be expanded into the form

$$U = \iint (dk)^4 (dl)^4 u(k_\mu, l^\mu) \exp(ik_\mu X^\mu) \times \delta(r_1 + l_1) \cdots \delta(r_4 + l_4);$$

so that any operator can be fully characterized by ~~$[p_\mu U]$~~ the operators coefficient $u(k_\mu, l^\mu)$. Thus the operators $[p_\mu U]$, $[x^\mu U]$ are characterized by the coefficient operators

$$k_\mu u(k_\mu l^\mu), \qquad l^\mu u(k_\mu, l^\mu).$$

The above postulate requires that only such ~~combination~~ coefficients of U with definite values of

$$k_\mu k^\mu \qquad l_\mu l^\mu \qquad k_\mu l^\mu$$

can be different from zero. In other words, U should be a solution of simultaneous operator equations

$$[x_\mu [x^\mu U]] = \Lambda \cdot U$$
$$[p_\mu [p^\mu U]] = M U$$
$$[p_\mu [x^\mu U]] = [x^\mu [p_\mu U]] = N U,$$

with arbitrary real coefficients Λ, M, N. We have shown that only when

$$\Lambda = \lambda^2 \geq 0, \quad M = -m^2 \leq 0$$
$$N = 0,$$

~~for $a' = b' =$~~ would reproduce an the quantized fields U ~~corresponds to~~ ~~the~~ assembly of elementary particles with the definite mass and radius.*

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* In general any operator U can be expressed as the superposition of fields with arbitrary ~~a', b', c'~~, Λ, M, N.
Whether further restriction with respect to the internal rotation of the elementary particle is necessary or not is an open question.

we further postulate that
Now the interaction of two or more fields ~~can~~ are also ~~be~~ characterized by the canonical invariants. For example, the interaction between scalar fields U, U^* and V can be characterized by

$\mathrm{Trace}(UVU^*)$, $\mathrm{Trace}(U^*VU)$

$\mathrm{Trace}([p^\mu U][p_\mu U^*]\cdot V)$ etc.

The reason for taking canonical invariants is ~~Then it can easily be shown~~ that as follows:
~~Now~~ First for any ~~ope~~ two operators F and G,

$$\mathrm{Trace}\,[F, G] = 0.$$

If we take $F = p_\mu$ and for any μ and G as any of the interaction operators.

$$\mathrm{Trace}\,[p_\mu, G] = 0.$$

or

$$\int\cdots\int \frac{\partial G(X, r)}{\partial X^\mu}(dX)^4(dr)^4 = 0.$$

Thus, ~~the order to two~~ if we divide the interaction operators into a product of ~~and~~ two operators in an arbitrary way, the change of order of these two factors does not alter the trace of the original operator, so ~~among such~~ for example such operators as

FGH, GHF, HFG

are equivalent to one another. Further, if U, ~~U^*~~ (and U^*) and V are commutative,

$$
\begin{array}{llll}
(\mathrm{I}) & UVU^* & U^*UV & VU^*U \\
(\mathrm{II}) & UU^*V & U^*VU & VUU^*
\end{array}
$$

are all equivalent to one another.*

Further for any operator of the form $\sum UVW\cdots$,

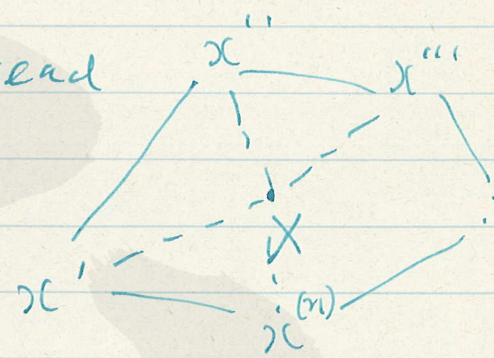
$$\mathrm{Trace}\left(\sum UVW\cdots\right) = \sum (x'|UVW\cdots|x')(dx')^4$$

$$= \sum \int\cdots\int (x'|UVW\cdots|x')(dx')^4$$

If $U, V, W \cdots$ contain the factors respectively

$$e^{ik^{(1)}\left(\frac{x'+x''}{2}\right)}, \quad e^{ik^{(2)}\left(\frac{x''+x'''}{2}\right)}, \quad \ldots \quad e^{ik^{(n)}\left(\frac{x^{(n)}+x'}{2}\right)},$$

we can integrate instead of $x', x'', \ldots x^{(n)}$, with respect to



$$X = \tfrac{1}{n}(x' + x'' + \cdots x^{(n)})$$

$$\xi^{(1)} = \cancel{X} \; x' - X$$

$$\xi^{(2)} = \cancel{X - x^{(2)}} \; x'' - X$$

$$\xi^{(n-1)} = \quad x^{(n-1)} - X.$$

* It should be noticed here that, for quantized fields U, U^*, V, the order of factors $b(k_\mu)$, $b^*(k_\mu)$ etc. can not be changed, so that the first group (I) and the second group (II) are only equivalent <u>except the order of these factors</u>.

Then ~~the~~ due to the factor

$$\int e^{i(k^{(1)}+k^{(2)}+\cdots k^{(n)})X}(dX)^4,$$

the trace vanishes unless

$$k_\mu^{(1)}+k_\mu^{(2)}+\cdots+k_\mu^{(n)}=0$$

$$\mu=1,2,3,4,$$

which ~~is~~ are just the conservation laws for total momentum and total kinetic energy. This can more easily be observed ~~by~~ from the canonical invariance of trace. Namely, the trace is invariant with respect to the displacement of the coordinate system of any amount in any direction. ~~Now~~ By the displacement of the origin ~~by~~ from O to a^μ, the trace is multiplied by a factor

$$e^{i(k_\mu^{(1)}+k_\mu^{(2)}+\cdots k_\mu^{(n)})a^\mu}$$

which should be 1 irrespective of a^μ. q.e.d.

Variation Principle in Nonlocalizable Field Theory

V.P.1

$$L = \mathrm{Trace}\left\{\frac{\lambda_0^2}{\hbar^2}[p_\mu U^*][p^\mu U] + \frac{1}{\lambda_0^2}[x_\mu U^*][x^\mu U]\right\}, \quad (1)$$

In order to apply classical variation principle, we have to consider the quantity L as an integral of functions of mutually commutative variables. For this purpose, as before, we put $(x'|U|x'') = U(X, r)$; $(x'|U^*|x'') = U^*(X, r)$. Then we obtain

$$L = \int\cdots\int (dX)^4 (dr)^4 \left\{ \frac{-\lambda_0^2}{\cancel{\hbar^2}} \frac{\partial U^*(X,r)}{\partial X_\mu} \frac{\partial U(X,-r)}{\partial X^\mu} - \frac{1}{\lambda_0^2} r_\mu r^\mu U^*(X,r) U(X,-r). \right\} \quad (2)$$

and from the variation principle

$$\delta L = 0, \quad (3)$$

Euler equations

$$\frac{\lambda_0^2}{\cancel{\hbar^2}} \frac{\partial^2 U(X,-r)}{\partial X_\mu \partial X^\mu} - \frac{1}{\lambda_0^2} r_\mu r^\mu U(X,-r) = 0 \quad (4)$$

or

$$\frac{\lambda_0^2}{\cancel{\hbar^2}} \frac{\partial^2 U(X,r)}{\partial X_\mu \partial X^\mu} - \frac{1}{\lambda_0^2} r_\mu r^\mu U(X,r) = 0. \quad (5)$$

If we assume a form

$$U(X, r) = \chi(X)\, u(r). \quad (6)$$

$$\frac{\partial^2 \chi}{\partial X_\mu \partial X^\mu} - \kappa^2 \chi = 0, \quad r_\mu r^\mu u - \lambda^2 u = 0 \quad (7)$$

$$\cancel{\kappa^2 =} \quad \lambda^2 = \frac{\kappa^2}{\cancel{\hbar^2}} \lambda_0^4 \quad (8)$$

The separation constant constant $\varkappa$ (and λ) determines the mass of the solution (and size).
We further impose the auxiliary condition

$$r_\mu \frac{\partial}{\partial x^\mu} U = 0 \qquad (9)$$

(This condition is certainly compatible with the equation (5'))

Then general solution of (5') takes the form

$$U(x, r) = \sum_{\substack{\varkappa^2=-\infty \\ \cancel{\varkappa=\text{const}} \\ k_\mu k^\mu + \varkappa^2 = 0}}^{+\infty} \iiiint \chi(x, k, \varkappa)\, u(r, k, \varkappa)\, (dk)^4$$

i) the ~~vault~~ solution for $\varkappa^2 > 0$, corresponds to the particle with the real mass m ($m = \frac{\varkappa \hbar}{c}$) and ~~size~~ real ~~size~~ radius $\lambda^2 = \varkappa^2 \lambda_0^2$ ($\lambda = \frac{mc}{\hbar}\lambda_0^2$)

ii) The solution for $\varkappa = 0$, corresponds to the particle with mass zero and indefinite size.

iii) the solution for $\varkappa^2 < 0$, corresponds to the particle with the imaginary mass and imaginary radius. *

---

* In order that any function of x, r can be expanded into series of ~~product~~ χu, we have to change the variation principle as follows: L should be extremum with the value of $\iint U^*(x, r)\, U(x, r)\, dx\, dr$ fixed. This is equivalent to choose $\varkappa$ and λ independently.

S-Matrix in Nonlocalizable Field Theory

When there are two or more fields interacting with each other, the conventional way of starting from Schrödinger equation (or any substitute for it) does not work for following reasons:

Nonlocal ~~??~~ field operators such as a scalar ~~quantity~~ U, a spinor ψ etc can be expressed as matrices with rows and columns characterized by x'_μ, $n'(k_\mu)$ etc. which are eigenvalues of operators x_μ (for all possible k_μ) $n(k_\mu) = b^*(k_\mu)\, b(k_\mu)$ etc.

Thus, it ~~is indeed possible to~~ seems possible at first to construct a probability amplitude for the system

$$\Psi(x'_\mu, n'(k_\mu), \ldots).$$

However, there is no such operator corresponding to the Hamiltonian $\overline{H}$, which appears in the Schrödinger equation

$$-\cancel{\hbar} i\hbar c \frac{\partial \Psi}{\partial x'_4} = \overline{H}\Psi.$$

More precisely, $\overline{H}$ is in general a operator of the form

$$\sum_{\substack{n''(k_\mu)\ (\text{for all possible } k_\mu) \\ x''_\mu}} (x'_\mu, n'(k_\mu), \ldots | H | x''_\mu, n''(k_\mu) \ldots)\, \Psi(x''_\mu, n''(k_\mu) \ldots)$$

Interaction of Two Fields

We consider the ~~of~~ simple case of a complex field $U(X, r)$, $U^*(X, r)$ coupled with a ~~real~~ real field $V(X, r)$.

$$L = \mathrm{Trace}\left\{ \frac{\lambda_0^2}{\hbar^2} [p_\mu U^*][p^\mu U] + \frac{1}{\lambda_0^2}[x_\mu U^*][x^\mu U] \right\}$$

$$+ \frac{1}{2}\mathrm{Trace}\left\{ \frac{\lambda_0^2}{\hbar^2} [p_\mu V][p^\mu V] + \frac{1}{\lambda_0^2}[x_\mu V][x^\mu V] \right\}$$

$$+ \varepsilon \,\mathrm{Trace}\, U^* V U$$

$$L = \int\cdots\int (dX)^4 (dr)^4 \Big\{ -\lambda_0^2 \frac{\partial^* U(X, r)}{\partial X_\mu} \frac{\partial U(X, -r)}{\partial X^\mu}$$

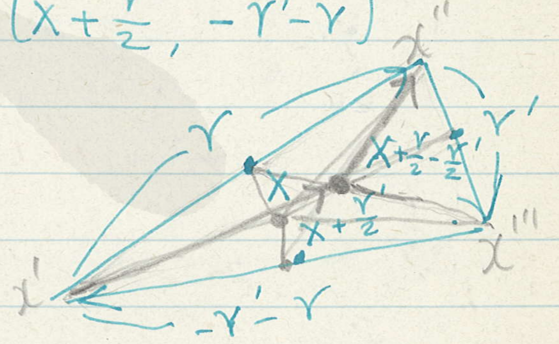
$$- \frac{1}{\lambda_0^2} r_\mu r^\mu U^*(X, r) U(X, -r) - \frac{1}{2}\lambda_0^2 \frac{\partial V(X, r)}{\partial X_\mu} \frac{\partial V(X, -r)}{\partial X^\mu}$$

$$- \lambda_0^2 r_\mu r^\mu V(X, r) V(X, -r) \Big\}$$

$$+ \varepsilon \int\cdots\int (dX)^4 (dr)^4 U^*(X, r) \int (dr')^4$$

$$\times V\left(X + \frac{r}{2} - \frac{r'}{2}, r'\right) U\left(X + \frac{r'}{2}, -r' - r\right)$$

$$\frac{\partial(x', x'', x''')}{\partial(X, r, r')} = 1.$$



$$\lambda_0^2 \frac{\partial^2 U(X, -r)}{\partial X \partial X} - \frac{1}{\lambda_0^2} r r U(X, -r)$$

$$+ \varepsilon \int (dr')^4 V\left(X + \frac{r}{2} - \frac{r'}{2}, r'\right) U\left(X + \frac{r'}{2}, -r' - r\right)$$

$$= 0$$

$$\lambda_0^2 \frac{\partial^2 U^*(X, r)}{\partial X \partial X} - \frac{1}{\lambda_0^2} r r U^*(X, r)$$

$$+ \varepsilon \int (dr')^4 U^*\left(X - \frac{r}{2} - \frac{r'}{2}, r'\right) V\left(X + \frac{r'}{2}, -r' + r\right)$$

$$= 0$$

$$\lambda_0^2 \frac{\partial^2 V(X, r)}{\partial X \partial X} = \frac{1}{\lambda^2} r r\, V(X, r)$$

$$+ \varepsilon \int (dr')^4\, U^*\left(X + \frac{r'}{2}, -r' + r\right) U\left(X - \frac{r}{2} - \frac{r'}{2}, r'\right)$$

$$= 0$$

Now, for free fields U and V, which satisfy the equations considered above, the eigenfunctions $u_k(X, r)$, $u^*_{k'}(X, r)$ have the orthogonality properties:

$$\iiiint u^*_{k'}(X, r)\, u_k(X, r)\, (dX)^4 (dr)^4$$

$$= 0 \qquad \text{for } k' \neq k.$$

Hence

$$\mathrm{Trace}\, [p_\mu U^*][p^\mu U] = \sum_k \underbrace{\mathrm{Trace}\, [p_\mu, u_k^*][p^\mu, u_k]}_{a^*(k_\mu) b(k_\mu)}$$

In the similar way, other terms can be written ~~as~~ in the form $\sum_k \ldots$ Thus L for free field is 0, provided that U, U^* are sums of solution of equations considered at previously.

$*$ $\mathrm{Trace}\, [p_\mu U^*][p^\mu U] = -\mathrm{Trace}\, U^* [p_\mu [p^\mu U]]$

$= -\mathrm{Trace}\, [p_\mu [p^\mu U^*]] U$

Substitute for Schrödinger Eq.

→ S-Matrix *

$$i\frac{\partial\Psi(X_4)}{\partial X_4} = \int\!\!\int\!\!\int H\,dX_1dX_2dX_3\,\Psi(X_4)$$

$$\Psi(X_4+dX_4) = \left(1+i\int_{X_4}^{X_4+dX_4}\int\cdot\int H(dX)^4\right)\Psi(X_4)$$

$$\Psi(X_4+2dX_4) = \left(1+i\int_{X_4+dX_4}^{X_4+2dX_4}\int\cdots\int H(dX)^4\right)\Psi(X_4+dX_4)$$

$$\Psi(+\infty) = \Big\{1+\int\cdots\int H(dX)^4$$

$dx^{(j)}$

$$+(i)^2\int\cdots\int_{X_4^{(j)}>X_4^{(i)}} H(X_\mu^{(j)})\,H(X_\mu^{(i)})\,(dX^{(j)})^4(dX^{(j)})^4$$

$$+\int\cdots\int \cancel{H(X_\mu^{(i)})}$$

$dx^{(i)}$

$$+\cdots$$

$$= 1+i\int\cdots\int H(dX)^4 + \frac{1}{2}(i)^2\int\cdots\int P(H^{(j)},H^{(i)})\,(dX^{(i)})^4(dX^{(j)})^4$$

$$+\cdots$$

$$\to\ 1+\mathrm{Trace}(iH)+\frac{1}{2}\mathrm{Trace}\langle iH;iH\rangle^{+}$$

$$+\cdots$$

---

* Dyson, Phys. Rev. 75 (1949), 486.

Trace $\langle iH(x);iH(x')\rangle$ = Dia Sum of $\begin{cases} H(X_4)\cdot H(X_4') & \text{for } X_4>X_4' \\ \text{and } H(X_4')H(X_4) & \text{for } X_4'<X_4 \end{cases}$

$$S = 1 + \mathrm{Trace}(iL)$$
$$+ \mathrm{Trace}\,\tfrac{1}{2}(iL)(iL)$$
$$+ \cdots$$

Trace $L = 0$ for free fields.

Trace $(iL)(iL)$? for free fields

$L \Rightarrow$ Lagrangian density in classic field theory.

$$L\,(x_\mu,\ p_\mu,\ b(k_\mu),\ b^*(k_\mu) \\ \cdots\ a(k_\mu)\ a^*(k_\mu) \cdots\ )$$

$$\frac{\partial U}{\partial X^4} \rightarrow [p^4, U]$$

$$H = \frac{\partial L}{\partial [p^4, U]}[p^4 U] - L$$

$$i\hbar \frac{\partial \Psi}{\partial X^4} = \iint H\, dX_1 dX_2 dX_3 \\ dV_1 dV_2 dV_3 dV_4\, \Psi$$

$$W = \underline{\sum_n \frac{(i/\hbar)^n}{n!} \underline{\text{Trace}}\, L^n}$$

$$\text{Trace}\, L^2 = \iint_{x'x''} (x'|L|x'')(x''|L|x')$$

$[\quad][\quad]$

x'

space like)
time like)

$W = \exp iS/\hbar \qquad S = \text{const.}\, L$

$(\cancel{x'}, n'(k') \mid W \mid x'', n''(k''))$

L

on another

on one space like surface

Trace → three dimensional infinite integral, ~~time~~ finite integral in time direction

In order that,

$$\mathrm{Trace}\, L \neq 0$$

if we add interaction, the ~~latter~~ interaction Lagrangian should contain real transition term, or in the first approximation

$$W = 1 + \frac{i}{\hbar}\,\mathrm{Trace}\, L$$

In the second approximation

$$W = 1 + \frac{i}{\hbar}\,\mathrm{Trace}\, L + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 (\mathrm{Trace}\, L)^2$$

which should be changed to

$$W = 1 + \frac{i}{\hbar}\,\mathrm{Trace}\, L + \frac{1}{2}\left(\frac{i}{\hbar}\right)^2 \mathrm{Trace}\, L^2$$

In general, instead of

$$W = \exp\left(\frac{i}{\hbar}\,\mathrm{Trace}\, L\right)$$

$$W = \sum_n \frac{1}{n!}\left(i/\hbar\right)^n \mathrm{Trace}\,(L^n)$$

$$\tilde{W} = \sum_n \frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n \mathrm{Trace}\,(L^n)$$

$$W\tilde{W} = 1 + \frac{4}{1}\left(\frac{i}{\hbar}\right)^2 \mathrm{Trace}\, L^2 - \left(\frac{i}{\hbar}\right)^2 (\mathrm{Trace}\, L)^2$$

Fundamental Equation of ~~Lagrangian~~ nonlocalizable Field Theory.

classical point dynamics:

$$H = \sum_i \dot{q}_i p_i - L(q, p). \qquad (1)$$

quantum mechanics

$$i\hbar \frac{\partial \Psi(q,t)}{\partial t} = H(q, -i\hbar\frac{\partial}{\partial q})\Psi \qquad (2)$$

or

$$i\hbar \frac{\partial \Psi}{\partial t} = -i\hbar \frac{dq}{dt}\frac{\partial \Psi}{\partial q} - L\Psi \qquad (3)$$

$\cancel{Hq+}$ $qH + Hq$

$$i\hbar \frac{D\Psi}{Dt} = -L\Psi, \qquad (4)$$

If we consider Ψ is a function of q's which change ~~according~~ to with time, and Ψ do not depend explicitely on t, instead of considering Ψ as a function of q and t,* wave equation takes the form

$$L\Psi = 0. \qquad (5)$$

In nonrelativistic quantum mechanics for a particle with the Lagrangian

$$L = \frac{1}{2m}p^2 - V(q)^{**}, \qquad (6)$$

~~This is eq.~~ (5) is

* This means to go over from Schrödinger to Heisenberg representation.

** $L = \frac{dq}{dt}p - \frac{1}{2m}p^2 - V(q).$

$$-i\hbar\frac{\partial\Psi'}{\partial t}+\frac{\hbar}{2m}\frac{\partial\Psi'}{\partial q^2}=V\Psi'=$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial q^2}-V\Psi=0, \qquad (7)$$

where q depends on time according to

$$\frac{d^2q}{dt^2}=-\frac{\partial V}{\partial q}.$$

For instance,

$$V=\frac{1}{2}\omega^2q^2$$

$$\frac{d^2q}{dt^2}=-\omega^2q \qquad \frac{dq}{dt}=\omega q_0\cos(\omega t+\varepsilon)$$

$$q=q_0\sin(\omega t+\varepsilon).$$

$$\cancel{\frac{d\Psi}{dq}}\ \frac{d\Psi}{dt}=\frac{dq}{dt}\frac{\partial\Psi}{\partial q}=\sqrt{\omega^2q_0^2-2V}\cdot\frac{\partial\Psi}{\partial q}.$$

$$\frac{d^2\Psi}{dt^2}=\left(\frac{dq}{dt}\right)^2\frac{\partial^2\Psi}{\partial q^2}+\frac{d^2q}{dt^2}\frac{\partial\Psi}{\partial q}$$

$$=(\omega^2q_0^2-\omega^2q^2)\frac{\partial^2\Psi}{\partial q^2}-\omega^2q\frac{\partial\Psi}{\partial q}$$

$$\frac{\partial^2\Psi}{\partial q^2}=\frac{\frac{d^2\Psi}{dt^2}-\frac{d^2q}{dt^2}\frac{\partial\Psi}{\partial q}}{\left(\frac{dq}{dt}\right)^2}$$

$$=\frac{\frac{d^2\Psi}{dt^2}-\frac{d^2q}{dt^2}\frac{d\Psi}{dt}\Big/\frac{dq}{dt}}{\left(\frac{dq}{dt}\right)^2}$$

classical field theory:

$$\overline{H}_{(\psi,\pi)} = \int \frac{\partial\psi}{\partial t}\pi\, dx - \int L\, dx$$

$$i\hbar \frac{\partial \Psi[\psi(x), t]}{\partial t} = \overline{H}\left(\psi, \pi \to -i\hbar\frac{\delta}{\delta\psi}\right) \Psi[\psi(x), t]$$

π ~~=~~

$$i\hbar \frac{\partial \Psi}{\partial t} + i\hbar \int \frac{\partial\psi}{\partial t}\frac{\delta\Psi}{\delta\psi} = -\overline{L}\,\Psi$$

$$\overline{L} = \int L\, dx$$

$$i\hbar \frac{d\Psi}{dt} = -\overline{L}\,\Psi,$$

$$i\hbar \cancel{\frac{d\Psi}{dt}} \quad \overline{L}\,\Psi = 0$$

(In the above consideration, Lagrangian should contain $\dot{q}$, p in the combination $\dot{q}p$.)

In nonlocalizable field theory, we start from

$$L\,\Psi = 0,$$

where L is the Lagrangian operator and Ψ is a function of x_μ', as well as the field variables $u'(k)$.*

$$* \sum_{u'' x''} L\,(u'(k)\, x_\mu' \mid \cdots u''(k) \cdots x_\mu'' \cdots)\, \Psi(u''(k) \cdots x_\mu'' \cdots) = 0$$

For ~~free~~ scalar field $\underbrace{}$ $U(\eta)$, depending only on η

$$L = \frac{1}{2}[\zeta_\mu U][\zeta^\mu U]$$

$$= \frac{1}{2}\Big[\sum_{k_\mu(k_0>0)} (2k_0)^{-1/2}\Big\{ (-2k_\mu)\, b(k_\mu) \exp(ik_\mu \eta^\mu)$$

$$\mp b^*(k_\mu)(2k_\mu) \exp(-ik_\mu\eta^\mu)\Big\}\Big]$$

$$\times \Big[\sum_{k'_\mu(k'_0>0)} (2k'_0)^{-1/2}\Big\{(-2k'_\mu)\, b(k'_\mu) \exp(ik'_\mu\eta^\mu)$$

$$+ (2k'_\mu)\, b^*(k'_\mu) \exp(-ik'_\mu \eta^\mu)\Big\}\Big]$$

$$= \sum_{k_\mu = k'_\mu}$$

Thus L has nondiagonal elements in space-time (as well as momentum-energy) representation, so that Trace $L = 0$, which can also be obvious from the fact*

$$\text{Trace } [\zeta_\mu U]\{\zeta^\mu U]$$

$$= \text{Trace}[[\zeta_\mu U]\zeta^\mu]\cdot U = 0$$

on account of field equation.

---

* The reason for choosing the above L instead of $\frac{1}{2}[[\zeta_\mu U]\zeta^\mu]U$ is ~~not~~ only for closer correspondence with the ordinary field theory.

If there ~~is some~~ are two kinds of scalar fiels U and V

$$L = \frac{1}{2}[\partial_\mu U][\partial^\mu U] + \frac{1}{2}[\partial_\mu V][\partial^\mu V]$$

$$+ gVUV$$

~~Application of~~
Conventional method:

$$\frac{\partial \Psi(x^{\mu'} \ldots)}{\partial x^{\mu}} = \frac{1}{\Delta}\{\Psi(\ldots x^{\mu'}+\Delta \ldots) - \Psi(\ldots x^{\mu'} \ldots)\}$$

$$= \frac{1}{\Delta}\{e^{i p_\mu \Delta/\hbar} - 1\}\Psi$$

$$\Psi(\ldots x^{\mu'}+\Delta \ldots) = e^{i\bar{H}\Delta} \cdot \Psi(\ldots x^{\mu'} \ldots)$$

$$e^{i p_\mu \Delta/\hbar}\Psi = e^{i\bar{H}\Delta}\Psi$$

$$\Psi = e^{-i p_\mu \Delta/\hbar} e^{i\bar{H}\Delta}\Psi$$

$\Delta \to 0$,

$$(-i p_\mu/\hbar + \bar{H})\Psi = 0$$

$$\bar{H} = \mathrm{Trace}\, H$$

$$H = \sum \dot{q}\frac{\partial L}{\partial \dot{q}} - L$$

Chang, Relativistic Field Theories (Phys. Rev. 75 (1949), 967)

S : any space-like surface with the curvilinear coordinates u.

$\langle q(u), S|\rangle$: Schrödinger functional on it.

Schrödinger functional

$$\hbar i\{\langle q(u), S'|\rangle - \langle q(u), S|\rangle\} = J\langle q(u), S|\rangle$$

$$S' : x_\mu = b'_\mu(u) = b_\mu(u) + \Delta x_\mu(u)$$

$$S : x_\mu = b_\mu(u)$$

J : Hermitian operator operating on $q(u)$ of Ψ and may be looked upon as ~~a~~ a matrix with both rows and columns labelled by functions $q(u)$

Chang's general formalism of relativistic field theory can be further extended by considering S as a ~~p~~ surface in 8-dimensional (quantized) phase space instead of space-like surface in 4-dimensional space-time world. Namely, if take S as a surface

$$\eta_4 = \text{const},$$

field quantities can be ~~or~~ regarded as ~~fun~~ operators with $\{\eta_1, \eta_2, \eta_3\}$ $\{u_1, u_2, u_3\}$ as parameters and the Schrödinger functional

$$\langle q(\eta_1 \eta_2 \eta_3)\ (u_1 u_2 u_3), S | \rangle \equiv \Psi$$

satisfies Schrödinger equation

$$\hbar \Delta \Psi = \bar{G} \Delta \Psi$$

with

$$\bar{G} = \int G_\mu \Delta\eta^\mu . du$$

$$G_\mu = L N_\mu - \sum_\alpha p_\alpha q_\mu^\alpha$$

$$p_\alpha = N_\mu \left(\frac{\partial L}{\partial q_\mu^\alpha}\right) \qquad q_\mu^\alpha = \frac{\partial q^\alpha}{\partial \eta_\mu}$$

$$N_{\mu} = \varepsilon_{\lambda\mu\nu\kappa} \frac{\partial \eta_\lambda}{\partial u_1} \frac{\partial \eta_\mu}{\partial u_2} \frac{\partial \eta_\nu}{\partial u_3}$$

$$G_\mu \Delta\eta^\mu \rightarrow L\,(N_\mu \Delta\eta^\mu) - \underline{\sum_\alpha p_\alpha (q_\mu^\alpha \Delta\eta^\mu)}$$

$$\left(x' \left| \overset{n_k', \, n_k''}{U} \right| x''\right) \Psi(n_k'', x'') = 0$$

$$J = U^* V U.$$

$$J = e\psi^\dagger \gamma^\mu A_\mu \psi$$

$$U[\zeta_\mu U^\mu] \pm U^\mu[\zeta_\mu U]$$

$$\left.\begin{aligned}&[\zeta_\mu U^\mu] = -\kappa U \\ &[\zeta^\mu U] = \kappa U^\mu\end{aligned}\right\}$$

$$\left.\begin{aligned}&[\zeta_\mu U^\mu] + [\zeta_5 U^5] = 0 \\ &[\zeta^\mu U^5] - [\zeta_5 U^\mu] = 0\end{aligned}\right\}$$

$$[\zeta^\mu[\zeta_\mu U^{\mu\nu}]] = -\kappa^2 U^\nu$$

$$\left.\begin{aligned}&[\zeta_\mu U_\nu^*] - [\zeta_\nu U_\mu^*] = \kappa U_{\mu\nu} \\ &[\zeta_\mu U^\mu] + [\zeta_5 U^5] = 0 \\ &[\zeta_\mu U_5] - [\zeta_5 U_\mu] = \kappa U_{\mu 5} \\ &[\zeta^\mu U_{\mu\nu}] + [\zeta^5, U_{5\nu}] = 0\end{aligned}\right\}$$

$$= \kappa U_\nu^*$$

λ-limiting Process

W. Pauli, Rev. Mod. Phys. 15 (1943), 175
J. M. Jauch, ~~Phys.~~ Phys. Rev. 63 (1943), 334.
W. Pauli, ~~Rev. Mod. Phys.~~ Phys. Rev. 64 (1943), 332

~~General~~

1. Defects of the λ-limiting process
   i) Positron theoretical self-energy, which cannot be convergent even with the additional assumption of negative energy photon
   ii) Anomalous magnetic moments of nucleons due to vector and pseudo-scalar mesons ~~to~~ have opposite sign.
   (this can be remedied by considering pseudovector mesons as pointed out by Araki, Prog. Theor. Phys. 4 in press (1949).)

2. Similarity of λ-limiting process with nonlocalizable field

λ-limiting:

$$[A_\mu(X), A_\nu(X')] = \frac{-i}{2}\delta_{\mu\nu}[D(X-X'+\lambda) + D(X-X'-\lambda)$$

$$\cancel{\lambda_0^2 - \lambda^2}$$

$$\lambda_\mu \lambda^\mu < 0.$$

$$A_\mu = V^{-\frac{1}{2}} \Sigma_k (2k_0)^{-1/2} [a_\mu(k, x_0) \exp(ikx)$$
$$+ a_\mu^*(k, x_0) \exp(-ikx)]$$

$$[a_\mu(k, x_0), a_\nu^*(k', x_0)] = \delta_{\mu\nu} \delta_{kk'}$$

$$\cos(\lambda_0 k_0 - \lambda k)$$

~~nonlocalizable field:~~

$$A_\mu = V^{-\frac{1}{2}} \Sigma_k (2k_0)^{-1/2} [a_\mu(k) \exp(i\lambda k^\mu p_\mu/\hbar)$$
$$\exp(ik_\mu x^\mu/\lambda)$$

$$+ a_\mu^*(k) \exp(-i\lambda k^\mu p_\mu/\hbar)$$
$$\exp(-ik_\mu x^\mu/\lambda)]$$

$$a^{\nu *}$$

$$[a_\mu(k), a_\nu^*(k')] = \delta_{\mu\nu} \delta_{kk'}$$

$$\Sigma (2k_0)^{-1/2}$$

$$[A_\mu(x), A_\nu(x')] = \Sigma' a_\mu(k) a_\nu^*(k') e \cancel{(i\lambda k^\mu p_\mu/\lambda)}$$

$$\exp(i\lambda k^\mu x^\mu/\lambda) \qquad \exp(-i\lambda k'^\mu x^\mu/\lambda)$$

$$[\exp i\lambda(k^\mu - k'^\mu) p_\mu/\hbar - \exp(-i\lambda(k^\mu - k'^\mu) p_\mu/\hbar)]$$

$$\exp + i\lambda k^\mu x^\mu/\lambda \qquad \exp(-i\lambda k'$$

$$(x' | [A_\mu, A_\nu] | x'') = a_\mu(k) a_\nu^*(k')$$

$$(x' | a_\mu^{(*)} \exp \cdot \exp | x'') = \delta(r_\mu + \lambda k_\mu) e^{ik_\mu X^\mu}$$

$$(x' | a_\mu^* \exp \exp) x'') = \delta(r_\mu - \lambda k_\mu) e^{-ik_\mu X^\mu}$$

λ-limiting Process

Phys. Rev. 63 (1943), 334.
J. M. Jauch, Meson Theory of the Magnetic Moment of Proton and Neutron

$\mu_p = +2.785 \pm 0.02$

$\mu_n = -1.935 \pm 0.02$

second order: $\int_0^\infty \frac{k^4}{(k^2+\mu^2)^2} dk \xrightarrow[\lambda\text{-process}]{} \int k^{2\cdots} dk$: convergent

higher order: $\int_0^\infty R(k_0, k)\, dk$ R: rational even fn of k.

$\xrightarrow[\text{photon}]{\text{neg.}} \frac{1}{2}\int_0^\infty [R(k_0,k) + R(-k_0,k)]\, dk$

$$\int \left(\frac{k}{k_0}\right)^4 \cos k_0 \lambda_0\, dk = \overset{0}{\overset{\prime\prime}{\int \cos k_0\lambda_0\, dk}}$$

$$- \int \frac{k_0^4 - k^4}{k_0^4} \cos k_0 \lambda_0\, dk$$

$$= -\frac{3\pi}{4}\mu.$$

Araki, Prog. Theor. Phys. 4, Nr. 2, 1949

S-N 1

S-Matrix and the Nonlocalizable Field

§1. General Considerations

S-Matrix scheme:

Heisenberg: I, Z.S. f. Phys. 120 (1943), 513.
II, ibid. 673
III, ibid 123 (1944), 93.

Møller: I, D. Kgl. Danske Vidensk. Selskab, Math.-fys. Medd. XXIII, Nr. 1 (1945)
II, ibid. XXII. Nr. 19 (1946)

Stueckelberg: Nature 153 (1944), 143.

Wentzel: Rev. Mod. Phys. 19 (1947), 1.

Dyson: I, Phys. Rev. 75 (1949), 486
II

Pauli: Rev. Mod. Phys. (1949)

S-matrix scheme proposed by Heisenberg and considered in detail by Møller is by itself only a framework, in which various types of theory of elementary particles can be put. ~~Stueckelberg and Dyson have shown that Schwinger, Tomonaga theory field theory can could be naturally fitted into this scheme.~~ Stueckelberg has shown ~~that~~ by starting from Schrödinger eq. for ~~the~~ interacting field that $\Psi(t) = e^{\alpha(t)}\Psi(-T)$ could be determined ~~su~~ by $\alpha(t)$ in successive approximation in powers of the coupling constant and that we could forget the ~~start~~ initial Schrödinger eq. ~~so as~~ in order to change $\alpha(t)$ the form of $\alpha(t)$ so as to be free from ~~singularity~~ ~~diffic~~ divergence difficulties. As shown by Dyson, this ~~corres~~ can be achieved in q. current q.e.d. by the procedure of charge and mass renormalizations, or by Pauli, the procedure becomes more satisfactory by ~~tak~~ introducing "regulators".

The obvious drawback ~~of~~ common to these attempts is the lack of "initial system" ~~from~~ where we ~~must~~ can start. In order words, some change in the "system" or the method of dealing with it had to be itself introduced ad hoc in the course of calculations, ~~but~~ not from the beginning but. If we confine our attention to q.e.d., ~~one~~ we may say that this is not a substancial change, but only a "reinterpretation" of the same object. However, if we want to ~~consider~~ construct a more general ~~field~~ theory of elementary particles including various kinds of mesons, we find that the necessary change is more than ~~a "reinterpretation"~~ formal. Here we recognize ~~the~~ again the necessity of a ~~better~~ starting point.

Now the restriction that the field quantities must be ~~always a~~ attached to each point of the space-time world does not seem ~~to~~ to be absolutely necessary, since ~~we can~~ obtain various type of "nonlocalizable" fields, at least in vacuum," starting from ~~a~~ simple relations between ~~spa~~ field quantities, ~~and~~ space-time coordinate operators ~~as well as between the four~~ and momentum-energy operators.

(S-N 7)

Certainly, one of the most difficult questions of nonlocal field theory is to find a substitute for Schrödinger equation*, ~~in order to deal with~~ which has been necessary for ~~the~~ dealing with the interaction between fields. ~~According to~~ This is because the Hamiltonian for nonlocal field theory, even if it exist, is very probably not a quantity referred to one time instant t. However, we can still assume ~~an~~ the existence of integral equation of the form†:

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = \int H(t, t')\, \Psi(t')\, dt \qquad (1)$$

Now the time instant t' may ~~either~~ be either earlier than t $(t'<t)$ or later $(t'>t)$.

Feynman-Dyson's way of solving Tom.-Schw. equation ~~was to solve T.-S. Eq~~ guaranteed automatically ~~the~~ ~~causal sequence~~ the requirement of causality in that the Schrödinger function $\Psi(t)$ at time t is determined solely by the form of $\Psi(t')$ at a instant t' earlier than t, provided that the form of $H(t')$ is given up to t'

† For simplicity, we consider only ~~we~~ the one-time formalism. Essentially the same argument can be made ~~for~~ in the case of ~~many~~ super-many-time formalism.

* ~~Here~~ the word "Schrödinger equation" is used in a very general sense, so as to include the equations ~~Dirac's many-time~~ for quant. elect. in Dirac's many-time formalism as well as ~~that~~. T.S. equation.

In nonlocalizable field theory, we can not easily decide whether the contribution of $\Psi(t')$ for $t'>0$ to the integral of the right hand side of (1) is to be discarded or not.

~~But if the theory is so constructed that the contribution of from $\Psi(t')$ is small except for small values of $|t'-t|$, the S-matrix~~

So we had better for the time being retain all the contribution from the future as well as from the past lest ~~in order that~~ the rel. invariance should ~~may not~~ be destroyed. Then, each $\Psi(t')$ on the right hand side of (1) is in turn connected with $\Psi(t'')$ of different time instant t'', which may either be earlier or later than t', by means of the ~~same~~ Schrödinger eq. of the same ~~form~~. ~~We make here an assumption that~~

If we repeat such a procedure infinitely many times, $\Psi(t')$ appearing on the right hand side of the iterated equation, will have the argument $t' = -\infty$ or $+\infty$, or t' referred to infinitely distant space-like points.*

If we go over to the limit $t \to +\infty$, the relation corresponding to

$$\Psi(+\infty) = S^{(+\infty,-\infty)}\Psi(-\infty),$$

will take the form

S, N 3

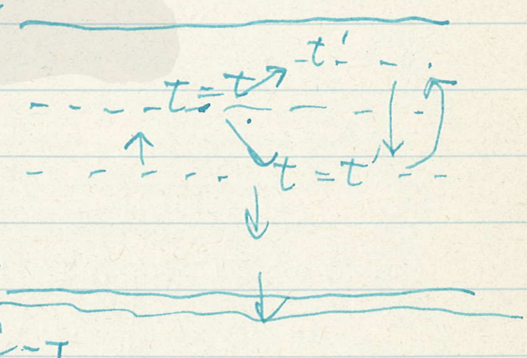
$$\Psi(C_{+\infty}) = \int_{C_{-\infty}} S(C_{+\infty}, C_{-\infty})\, \Psi(C_{-\infty}),$$

~~P_∞ is a point at $t=+\infty$ and~~

where C_∞ is a three dimensional closed surface,

More specifically, one can start in this way: ~~whenever we start from~~ We imagine two surfaces $t=+T$ and $t=-T$ for a given large positive value of T.

Whenever the argument $t = T$ t' of ~~arbitrary one term~~ $\Psi(t')$ appearing in the right hand side of the iterated equation becomes larger than $+T$ or smaller $t=-T$ than $-T$, we stop further iteration of this particular $\Psi(t')$ $(|t'| \geq T)$ and ~~only stops~~ continue ~~the iterate~~ procedure of iteration for other ~~term~~ $\Psi(t')$ with $|t'| < T$. Then we get an equation of the form



$$\Psi(t) = \int_{C_{-T}} S(t, C_{-T})\, \Psi(C_{-T}) + \int_{C_{+T}} S(t, C_{+T})\, \Psi(C_{+T})$$

where the form of S ~~will~~ must depend on t and $-T$. and $C_{\pm T}$ is a surface very near to $t=\pm T$ provided that the theory is nearly localizable. ~~Strictly speaking~~, $C_{\pm T}$ may not be a space-like surface microscopically, but it will be so to speak "macroscopically" or roughly space-like surface.

(i) ~~Further~~ Now if we ~~move~~ increase T to infinity, we get a relation

$$\Psi(t) = \int S(t, C_{-\infty})\,\Psi(C_{-\infty}) + \int S(t, C_{+\infty})\,\Psi(C_{+\infty}),$$

which means that the state of the system at a time instant t is determined, ~~by means of~~ with given $S(t, -\infty)$ and $S(t, +\infty)$, by the states of ~~$\Psi(-\infty)$~~ the system $t = -\infty$ and $t = +\infty$.

(ii) Alternatively, if we first increase t so as to be equal to $+T$ and then take the limit T to infinity, then we get a relation

$$\Psi(C_{+\infty}) = \int S(C_{+\infty}, C_{-\infty})\,\Psi(C_{-\infty}) + \int S(C_{+\infty}, C_{+\infty})\,\Psi(C_{+\infty})$$

If we further define an operator R

$$\left.\begin{aligned} R(C_{+\infty}) &= S(C_{+\infty}, C_{+\infty}) - 1 \\ R(C_{-\infty}) &= S(C_{+\infty}, C_{-\infty}) \end{aligned}\right\},$$

the above relation can be written as

$$\int R(C)\,\Psi(C) = 0. \quad (\mathrm{I})$$

or $\int R(C_{+\infty})\,\Psi(C_{+\infty}) = -\int R(C_{-\infty})\,\Psi(C_{-\infty})$ (II)'

S.N 4

Of course, the ~~po~~ limiting form of S or T at infinity will depend on the process of limiting. However, we can conversely postulate that in the correct theory S or R is uniquely determined from the ~~field~~ relations between field operators ~~and those~~ space-time and momentum-energy operators.

Although ~~Now~~ in contrast to current field theory, the state of the system ~~is not~~ at a time instant t ~~in~~ depends not only on the state in remote past at t=-∞, but also on that ~~at t=~~ in remote future, still ~~we can~~ if we go over to the limit t=+∞, $\Psi(C_{+\infty})$ can be determined by solving an equation of the form (II), provided that both $S(C_{+\infty}, C_{-\infty})$ and $S(C_{+\infty}, C_{+\infty})$ are given.

Thus one of the most important problem of nonlocalizable field theory is to find first ~~the~~ general ~~character~~ properties of S-operator and then to find the way of ~~obtaining S-o in~~ determining the actual form of S-operator.

As to the general properties of S-operators, ~~extensive~~ investigations ~~were~~ made by Møller ~~as~~ an extension of Heisenberg's considerations. However, they are concerned essentially with $S(C_{+\infty}, C_{-\infty})$ and not with $S(C_{+\infty}, C_{+\infty})$, the latter being a new ~~few~~ operator important in nonlocalizable field theory.

---

* There will be terms other than that considered ~~in the text~~ above, which, as the result of reiteration, ~~do neither go~~ tend neither to time-like infinity, nor space-like infinity. ~~Actually~~ Actually, we can expect that, ~~w~~ after by repeated multiplication of field quantities, at some stage $\Psi(t)$ on the right-hand side. ~~This gives rise to an ambiguity as to whether we go further continue~~ Once such a term appears, ~~it will never tend to~~ at least some term will ~~go~~ remain for ever even if we repeat the iteration for infinitely many times. So we had better stop ~~the~~ further iterations for this term and ~~regard~~ replace it to the left hand side. This is a general kind of renormalization process. We can expect that in nonlocalizable field theories, the renormalization factors will not be infinite.

§2. S-matrix for ~~a~~ simple nonlocalizable fields

(S, N5)

As a very simple example, we consider two kinds of scalar fields interacting in a simplest possible way. As shown previously[1], set of eq.

$$[x^\mu, p_\nu] = i\hbar\delta_{\mu\nu}$$
$$[x_\mu, [x^\mu, U]] = 0 \qquad (I)$$
$$[p_\mu, [p^\mu, U]] = 0$$

has a solution of the type*

$$U(x_\mu, r_\mu) = \sum_{k^0>0} (2k^0)^{-1/2}\{b(k_\mu)\delta(r_\mu+\lambda k_\mu)$$
$$\cdot\exp(ik_\mu x^\mu/\lambda) + b^*(k_\mu)\delta(r_\mu-\lambda k_\mu)\cdot\exp(-ik_\mu x^\mu/\lambda)\} \quad (1)$$

~~The eqs~~ We consider another complex ~~for~~ scalar field V (with the mass) satisfying the relations[2]

$$\left.\begin{array}{l}[x^\mu[x_\mu, V]] = ~~0~~ -l^2 V \\ [p_\mu[p^\mu, V]] = ~~0~~ -\kappa^2 V,\end{array}\right\} \quad (II)$$

$\kappa = \frac{mc}{\hbar}$, $m = \frac{\hbar\kappa}{c}$

or

$$\left.\begin{array}{l}(r_\mu r^\mu + l^2)V(x_\mu, r_\mu) = 0 \\ \left(\frac{\partial^2}{\partial x_\mu \partial x^\mu} - \kappa^2\right)V(x_\mu, r_\mu) = 0\end{array}\right\} \quad (II)'$$

the first relation or

$$V(x_\mu, r_\mu) = \delta(r_\mu r^\mu + l^2),$$

---

(1) Yukawa, Prog. Theor. Phys. 3 (1949), 452.

(2) This case was considered by Pais, unpublished.

* More generally

$$U(x_\mu, r_\mu) = \sum_{k^0>0} (2k^0)^{-1/2}\{b(k_\mu)\delta(r_\mu+\Lambda k_\mu)$$
$$\cdot\exp(ik_\mu x^\mu/\Lambda) + b^*(k_\mu)\delta(r_\mu-\lambda k_\mu)\exp(-ik_\mu x^\mu/\lambda)\}$$

This means that those points ~~(x', x'')~~ x' and x'', which are time-like with each other, are connected by the field operator V. Set of equations (II) or (II)' has the simple solutions

$$V(X_\mu, r_\mu) = \sum (2k_0)^{-1/2} \{ c(k_\mu) \, \delta(r_\mu + \lambda' k_\mu) \cdot \exp(i k_\mu X^\mu / \lambda') + d^*(k_\mu) \, \delta(r_\mu - \lambda' k_\mu) \cdot \exp(-i k_\mu X^\mu / \lambda') \}$$

and

$$V^*(X_\mu, r_\mu) = \sum (2k_0)^{-1/2} \{ d(k_\mu) \, \delta(r_\mu + \lambda' k_\mu) \cdot \exp(i k_\mu X^\mu / \lambda') + c^*(k_\mu) \, \delta(r_\mu - \lambda' k_\mu) \cdot \exp(-i k_\mu X^\mu / \lambda') \},^{\dagger\dagger}$$

where

$$\left.\begin{aligned} \lambda'^2 k_\mu k^\mu + l^2 &= 0 \\ \frac{k_\mu k^\mu}{\lambda'^2} + \kappa^2 &= 0 \end{aligned}\right\}$$

so that

$$\lambda' = \sqrt{l/\kappa} \quad \text{or} \quad \lambda'^2 = \frac{l}{\kappa}.$$

$$k_\mu k^\mu = -l\kappa.$$

The interaction between U and V, V^* is assumed to be

$$H = \varepsilon V^* U V.$$

---

†† $V(X_\mu, r_\mu)$ and $V^*(X_\mu, r_\mu)$ can be written as functions of x'_μ, x''_μ as in the case of U.

SN 6

we tentatively assume the equation for the probability amplitude

$$\Psi(\underbrace{n(k^{(1)}), n(k^{(2)}), \cdots ;}\ \underbrace{;\ x_1', x_2', x_3', x_0'}) $$

$$N(K^{(1)}), N(K^{(2)}), \cdots$$

(where k's satisfy

$$k_\mu^{(n)} \quad k_\mu k^\mu = 0$$

and K's satisfy

$$K_\mu K^\mu = -l\kappa = -\lambda'^2\kappa^2);$$

$$i\hbar c \frac{\partial \Psi}{\partial x_0'} = \sum_n \int H\,\Psi.$$

where H is a matrix with ~~colo~~ rows and columns characterized by

$$n'(k^{(1)}), n'(k^{(2)}) \cdots N'(K^{(1)}), N(K^{(2)}), \ldots x_1', x_2', x_3', x_0'$$

and

$$n''(k^{(1)}), n''(k^{(2)}) \cdots N''(K^{(1)}), N''(K^{(2)}) \cdots x_1'', x_2'', x_3'', x_0''$$

or, in ~~T.S. eq~~ the form similar to T.S. eq.:

$$i\hbar \frac{\delta \Psi}{\delta \sigma_{x}(x')} = H\Psi.$$

(S.N 67)

§3. Direct Integration of Nonlocalizable Field Equations.

$$[x^\mu, p_\nu] = i\hbar\delta_{\mu\nu}$$

$$[x_\mu[x^\mu, U]] = \cancel{\varepsilon V^* V}\ ?\ A\ ?$$

$$[p_\mu[p^\mu, U]] = \hbar^2 B\ \cancel{\varepsilon V^* V}\ ?$$

$$[x^\mu.[x^\mu, V]] + l^2 V = C \quad ?$$

$$[p_\mu[p^\mu, V]] + \varkappa^2 V = \hbar^2 D \quad ?$$

$$r_\mu r^\mu U(X_\mu, r_\mu) = A(X_\mu, r_\mu)$$

$$\frac{\partial^2 U}{\partial X_\mu \partial X^\mu} = + B(X_\mu, r_\mu)$$

$$r_\mu r^\mu B = -\frac{\partial^2 A}{\partial X_\mu \partial X^\mu}$$

$$B = \delta(r_\mu r^\mu) B' - \frac{1}{r_\mu r^\mu}\frac{\partial^2 A}{\partial X_\mu \partial X^\mu}$$

$$(r_\mu r^\mu + l^2) V = C$$

$$\frac{\partial^2 V}{\partial X_\mu \partial X^\mu} - \varkappa^2 V = D$$

Put:

$$B = \varepsilon V^* V$$

$$A =$$

$$U(x_\mu, r^\mu) = \sum u(k_\mu, r_\mu)\, e^{ik_\mu x^\mu}$$

$$V(x_\mu, r^\mu) = \quad \cdot \quad \cdot$$

$$A(x_\mu, r^\mu) = \sum a(k_\mu, r_\mu)\, e^{ik_\mu x^\mu}$$

$$\vdots \qquad\qquad \vdots$$

$$\left.\begin{aligned} r_\mu r^\mu\, u(k_\mu, r_\mu) &= a(k_\mu, r_\mu) \\ k_\mu k^\mu\, u(k_\mu, r_\mu) &= b(k_\mu, r_\mu) \\ (r_\mu r^\mu + l^2)\, v(k_\mu, r_\mu) &= c(k_\mu, r_\mu) \\ (k_\mu k^\mu + \kappa^2)\, v(k_\mu, r_\mu) &= d(k_\mu, r_\mu) \end{aligned}\right\}$$

$$\left.\begin{aligned} r_\mu r^\mu\, b &= k_\mu k^\mu\, a \\ (r_\mu r^\mu + l^2)\, d &= (k_\mu k^\mu + \kappa^2)\, c \end{aligned}\right\}$$

$$B(x_\mu, r_\mu) = \frac{1}{r_\mu r^\mu} \sum k_\mu k^\mu\, a\, e^{ik_\mu x^\mu}$$

$$= \frac{1}{r_\mu r^\mu}\left(-\frac{\partial^2 A}{\partial x_\mu \partial x^\mu}\right)$$

$$B = \overbrace{D(r)}^{B'} + \mathrm{const}\, D_1(r) \cdot D_x A$$

$$C = \overbrace{\Delta(r)}^{C'} + \mathrm{const}\, \Delta_1(r)\, \Delta_x \cancel{A}\, C$$

Simplest case

S, N 8

$$(x'|V|x'') = \sum (2k_0)^{-1/2} \{ c(k,\mu)\, \delta(x'-x''+\lambda' k) \exp\left(ik\cdot\frac{x'+x''}{2\lambda'}\right) + d^*(k)\, \delta(x'-x''-\lambda' k) \exp\left(-ik\frac{x'+x''}{2\lambda'}\right)\}$$

$$(x'|V^*|x'') = \sum (2k_0)^{-1/2} \{ d^*(k,\mu)\, \delta(x'-x''+\lambda' k) \exp\left(ik\frac{x'+x''}{2\lambda'}\right) + \cancel{d}\, c^*(k)\, \delta(x'-x''-\lambda' k) \times \exp\left(-ik\frac{x'+x''}{2\lambda'}\right)\}$$

$$(x'|V^*V|x'') = \sum\sum \{(2k_0)^{-1/2}(2k_0')^{-1/2} \times \{ c(k,\mu)\, d(k',\mu)\, \delta(x'-x'''+\lambda' k)\, \delta(x'''-x''+\lambda' k') \exp\left\{ik\left(\frac{x'+x'''}{2\lambda'}\right) + ik'\left(\frac{x'''+x''}{2\lambda'}\right)\right\}$$

$$+ \cancel{d^*}\, c(k,\mu)\, c^*(k',\mu)\, \delta(x'-x'''+\lambda' k)\, \delta(x'''-x''-\lambda' k') \exp\left\{ik\left(\frac{x'+x'''}{2\lambda'}\right) - ik'\left(\frac{x'''+x''}{2\lambda'}\right)\right\}$$

$$+ d^*(k)\, d(k')\, \delta(x'-x'''-\lambda' k)\, \delta(x'''-x''+\lambda' k') \exp\left\{-ik\left(\frac{x'+x'''}{2\lambda'}\right) + ik\left(\frac{x'''+x''}{2\lambda'}\right)\right\}$$

$$+ d^*(k)\, c^*(k')\, \delta(x'-x''-\lambda' k)\, \delta(x'-x''-\lambda' k') \exp\left\{-ik\frac{x'+x'''}{2\lambda'} - ik'\frac{x'''+x''}{2\lambda'}\right\}\}$$

$$= \sum\sum (2k_0)^{-1/2}(2k_0')^{-1/2} \{ c(k,\mu)\, d(k',\mu) \cdot \delta(x'-x''+\lambda'(k+k')) \exp\left\{\frac{x'+x''}{2\lambda'}\, i(k+k')\right\}$$

$+ \cdots$

or $V^*V(X,x) = \sum\sum (2k_0)^{-1/2}(2k'_0)^{-1/2}$

$\times \{ c(k_\mu) d(k'_\mu)\, \delta(r+\lambda'(k+k')) \exp(i(k+k')\frac{X}{\lambda'})$

$+ c(k_\mu) c^*(k'_\mu)\, \delta(r+\lambda'(k-k')) \exp(i(k-k')\frac{X}{\lambda'})$

$+ d^*(k) d(k')\, \delta(r-\lambda'(k-k')) \exp(-i(k-k')\frac{X}{\lambda'})$

$+ d^*(k) c^*(k')\, \delta(r-\lambda'(k+k')) \exp(-i(k+k')\frac{X}{\lambda'}) \}$

Thus

$r_\mu r^\mu V^*V(X,x) = \sum\sum (2k_0)^{-1/2}(2k'_0)^{-1/2}$

$\times \{ c(k_\mu) d(k'_\mu)\, \lambda'^2 (k_\mu+k'_\mu)(k^\mu+k'^\mu) \exp \cdots$

$+ \cdots \}$

$$\frac{\partial^2 (V^*V)}{\partial X_\mu \partial X^\mu} = -\frac{1}{\lambda'^4} r_\mu r^\mu V^*V.$$

The equations are compatible at least, in the first approx. for

$$r_\mu r^\mu U = \frac{\lambda'^2}{\hbar^2} \varepsilon V^*V$$

$$\frac{\partial^2 U}{\partial X_\mu \partial X^\mu} = -\frac{1}{\lambda'^2} \varepsilon V^*V$$

and $\lambda' = \lambda = \sqrt{l/\kappa}$

$$r_\mu r^\mu V + l^2 V = \frac{\lambda'^2}{\hbar^2} \varepsilon U V$$

$$\frac{\partial^2 V}{\partial X_\mu \partial X^\mu} + \kappa^2 V = -\frac{1}{\lambda'^2} \varepsilon U V$$

(S, N 9)

$$V^*V(X, x) = \sum (2k_0)^{-1/2}(2k_0')^{-1/2}$$
$$\times \{ c(k_\mu) d(k_\mu') \delta(r + \lambda'(k+k')) \exp(i(k+k')\tfrac{X}{\lambda'})$$
$$+ c(k_\mu) c^*(k_\mu') \delta(r + \lambda'(k-k')) \exp(i(k-k')\tfrac{X}{\lambda'})$$
$$+ d^*(k) d(k') \delta(r - \lambda'(k-k')) \exp(-i(k-k')\tfrac{X}{\lambda'})$$
$$+ d^*(k) c^*(k') \delta(r - \lambda'(k+k')) \exp(-i(k+k')\tfrac{X}{\lambda'}) \}$$

$$\bigcirc \ (2k_0)^{-1/2}(2k_0')^{-1/2} c(k_\mu) d(k_\mu') \delta(r + \lambda'(k+k')) \exp(i(k+k')\tfrac{X}{\lambda'})$$

$$\left.\begin{aligned} K &= (k + k') \\ K' &= (k - k') \end{aligned}\right\} \qquad k = \frac{K+K'}{2}, \quad k' = \frac{K-K'}{2}$$

$$\frac{\partial(K, K')}{\partial(k, -k')} = 1$$

$$\sum_K \sum_{K'} g(K, K') c\left(\frac{K+K'}{2}\right) d\left(\frac{K-K'}{2}\right) \delta(r + \lambda' K)$$
$$\exp(iKX/\lambda')$$

$$k_\mu k^\mu u(k) = \frac{\lambda'^2}{\hbar^2} \varepsilon \sum_{k'} v^*(k'-k) v(k') \cancel{\delta(r+\lambda' k)}$$
$$\cancel{\times \exp(iKX/\lambda')}$$

This has just the form considered by Heisenberg (III); ~~with~~

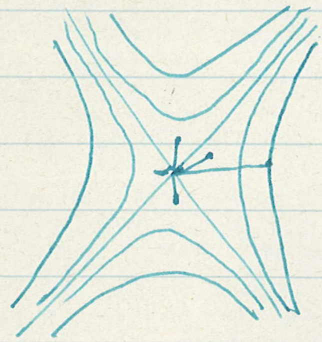
$$A(k)\varphi(k) + \int dk' B(k', k-k')\varphi(k')\varphi(k-k')$$
$$+ \cdots = 0$$

where ~~φ and~~ φ should be replaced by u and v, and $A = k_\mu k^\mu$ $B = \delta$ const.

Thus $\lambda = \lambda'$, there is not essential change from the ordinary localizable theory.
The difference is, ~~only~~ to regard X instead of x as space-time coordinates observed.

Direct Integration of Field Equations

(1)



$$f(s^2) = \int e^{-\kappa s} \quad e^{-\kappa^2 s^2} \qquad s^2 > 0$$

$$f(s^2) = 0 \qquad \text{for } s^2 < 0$$

$$\delta(x_\mu) = \iiint e^{ik_\mu x^\mu} (dk_\mu)^4$$

$$\frac{1}{2} dK = \frac{1}{2} d(\kappa^2) = \kappa\, d\kappa = -k_0\, dk_0$$

$$\delta(x_\mu) = \int_{-\infty}^{+\infty} \frac{1}{2k_0} \iiint_{-\infty}^{+\infty} e^{ik_\mu x^\mu}\, dK\, (dk_\mu)^3$$

$$= \frac{1}{2} \int_{-\infty} dK\, D_1(x_\mu, K)$$

$$\psi^\dagger(x_\mu)$$

$$[x_\mu, p_\nu] = i\hbar\, \delta_{\mu\nu}$$

$$\times\ [x^\mu, p^\nu] = i\hbar\, g^{\mu\nu}$$

Direct Integration of Field Equations. I.

~~Gage~~ Instead of the usual formalism, which start from the Lagrangian, an alternative method of integrating the field equations is recommended because the latter are obviously gage invariant. (The separation of into ~~free~~ free part and interaction part of the Hamiltonian, is not gage invariant.) ~~in~~ ~~not~~ T. S. method

$$\gamma^\mu p_\mu \psi + \varkappa \psi + \frac{e}{c}\gamma^\mu A_\mu \psi = 0$$

$$\Box A_\mu = -4\pi e\, \psi^\dagger \gamma_\mu \psi$$

$$\psi = \sum_{k,\cancel{x}} u(k,\cancel{x})\, e^{ik_\mu x^\mu}$$

$$A_\mu = \sum_{k,\cancel{x}} a_\mu(k,\cancel{x})\, e^{ik_\mu x^\mu}$$

$$\gamma^\mu k_\mu u(k,\cancel{x}) + \varkappa u(k,\cancel{x}) + \frac{e}{\hbar c}\gamma^\mu a_\mu(k')u(k-k') = 0$$

~~$k_\mu k^\mu$~~

$$k^\nu k_\nu a_\mu(k) + 4\pi e\, u^\dagger(k')\gamma_\mu u(k-k') = 0$$

$$\gamma^\mu k_\mu u(k) + \varkappa u(k) - 4\pi\alpha \overbrace{\{u^\dagger(k'')\gamma_\mu u(k'-k'')\}\gamma^\mu u(k-k')} = 0$$

$$[U(X_\mu, r_\mu), U(X'_\mu, r'_\mu)]$$
$$= \sum_{kk'} [b(k_\mu), b^*(k'_\mu)] (2k_0)^{-1/2} (2k'_0)^{-1/2}$$
$$\delta(r_\mu + \lambda k_\mu) \exp(ik_\mu X^\mu/\lambda)\, \delta(r'_\mu - \lambda k'_\mu) \exp(-ik'_\mu X'^\mu/\lambda)$$
$$+ \sum_{kk'} [b^*(k_\mu), b(k'_\mu)] (2k_0)^{-1/2} (2k'_0)^{-1/2}$$
$$\delta(r_\mu - \lambda k_\mu) \exp(-ik_\mu X^\mu/\lambda)\, \delta(r'_\mu + \lambda k'_\mu) \exp(ik'_\mu X'^\mu/\lambda)$$
$$= \sum_k \frac{1}{2k_0} \{ \delta(r_\mu + \lambda k_\mu) \delta(r'_\mu - \lambda k_\mu) \exp(ik_\mu(X^\mu - X'^\mu)/\lambda)$$
$$- \delta(r_\mu - \lambda k_\mu) \delta(r'_\mu + \lambda k_\mu) \exp(-ik_\mu(X^\mu - X'^\mu)/\lambda) \}$$

$\lambda \to 0$, $\quad \delta(r_\mu \pm \lambda k_\mu) = \delta(r'_\mu \pm \lambda k_\mu) = 1$ for $r_\mu = r'_\mu = 0$

$= 0$ for $r_\mu \neq 0$, $r'_\mu \neq 0$ }

$$[U(X_\mu), U(X'_\mu)] = \sum_k \frac{1}{2k_0} \{ \exp(ik_\mu(X^\mu - X'^\mu)/\lambda)$$
$$- \exp(-ik_\mu(X^\mu - X'^\mu)/\lambda) \}$$

$$\bar{U}(X_\mu) = \int U(X_\mu, r_\mu) f(r_\mu) (dr_\mu)^4$$

$$[\bar{U}(X_\mu), \bar{U}(X'_\mu)] = \sum_k \frac{1}{2k_0} \{ f(-\lambda k_\mu) f(\lambda k_\mu) \exp(ik_\mu(X_\mu - X'_\mu)/\lambda)$$
$$- f(-\lambda k_\mu) f(\lambda k_\mu) \exp(-ik_\mu(X_\mu - X'_\mu)/\lambda) \}$$

$$f(-\lambda k_\mu) = f(\lambda k_\mu)$$

$$[U(x_\mu), U(x'_\mu)]_+ = \sum_{\substack{kk' \\ (k_0, k'_0 > 0)}} \Big\{ [b(k_\mu), b^*(k'_\mu)]_+$$

$$\frac{1}{\sqrt{2k_0}} \frac{1}{\sqrt{2k'_0}} \exp(ik_\mu x_\mu) \exp(-ik'_\mu x'_\mu)$$

$$+ [b^*(k_\mu), b(k_\mu)]_+ \frac{1}{\sqrt{2k_0}} \frac{1}{\sqrt{2k'_0}} \exp(-ik_\mu x_\mu)$$

$$\times \exp(+ik'_\mu x'_\mu) \Big\}$$

$$= \sum_{\substack{k \\ (k_0 > 0)}} \frac{1}{2k_0} \{\exp\{ik_\mu(x_\mu - x'_\mu)\} + \exp\{-ik_\mu(x_\mu - x'_\mu)\}\}$$

$$L = \frac{1}{2} \Big\{ \sum k_\mu k^{\nu'} b(k_\mu) b^*(k'_\mu) \frac{1}{\sqrt{2k_0}} e^{ik_\mu x_\mu} \frac{1}{\sqrt{2k'_0}} e^{-ik'_\mu x_\mu}$$

$$+ \sum k_\mu k^{\nu'} b^*(k_\mu) b(k'_\mu) \frac{1}{\sqrt{2k_0}} e^{-ik_\mu x_\mu} \frac{1}{\sqrt{2k'_0}} e^{ik'_\mu x_\mu}$$

$$\cancel{T_{\mu\nu}}$$

$$+ \sum k_\mu k^{\mu'} b(k_\mu) b(k'_\mu) \cdots$$

$$+ \sum k_\mu k^{\mu'} b^*(k_\mu) b^*(k'_\mu) \cdots \Big\}$$

$$T_\mu^{\nu} = \sum k_\mu k^{\nu'} b(k_\mu) b^*(k'_\mu) \cdots$$

$$+ \cdots$$
$$+ \cdots$$
$$+ \cdots - L\delta_{\mu\nu}$$

$$T_0^{\ 0} =$$

Direct Integration of Field Equations. II. (3)

$$\square A_\mu = 4\pi e \, \underline{\psi^\dagger \gamma_\mu \psi} \quad \dagger$$

$$(\gamma_\mu p^\mu + mc)\psi = -\frac{e}{c}\gamma_\mu \underline{A^\mu \psi}$$

$$p^\mu A_\mu = 0$$

$$\gamma_\mu \gamma^\nu + \gamma^\nu \gamma_\mu = -2\delta_{\mu\nu}$$

$$x^0 = -x_0 = ct \; ; \quad \gamma^0 = -\gamma_0 = \rho_3$$

$$\gamma^1 = \gamma_1 = \rho_3\rho_1\sigma_x = i\sigma_2\sigma_x \quad \text{etc}$$

$$[x^\mu, p_\nu] = i\hbar\delta_{\mu\nu}$$

$$A_\mu(x) = \sum_k A_\mu(k)\exp(ikx) \quad *$$

$$\psi(x) = \sum_k \psi(k)\exp(ikx)$$

$$\psi^\dagger(x) = \sum_k \psi^\dagger(k)\exp(ikx)$$

$$\kappa = \frac{mc}{\hbar}$$

$$\begin{cases} k_\mu k^\mu A_\nu(k) = -4\pi e \sum_l \psi^\dagger(\cancel{k}\, l)\gamma_\nu \psi(k-l) \\ \cancel{k_\mu \gamma^\mu}(\gamma_\mu k^\mu + \kappa)\psi(k) = \frac{e}{\hbar c}\sum_l \gamma_\mu A^\mu(l)\psi(k-l) \\ k^\mu A_\mu(k) = 0 \end{cases}$$

$$\left.\begin{aligned} A_\nu(k) &= a_\nu(k)\,\delta(k_\mu k^\mu) + b_\nu(k) \\ \psi(k) &= (\gamma_\mu k^\mu - \kappa)\,u(k)\,\delta(k_\mu k^\mu + \kappa^2) + v(k) \\ \psi^\dagger(k) &= u^\dagger(k)(\gamma_\mu k^\mu - \kappa)\,\delta(k_\mu k^\mu + \kappa^2) + v^\dagger(k) \end{aligned}\right\}$$

---

† Underline means that we may need some procedures such as symmetrization, or change of order etc.

* k_μ extend over all ~~k-space~~ 4-dimensional k-space.

$$k_\mu k^\mu b_\nu(k) = \overset{-4\pi}{e}\sum_{\text{spin}}\sum_{l} u^+(l)(\gamma_\mu l^\mu - \kappa)\,\delta(l_\mu l^\mu + \kappa^2)$$
$$\times \gamma_\nu\{\gamma_\lambda(k^\lambda - l^\lambda) - \kappa\}u(k-l)\,\cdot$$
$$\times\,\delta\left((k_\mu - l_\mu)(k^\mu - l^\mu) + \kappa^2\right) + O(e^2)$$

$$(\gamma_\mu k^\mu + \kappa)v(k) = \frac{e}{\hbar c}\sum_{\substack{\text{spin}\\ \text{pol.}}}\sum_{l}\gamma_\mu a^\mu(l)\,\delta(l_\mu l^\mu)$$
$$\times(\gamma_\mu k^\mu - \kappa)\,u(k-l)\,\delta\{(k_\mu - l_\mu)(k^\mu - l^\mu) + \kappa^2\}$$

$$b_\nu(k) = \overset{-4\pi}{e}a_\nu^{(1)}(k)\,\delta(\underset{\overset{\|}{0}}{k_\mu k^\mu})$$
$$+\ e\frac{\sum_{\text{spin}}\sum u^+(l)(\gamma_\mu l^\mu - \kappa)\,\delta(l_\mu l^\mu + \kappa^2)\,\gamma_\nu\{\gamma_\lambda(k^\lambda - l^\lambda) - \kappa\}}{k_\mu k^\mu}$$
$$\underline{\times\,u(k-l)\,\delta((k_\mu - l_\mu)(k^\mu - l^\mu) + \kappa^2)} + O(e^2)$$

$$v(k) = \left(\frac{e}{\hbar c}\right)u^{(1)}(k)\,\delta(\underset{\overset{\|}{0}}{k_\mu k^\mu + \kappa^2})$$
$$+\ \frac{e}{\hbar c}\,\frac{1}{k_\mu k^\mu + \kappa^2}\sum_{\substack{\text{spin}\\ \text{pol.}}}\sum_{l}(\gamma_\mu k^\mu - \kappa)\,\gamma_\mu\,a^\mu(l)\,\delta(l_\mu l^\mu)$$
$$\times(\gamma_\mu k^\mu - \kappa)\,u(k-l)\,\delta\{(k_\mu - l_\mu)(k^\mu - l^\mu) + \kappa^2\}$$

$$k^\mu a_\mu(k) = 0$$

$$\bigcirc \cdot\ k^\mu b_\mu(k) = 0\ !$$

$$k^\nu b_\nu(k) = \sqrt[-4\pi]{e}\,\frac{1}{k_\mu k^\mu}\sum_{spin}\sum_{l} u^+(l)(\gamma_\mu l^\mu - \varkappa)\,\delta(l_\mu l^\mu + \varkappa^2) \tag{4}$$

$$\times \gamma_\nu k^\nu \{\gamma_\lambda (k^\lambda - l^\lambda) - \varkappa\}\, u(k-l)\, \delta\{(k_\mu - l_\mu)(k^\mu - l^\mu) + \varkappa^2\}$$

$$= 0$$

Detailed characterization of 0-order solutions

~~Char~~

(May 19, 1949)

Young: Integration in x-space

$$\Box A_\mu = +4\pi e(\psi^\dagger \gamma_\mu \psi - \psi \gamma_\mu \psi^\dagger)$$

$$(\gamma_\mu p^\mu + mc)\psi = -\frac{e}{c}\gamma_\mu A^\mu \psi$$

$$A_\mu = A^{out} + 4\pi e \int D^{adv} \psi'^\dagger \gamma_\mu \psi' dx'$$

$$= A^{in} + 4\pi e \int D^{ret} \psi'^\dagger \gamma_\mu \psi' dx'$$

$$\psi = \psi^{in} + \frac{e}{\pi c}\int S^{ret}(x-x') A' \psi' dx'$$

$$A^{out} = S^{-1} A^{in} S$$

$$\psi^{out} = S^{-1} \psi^{in} S$$

~~Con~~ Form of the outgoing wave[†]

$$\frac{1}{W'-W} - i\pi\delta(W'-W)$$

incoming wave

$$\frac{1}{W'-W} + i\pi\delta(W'-W)$$

[†] Dirac, Q.M. p. 198.

$$\frac{d}{dx}\log x = \frac{1}{x} - i\pi\delta(x)$$

Direct Integration of Nonlocalizable Field Equations. I

(5)

$$\left.\begin{aligned} [p^\nu[p_\nu, A_\mu]] &= -4\pi e \hbar^2 \psi^\dagger \gamma_\mu \psi \\ \gamma_\mu[p^\mu, \psi] &= -\frac{e}{c}\gamma_\mu A^\mu \psi \\ [p^\mu, A_\mu] &= 0 \end{aligned}\right\}$$

~~and~~

$$\left.\begin{aligned} [x^\nu[x_\nu, A_\mu]] &= \mp 4\pi e \cancel{\hbar}\lambda^2 \psi^\dagger \gamma_\mu \psi \\ \gamma_\mu[x^\mu, \psi] &= \mp \frac{e\lambda^2}{\hbar}\gamma_\mu A^\mu \psi \\ [x^\mu, A_\mu] &= 0, \end{aligned}\right\}$$

where $\mp$ correspond to two types of non-localizable fields considered recently*
The first type ~~is not~~ does not differ essentially from the usual ~~theor type~~ localizable field, because in η-space, where

$$\eta_\mu = \frac{x_\mu}{\lambda} + \frac{\lambda p_\mu}{\hbar},$$

everything takes the same form as in the usual theory.

In the second case, A_μ and $\psi, \psi^\dagger$ are considered as ~~functions~~ sums of functions of ξ and ξ^*. Namely

$$A_\mu = \frac{1}{2}\{A_\mu(\xi_\mu) + A^*_\mu(\xi^*_\mu)\}$$

$$\psi = \frac{1}{2}\{\psi(\xi_\mu) + \varphi^*(\xi^*_\mu)\},$$

* Yukawa, Prog. Theor. Phys. 3 (1948), 452.

$$\psi^* = \psi^\dagger\beta = \frac{1}{2}\{\varphi(\xi_\mu) + \psi^*(\xi^*_\mu)\}$$

where $A_\mu(\xi_\mu)$, $\psi(\xi_\mu)$, $\varphi(\xi_\mu)$ contain only positive frequency terms, while $A_\mu^*(\xi_\mu^*)$, $\psi^*(\xi_\mu^*)$, $\varphi^*(\xi_\mu^*)$ only negative ~~…~~ frequency terms.

In other words

Finished

Oct. 13, 1949
For correction
Yukawa

Quantum Theory of Nonlocal Fields

Part I. Free Fields

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Abstract

The possibility of a theory of nonlocal fields, which is free from the restriction that field quantities are always point functions in the ordinary space, is investigated. Certain types of nonlocal fields, each satisfying a set of mutually compatible commutation relations, which can be obtained by extending familiar field equations for local fields in conformity with the principle of reciprocity, are considered in detail. Thus a scalar nonlocal field is obtained, which represents an assembly of particles with the mass, radius and spin 0, provided that the field is quantized according to the procedure similar to the method of second quantization in the usual field theory. Nonlocal vector and spinor fields corresponding to assemblies of particles with the finite radius and the spins 1 and 1/2 respectively are obtained in the similar way.

On leave of absence from Kyoto University, Kyoto, Japan.

I. Introduction

It has been generally believed for years that wellknown divergence difficulties in quantum theory of wave fields could be solved only by taking into account the finite size of the elementary particles consistently. Recent success of quantum electrodynamics, which took advantage of the relativistic covariance to the utmost,(1) however, seemed to have weakened to some extent the necessity of introducing socalled universal length or any substitute for it into field theory. In fact, all infinities which had been familiar in previous formulations of quantum electrodynamics were reduced to unobservable renormalization factors for the mass and the electric charge in the newer formalism. Furthermore, in order to get rid of the remaining difficulties that these renormalization factors were still infinite or indefinite, main efforts were concentrated in the direction of introducing various kinds of auxiliary fields, either real or only formal, rather than in the direction of introducing explicitly the universal length or the finite radius of the elementary particles. So far as the results of the investigations in the former direction are concerned, however, the prospect is not so encouraging. Namely, an ingenious method of regulators, which was investigated by Pauli extensively,(2) can be regarded as a formalistic generalization of the theory of mixed fields,(3) but cannot be replaced by a combination of neutral vector fields and charged spinor fields with different masses, unless we admit the introduction of Bosons with negative energies and Fermions with imaginary charges as pointed out by Feldman.(4) More generally, according to recent investigations by Umezawa and others(5) and by Feldman,(4) no combination of quantized fields with spins 0, 1/2 and 1 can be free from all of the divergence difficulties, as long as only <u>positive</u> energy states for Bosons and real coupling constants for the interactions between Fermions and Bosons are taken into account.

Nevertheless, the difficulties remaining in quantum electrodynamics are not so serious as those which appear in meson theory. In the latter case, we know that straightforward calculations very often lead to divergent results for directly observable quantities such as the probabilities of certain types of meson decay. Although the application of Pauli's regulators to meson theory was found useful for obtaining finite results,(6) it can hardly be considered as a satisfactory solution of the problem for reasons mentioned above. It seems to the present author that, at least, a part of the defect of the present meson theory is due to the lack of a consistent method of dealing with the finite extension of the elementary particle such as the nucleon, whereas the effect of the finite extension is usually very small so far as electrodynamical phenomena in the narrowest sense are concerned, except for its decisive effect on the renormalizations of the mass and the electric charge.

Under these circumstances, it seems worthwhile to investigate again the possibility of extension of the present field theory in the direction of introducing the finite radius of the elementary particle. In this paper, as the continuation of the preceding papers,(7) the possibility of a theory of quantized nonlocal fields, which is free from the restriction that field quantities are always point functions in the ordinary space, will be discussed in detail. One may be very sceptical about the necessity of such a drastic change in field theory, because other possibilities such as the introduction of local fields corresponding to particles with spins higher than 1 are not yet fully investigated. However, present theory of elementary particles with spins higher than 1 suffers from the difficulty associated with the necessity of auxiliary conditions, and even if this is overcome by some revision of the formalism as proposed by Bhabha,[8] we can hardly expect a satisfactory solution of the whole problem, because the admixture of higher spin fields may well give rise to newer types of divergence in return for the elimination of more familiar ones. Moreover, it does not seem to the present author that the theory of nonlocal fields is necessarily contradictory to the theory of mixed local fields. They can rather be complementary to each other in that a nonlocal field may well happen to be approximately equivalent to some mixture of local fields. The most essential point, which is in favour of the nonlocal field, is that the convergence of field theory can be guaranteed by introducing a new type of irreducible field instead of a mixture, which is reducible.

In this paper, as in the preceding papers, we confine our attention to certain types of nonlocal field, each satisfying a set of mutually compatible commutation relations, which can be obtained by extending familiar field equations for local fields in conformity with the principle of reciprocity. The solutions of these operator equations can be interpreted as a field-theoretical representation of assemblies of elementary particles, each having a definite mass and a definite radius. In this connection, a recent attempt by Born and Green[9] is interesting particularly in that they made use of the principle of reciprocity as a postulate for determining possible masses of elementary particles of various types. However, it is not yet clear whether their method of density operator contains something essentially different from the usual theory of mixture of local fields

The most important question of the interaction of two or more nonlocal fields will be discussed in Part II. of this paper.

II. An Example of the Nonlocal Scalar Field

In order to see what comes out by generalizing a field theory so as to include nonlocal fields, we start from a particular case of the nonlocal scalar field. A scalar operator, U, which is supposed to describe a nonlocal scalar field, can be represented, in general, by a matrix with rows and columns, each characterized by a set of values of space and time coordinates. Alternatively, we can regard this operator U as a certain function of four space-time operators x^μ ($x^1 = x_1 = x$, etc. $x^4 = -x_4 = ct$) as well as of four space-time displacement operators p_μ, which satisfy wellknown commutation relations

$$[x^\mu, p_\mu] = i\hbar\delta_{\mu\nu} \tag{1}$$

where

$$[A, B] \equiv AB - BA \tag{2}$$

for any two operators A and B. Usual local fields are included as the particular case in which the field operator U is a function of x^μ alone, so that it can be represented by a diagonal matrix in the representation, in which the operators x^μ themselves are diagonal. In this particular case, it is customary to start from the second order wave equation

$$\left(\frac{\partial^2}{\partial x_\mu \partial x^\mu} - \kappa^2\right) U(x^\mu) = 0, \quad \kappa = mc/\hbar \tag{3}$$

for the local field $U(x^\mu)$, in order that it can reproduce, when quantized, an assembly of identical particles with a definite mass m and the spin 0. The equation (3) is equivalent to the relation between the operator U and the operators p

$$[\gamma_\mu[p^\mu, U]] + m^2c^2U = 0 \tag{4}$$

for this case. We assume that the nonlocal scalar field U in question satisfies the commutation relation of the same form as (4). However, in our case, we need further the commutation relation between U and x^μ, in contrast to the case of local field, in which U and x^μ are simply commutative with each other. In order to guess the correct form for it, some heuristic idea is needed. The principle of reciprocity seems to be very useful for this purpose. Namely, we assume that the commutation relation between U and x^μ has a form

$$[x_\mu[x^\mu, U]] - \lambda^2 U = 0, \tag{5}$$

where λ is a constant with the dimension of length and can be interpreted as the radius of the elementary particle in question, as will be shown below. The relations (4) and (5) are not exactly the same in form, but differ from each other by plus and minus signs of the last terms on the left hand sides of (4) and (5). Thus, the two operator equations (4) and (5) can be said to be mutually reciprocal rather than perfectly symmetrical, indicating that the radius of the elementary particle λ must be introduced as something reciprocal to the mass m.

Now the operator U can be represented by a matrix $(x'_\mu|U|x''_\mu)$ in the representation, in which x_μ are diagonal matrices. The matrix elements, in turn, can be considered as a function $U(X_\mu, r_\mu)$ of two sets of real variables

$$X_\mu = \frac{1}{2}(x'_\mu + x''_\mu), \qquad r_\mu = x'_\mu - x''_\mu \tag{6}$$

Accordingly, the relations (4) and (5) can be replaced by

$$\left(\frac{\partial^2}{\partial X_\mu \partial X^\mu} - \kappa^2\right) U(X_\mu, r_\mu) = 0 \tag{7}$$

$$(r_\mu r^\mu - \lambda^2) U(X_\mu, r_\mu) = 0 \tag{8}$$

respectively. The equations (7) and (8) are obviously compatible with each other and the former implies that $U(X_\mu, r_\mu)$ is, in general, a superposition of plane waves of the form $\exp i k_\mu X^\mu$ with k_μ satisfying the condition

$$k_\mu k^\mu + \kappa^2 = 0, \tag{9}$$

whereas the latter implies that $U(X_\mu, r_\mu)$ can be different from zero only for those values of r_μ, which satisfy the condition

$$r_\mu r^\mu - \lambda^2 = 0. \tag{10}$$

Thus the most general solution of the simultaneous equations (7) and (8) has the form

$$U(X_\mu, r_\mu) = \int\cdot\cdot\int (dk)^4 u(k_\mu, r_\mu)\, \delta(r_\mu r^\mu - \lambda^2) \times \times \delta(k_\mu k^\mu + \kappa^2) \exp(i k_\mu X^\mu) \tag{11}$$

where $u(k_\mu, r_\mu)$ is an arbitrary function of two sets of variables k_μ and r_μ.

The above considerations suggest us that one set X_μ of the real variables could be identified with the conventional space and time coordinates of the elementary particle regarded as a material point in the limit of $\lambda \to 0$, whereas the other set r_μ could be interpreted as variables describing the internal motion in general case, in which the finite extension of the elementary particle in question could not be ignored. Thus, we might expect that the field U of the above type is equivalent to an assembly of elementary particles with the mass m, the radius λ and the spin 0, if it is further quantized according to the familiar method of second quantization. However, we can easily anticipate that the equivalence is incomplete, because $U(X_\mu, r_\mu)$ is different from zero for arbitary large values of r_μ, so far as they satisfy the condition (10), even when only one term of the right hand side of (11) corresponding to a definite set of values of k_μ is taken into account. In other words, we need another condition for restricting the possible form of $U(X_\mu, r_\mu)$ or $u(k_\mu, r_\mu)$ in order to complete the equivalence above mentioned. For this purpose, we introduce an auxiliary condition

$$[p_\mu [x^\mu, U]] = 0, \tag{12}$$

which can be said to be self-reciprocal in that the relation

$$[x^\mu [p_\mu, U]] = 0 \tag{13}$$

can be deduced from (12) immediately on account of the commutation relation (1). Both of (12) and (13) are equivalent to the condition

$$r^\mu \frac{\partial U(X_\mu, r_\mu)}{\partial X^\mu} = 0 \tag{14}$$

for $U(X_\mu, r_\mu)$, or the restriction that $u(k_\mu, r_\mu)$ should be zero unless k_μ and r_μ satisfy the condition

$$k_\mu r^\mu = 0. \tag{15}$$

Thus the most general form of $U(X_\mu, r_\mu)$ which satisfies all the relations (7), (8) and (14) is

$$U(X_\mu, r_\mu) = \int\cdots\int (dk)^4 u(k_\mu, r_\mu)\, \delta(k_\mu k^\mu + \kappa^2)\, \delta(k_\mu r^\mu) \times \delta(r_\mu r^\mu - \lambda^2) \exp(i k_\mu X^\mu) \tag{16}$$

where $u(k_\mu, r_\mu)$ is again an arbitrary function of k_μ and r_μ.[(10)]

Now a simple physical interpretation can be given to the non-local field of the form (16) by considering the corresponding particle picture: Suppose that the particle is at rest with respect to a certain reference system. In this particular case, the motion of the particle as a whole, or the motion of its center of mass, can be represented presumably by a plane wave in X-space with the wave vector $k_1 = k_2 = k_3 = 0$, $k_4 = -\kappa$. The corresponding form of $U(X_\mu, r_\mu)$ is, apart from the factor independent of X_μ, r_μ,

$$u(0,0,0,-\kappa;r_\mu)\,\delta(r_\mu r^\mu-\lambda^2)\,\delta(\kappa r_4)\exp(-i\kappa X^4) \qquad (17)$$

which is different from zero only for those values of r_μ which satisfy the conditions

$$r_1^2+r_2^2+r_3^2=\lambda^2, \qquad r_4=0, \qquad (18)$$

Thus, the form of $U(X_\mu, r_\mu)$ in this case is determined completely by giving $u(0,0,0,-\kappa;r_\mu)$ as defined on the surface of the sphere with the radius λ in r-space. In other words, the internal motion can be described by the wave function $u(\theta,\varphi)$ depending only on the polar angles θ, φ, which are defined by

$$r_1 = r\sin\theta\cos\varphi, \quad r_2 = r\sin\theta\sin\varphi, \quad r_3 = r\cos\theta. \qquad (19)$$

In general, $u(\theta,\varphi)$ can be expanded into series of spherical harmonics:

$$u(\theta,\varphi)=\sum_{l,m} c(0,0,0,-\kappa;l,m)\,P_l^m(\theta,\varphi), \qquad (20)$$

which is equivalent to decompose the internal rotation into various states characterized by the azymuthal quantum number l and the magnetic quantum number m.

In case, when the center of mass of the particle is moving with the velocity v_x, v_y, v_z, it can be described by a plane wave in X-space with the wave vector k_μ, which is connected with the velocity by the relations

$$v_x = -k_1 c/k_4, \quad v_y = -k_2 c/k_4, \quad v_3 = -k_3 c/k_4, \quad k_4 = -\sqrt{k^2+\kappa^2} \tag{21}$$

In this case, $U(X_\mu, r_\mu)$ has the form

$$u(k_\mu, r_\mu)\,\delta(r_\mu r^\mu - \lambda^2)\,\delta(k_\mu r^\mu)\exp(ik_\mu X^\mu), \tag{22}$$

which is different from zero only on the surface of the sphere with the radius λ in r-space, the sphere itself moving with the velocity v_x, v_y, v_z. Accordingly, we perform first the Lorentz transformation

$$x'_\mu = a_{\mu\nu} x_\nu \tag{23}$$

with the transformation matrix

$$(a_{\mu\nu}) \equiv \begin{pmatrix} 1+(k_1/K)^2 & k_1k_2/K^2 & k_1k_3/K^2 & k_1/\kappa \\ k_1k_2/K^2 & 1+(k_2/K)^2 & k_2k_3/K^2 & k_2/\kappa \\ k_1k_3/K^2 & k_2k_3/K^2 & 1+(k_3/K)^2 & k_3/\kappa \\ k_1/\kappa & k_2/\kappa & k_3/\kappa & -k_4/\kappa \end{pmatrix} \tag{24}$$

where $K = \sqrt{\kappa(\kappa - k_4)}$. Then the wave function for the internal motion can be described by a function $u'(\theta', \varphi')$ of the polar angles θ', φ' defined by

$$r'_1 = a_{1\nu} r_\nu = r' \sin\theta' \cos\varphi', \qquad r'_2 = a_{2\nu} r_\nu = r' \sin\theta' \sin\varphi'$$

$$r'_3 = a_{3\nu} r_\nu = r' \cos\theta', \qquad r'_4 = a_{4\nu} r_\nu = k_\mu r^\mu/\kappa. \tag{25}$$

Incidentally, r'_4 as defined by the last expression in (25) is nothing but the proper time, multiplied by $-c$, for the particle, which is moving with the velocity v_x, v_y, v_z. Again $u'(\theta', \varphi')$ can be expanded into series of spherical harmonics:

$$u'(\theta', \varphi') = \sum_{l,m} c(k_\mu, l, m)\, P_l^m(\theta', \varphi'). \tag{26}$$

Since the above arguments are in conformity with the principle of relativity perfectly, the nonlocal field in question can be regarded as a field-theoretical representation of a system of identical particles each with the mass m, the radius and the spin 0, which can rotate as the relativistic rigid sphere without any change in shape other than the Lorentz contraction associated with the change of the proper time axis.

The nonlocal field U given by (16) reduces to the ordinary local scalar field in the limit $\lambda \to 0$, as it should be, provided that the rest mass m is different from zero. Namely, $(x'_\mu | U | x''_\mu)$ is different from zero only for $x'_\mu = x''_\mu$, because the only possible solution of the simultaneous equations (9), (11) and (15) with $m \neq 0$ and $\lambda = 0$ is $r_1 = r_2 = r_3 = r_4 = 0$. On the contrary, the case of the zero rest mass $m = 0$ is exceptional in that the nonlocal field U does not necessarily reduce to the local field in the limit $\lambda = 0$. This is because the simultaneous equations (9), (11) and (15) with $m = 0$ and $\lambda = 0$ have solutions of the form

$$r_\mu = \pm(\lambda')^2 k_\mu , \qquad k_4 = \pm\sqrt{k_1^2 + k_2^2 + k_3^2} , \tag{27}$$

where λ' is an arbitrary constant with the dimension of length. More generally, the simultaneous equations with $m = 0$ and $\lambda \neq 0$ has the general solution of the form

$$r_\mu = r'_\mu \pm(\lambda')^2 k_\mu , \qquad k_4 = \pm\sqrt{k_1^2 + k_2^2 + k_3^2} , \tag{28}$$

where r'_μ is any particular solution of the same equations. Thus the radius of the particle without the rest mass cannot be defined so naturally as in the case of the particle with the rest mass, corresponding to the circumstance that there is no rest system in the former case. Detailed discussions of this particular case will be made elsewhere.

III. Quantization of Nonlocal Scalar Field

In order to show that the nonlocal field above considered represent exactly the assembly of identical particles with the finite radius, we have to quantize the field on the same lines as the method of second quantization in ordinary field theory. For this purpose, it is convenient to write (16) in another form

$$U(X_\mu, r_\mu) = \int\cdots\int (dk)^4 (dl)^4 u(k_\mu, l_\mu)\,\delta(k_\mu k^\mu + \kappa^2)\,\delta(k_\mu l^\mu) \times \delta(l_\mu l^\mu - \lambda^2)\exp(ik_\mu X^\mu)\prod_\mu \delta(r_\mu + l_\mu) \tag{29}$$

where l_μ is a four vector. The integrand is different from zero only for those values of k_μ, l_μ which satisfy the relations

$$k_\mu k^\mu + \kappa^2 = 0, \qquad l_\mu l^\mu - \lambda^2 = 0, \qquad k_\mu l^\mu = 0. \tag{30}$$

Accordingly, the matrix elements for the operator U are

$$(x'_\mu|U|x''_\mu) = \int\cdots\int (dk)^4 (dl)^4 u(k_\mu, l_\mu)\,\delta(k_\mu k^\mu + \kappa^2) \times \delta(k_\mu l^\mu)\,\delta(l_\mu l^\mu - \lambda^2)\exp(ik^\mu x'_\mu/2) \times \prod_\mu \delta(x'_\mu - x''_\mu + l_\mu)\cdot\exp(ik^\mu x''_\mu/2), \tag{31}$$

which is equivalent to the relation

$$U = \int\cdots\int (dk)^4 (dl)^4\, \bar{u}(k_\mu, l_\mu)\exp(ik_\mu x^\mu/2)\exp(il^\mu p_\mu/\hbar)\exp(ik_\mu x^\mu/2) \tag{32}$$

between the operators x^μ, p_μ and U, where

$$\bar{u}(k_\mu, l_\mu) = u(k_\mu, l_\mu)\,\delta(k_\mu k^\mu + \kappa^2)\,\delta(k_\mu l^\mu)\,\delta(l_\mu l^\mu - \lambda^2). \tag{33}$$

As the operators $k_\mu x^\mu$ and $l^\mu p_\mu$ in the same term on the right hand side of (32) are commutative with each other on account of the relations (1) and (30), (32) can also be written in the form

$$U = \int\cdots\int (dk)^4 (dl)^4 \cdot \bar{u}(k_\mu, l_\mu) \exp(ik_\mu x^\mu) \exp(il^\mu p_\mu/\hbar) \quad (32')$$

Similarly the operator U^*, which is the Hermitian conjugate of U, can be written in the form

$$U^* = \int\cdots\int (dk)^4 (dl)^4 \bar{u}^*(k_\mu, l_\mu) \exp(-ik_\mu x^\mu) \exp(-il^\mu p_\mu/\hbar) \quad (34)$$

Now the method of second quantization can be applied to our casein the following way: $\bar{u}(k_\mu, l_\mu)$ and $\bar{u}^*(k_\mu, l_\mu)$ in (32') and (34) are regarded as operators, which are Hermitian conjugate to each other and are noncommutative in general. The fact that the operators defined by

$$\left.\begin{aligned} U(k_\mu, l_\mu) &\equiv \exp(ik_\mu x^\mu) \exp(il^\mu p_\mu/\hbar) \\ U^*(k_\mu, l_\mu) &\equiv \exp(-ik_\mu x^\mu) \exp(-il^\mu p_\mu/\hbar) \end{aligned}\right\} \quad (35)$$

are unitary, i.e., satisfy the relation

$$U(k_\mu, l_\mu) U^*(k_\mu, l_\mu) = U^*(k_\mu, l_\mu) U(k_\mu, l_\mu) = 1, \quad (36)$$

suggests us the commutation relations

$$\left.\begin{aligned} [\bar{u}(k_\mu, l_\mu), \bar{u}^*(k'_\mu, l'_\mu)] &= \delta(k_\mu k^\mu + \varkappa^2)\delta(k_\mu l^\mu)\delta(l_\mu l^\mu - \lambda^2) \\ &\quad \times \prod_\mu \delta(k_\mu - k'_\mu)\delta(l_\mu - l'_\mu), \\ [\bar{u}(k_\mu, l_\mu), \bar{u}(k'_\mu, l'_\mu)] &= 0, \\ [\bar{u}^*(k_\mu, l_\mu), \bar{u}^*(k'_\mu, l'_\mu)] &= 0, \end{aligned}\right\} \quad (37)$$

$\frac{k^4}{|k^4|}$

which are obviously invariant with respect to the whole group of Lorentz transformations. In order to make the physical meaning of the relations (37) clear, we suppose the field in a cube with the edges of the length L, which is very large compared with λ. Then the effects of nonlocalizability of the field are negligible, because they are confined to small regions very near the surface of the cube.(12) In this case, the integrations with respect to k_μ on the right hand side of (32') and (34) are replaced by the summations with respect to k_μ, which take the values

$$k_1 = \frac{2\pi n_1}{L}, \quad k_2 = \frac{2\pi n_2}{L}, \quad k_3 = \frac{2\pi n_3}{L}, \quad k_4 = \pm\sqrt{k^2+\kappa^2}, \tag{38}$$

where n_1, n_2, n_3 are integers, either positive or negative, including zero. The integrations with respect to l_μ with k_μ fixed are replaced by those with respect to l'_μ defined by

$$l'_\mu = a_{\mu\nu} l_\nu \tag{39}$$

where $a_{\mu\nu}$ are given by (24). Further, we introduce the polar angles Θ, Φ, which are connected with l_1, l_2, l_3 just as θ, φ are connected with r_1', r_2', r_3' by the relations (25). Thus we obtain

$$U = \sum_{k_1 k_2 k_3} \iint \left(\frac{2\pi}{L}\right)^3 \frac{\lambda \sin\Theta \, d\Theta \, d\Phi}{4\kappa\sqrt{k^2+\kappa^2}} \times \{u(\underline{k}, \Theta, \Phi)U(\underline{k}, \Theta, \Phi) + v^*(\underline{k}, \Theta, \Phi)U^*(\underline{k}, \Theta, \Phi)\}, \tag{40}$$

where

$$\left.\begin{aligned}
u(\underline{k}, \Theta, \Phi) &\equiv u(k_1, k_2, k_3, -\sqrt{k^2+\kappa^2}; l_\mu) \\
v^*(\underline{k}, \Theta, \Phi) &\equiv u(-k_1, -k_2, -k_3, \sqrt{k^2+\kappa^2}; -l_\mu) \\
U(\underline{k}, \Theta, \Phi) &\equiv \exp(i\underline{k}\underline{x} + i\sqrt{k^2+\kappa^2}\cdot x_4)\exp(il^\mu p_\mu/\hbar) \\
U^*(\underline{k}, \Theta, \Phi) &\equiv \exp(-i\underline{k}\underline{x} - i\sqrt{k^2+\kappa^2}\cdot x_4)\exp(-il^\mu p_\mu/\hbar)
\end{aligned}\right\} \tag{41}$$

Finally, by expanding u and v^* into series of spherical harmonics, we obtain

$$U = \sum_{k_1 k_2 k_3} \sum_{l,m} \left(\frac{2\pi}{L}\right)^3 \frac{\lambda}{4\kappa\sqrt{k^2+\kappa^2}} \{ u(\underline{k}, l, m) U(\underline{k}, l, m) + v^*(\underline{k}, l, m) U^*(\underline{k}, l, m) \} \tag{42}$$

where

$$u(\underline{k}, l, m) \equiv \iint u(\underline{k}, \Theta, \Phi) \tilde{P}_l^m(\Theta, \Phi) \sin\Theta \, d\Theta \, d\Phi$$
$$v^*(\underline{k}, l, m) \equiv \iint v^*(\underline{k}, \Theta, \Phi) P_l^m(\Theta, \Phi) \sin\Theta \, d\Theta \, d\Phi, \tag{43}$$

$$U(\underline{k}, l, m) \equiv \iint U(\underline{k}, \Theta, \Phi) P_l^m(\Theta, \Phi) \sin\Theta \, d\Theta \, d\Phi$$
$$U^*(\underline{k}, l, m) \equiv \iint U^*(\underline{k}, \Theta, \Phi) \tilde{P}_l^m(\Theta, \Phi) \sin\Theta \, d\Theta \, d\Phi, \tag{44}$$

assuming that the spherical harmonics $P_l^m(\Theta, \Phi)$ and their complex conjugate $\tilde{P}_l^m(\Theta, \Phi)$ are normalized according to the rule:

$$\iint \tilde{P}_l^m(\Theta, \Phi) P_l^m(\Theta, \Phi) \sin\Theta \, d\Theta \, d\Phi = 1. \tag{45}$$

Similarly, U^* is transformed into the form

$$U^* = \sum_{k_1 k_2 k_3} \sum_{l,m} \left(\frac{2\pi}{L}\right)^3 \frac{\lambda}{4\kappa\sqrt{k^2+\kappa^2}} \{ v(\underline{k}, l, m) U(\underline{k}, l, m) + u^*(\underline{k}, l, m) U^*(\underline{k}, l, m) \}, \tag{46}$$

where

$$u^*(\underline{k}, l, m) \equiv \iint u^*(\underline{k}, \Theta, \Phi) P_l^m(\Theta, \Phi) \sin\Theta \, d\Theta \, d\Phi$$
$$v(\underline{k}, l, m) \equiv \iint v(\underline{k}, \Theta, \Phi) \tilde{P}_l^m(\Theta, \Phi) \sin\Theta \, d\Theta \, d\Phi. \tag{47}$$

By the same transformation, we obtain from (37) the commutation

$$
\left.
\begin{aligned}
&[a(\underline{k},l,m),\ a^*(\underline{k}',l',m')] = \delta(\underline{k},\underline{k}')\,\delta(l,l')\,\delta(m,m') \\
&[b(\underline{k},l,m),\ b^*(\underline{k}',l',m')] = \delta(\underline{k},\underline{k}')\,\delta(l,l')\,\delta(m,m') \\
&[a(\underline{k},l,m),\ b(\underline{k}',l',m')] = 0
\end{aligned}
\right\} \qquad (48)
$$

etc.

for the operators defined by

$$
\left.
\begin{aligned}
a(\underline{k},l,m) &\equiv \sqrt{\left(\frac{2\pi}{L}\right)^3 \frac{\lambda}{4\kappa\sqrt{\underline{k}^2+\kappa^2}}}\cdot u(\underline{k},l,m) \\
a^*(\underline{k},l,m) &\equiv \sqrt{\quad {''} \quad}\cdot u^*(\underline{k},l,m) \\
b(\underline{k},l,m) &\equiv \sqrt{\quad {''} \quad}\cdot v(\underline{k},l,m) \\
b^*(\underline{k},l,m) &\equiv \sqrt{\quad {''} \quad}\cdot v^*(\underline{k},l,m),
\end{aligned}
\right\} \qquad (49)
$$

Hence, each of the operators defined by

$$
\begin{aligned}
n^+(\underline{k},l,m) &\equiv a^*(\underline{k},l,m)\,a(\underline{k},l,m) \\
n^-(\underline{k},l,m) &\equiv b^*(\underline{k},l,m)\,b(\underline{k},l,m)
\end{aligned}
\qquad (50)
$$

has eigenvalues 0, 1, 2, ... and can be interpreted as the number of particles in the state characterized by the quantum numbers $\underline{k}$, l, m with either positive or negative charge. Thus the nonlocal field above considered corresponds to the assembly of charged Bosons with the mass m, the radius λ and the spin 0. It can easily be shown that in the limit $\lambda \rightarrow 0$, U reduces to the familiar quantized local field for Bosons apart from the extra factor

$$
\delta(x_1'-x_1'')\,\delta(x_2'-x_2'')\,\delta(x_3'-x_3'')\,\delta(x_4'-x_4''), \qquad (51)
$$

which must be omitted, whenever we go over from nonlocal to local field.

The nonlocal neutral field can be obtained, if we assume that the field ~~of~~ U is Hermitian, i.e., $U = U^*$. In this case, we cannot discriminate between a and b, so that we have instead of (42) and (46) the relation

$$U = \sum_{k_1, k_2, k_3} \sum_{l, m} \left(\frac{2\pi}{L}\right)^3 \frac{\lambda}{4\kappa\sqrt{k^2+\kappa^2}} \{ u(\underline{k}, l, m) U(\underline{k}, l, m) + u^*(\underline{k}, l, m) U^*(\underline{k}, l, m) \}. \tag{52}$$

It should be noticed, further, that we could start from the commutation relations

$$\begin{aligned} [\bar{u}(k_\mu, l_\mu), \bar{u}^*(k'_\mu, l'_\mu)]_+ &= \delta(k_\mu k^\mu + \kappa^2)\, \delta(k_\mu l^\mu)\, \delta(l_\mu l^\mu - \lambda^2) \\ &\quad \times \prod_\mu \delta(k_\mu - k'_\mu)\, \delta(l_\mu - l'_\mu) \\ [\bar{u}(k_\mu, l_\mu), \bar{u}(k'_\mu, l'_\mu)]_+ &= 0 \\ [\bar{u}^*(k_\mu, l_\mu), \bar{u}^*(k'_\mu, l'_\mu)]_+ &= 0 \end{aligned} \tag{37'}$$

instead of (37), where

$$[A, B]_+ \equiv AB + BA \tag{53}$$

for any two operators A and B. However, in this case, we arrive at the wellknown contradiction in the limit of $\lambda \to 0$, which prohibits the elementary particles with spin 0 from obeying Fermi statistics.

IV. Nonlocal Spinor Field

The above considerations can easily be extended to the nonlocal vector field without introducing anything essentially new which needs detailed discussion. On the contrary, the case of the nonlocal spinor field must be investigated from the beginning. We start from the spinor operator ψ with four components, which transform as the components of Dirac wave function. Each of these components can be considered as a nonlocal operator just like the operator U in the case of the scalar field. As an extension of Dirac's wave equations for the local spinor field, we assume the relations between the operators x^μ, p_μ and ψ:

$$\gamma^\mu [p_\mu, \psi] + mc\psi = 0 \tag{54}$$

$$\beta_\mu [x^\mu, \psi] + \lambda\psi = 0 \tag{55}$$

where γ^μ are wellknown Dirac matrices forming a four vector, which satisfy the commutation relations among themselves:

$$[\gamma^\mu, \gamma_\nu]_+ = -2\delta_{\mu\nu}. \tag{56}$$

We assume similar commutation relations for matrices β_μ:

$$[\beta^\mu, \beta_\nu]_+ = 2\delta_{\mu\nu}. \tag{57}$$

Then, we obtain by iteration the relations

$$[p^\mu [p_\mu, \psi]] + m^2c^2\psi = 0 \tag{58}$$

$$[x_\mu [x^\mu, \psi]] - \lambda^2\psi = 0, \tag{59}$$

which have the same form as the relations (4) and (5) for the scalar field. However, the matrices β_μ have to be so chosen as to satisfy the demand that the relations (54) and (55) are compatible with each other. Namely, the relations

$$\beta_\mu \gamma^\nu [x^\mu [p_\nu, \psi]] = \lambda mc\psi \tag{60}$$

$$\gamma^\nu \beta_\mu [p_\nu [x^\mu, \psi]] = \lambda mc\psi \tag{61}$$

which can be readily obtained by combining (54) and (55) must have the same form so that β_μ must satisfy an additional condition:

$$[\beta_\mu, \gamma^\nu][x^\mu [p_\nu, \psi]] = 0. \tag{62}$$

This condition reduces to the form

$$[x^\mu [p_\mu, \psi]] = 0, \tag{63}$$

which is the same as the condition (12) or (13) for the scalar field, if β_μ are so chosen as to satisfy the commutation relations

$$[\beta_\mu, \gamma^\nu] = c' \delta_{\mu\nu}, \tag{64}$$

where c' is a matrix with the determinant different from zero. (64) can be satisfied by matrices γ^μ, β_μ which are expressed in the form

$$\gamma^1 = i\rho_2\sigma_1, \quad \gamma^2 = i\rho_2\sigma_2, \quad \gamma^3 = i\rho_2\sigma_3, \quad \gamma^4 = \rho_3 \tag{65}$$

$$\beta_1 = \rho_3\sigma_1, \quad \beta_2 = \rho_3\sigma_2, \quad \beta_3 = \rho_3\sigma_3, \quad \beta_4 = -i\rho_2 \tag{66}$$

in terms of sets of mutually independent Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ and ρ_1, ρ_2, ρ_3. It is wellknown that the matrices as given by (66) do not form an ordinary vector, but a pseudovector. Thus, if we confine our attention to the proper Lorentz transformation, the relations (54) and (55) are both invariant. However, if we perform the improper Lorentz transformation, for which the determinant of the transformation matrix has the value -1 instead of +1, the form of the relation (55) changes into

$$\beta_\mu [x^\mu, \psi] - \lambda \psi = 0, \tag{67}$$

whereas the relation (54) is invariant.* In other words, the fundamental equations for the nonlocal spinor field, which has similar properties as the nonlocal scalar field considered in the preceding sections, can be constructed so as to be invariant with respect to the whole group of Lorentz transformations including reflections, only if both forms (55) and (67) are put together into one relation for one spinor field with the components twice as many as the four components for the usual spinor field.* This is equivalent to introduce one more independent set of Pauli matrices ω_1, ω_2, ω_3 and to assume that all of the matrices γ^μ, β_μ have each eight rows and columns characterized by eight combinations of eigenvalues of $\sigma_3, \rho_3, \omega_3$. Therewith the spinor must have eight components, first four components and the remaining four corresponding respectively to the eigenvalues +1 and -1 of ω_3.

In order to establish the invariance of fundamental laws for the nonlocal spinor field with respect to the whole group of Lorentz transformations, we assume further that ω_2 and ω_3 change sign under improper Lorentz transformation, whereas ω_1 does not. We can now adopt the relation

$$\beta_\mu [x^\mu, \psi] + \omega_3 \psi = 0 \tag{68}$$

in place of (55). It is clear from the above arguments that the

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* However, there is a more satisfactory way of removing this difficulty without increasing the number of components. See H. Yukawa, Phys. Rev. 76, , (1949)

fundamental equations (54) and (68) are invariant with respect to the whole group of Lorentz transformations. However, for the purpose of proving it more explicitly, we consider the transformation properties of ψ with respect to the Lorentz transformation, whereby we assume that the matrices γ^μ, β_μ have prescribed forms as defined by (65), (66) independent of the coordinate system. In the usual theory, in which the spinor field ψ has four components, we have the linear transformation

$$\psi' = S\psi \tag{69}$$

associated with each of the Lorentz transformations for the coordinates:

$$x_\mu' = a_{\mu\nu} x_\nu \tag{70}$$

where S is a matrix with four rows and four columns.[14] In our case, in which the spinor ψ has eight components, we assume the same form for S in (69) except that the numbers of rows and columns are doubled, when (70) is a proper Lorentz transformation with the determinant +1, whereas we have to replace (69) by

$$\psi' = \omega_1 S \psi, \tag{71}$$

when (70) is an improper Lorentz transformation with the determinant -1. This guarantees the invariance of the relation (68) with respect to improper as well as proper Lorentz transformations.

Now the question is, what is the new degree of freedom which corresponds to the eigenvalues +1 and -1 of ω_3. We are naturally induced to connect the eigenvalues +1 and -1 with the neutral and charged states of the spinor particle, as was customary in the theory of the nucleon. This is, however, misleading because the operator

$\frac{1-\omega_3}{2}$ is not an invariant and cannot be identified with the operator, which sorts out the charged states. Thus, we have to consider the possibility of associating the quantities ω_1, ω_2, ω_3 with the internal motion of the spinor particle.

Now, each component ψ_i $(i=1,2,\cdots 8)$ of the spinor ψ can be represented as a matrix $(x'_\mu|\psi_i|x''_\mu)$ in the representation in which x_μ are diagonal. $(x'_\mu|\psi_i|x''_\mu)$ can be regarded, in turn, as a function $\psi_i(X_\mu, r_\mu)$ of X_μ, r_μ, where X_μ, r_μ are defined by (6). Therewith the relations (54) and (68) can be represented by

$$\gamma^\mu \frac{\partial \psi(X_\mu, r_\mu)}{\partial X^\mu} - i\kappa\psi(X_\mu, r_\mu) = 0 \tag{72}$$

$$\beta_\mu r^\mu \psi(X_\mu, r_\mu) + \omega_3 \lambda \psi(X_\mu, r_\mu) = 0 \tag{73}$$

respectively, where $\psi(X_\mu, r_\mu)$ is a spinor with eight components $\psi_i(X_\mu, r_\mu)$ $(i=1,2,\cdots 8)$. The simultaneous equations (72), (73) for $\psi(X_\mu, r_\mu)$ have a particular solution of the form

$$\psi(X_\mu, r_\mu) = \bar{u}(k_\mu, r_\mu)\exp(ik_\mu X^\mu) \tag{74}$$

where $\bar{u}(k_\mu, r_\mu)$ is a spinor with eight components satisfying

$$\left.\begin{aligned} &\gamma^\mu k_\mu \bar{u} - \kappa\bar{u} = 0 \\ &\beta_\mu r^\mu \bar{u} + \lambda\omega_3 \bar{u} = 0. \end{aligned}\right\} \tag{75}$$

It follows immediately from (75) that $\bar{u}$ must satisfy

$$\left.\begin{aligned} &(k_\mu k^\mu + \kappa^2)\bar{u} = 0 \\ &(r_\mu r^\mu - \lambda^2)\bar{u} = 0 \\ &k_\mu r^\mu \bar{u} = 0 \end{aligned}\right\} \tag{76}$$

so that u can be written in the form

$$\bar{u} = u(k_\mu, r_\mu)\,\delta(k_\mu k^\mu + \kappa^2)\,\delta(r_\mu r^\mu - \lambda^2)\,\delta(k_\mu r^\mu) \tag{77}$$

Each of eight components of u can be expanded in the same way as the scalar operator u in the preceding sections. The second quantization can be performed by assuming commutation relations of the type (37) between field quantities, so that the nonlocal field represents an assembly of Fermions with the mass m, the radius λ and the spin 1/2. Further analysis of the nonlocal spinor field will be made in Part II of this paper. At any rate it is now clear that there exist nonlocal scalar, vector and spinor fields, each corresponding to the assembly of particles with the mass, radius and the spin 0, 1 and 1/2.

Now the question, with which we are met first, when we go over to the case of two or more nonlocal fields interacting with each other, is whether we can start from Schrödinger equation for the total system (or any substitute for it), thus retaining the most essential feature of quantum mechanics. We know that Schrödinger equation in its simplest form is not obviously relativistic in that it is a differential equation with the time variable as independent variable, space coordinates being regarded merely as parameters. It can be extended to a relativistic form as in Dirac's many-time formalism or, more satisfactorily, in Tomonaga-Schwinger's super-many-time formalism, as long as we are dealing with local fields satisfying the infinitesimal commutation relations. However, if we introduce the nonlocal fields or the nonlocalizability in the interaction between local fields, the clean-cut distinction between space-like and time-like directions is impossible in general. This is because the interaction term in the Lagrangian or Hamiltonian for the system of nonlocal fields contains the displacement operators in the time-like directions as well as those in the space-like directions. Thus, even if there exists an equation of Schrödinger type, it cannot be solved, in general, by giving the initial condition at a certain time in the past. Under these circumstances, we must have recourse to more general formalism such as the S-matrix scheme, which was proposed by Heisenberg.[13] In other words, we had better start from the integral formalism rather than the differential formalism. In local field theory, the integral formalism such as that which was developed by Feynman can be deduced from the ordinary differential formalism.[14] [15] In nonlocal field theory, however, it may well happen that we are left only with some kind of integral formalism. In fact it will be shown in Part II that the nonlocal fields above considered can be fitted into the S-matrix scheme.

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FOOTNOTES

(1) As to the list of recent works by Tomonaga, Schwinger and others, see V. Weisskopf, Rev. Mod. Phys. 21, 305 (1949).

(2) W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949). The method of regulators is an extension of cut-off procedures by R. P. Feynman, Phys. Rev. 74, 1430 (1948) and by D. Rivier and E.C.G. Stueckelberg, Phys. Rev. 74, 218 (1948).

(3) Field theories by Bopp, Podolsky, Dirac and others are more formalistic in that negative energy Bosons are taken into account, whereas those by Pais, Sakata and Hara are more realistic.

(4) D. Feldman, Phys. Rev. 76, (1949). The author is indebted to Dr. Feldman for discussing the subject before publication of his paper.

(5) H. Umezawa, J. Yukawa and E. Yamada, Prog. Theor. Phys. 4, 25, 113 (1949).

(6) H. Fukuda, and Y. Miyamoto, Prog. Theor. Phys. 4, 235, (1949); S. Sasaki, S. Oneda and S. Ozaki, Prog. Theor. Phys. 4, (1949); J. Steinberger, Phys. Rev. 76, (1949). See further a comprehensive survey of recent works on meson theory by H. Yukawa, Rev. Mod. Phys. 21, 474 (1949).

(7) A preliminary account of the content of this paper was published in H. Yukawa, Phys. Rev. 76, 300 (1949), which will be cited as I.

(8) H. J. Bhabha, Proc. Ind. Acad. Sci. A 21, 241 (1945); Rev. Mod. Phys. 17, 200 (1945).

(9) M. Born, Nature 163, 207 (1949); H. S. Green, Nature 163, 208 (1949); M. Born and H. S. Green, Proc. Roy. Soc. Edin. A 92, 470 (1949).

(10) $U(X_\mu, r_\mu)$ as given by the expression (6) in I was not the most general form in that the coefficients $b(k_\mu)$ were independent of l_μ, which corresponded to ignore the internal rotation. The author is indebted to Professor R. Serber for calling attention to this point.

(11) More precisely, L must be large compared with $\lambda/\sqrt{1-\beta^2}$, where βc is the maximum velocity of particles in consideration.

(12) See, for example, W. Pauli, Handb. d. Phys. 24, Part 1, 83 (1933).

(13) W. Heisenberg, Zeits. f. Phys. 120, 513, 673 (1943); 123, 93 (1944); C. Møller, K. Danske Vidensk. Selsk. 23, Nr. 1 (1945); 22, Nr. 19 (1946).

(14) R. P. Feynman, Phys. Rev. 76, 749, 769 (1949).

(15) F. J. Dyson, Phys. Rev. 75, 486, 1736 (1949). See also many papers by E.C.G. Stueckelberg, which appeared mainly in Helv. Phys. Acta.

Remarks on Nonlocal Spinor Field

The problem of invariance of the relation (55) with respect to improper Lorentz transformation can be solved without introducing extra components to the spinor field. Namely, we take advantage of the antisymmetric tensor of the fourth rank with the components $\varepsilon_{\kappa\lambda\mu\nu}$, which are +1 or -1 according as $(\kappa, \lambda, \mu, \nu)$ are even or odd permutations of (1, 2, 3, 4) and 0 otherwise. Further we take into account the relations

$$i\beta_\nu = \gamma^\kappa\gamma^\lambda\gamma^\mu \tag{78}$$

where $(\kappa, \lambda, \mu, \nu)$ are even permutations of (1, 2, 3, 4). Then (55) can be written in the form

$$\frac{1}{6}\sum_{\kappa\lambda\mu\nu} \varepsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu[x^\nu, \psi] + i\lambda\psi = 0, \tag{79}$$

which is obviously invariant with respect to the whole group of Lorentz transformations. The invariance can be proved more explicitly by associating a linear transformation

$$\psi' = S\psi \tag{80}$$

with each of the Lorentz transformations (70), where S is a matrix with four rows and columns satisfying the relations

$$S\gamma^\mu S^{-1} = a_{\mu\nu}\gamma^\nu. \tag{81}$$

It should be noticed, however, that the relation (79) is a unification of the relations (55) and (67) rather than the simple reproduction of (55), because (79) must be identified with (67) in the coordinate system, which is connected with the original coordinate system by an improper Lorentz transformation with the determinant -1.

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Remarks on Non-local Spinor Field

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October 18, 1949

IN a recent letter to the editor,[1] it was shown that quantized non local fields could be so constructed as to represent assemblies of particles with the definite mass and radius. In a paper, which will appear very soon,[2] detailed account is given together with the elucidation of most of the points, on which the author was not very sure when he wrote the above letter.[1] However, there is still one point, which seems to the author to be unsatisfactory. Namely, in the case of non-local spinor field, we assumed the commutation relation

$$\beta_\mu[x^\mu, \psi]+\lambda\psi=0 \tag{1}$$

between the space-time operators x^μ and the non-local spinor operator ψ, in addition to the commutation relation

$$\gamma^\mu[p_\mu, \psi]+mc\psi=0 \tag{2}$$

between ψ and the space-time displacement operators p_μ. Further, we assumed that γ^μ, β_μ, which were matrices with four rows and columns, were defined by

$$\left.\begin{array}{llll}\gamma^1=i\rho_2\sigma_1, & \gamma^2=i\rho_2\sigma_2, & \gamma^3=i\rho_2\sigma_3, & \gamma^4=\rho_3 \\ \beta_1=\rho_3\sigma_1, & \beta_2=\rho_3\sigma_2, & \beta_3=\rho_3\sigma_3, & \beta_4=-i\rho_2\end{array}\right\}. \tag{3}$$

Now the difficulty was that, in contrast to (2), the relation (1) was not invariant with respect to the improper Lorentz transformation with the determinant -1, but was to change itself into the form

$$\beta_\mu[x^\mu, \psi]-\lambda\psi=0. \tag{4}$$

In the paper mentioned above,[2] a way of removing this difficulty was indicated, but was very unsatisfactory in that the number of components of the spinor ψ was to be increased from 4 to 8 without any immediate physical interpretation for the extra degree of freedom. It came to the author's notice very recently that the following alternative way was far more acceptable in that no extra components of the spinor were introduced. Namely, we take advantage of the antisymmetric tensor of the fourth rank with the components $\epsilon_{\kappa\lambda\mu\nu}$ which are $+1$ or -1 according as $(\kappa, \lambda, \mu, \nu)$ are even or odd permutations of (1, 2, 3, 4) and 0 otherwise.[3] Further we take into account the relations

$$i\beta_\nu=\gamma^\kappa\gamma^\lambda\gamma^\mu, \tag{5}$$

where $(\kappa, \lambda, \mu, \nu)$ are even permutations of (1, 2, 3, 4). Then (1) can be written in the form

$$\frac{1}{6}\sum_{\kappa\lambda\mu\nu}\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu[x^\nu, \psi]+i\lambda\psi=0, \tag{6}$$

which is obviously invariant with respect to the whole group of Lorentz transformations. However, the invariance of (6) can be proved more explicitly by transforming ψ, while the matrices γ^μ are assumed to retain their prescribed forms as defined by (3) independent of the coordinate system. Namely, we can associate a linear transformation

$$\psi'=S\psi \tag{7}$$

with each of the Lorentz transformation

$$x_\mu'=a_{\mu\nu}x_\nu, \tag{8}$$

where S is a matrix with four rows and columns satisfying the relations

$$S\gamma^\mu S^{-1}=a_{\nu\mu}\gamma^\nu. \tag{9}$$

If we insert (7), (8) and (9) in (6) and take advantage of the fact that $\epsilon_{\kappa\lambda\mu\nu}$ are components of a tensor of the fourth rank, we obtain the commutation relation

$$\frac{1}{6}\sum_{\kappa\lambda\mu\nu}\epsilon'_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu[x'^\nu, \psi']+i\lambda\psi'=0, \tag{10}$$

which has the same form as (6).

It should be noticed, however, that the relation (6) is to be regarded as a unification of (1) and (4) rather than the mere reproduction of (1), because (6) must be identified with (4) in the coordinate system, which is connected with the original coordinate system by an improper Lorentz transformation with the determinant -1.

[1] H. Yukawa, Phys. Rev. **76**, 300 (1949).
[2] H. Yukawa, Phys. Rev. (to be published).
[3] The antisymmetric tensor $\epsilon_{\kappa\lambda\mu\nu}$ was useful for unifying the scalar and pseudoscalar fields as well as the vector and the pseudovector fields as shown by M. Schoenberg, Phys. Rev. **60**, 468 (1941).

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On the Radius of the Elementary Particle

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June 2, 1949

RECENTLY it was shown[1] that the present theory of quantized fields could be generalized in conformity with the idea of reciprocity proposed by Born[2] to the case, in which field quantities were no longer functions of time and space coordinates alone.[3] The possibility of a theory of such non-localizable fields was illustrated by simple types of scalar neutral zero-mass fields in vacuum. It was further indicated that the theory could be extended to the particle with non-zero rest mass by considering non-localizable fields in five-dimensional space. In this connection, Pais[4] pointed out that the theory of non-localizable fields with non-zero mass could also be formulated in the usual four-dimensional space. In fact, as will be shown in the following paragraphs, the classical rigid sphere model for the electron, which has been believed to be much too classical to be incorporated into quantum theory of fields, can be reproduced as a very simple type of quantized non-localizable fields satisfying postulates of reciprocity.

First we consider a scalar non-localizable field with the mass m satisfying a set of reciprocal commutation relations

$$[x^\mu, p_\nu] = i\hbar\delta_{\mu\nu}, \tag{1}$$

$$[x_\mu[x^\mu, U]] - \lambda^2 U = 0, \tag{2}$$

$$[p_\mu[p^\mu, U]] + m^2c^2 U = 0, \tag{3}$$

$$[p_\mu[x^\mu, U]] = 0, \tag{4}$$

where x^μ denote contravariant space-time operators with $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = ct$, and p_μ ($\mu = 1, 2, 3, 4$) are covariant space-time displacement operators. For any two operators A and B, we write

$$[A, B] \equiv AB - BA. \tag{5}$$

λ is a constant with the dimension of length, which will be interpreted as the radius of the elementary particle described by the field U.[5]

Now the scalar operator U can be expressed as a matrix $(x_\mu'|U|x_\mu'')$ in the representation in which operators x_μ are diagonal. The matrix element $(x_\mu'|U|x_\mu'')$ can alternatively be regarded as a function $U(X_\mu, r_\mu)$ of $X_\mu = \frac{1}{2}(x_\mu' + x_\mu'')$ and $r_\mu = x_\mu' - x_\mu''$. Then (2), (3), and (4) reduce to

$$(r_\mu r^\mu - \lambda^2)U = 0, \tag{2'}$$

$$\left(\frac{\partial^2}{\partial X_\mu \partial X^\mu} - \kappa^2\right)U = 0, \quad \kappa = mc/\hbar, \tag{3'}$$

$$r^\mu(\partial U/\partial X^\mu) = 0. \tag{4'}$$

General form of U satisfying these equations simultaneously is

$$U(X_\mu, r_\mu) = \int\cdots\int (dk)^4(dl)^4\delta(k_\mu k^\mu + \kappa^2)\delta(k_\mu l^\mu)\delta(l_\mu l^\mu - \lambda^2) \times b(k_\mu)\exp(ik_\mu X^\mu)\delta(r_1 + l_1)\delta(r_2 + l_2)\delta(r_3 + l_3)\delta(r_4 + l_4). \tag{6}$$

δ-functions in (6) are equivalent to restrict two four-dimensional integrations with respect to two vectors k_μ and l^μ to domains satisfying the conditions

$$k_\mu k^\mu + \kappa^2 = 0, \quad l_\mu l^\mu - \lambda^2 = 0, \quad k_\mu l^\mu = 0. \tag{7}$$

A simple physical interpretation can be given to these restrictions. Suppose that the particle is at rest represented by the plane wave vector $k_1 = k_2 = k_3 = 0$, $k_4 = \pm\kappa$. In this particular case, the conditions (7) reduce to

$$l_1^2 + l_2^2 + l_3^2 = \lambda^2, \quad l_4 = 0, \tag{8}$$

which mean that the contribution of the particle at rest to the matrix elements of U is restricted to values of r_μ satisfying

$$r_1^2 + r_2^2 + r_3^2 = \lambda^2, \quad r_4 = 0. \tag{9}$$

The operator U has a character of the wave amplitude rather than the density itself. However, the density operator, which is quadratic or bilinear in U, has again the matrix elements, which are different from zero only for

$$(r_{1/2})^2 + (r_{2/2})^2 + (r_{3/2})^2 \leqq \lambda^2. \tag{10}$$

Since the above formulation is perfectly relativistic, the whole field U can be regarded as a field theoretical representation of an assembly of elementary particles with the mass m and the radius λ, which move without any change in their form except the Lorentz contraction associated with the change of the proper time axis. Of course, $b(k_\mu)$ should be quantized in the usual way in order to make the correspondence with the particle picture complete.

The above arguments can be extended to the vector field with no essential alteration. The case of the Dirac particle is more interesting. We start from the relations

$$\gamma_\mu'[x^\mu, \psi] + \lambda\psi = 0, \tag{11}$$

$$\gamma^\mu[p_\mu, \psi] + mc\psi = 0, \tag{12}$$

instead of (2) and (3), where ψ denotes a non-localizable spinor field with four components. γ^μ are well known Dirac matrices

$$\gamma^1 = i\rho_2\sigma_x, \quad \gamma^2 = i\rho_2\sigma_y, \quad \gamma^3 = i\rho_2\sigma_z, \quad \gamma^4 = \rho_3, \tag{13}$$

forming a four-vector and γ_μ' are also four Dirac matrices

$$\gamma_1' = \rho_3\sigma_x, \quad \gamma_2' = \rho_3\sigma_y, \quad \gamma_3' = \rho_3\sigma_z, \quad \gamma_4' = -i\rho_2, \tag{14}$$

forming a pseudovector.[6] If we again represent ψ by four functions of X_μ and r_μ, we can deduce from (11) and (12) by taking account of the well-known commutation relations between Dirac matrices the result that each of four functions must have the form similar to $U(X_\mu, r_\mu)$ in (6).

Whether a consistent field theory can be constructed by starting from the above model is, of course, an open question, because the introduction of any kind of extended source is almost inevitably accompanied by the departure from the Schrödinger equation. Probably a formalism, which is more or less similar to Heisenberg's S-matrix scheme,[7] will be needed for further developments.

Detailed accounts will be made in later issues of this journal and Progress of Theoretical Physics.

The author wants to express his hearty thanks to Professor Oppenheimer for giving him the opportunity of staying at the Institute for Advanced Study, Princeton, and also to Professor Oppenheimer, Professor Uhlenbeck, and Dr. Pais for stimulating discussions.

* On leave of absence from Kyoto University, Kyoto, Japan.

[1] H. Yukawa, Prog. Theor. Phys. **2**, 209 (1947); **3**, 205, 452 (1948).

[2] M. Born, Proc. Roy. Soc. **A165**, 291 (1938); **A166**, 552 (1938). The idea of reciprocity has been developed since then by Born, Landé, and others in various ways. Very recently, attempts at the problem of masses of elementary particles were made by M. Born, Nature **163**, 207 (1949); H. Green, Nature **163**, 208 (1949).

[3] Non-localizable fields were first considered by Markow, J. Phys. **2**, 453 (1940). More general considerations were made by Born and Peng, Proc. Roy. Soc. Edinburgh **62**, 40, 92, 127 (1944).

[4] The author is indebted to Dr. Pais for informing him of unpublished results.

[5] From the relations (1) and (4), it follows immediately that

$$[x^\mu[p_\mu, U]] = 0.$$

[6] λ is not a pure scalar, but a pseudoscalar, which changes sign by reflection. However, the form of ψ is not altered by the change of sign of λ.

[7] W. Heisenberg, Zeits. f. Physik **120**, 513, 673 (1943); **123**, 93 (1944).

A copy of the letter to the Phys. Rev. 76 (1949), 300 (July 15)

ON THE RADIUS OF THE ELEMENTARY PARTICLE

Hideki Yukawa*

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May 30, 1949

Recently it was shown[1] that the present theory of quantized fields could be generalized in conformity with the idea of reciprocity proposed by Born[2] to the case, in which field quantities were no more functions of time and space coordinates alone.[3] The possibility of a theory of such nonlocalizable fields was illustrated by simple types of scalar neutral zero-mass fields in vacuum. It was further indicated that the theory could be extended to the particle with non-zero rest mass by considering non-localizable fields in five dimensional space. In this connection, Pais[4] pointed out that the theory of nonlocalizable fields with non-zero mass could also be formulated in the usual four dimensional space. In fact, as will be shown in the following paragraphs, the classical rigid sphere model for the electron, which has been believed to be much too classical to be incorporated into quantum theory of fields, can be reproduced as a very simple type of quantized nonlocalizable fields satisfying postulates of reciprocity.

First we consider a scalar nonlocalizable field U with the mass m satisfying a set of reciprocal commutation relations

$$[x^{\mu}, p_{\nu}] = i\hbar\delta_{\mu\nu} \qquad (1)$$

$$[x_{\mu}, (x^{\mu}, U)] - \lambda^2 U = 0 \qquad (2)$$

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$$\left[p_\mu (p^\mu, U)\right] + m^2c^2\, U = 0 \tag{3}$$

$$\left[p_\mu (x^\mu, U)\right] = 0 \tag{4}$$

where x^μ denote contravariant space time operators with $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = ct$ and $p_\mu (\mu = 1, 2, 3, 4)$ are covariant space-time displacement operators. For any two operators A and B, we write

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Now the scalar operator U can be expressed as a matrix $(x'_\mu | U | x''_\mu)$ in the representation in which operators x_μ are diagonal. The matrix element $(x'_\mu | U | x''_\mu)$ can alternatively be regarded as a function $U(X_\mu, r_\mu)$ of $X_\mu = 1/2(x'_\mu + x''_\mu)$ and $r_\mu = x'_\mu - x''_\mu$. Then (2), (3) and (4) reduce to

$$(r_\mu r^\mu - \lambda^2)\, U = 0 \tag{2$'$}$$

$$\left(\frac{\partial^2}{\partial X_\mu \partial X^\mu} - \kappa^2\right) U = 0 , \qquad \kappa = \frac{mc}{\hbar} \tag{3$'$}$$

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General form of U satisfying these equations simultaneously is

$$U(X_\mu, r_\mu) = \int \cdots \int (dk)^4 (d\ell)^4\, \delta\left[k_\mu k^\mu + \kappa^2\right] \delta(k_\mu \ell^\mu)\, \delta(\ell_\mu \ell^\mu - \lambda^2)\ \times$$
$$\times\ b(k_\mu) \exp(ik_\mu X^\mu)\, \delta(r_1 + \ell_1)\, \delta(r_2 + \ell_2)\, \delta(r_3 + \ell_3)\, \delta(r_4 + \ell_4) \tag{6}$$

δ-functions in (6) are equivalent to restrict two four dimensional integrations with respect to two vectors k_μ and ℓ_μ to domains satisfying the conditions

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A simple physical interpretation can be given to these restrictions. Suppose that the particle is at rest represented by the plane wave vector $k_1 = k_2 = k_3 = 0$, $k_4 = \pm \kappa$. In this particular case, the conditions (7) reduce to

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which mean that the contribution of a particle at rest to the matrix elements of U is restricted to values of r_μ satisfying

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forming a four vector and γ'_μ are also four Dirac matrices

$$\gamma'_1 = \rho_3\sigma_x, \quad \gamma'_2 = \rho_3\sigma_y, \quad \gamma'_3 = \rho_3\sigma_z, \quad \gamma'_4 = -i\rho_2 \qquad (14)$$

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FOOTNOTES

1. Yukawa, Prog. Theor. Phys. 2, 209 (1947); 3, 205, 452 (1948).

2. Born, Proc. Roy. Soc. A 165, 291 (1938); 166, 552 (1938). The idea of reciprocity has been developed since then by Born, Landé and others in various ways. Very recently, attempts at the problem of masses of elementary particles were made by Born, Nature 163, 207 (1949); Green, Nature 163, 208 (1949).

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7. Heisenberg, ZS. f. Phys. 120, 513, 673 (1943); 123, 93 (1944).

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