D(2)

$$\Psi(t') = \frac{1}{T}\int_0^T dt \int (t'|D_t|t'')\,\Psi(t'')\,dt''$$

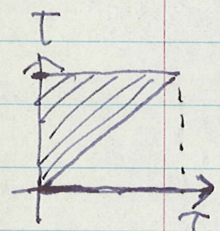
$$+\frac{1}{T}\int_0^T dt \int_{t>\tau} d\tau\, (t'|D_{t-\tau}|t''')\,\bar{H}'(t''')\,dt'''\,(t'''|D_\tau|t'')\,dt''$$

$$+\frac{1}{T}\int_0^T dt \iint_{t>\tau+\tau'} (t'|D_{t-\tau-\tau'}|t''')\,\bar{H}'(t''')\,dt''' \times (t'''|D_{\tau'}|t^{iv})\,\bar{H}'(t^{iv})\,dt^{iv} \times (t^{iv}|D_\tau|t'')\,dt''$$

$$\frac{1}{T}\int_0^T (t'|D_t|t'')\,dt$$

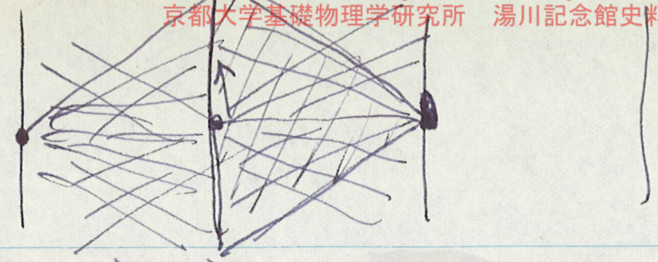
$$+\frac{1}{T}\int_0^T dt \int_0^t d\tau\,(t'|D_{t-\tau}|t''')\,\underbrace{\bar{H}'(t''')\,dt'''}_{dt'''}\,(t'''|D_\tau|t'')\,\cancel{dt''}$$

$$+\frac{1}{T}\int_0^T dt \int_0^t d\tau \int_0^{t-\tau} d\tau'\,(t'|D_{t-\tau-\tau'}|t'')\,\bar{H}'(t'')\,dt'''\,(t''|D_{\tau'}|t^{iv})\,\underbrace{\bar{H}'(t^{iv})}_{dt^{iv}}(t^{iv}|D_\tau|t'')\,\cancel{dt''}$$



$$\frac{1}{T}\int_0^T D_t\,dt + \frac{1}{T}\int_0^T dt \int_0^t d\tau\,\{D_{t-\tau}\,\bar{H}'\cdot D_\tau$$

$$+\frac{1}{T}\int_0^T dt \int_0^t d\tau \int_0^{t-\tau} d\tau'\, D_{t-\tau-\tau'}\,\bar{H}'\,D_{\tau'}\,H'\,D_\tau$$



In the limit of $T \to \infty$

$$D_\infty + D_\infty \bar{H}' D_\infty + D_\infty \bar{H}' D_\infty \bar{H}' D_\infty$$
$$D_s H' D_s$$
$$+ \cdots$$

This can be extended into a relativistically invariant form

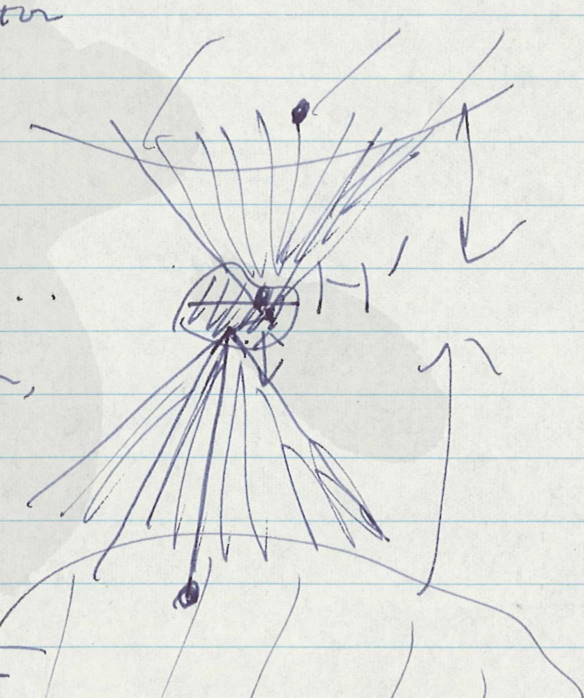
$$S = D_{FP} + D_{FP} H' D_{FP} + D_{FP} H' D_{FP} H' D_{FP}$$
$$+ \cdots$$

$$D_{FP} = \sum_{l_\mu} \exp i(l^\mu x_\mu/\hbar)$$

l_μ: future-like vector

or simply

$$S = D + DH'D$$
$$+ DH'DH'D + \cdots$$



denote the operator, which transform ~~the probabi-~~ Ψ in the past into Ψ in the future.*

* However, ~~it is not~~ S, itself, can connect $\Psi(n, x_\mu)$ in the past with that in ~~$\Psi(n,$~~ the future, so that it can not claim an immediate physical meaning.

Part II.

I. General Properties of Nonlocal Operators

The most general ~~type~~ of operators B which may ~~, which come into play~~ appear in nonlocal field theory, is those can be expressed in the form

$$A ~~B~~ = \sum_{n_1, n_2 \cdots n_\alpha} C_{n_1 n_2 \cdots n_\alpha} U_1^{n_1} U_2^{n_2} \cdots U_\alpha^{n_\alpha}, \qquad (1)$$

where $n_1, n_2, \cdots n_\alpha = 0, 1, 2, \cdots$ each of $U_1, U_2 \cdots U_\alpha$ denotes either one ~~any~~ of the field operators U, U^* etc appearing in the theory, or any one of the invariant operators such as Ω, Θ. For convenience, ~~we~~ ~~the~~ those ~~such~~ operators, which do not contain invariant operators Ω, Θ, and is ~~but~~ ~~are~~ expressible as a sums of ~~powers~~ products of field operators U, U^* etc. alone, will be called the pure field operators ~~of the first kind~~ and symbols ~~like~~ P, P_1, P_2, P_3 etc. will be used for them. Those operators, which contain both the invariant operators Ω, Θ and the field operators U, U^* etc, will be called the mixed operators ~~of the second kind~~ and symbols M, M_1, M_2, M_3 etc will be used for them. Further, single ~~We start from the~~ invariablent operators, which are constant or independent of field quantities, will be called field independent operators and symbols $I, I_1, I_2, \cdots$ will be used for them.

First, we consider an arbitrary pure operator P, which can be expanded in the form

$$P = \sum_{n_1 n_2 \cdots} c_{n_1 n_2 \cdots n_\alpha} U_1^{n_1} U_2^{n_2} \cdots U_\alpha^{n_\alpha},$$

where $U_1, U_2, \cdots U_\alpha$ denote the nonlocal fields. Now the most general scalar nonlocal field U can be represented by an arbitrary function $U(X_\mu, r_\mu)$ of two sets of parameters X_μ, r_μ, which were defined in Part I of this paper. ~~$U(X$~~ Since $U(X_\mu, r_\mu)$ can be expanded, in general, into the double series

$$U(X_\mu, r_\mu) = \int \cdots \int \underbrace{\quad}_{u(k_\mu, l_\mu)} \exp(ik_\mu X^\mu) \prod_\mu \delta(r_\mu + l_\mu) (dk_\mu)^4 (dl^\mu)^4$$

or

$$(x'_\mu | U | x''_\mu) = \underbrace{\int \cdots \int u(k_\mu, l_\mu)} \exp(ik_\mu x'^\mu/2) \prod_\mu \delta(x'_\mu - x''_\mu + l_\mu)$$
$$\times \exp(ik^\mu x''_\mu/2) (dk_\mu)^4 (dl^\mu)^4.$$

Thus the most general scalar nonlocal field can be written in the form

$$U = \int \cdots \int u(k_\mu, l_\mu) (dk_\mu)^4 (dl^\mu)^4$$

$$\exp(ik_\mu x^\mu/2) \cdot \exp(i l^\mu p_\mu) \exp(ik_\mu x^\mu/2)$$

~~So the product of two such operators~~

$(x'_\mu | U | x''_\mu)$ can alternatively be written as

$$(x'_\mu | U | x''_\mu) = \int\cdots\int u(k_\mu, l_\mu) \exp(i k^\mu x'_\mu/2)$$

$$\exp(i k^\mu x''_\mu/2) \exp(i k_\mu l^\mu/2) \prod_\mu \delta(x'_\mu - x''_\mu + l_\mu)$$
$$\times (dk_\mu)^4 (dl^\mu)^4$$

So U can be written alternatively as

$$U = \int\cdots\int u(k_\mu, l_\mu) \exp(i k_\mu l^\mu/2) (dk_\mu)^4 (dl^\mu)^4$$
$$\times \exp(i k_\mu x^\mu) \exp(i l^\mu p_\mu/\hbar).$$

$$\sim U = \int\cdots\int u'(k_\mu, l_\mu) (dk_\mu)^4 (dl^\mu)^4 \exp(i k_\mu x^\mu)$$
$$\times \exp(i l^\mu p_\mu/\hbar),$$

where u and u' are connected by the simple relation

$$u'(k_\mu, l_\mu) = u(k_\mu, l_\mu) \exp(i k_\mu l^\mu/2)$$

Similarly, we obtain another expression

$$U = \int\cdots\int u''(k_\mu, l_\mu)(dk_\mu)^4 (dl^\mu)^4 \exp(i l^\mu p_\mu/\hbar)$$
$$\exp(i k_\mu x^\mu),$$

where

$$u''(k_\mu, l_\mu) = u(k_\mu, l_\mu) \exp(-i k_\mu l^\mu/2)$$
$$= u'(k_\mu, l_\mu) \exp(-i k_\mu l^\mu)$$

Further, the product of two operators

$$U_1 = \exp(i k_\mu^{(1)} x^\mu) \exp(i l_\mu^{(1)} p^\mu/\hbar)$$

$$U_2 = \exp(i k_\mu^{(2)} x^\mu) \exp(i l_\mu^{(2)} p^\mu/\hbar)$$

becomes

$$U_1 U_2 = \exp(i k_\mu^{(1)} x^\mu) \exp(i l_\mu^{(1)} p^\mu/\hbar) \times \exp(i k_\mu^{(2)} x^\mu) \exp(i l_\mu^{(2)} p^\mu/\hbar)$$

$$= \exp\{i (k_\mu^{(1)} + k_\mu^{(2)}) x^\mu\} \exp\{i (l_\mu^{(1)} + l_\mu^{(2)}) p^\mu/\hbar\} \exp\{i k_\mu^{(2)} l^{\mu(1)}\}$$

So the product of two more general operators

$$U_1 = \int \cdots \int u_1'(k_\mu^{(1)}, l_\mu^{(1)}) \exp(i k_\mu^{(1)} x^\mu) \exp(i l_\mu^{(1)} p^\mu/\hbar) (dk_\mu^{(1)})^4 (dl^{\mu(1)})^4$$

$$U_2 = \int \cdots \int u_2'(k_\mu^{(2)}, l_\mu^{(2)}) \exp(i k_\mu^{(2)} x^\mu) \exp(i l_\mu^{(2)} p^\mu/\hbar) (dk_\mu^{(2)})^4 (dl^{\mu(1)})^4$$

is

$$U_1 U_2 = \int \cdots \int u_1'(k_\mu^{(1)}, l_\mu^{(1)}) u_2'(k_\mu^{(2)} l_\mu^{(2)}) \exp(i k_\mu^{(2)} l^{\mu(1)}) \exp\{i K_\mu x^\mu\} \exp\{i L_\mu p^\mu/\hbar\} (dk_\mu^{(1)})^4 (dl^{\mu(1)})^4 (dk_\mu^{(2)})^4 (dl^{\mu(2)})^4$$

where

$$\left.\begin{aligned} K_\mu &= k_\mu^{(1)} + k_\mu^{(2)} \\ L_\mu &= l_\mu^{(1)} + l_\mu^{(2)} \end{aligned}\right\}$$

More generally, product of any number of such operators $U_1, U_2, \cdots U_\alpha$ can be written in the form:

$$U_1 U_2 \cdots U_\alpha = \int\cdots\int u_1'(k_\mu^{(1)}, l_\mu^{(1)})\, u_2'(k_\mu^{(2)}, l_\mu^{(2)}) \cdots u_\alpha'(k_\mu^{(\alpha)}, l_\mu^{(\alpha)})$$

$$\exp\{i(k_\mu^{(1)} l^{\mu(1)} + \cdots\cdots)\} \exp(i K_\mu x^\mu) \exp(i L_\mu p^\mu)$$

where

$$\left.\begin{aligned} K_\mu &= \sum_{r=1}^{\alpha} k_\mu^{(r)} \\ L_\mu &= \sum_{r=1}^{\alpha} l_\mu^{(r)} \end{aligned}\right\}$$

In general, each of $u_1', u_2' \cdots$ contain, when quantized, a ~~factor~~ an operator, which decreases or increases the number of first, second ... or α-th ~~pa~~particle with the energy ~~k~~ ~~$\hbar k$~~ $\hbar c|k_4^{(1)}|$ etc and the momentum $\pm(\hbar k_1^{(1)}, \hbar k_2^{(1)}, \hbar k_3^{(1)})$* and it can easily be observed that

$$K_\mu = 0,$$

only when the total kinetic energy and momentum of the system is unchanged by the change in numbers of particles ~~of~~ different kinds in different states.

* For $k_4^{(1)} < 0$ + sign. decrease
$k_4^{(1)} > 0$ − sign. increase

In other words, if we ~~take up~~ pick expand the most general pure field operator P into the series of the form

$$P(x_\mu, p_\mu) = \int \!\!\! \sum_{k_\mu, l_\mu} P(k_\mu, l_\mu) \exp(i k_\mu x^\mu) \times \exp(i l^\mu p_\mu)(dk_\mu)^4 (dl^\mu)^4$$

P(0,0) corresponds to the real transition of the whole system, in which the energy and momentum are conserved.

In matrix representation $(x'|P|x'')$ or $P(X, r)$, if we perform the integration with respect to X, ~~we obtain that~~ only the term, which is proportional to P(0,0), remains as ~~to be~~ different from zero.

Thus it seems reasonable to define the S-matrix in nonlocal field theory as the integral

$$\iint (x'|P|x'')(dx')^4(dx'')^4$$
$$= \iint P(X, r)(dX)^4(dr)^4$$

The above consideration is not yet general enough, however, in that we have also to consider the properties of mixed operators.

However, any ~~operator~~ field independent operator can be written again in the form

$$I = \iint u(k_\mu, l_\mu)(dk_\mu)^4 (dl^\mu)^4 \exp(ik_\mu x^\mu/2) \times$$
$$\times \exp(il^\mu p_\mu/\hbar) \exp(ik_\mu x^\mu/2)$$

or

$$I = \int\cdots\int u'(k_\mu, l_\mu)(dk_\mu)^4 (dl^\mu)^4 \exp(ik_\mu x^\mu)$$
$$\times \exp(il^\mu p_\mu/\hbar),$$

where $u'(k_\mu, l_\mu) = u(k_\mu, l_\mu)\exp(ik_\mu l^\mu/2)$

The only difference between the field independent operator and the field dependent operator is that $u(k_\mu, l_\mu)$ ~~and~~ or $u'(k_\mu, l_\mu)$ for the former is an ordinary fn. of k_μ and l_μ.

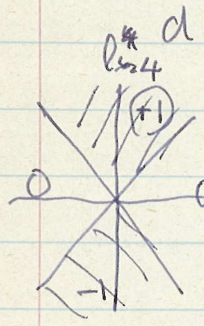
If ~~they~~ $u(k_\mu, l_\mu)$ is independent of k_μ, l_μ, $I(X_\mu, r_\mu)$ is ~~in~~ completely independent of X_μ, r_μ or $(x'_\mu | I | x''_\mu)$ is everywhere constant for any ~~value~~ set of values of x'_μ, x''_μ.

(i) For convenience, ~~let us call~~ we introduce an operator E, ~~which~~ with ~~is represented by a~~ the matrix $(x'_\mu | E | x''_\mu)$, which is 1 everywhere, with the coef. $u(k_\mu, l_\mu) = \Pi\delta(k_\mu)$

(ii) Further, we introduce another operator D

$$D = \iint d(k_\mu, l_\mu)(dk_\mu)^4 (dl^\mu)^4 \exp(ik_\mu x^\mu/2)$$
$$\times \exp(il^\mu p_\mu/\hbar) \exp(ik_\mu x^\mu/2),$$

where $d(k_\mu, l_\mu)$ ~~is independent of k_μ and~~ has the form



$$d(k_\mu, l_\mu) = \underbrace{\mathfrak{E}(l_\mu)}_{\Pi\delta(k_\mu)} = \left.\begin{matrix} 1 \\ 0 \\ -1 \end{matrix}\right\} \text{ for } \begin{cases} l_4^* > l \\ l > l_4^* > -l \\ -l > l_4^* \end{cases}$$

The matrix element $(x'|D|x'')$ or $D(X, r)$ has the form

$$D(X, r) = \begin{Bmatrix} \cancel{D(r)} \\ \varepsilon(r_\mu) \end{Bmatrix} = \left.\begin{matrix} 1 \\ 0 \\ -1 \end{matrix}\right\} \text{ for } \begin{cases} r^4 > r \\ r > r^4 > -r \\ -r > r^4 \end{cases}$$

Any operator I, which is independent of field quantities and is invariant, has the coefficient $u(k_\mu, l_\mu)$, which ~~is depend~~ only on $k_\mu k^\mu$, $k_\mu l^\mu$, $l_\mu l^\mu$ (and also may depend on the signatures of k_μ and l^μ *) [$\varepsilon(k_\mu)$, $\varepsilon(l_\mu)$]

For example, the operator with the coefficient

$$u(k_\mu, l_\mu) = \delta(k_\mu k^\mu + \kappa^2)\,\delta(k_\mu l^\mu)\,\delta(l_\mu l^\mu - \lambda^2)$$

and

$$u(k_\mu, l_\mu) = \delta(\cdot\cdot)\,\delta(\cdot\cdot)\,\delta(\cdot\cdot)\,\varepsilon(l_\mu)$$

---

* By a signature [$\varepsilon(k_\mu)$] of a four vector k_μ, we mean $\mathfrak{E}(k_\mu) = +1$, ~~or~~ for $k_4 > k$, $\mathfrak{E}(k_\mu) = 0$, for $k > k_4 > -k$, $\mathfrak{E}(k_\mu) = -1$ for $-k > k_4$.

~~invariant~~

In general, ~~an~~ function ~~of~~ $U(X, r)$, which is independent of X_μ and invariant with respect to (homogeneous) Lorentz transformation, is also invariant with respect to inhomogeneous Lorentz transformation including the displacement of the coordinate origin:

$$x'_\mu = a^{(0)}_\mu + a_{\mu\nu} x_\nu .$$

On the contrary, if $U(X, r)$ ~~is~~ is an invariant (with respect to (homogeneous) L. trans.) function of both X_μ and r_μ, it does not follow necessarily that $U(X, r)$ is invariant with respect to inhomogeneous L. Transf., because ~~by translation~~ we have

$$\left.\begin{aligned} X'_\mu &= a^{(0)}_\mu + a_{\mu\nu} X_\nu \\ r'_\mu &= a_{\mu\nu} r_\nu \end{aligned}\right\}$$

invariant

So we postulate that ~~all~~ the field indep. operators appearing in ~~to~~ nonlocal field theory, must be represented by a fn $U(X, r)$, which is invariant with respect to inhomog. L. T. of X and with respect to homog. L. T. of r.
~~So, for example, the signature $\varepsilon(k_\mu)$ destroys the invariance in this sense.~~

This condition is satisfied only if $u(k_\mu, l_\mu)$ has a factor $\prod_\mu \delta(k_\mu)$.

In other words, the appearance of field independent invariant operators should not and actually does not ~~change the~~ affect the energy and momentum of the total system, because ~~the~~ field independent ~~ounde~~ operators have always only those terms corresponding to $k_\mu = 0$.

On the contrary, ~~and~~ an identical operator in x-space, which can be represented by the matrix

$$\delta_x \rightarrow \delta(x', x'') \cancel{\propto} (x' | \int \cdots \int \overbrace{\exp(ik_\mu x^\mu)}^{\delta(k_\mu)\delta(l_\mu)} \underbrace{(dk_\mu)^4}_{\exp(il^\mu p_\mu/\hbar)} | x'')$$

is nothing but the operator

$$\cancel{E_p \rightarrow \Delta(p', p'') = \int \cdots \int \delta(p_\mu' - p_\mu'' - k_\mu)(dk_\mu)^4}$$

$$\delta_p \rightarrow \delta(p', p'')$$

in p-space. ~~Thus there is no such operator as the identity operator~~ ~~More generally,~~ E_x and E_p ~~are the particular case of the most general operator~~

$$\cancel{U(x^\mu, p_\mu) = \int \cdots \int (dk_\mu)^4 (dl^\mu)^4 u(k_\mu, l_\mu) \exp(ik_\mu x^\mu) \exp(il_\mu p^\mu)}$$

~~with $u(k_\mu, l_\mu) = \prod_\mu \delta(k_\mu)$ and $\prod_\mu \delta(l_\mu)$ respectively.~~

* There is a very interesting connection between matrices representations of the same operator in x- and p-spaces. For example,

$$\delta_p \rightarrow \delta(p', p'') \cancel{\propto} (p' | \iiint \exp(il^\mu p_\mu)(dl^\mu)^4 | p'')$$

$$\overset{\parallel}{E_x} \rightarrow \Delta(x', x'') \propto \int \cdots \int \delta(x_\mu + l_\mu)(dl_\mu)^4,$$

from this it is clear that the operator E_{k_μ} ~~gives rise to contribute~~ nothing to the energy and momentum of ~~the~~ system, because ~~the~~ it does not ~~gives~~ rise to any change in energy, momentum, More generally, any field independent invariant operator I, which has the form

$$I(x,p) = \int u(l_\mu) \exp(i l^\mu_\mu p_\mu/\hbar)(dl^\mu)^4$$

$$= \int u(k_\mu, l_\mu) \prod_\mu \delta(k_\mu) \exp(i k_\mu x^\mu)$$

$$\times \exp(i l^\mu p_\mu/\hbar)(dk_\mu)^4 (dl^\mu)^4,$$

can be represented in x-space as a matrix

$$(x'|I|x'') = \int u(l_\mu) \prod_\mu \delta(x'-x''+l_\mu)(dl^\mu)^4$$

$$= u(x''-x'),$$

while in p-space as

$$(p'|I|p'') = \int u(l_\mu) \exp(i l^\mu p'_\mu/\hbar)(dl^\mu)^4$$

$$\times \delta(p', p'')$$

For example, for E_x

$$(x'|E_x|x'') = 1. \quad \text{or} \quad u(l_\mu) = 1.$$

$$(p'|E_x|p'') = \delta\left(\frac{p'+p''}{2}\right)\delta(p', p'')$$

$$= \delta(p')\delta(p'')$$

Conversely

$$\left.\begin{aligned} (x'|E_p|x'') &= \delta(x')\delta(x'') \\ (p'|E_p|p'') &= 1 \end{aligned}\right\}$$

Another simple example

$$(x'|U_4|x'') = U_4(x_4''-x_4') = \mathcal{U}(-r_4)$$

$$= -1 \quad \text{for } R < r_4 < 0$$

$$+1 \quad \text{for } 0 < r_4 < R$$

$$0 \quad \text{for } |r_4| > R$$

$$\underbrace{\mathcal{U}(-r_4)}_{=-\mathcal{U}(r_4)} = \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{d\omega}{\omega}\left(1-\cos\frac{\omega R}{2}\right)e^{i\omega r_4}$$

$$(p'|U_4|p'') = \int \mathcal{U}(l_\mu)\exp(i l_\mu p_\mu/\hbar)(dl_\mu)^4$$

$$\times\,\delta(p',p'')$$

$$\cancel{= -\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{dP}{P_4}\exp(i\omega P)\frac{dP}{d\omega}}$$

$$= -\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{\omega}\left(1-\cos\frac{\omega R}{2}\right)e^{i\omega l_4}$$

$$\times\exp(-i l_4 P^4/\hbar)\,dl_4$$

$$\times\,\delta(P_1)\delta(P_2)\delta(P_3)\Pi\delta(\pi_\mu)$$

$$= -\frac{i}{\pi}\frac{1}{P^4}\left(1-\cos\frac{P_4 R}{2}\right)$$

$$\times\,\delta(P_1)\delta(P_2)\delta(P_3)\Pi\delta(\pi_\mu)$$

$$= \cancel{2\pi\hbar} \cdot \frac{2i}{\pi}\,\frac{1}{p^4}\,\sin^2(p^4 R/2)$$

$$(p'|U_4|p'') = -\frac{i}{\pi}\,\frac{1}{p^4}\,\cancel{\sin}$$

More generally, for the ~~fuel~~ field independent operator

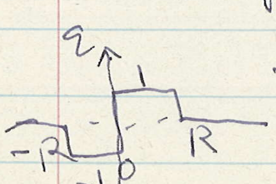
$$I(x,p) = \int u(k_\mu, l_\mu)\exp(i k_\mu x^\mu) \times \exp(i l^\mu p_\mu/\hbar)(d k_\mu)^4 (d l^\mu)^4$$

$$(x'|I|x'') = \int u(k_\mu, l_\mu)\exp(i k_\mu x'^\mu) \times \delta(x'_\mu - x''_\mu + l_\mu)(d k_\mu)^4 (d l^\mu)^4$$

$$(p'|I|p'') = \int u(k_\mu, l_\mu)\,\cancel{\exp}\;\delta(p' - p'' - k) \times \exp(i l^\mu p''_\mu/\hbar)(d k_\mu)^4 (d l^\mu)^4$$

$$I(x,p) = \iint \frac{u(k_\mu, l_\mu)}{k_4}\exp(i k_\mu x^\mu)\exp(i l^\mu p_\mu/\hbar) \times (d k_\mu)^4 (d l^\mu)^4$$

~~$I(x,p) = \int \cdots \int$~~



$$\frac{1}{k_4}\sin^2(k_4 R/2) \propto \int \frac{\exp(i k_4 l^4)}{\varepsilon(l^4)}\, d l^4$$

II. Generalization of Schrödinger Equation to Nonlocal Fields (II 8)

In local field theory, any operator H can be written as a matrix

(i) either with rows and columns characterized by eigenvalues of $n'(x) = \psi^*(x)\psi(x)$ etc

$$(\cdots n'(x) \cdots | H | \cdots n''(x) \cdots)$$

and Schrödinger equation can be written as

$$i\hbar \frac{\partial \Psi(\cdot\cdot n'(x) \cdots, t)}{\partial t} = \sum_{n''(x)} (\cdot\cdot n'(x) \cdot\cdot | H | \cdot\cdot n''(x) \cdot\cdot) \times \Psi(\cdot\cdot n''(x) \cdot\cdot, t)$$

(ii) or with r. and c. characterized by ~~n~~ eigenvalues of $n(k)$ etc

$$(\cdots n'(k) \cdot\cdot | H | \cdot\cdot n''(k) \cdot\cdot)$$

and S. eq. can be written as

$$i\hbar \frac{\partial \Psi(\cdot n'(k) \cdot\cdot, t)}{\partial t} = \sum_{n''(k)} (\cdot\cdot n'(k) \cdot\cdot | H | \cdot\cdot n''(k) \cdot\cdot) \times \Psi(\cdot\cdot n''(k) \cdot\cdot, t)$$

In nonlocal field theory, however, ~~the former~~ there is neither ~~no exact form~~ the expression for the operator ~~in~~ corresponding to (i), ~~since~~ nor to (ii), because the nonlocal operators are always represented by matrices with rows and columns characterized by both by

$$\cdots n(k,l) = a^*(k,l)\, a(k,l) \cdots$$

and x_μ. Thus, the matrix for H is

$$(\cdots n'(k,l) \cdots, x'_\mu | H | \cdots n''(k,l) \cdot\cdot, x''_\mu)$$

So, if we assume that the Schrödinger function Ψ exists, it ~~be~~ must have the form

$$\Psi(\cdots n'(k,l)\cdots, x'_\mu)$$

We assume tentatively
~~and~~ the Schr. eq. ~~is must be~~ for Ψ:

$$i\hbar c \frac{\partial \Psi(\cdots n'(k,l)\cdots, x'_\mu)}{\partial x^4}$$

$$= \int\cdots\int (dx''_\mu)^4 \sum_{n''(k,l)} (\cdots n'(k,l)\cdots, x'_\mu | H | \cdots n''(k,l)\cdots, x''_\mu)$$

$$\times \Psi(\cdot n''(k,l)\cdots, x''_\mu). \qquad (1)$$

~~Then~~ In the limit $\lambda \to 0$, there will be no way of discriminating between $n(k,l)$ with the same k and different l, because all l_μ appearing in the above formulation reduce to zero vector, ~~and the summation (or integration) with respect to all l_μ will be replaced by $\prod_\mu \delta(r_\mu)$, which must be omitted whenever we go back to the usual field theory~~, which gives rise to ~~$\prod_\mu \delta(r_\mu)$~~ the appearance of a common factor $\prod_\mu \delta(r_\mu)$ everywhere. The Schrödinger eq. will be

$$i\hbar c \frac{\partial \Psi(\cdot\cdot n'(k)\cdot\cdot, x'_\mu)}{\partial x^4}$$

$$= \cancel{\int\cdots\int dx} \sum_{n''(k,l)} (\cdot\cdot n'(k,\cancel{l})\cdot, \cancel{x'_\mu} | H(x'_\mu) | \cdot\cdot n''(k)\cdot\cdot)$$

$$\times \Psi(\cdot\cdot n''(k), \cdot\cdot x_\mu)$$

(II 9)

This ~~is certainly different from~~ does not agree with the usual Schrödinger eq. for the wave fn $\Psi(\cdots x'(k)\cdots, x_4')$

$$i\hbar c \frac{\partial \Psi(\cdots x'(k)\cdots, x_4')}{\partial x'^4}$$

$$= \iiint (\cdots x'(k)\cdots | H | x_\mu') | \cdots x''(k)\cdots) dx_1' dx_2' dx_3'$$

$$\times \Psi(\cdots x''(k), \cdots x_4')$$

So the Schrödinger eq. for the nonlocal field (1) must be changed into the form:

$$i\hbar c \frac{\partial \Psi(\cdots x^0(k,l)\cdots, x_\mu^0)}{\partial x^4}$$

$$= \int\cdots\int (dx_\mu'')^4 (dx_\mu''')^4 \sum_{x''(k,l)} \sum_{x'''(k,l)} \overbrace{(x^0, x_\mu^0 | D | x', x_\mu')}(\cdots x'(k,l)\cdots, x_\mu' | H | \cdots x''(k,l)\cdots x_\mu'')$$

$$\times (\cdots x''(k,l)\, x_\mu'' D | \cdots x'''(k,l)\cdots x_\mu''') \Psi(\cdots x'''(k,l)\cdots x_\mu''')$$

~~and~~ and define

$$\Psi(\cdots x'(k,l)\cdots, x_4') = \sum_{x''} \iiiint \overbrace{(\cdots x'\cdots x_\mu' | D | \cdots x''\cdots x_\mu'')} \Psi(\cdots x''(k,l)\cdots x_\mu'') dx_1'' dx_2'' dx_3'' dx_4''$$

$$(\cdots x', x_\mu' D | \cdots x''\cdots x_\mu'') = \delta(x' x'')\cdots \delta(x_4' - x_4'')$$

$$\text{Then } i\hbar c \frac{\partial \Psi(\cdots x^0(k,l)\cdots, x_4^0)}{(dx')^4 \partial t}$$

$\int \cdot \Psi$

$$= \sum_{x' x''} \iiiint (dx_\mu')^4 \sum_{x''} \overbrace{(x^0 | D | x')} (\cdots x', x_\mu^0 | H | \cdots x''\cdots x_\mu'')$$

$$\times \iiint (dx_\mu'')^4\, \cancel{(\cdots x''\cdots x_\mu'')} \Psi(\cdots x''\cdots x_\mu'' x_2'' x_3'' x_4'')$$

$$= \sum_{x''} \int\cdots \iiint (dx_\mu'')^{4,3} \sum_{x''} (x'| D | x'') (\cdots x^0, x_\mu^0 | H | \cdots x''\cdots x_\mu'')_{x_4^0\, x_4''} \Psi(x'(k,l), x_4'')$$

$(dx_\mu'')^3$

In the limit of local field:

$$i\hbar c \frac{\partial \overline{\Psi}(\cdots x' x_4')}{\partial x_4'} = \cancel{H}(x', x_4' | \overline{H} | x'', x_4') \times \overline{\Psi}(\cdots x'', x_4'')$$

Thus, the Schrödinger function $\overline{\Psi}$ for the total system is nothing but the space integral of $\Psi(\cdots x' \cdots x_\mu')$ and have to satisfy the eq. (in general)

$$i\hbar c \frac{\partial \overline{\Psi}(\cdots x' \cdots x_4')}{\partial x_4'} = (\cdots x' \cdots x_4' | \overline{H} | \cdots x'' \cdots x_4'') \times \overline{\Psi}(\cdots x'' \cdots, x_4'') \qquad (2)$$

Now the question is to construct $\overline{H}$ ~~and~~ from a certain operator H so that the equation (2) is invariant.

We can write formally

$$-c p_4 \Psi = H D \Psi$$

and

$$-c p_4 D\Psi = DHD\Psi.$$

or $-c p_4 \overline{\Psi}(x_4') = \cancel{(DH)} \cdot \underbrace{\overline{H} \, \overline{\Psi}(x_4'')}_{(x_4', x_4'')}$ *

$$p_4 D - D p_4 = 0.$$

---

* $HD \neq DH$ $\therefore \int\cdots\int (x'|H|x''')(x'''|D|x'')\, dx'''$

$DH = \int\cdots\int (x'|H|x_1'' x_2'' x_3'' x_4')(dx'')^3 = \int\cdots\int (x_\mu' x_4'|H|x_\mu'' x_4'')(dx')^3$

$DHD = \int\cdots\int \underbrace{(x_\mu'|H|x_\mu'')}_{(dx')^3 (dx'')^3}$

$$(x'|D|x'') = \delta(x'_4 - x''_4)$$

$$(x'|D^2|x'') = \int\!\cdots\!\int (dx''')^4 (x'|D|x''')(x'''|D|x'')$$

$$= \iiint (x'|D|x'')(dx'')^3$$

$$(x'|H\cdot D\cdot H'|x'') = \int\!\cdots\!\int (x'|H|x''')(x'''|D|x^{iv})(x^{iv}|H'|x'') \, (dx'''_\mu)^4 (dx^{iv}_\mu)^4$$

$$= \int\!\cdots\!\int (x'|H|x''')\,\delta(x'''_4 - x^{iv}_4)\underbrace{(dx'''_\mu)^4 (dx^{iv}_\mu)^4}_{(x^{iv}|H'|x'')}$$

$$= \int\!\!\int (x'|H|x'''_j)(dx''')^3 \cdot \int\!\!\int (x_j''|H'|x'')(dx^{iv})^3$$
x'''_4 $\quad$ x'''_4

~~x'''~~

Thus HD must ~~be~~ have the same dimension as p_4, or H must have the same dimension as $c p_4^* D^{-1}$! D is an operator with the dimension $(L)^3$ and p_4, $\hbar c/L$ so the dimension of H is $\sim \hbar c/L^4$ *. H

$$D = \int\!\!\int (dl^\mu) \exp i l^\mu p_\mu/\hbar \; \delta(l^4_*).$$

---

* δ-fn: $\prod_\mu \delta(x'_\mu - x''_\mu)$ has dimension 0.

$$DFD = \int (x'|F|x'')(dx')^3(dx'')^3 \sim F L^6$$

$$-c\,p_4\,\Psi = H D_4 \Psi$$

$$D_4 = \int\cdots\int (dl^\mu)^4 \exp(i l^\mu p_\mu/\hbar)\,\delta(l^4).$$

$$[p_4, D_4] = 0.$$

$$p_\mu \Psi = L \cdot D_\mu \Psi$$

$$L = -H/c$$

$$D_\mu = \int\cdots\int (dl^\mu)^4 \exp(i l^\mu p_\mu/\hbar)\,\delta(l^\mu)$$

$$\sum_\mu \int D_\mu \exp(i l^\mu \cdot p_\mu/\hbar)(dl^\mu)$$

$$\Psi = \left\{\delta(p_4) - \frac{1}{c\,p_4^4} H D\right\}\Psi_0$$

$$-c\,p_4'\,(p_\mu'|\Psi|p_\mu'') = (p_\mu'|HD|p_\mu'')(p_\mu'|\Psi|p_\mu'')$$

$$-c\,p_4'\,\Psi(p_\mu') \simeq (p_\mu'|HD|p_\mu'')\Psi(p_\mu'')$$

$$\Psi(p_\mu') = \{\delta(p_4')\,\Psi_0(p_\mu') \cancel{\Psi_0(p_\mu}$$

$$- \frac{1}{c\,p_4'^4}(p_\mu'|HD|p_\mu'')\}\Psi_0(p_\mu'')$$

II 11

$$\Psi = \int\cdots\int u(n \cdots K'_\mu) \exp(i K'_\mu x'^\mu)(dK'_\mu)^4$$

$$(n'x'|H|n''x'') = \delta($$

$$-icK_4 u(n' \cdots K'_\mu) = \delta(K'_\mu - \sum n'_k k')$$

$$(n'|H|n'')\,\delta(K'''_\mu - \sum n''_k k'')$$

$$\cdot u(\cdots n'' \cdots K''_\mu)$$

$$D = \int\cdots\int (dl)^4 \exp(i l^\mu p_\mu/\hbar)\,\delta(l^4)$$

$$(x'|D|x'') = \delta(x'_4 - x''_4)$$

$$D\Psi = \int\cdots\int$$

$$-ic\int K_4 u(n'_k \cdots K'_\mu) \exp(i K'_\mu x'^\mu)(dK'_\mu)^4$$

$$= \sum_{n''}\int (n'_k \cdots x'|H|n''_k \cdots x'')(x''|D|x''')$$

$$(dx'')^4 (dx''')^4 dK''_\mu\; u(n''_k \cdots K''_\mu) \exp(i K''_\mu x''^\mu)$$

$$= \int \Big\{ \sum_{n''}\int (n'_k \cdots x'|H|n''_k \cdots x''' x''_4)(dx'')^3$$

$$\cdot \int u(n''_k \cdots K''_\mu) \exp(i K''_\mu x''^\mu)(dx'')^3$$

$$\times (dK''_\mu)^4 \Big\}\, dx''_4$$

The above considerations indicate that, in nonlocal field theory, we have to start from

$$i\hbar c \frac{\partial \overline{\Psi}(\cdot\cdot n'(k,l)\cdot\cdot, x_4')}{\partial t} = \sum_{n''}\int dx_4'' (\cdot\cdot n'\cdot\cdot x_4' | \overline{H} | \cdot\cdot n''\cdot\cdot x_4'')$$

$$\times \overline{\Psi}(\cdot\cdot\cdot n''\cdot\cdot x_4''),$$

where

$$\left.\begin{aligned} &\overline{\Psi}(\cdot\cdot n'(k,l)\cdot\cdot, x_4') = \underbrace{\iiint \Psi(\cdot\cdot n'(k,l)\cdot\cdot, x_1' x_2' x_3' x_4')(dx')^3} \\ &(\cdot\cdot n'\cdot\cdot x_4' | \overline{H} | \cdot\cdot n''\cdot\cdot x_4'') = \underbrace{\int\cdots\int (dx')^3 (dx'')^3 (\cdot\cdot n'\cdot\cdot x_\mu' | H | \cdot\cdot n''\cdot\cdot x_\mu'')} \end{aligned}\right\}$$

Ψ can be expanded into series

$$\overline{\Psi} = \int \Psi(\cdot\cdot n'(k,l)\cdot\cdot, K_4) \exp(iK_4 x^{4'})\, dK_4$$

~~$-\hbar c K_4 \Psi(\cdot\cdot n'(k,l)\cdot\cdot, K_4)$~~

and

$$(n' x_4' | \overline{H} | n'' x_4'') = \int dK_4' dK_4'' e^{iK_4' x^{4'}} (n' K_4' | \overline{H} | n'' K_4'') \times e^{-iK_4'' x^{4''}}$$

$$\int -\hbar c K_4 \Psi(\cdot\cdot n'\cdot\cdot K_4) \exp(iK_4 x^{4'})\, dK_4$$

$$= \sum_{n''}\int dK_4' dK_4'' e^{iK_4' x^{4'}} (n' K_4' | \overline{H} | n'' K_4'') e^{-iK_4'' x^{4''}}$$

$$\times \int dK_4''' \exp iK_4''' x^{4''}\, \Psi(n'', K_4''')$$

$$= \iint dK_4' dK_4'' \sum_{n''} e^{iK_4' x^{4'}} (n' K_4' | \overline{H} | n'' K_4'') \Psi(n'', K_4'')$$

$-\hbar c K_4' \Psi(\ldots n' \ldots K_4')$ $\qquad K_4'' = \sum n^0 k_4 - \sum n'' k_4$

$$= \sum_{n''} \int dK_4'' \, (n' K_4' | \bar{H} | n'' K_4'') \, \Psi(n'', K_4'')$$

However, the time dependence of $\bar{H}$ is *

$(n' \ldots \cancel{x_4'} | \bar{H} | \, n'' \ldots x_4'')$

$= (\ldots n'(\cancel{k}l) \ldots | \bar{H} | \ldots n''(kl) \ldots) \exp i(\sum n' k_4 \cancel{\sum n'' k_4}) x^4$

$\cancel{\cdot fn(p_4)}$

$-\hbar c K_4' \Psi(\ldots n' \ldots K_4') = \underbrace{(\ldots n' \ldots | \bar{H} | n'')}_{\delta(\sum n' k_4 \cancel{\sum n'' k_4} - K_4')} \cancel{fn(p_4)}$ $\delta(\sum n'' k_4 - K_4'')$ $\delta(\cancel{\sum n'' k_4 - K_4''})$

$\Psi(\ldots n'' \ldots K_4'')$ $\to K_4'$

$\cancel{\Psi(\ldots n' \ldots K_4') = \delta(\sum n' k_4 - K_4') \Psi(\ldots n' \ldots)}$

$K_4' \cancel{x} = \sum n^0 k_4^{\#} - \sum n' k_4^{\#}$

---

* suppose that

$H = U^* \cdot U \cdot U^*$ etc

$= \exp \underbrace{i k_4 x^4}_{(n_k' - n_k'')}$

$\ldots\ldots$

We assume:

$$(\cdots n'(k,l)\cdots|\overline{\Psi}|\cdots n^0(k,l)\cdots)\exp i\left(\sum n^0 k_4-\sum n' k_4\right)x'^4 \;!!!$$

$$(\cdots n'(kl)x'_4|\overline{H}|\cdots n''(kl)\cdot x''_4)=$$

$$(\cdots n'\cdots|\overline{H}|\cdots n''\cdots)\exp(-i n' k_4 x'^4) \;!!!$$
$$\exp(+i n'' k_4 x''^4)$$

$$\underbrace{-\hbar c\left(\sum n^0 k_4-\sum n' k_4\right)}_{\Psi(n'|\overline{\Psi}|n^0)}=\sum_{n''}(\cdots n'|\overline{H}|n'')(n''|\overline{\Psi}|n^0)$$

$$-\hbar c\sum n^0 k_4 = W_0$$
$$-\hbar c\sum n' k_4 = W'$$

$$(W_0-W')(n'|\overline{\Psi}|n^0)=\sum_{n''}(n'|\overline{H}|n'')(n''|\overline{\Psi}|n^0)$$

Møller's wave matrix: $(\cdots n'\cdots|\overline{\Psi}|\cdots n''\cdots)$

General wave fn

$$\cancel{\cdots n'|}\;\Psi(n',x'^4)=\sum_{n^0}f(n^0)(n'|\Psi|n^0)\exp i\left(\sum n^0 k_4-\sum n' k_4\right)x'^4$$

$$(n'|\overline{\Psi}|n^0)=1+\underbrace{\left(\frac{1}{W'-W^0}+\pi i\delta(W'-W^0)\right)}_{\sum_{n''}(n'|\overline{H}|n'')(n''|\overline{\Psi}|n^0)}\cancel{(n'|\overline{H}|n^0)}$$

$$S=1+R=1+\delta(W'-W^0)\sum_{n''}(n'|\overline{H}|n'')(n''|\overline{\Psi}|n^0)$$

or $S = 1 \cancel{+ \frac{1}{2\pi}} -2\pi i\, \delta(W'-W^0)\, \overline{H}$

$- 2\pi i\, \delta(W'-W^0)\, \overline{H} \cdot \delta_+(W'-W^0)(-2\pi i)\, \overline{H}\, \overline{\Phi}$

$= \dots$

$S = 1 + \delta(W'-W^0)\, \overline{V}$

$+ \delta(W'-W^0)\, \overline{V}\, \delta_+(W'-W^0)\, \overline{V}\, \overline{\Phi} \quad \dots$

<u>Time average</u>

<u>average over the internal motion (l)</u>