I. General Properties of Nonlocal Operators

In Part I of the paper under the same title, particular types of nonlocal fields were considered in detail. In order to cope with the problem of the interaction between nonlocal fields or that between a local field and a nonlocal field, however, we have to investigate more general properties common to various types of nonlocal operators. An arbitrary ~~operat~~ nonlocal operator A can be represented by a matrix $(x'_\mu | A | x''_\mu)$ with rows and columns, which are characterized by the eigenvalues of the space-time operators x_μ in the representation in x_μ-space. In general, the individual matrix element $(x'_\mu | A | x''_\mu)$ itself is not a pure number, but a sub-matrix with rows and columns, which are characterized by eigenvalues of numbers of particles in various possible states. In other words, $(x'_\mu | A | x''_\mu)$ can be regarded as an ensemble of a family of functions of x'_μ and x''_μ, or a family of functions $A(X_\mu, r_\mu)$ of X_μ and r_μ, where

$$X_\mu = \frac{1}{2}(x'_\mu + x''_\mu), \qquad r_\mu = x'_\mu - x''_\mu. \tag{1}$$

Now it is clear that any ~~function~~ one or a whole family of functions $A(X_\mu, r_\mu)$ can be written in the integral form

$$A(X_\mu, r_\mu) = \int\cdot\int (dk_\mu)^4 (dl_\mu)^4 \bar{A}(k_\mu, l_\mu) \exp(ik_\mu X^\mu) \prod_\mu \delta(r_\mu + l_\mu) \tag{2}$$

where k_μ and l_μ are arbitrary four vectors, which are independent of X_μ and r_μ. This is equivalent to the relation

$$(x'_\mu | A | x''_\mu) = \int\cdot\int (dk_\mu)^4 (dl_\mu)^4 \bar{A}(k_\mu, l_\mu) \exp(ik^\mu x'_\mu/2) \exp(ik^\mu x''_\mu/2) \prod_\mu \delta(x'_\mu - x''_\mu + l_\mu) \tag{3}$$

which, in turn, can be regarded as the matrix representation of the relation

$$A = \int\cdot\cdot\int (dk_\mu)^4 (dl_\mu)^4 \bar{A}(k_\mu, l_\mu) \exp(ik_\mu x^\mu/2) \exp(il^\mu p_\mu/\hbar) \exp(ik_\mu x^\mu/2) \tag{4}$$

between the operators A, x^μ and p_μ.

As the vectors k_μ and l_μ on the right hand side of (4) are, in general, independent of each other, two factors $\exp(ik_\mu x^\mu/2)$ and $\exp(il^\mu p_\mu/\hbar)$ are not commutative, but they satisfy the commutation relation

$$\exp(ik_\mu x^\mu/2)\cdot\exp(il^\mu p_\mu/\hbar) = \exp(-ik_\mu l^\mu/2)\cdot\exp(il^\mu p_\mu/\hbar)\cdot\exp(ik_\mu x^\mu/2) \qquad (5)$$

Thus, (4) can be written alternatively in the form

$$A = \int\cdots\int (dk_\mu)^4 (dl_\mu)^4 A'(k_\mu, l_\mu)\exp(ik_\mu x^\mu)\cdot\exp(il^\mu p_\mu/\hbar) \qquad (6)$$

or

$$A = \int\cdots\int (dk_\mu)^4 (dl_\mu)^4 A''(k_\mu, l_\mu)\exp(il^\mu p_\mu/\hbar)\cdot\exp(ik_\mu x^\mu) \qquad (7)$$

where A' and A'' are related with each other and with $\bar{A}$ by the simple relations $A'(k_\mu, l_\mu) = A''(k_\mu, l_\mu)\exp(ik_\mu l^\mu)$

$$A'(k_\mu, l_\mu) = \bar{A}(k_\mu, l_\mu)\exp(ik_\mu l^\mu/2),\quad A''(k_\mu, l_\mu) = \bar{A}(k_\mu, l_\mu)\exp(-ik_\mu l^\mu/2) \qquad (8)$$

In this way any nonlocal operator A can be expanded formally into Fourier series or expressed as Fourier integral in 8-dimensional x-p-space, in spite of the fact that x^μ and p_μ are noncommutative operators. The Fourier transform $\bar{A}(k_\mu, l_\mu)$ (or $A'(k_\mu, l_\mu)$ or $A''(k_\mu, l_\mu)$) of A is an ensemble of operators characterized by real parameters k_μ and l_μ.

Now there are various types of operators among the nonlocal operators in the most general sense. Among others, the following three classes of operators are of particular interest in connection with the formulation of the problem of interaction between nonlocal fields. The first class of operators consists of those which are relativistically invariant and are independent of the field quantities. More precisely, an operator A belongs to the first class, if the following two conditions are satisfied:

(i) The matrix element $(x'_\mu|A|x''_\mu)$ must be invariant with respect to <u>inhomogeneous</u> Lorentz group including the translation of the origin of coordinate system.

(ii) Each matrix element, which is by itself a submatrix with rows and columns characterized by the number of particles in various possible states, must be the unit submatrix multiplied by a certain function of x'_μ and x''_μ.

The condition (i) implies that $(x'_\mu|A|x''_\mu) = A(X_\mu, r_\mu)$ must be an invariant function of r_μ alone, because the parameters X_μ and r_μ are transformed into

$$X'_\mu = a_{\mu\nu}(X_\nu + b_\nu), \quad r'_\mu = a_{\mu\nu} r_\nu \tag{9}$$

respectively, by a general inhomogeneous Lorentz transformation

$$x'_\mu = a_{\mu\nu}(x_\nu + b_\nu) \tag{10}$$

of the space-time operators. The condition (ii) implies further that this function $A(r_\mu)$ is an ordinary complex function of $r_\mu r^\mu$ multiplied by the unit submatrix.

The second and more general class of operators, in which the operators of the first class are all included, consists of those which satisfy the condition (i), but do not necessarily satisfy the condition (ii). The nonlocal scalar field operators can be regarded as those which belong to the second class. For example, the scalar field operators U and U^*, which are defined by (32)' and (34) in Part I respectively, become invariant with respect to the inhomogeneous Lorentz transformation (10), if we assume that the operators $\bar{u}(k_\mu, l_\mu)$ and $\bar{u}^*(k_\mu, l_\mu)$ are transformed by the rule

$$\bar{u}'(k'_\mu, l'_\mu) = \exp(-ik_\mu b^\mu)\,\bar{u}(k_\mu, l_\mu), \quad \bar{u}'^*(k'_\mu, l'_\mu) = \exp(ik_\mu b^\mu)\,\bar{u}^*(k_\mu, l_\mu). \tag{11}$$

This gives rise to no change in the form of commutation relations (4), (5), (12) and (37) in Part I. Thus the fundamental laws for the nonlocal field are indeed invariant with respect to the whole group of inhomogeneous Lorentz transformations as it should be. Furthermore, the operator such as n(k, l, m), which represents the number of elementary particles in the quantum state characterized by the quantum numbers k, l, m, is invariant in the same sense.

Now, any operator A of the first class can be written in the form

$$(x'_\mu|A|x''_\mu) = \int\cdots\int (dl_\mu)^4 \bar{A}(l_\mu l^\mu) \prod_\mu \delta(x'_\mu - x''_\mu + l_\mu) \tag{12}$$

or

$$A = \int\cdots\int (dl_\mu)^4 \bar{A}(l_\mu l^\mu) \exp(i l^\mu p_\mu/\hbar), \tag{13}$$

where $\bar{A}(l_\mu l^\mu)$ is a certain function of $l_\mu l^\mu$ alone. More precisely, $\bar{A}$ is a unit submatrix multiplied by a numerical function of $l_\mu l^\mu$. In particular, if $\bar{A}$ has the form

$$\bar{A}(l_\mu l^\mu) = a \prod_\mu \delta(l_\mu) \tag{14}$$

~~where a is a number,~~ A itself is a unit matrix multiplied by a certain number $\int a$. Among more general types of the operator of the first class, we can discriminate the space-like displacement operators from the time-like displacement operators. The former can be written in the form

$$A = \int_0^\infty 2\lambda d\lambda \int\cdots\int (dl_\mu)^4 a(\lambda^2)\, \delta(l_\mu l^\mu - \lambda^2) \exp(i l^\mu p_\mu/\hbar) \tag{15}$$

where $a(\lambda^2)$ is an arbitrary function of λ^2. Such an operator A as given by (15) has, in general, matrix elements $(x'_\mu|A|x''_\mu)$, which are different from zero only for those values of $r_\mu = x'_\mu - x''_\mu$ satisfying the condition

$$r_\mu r^\mu \geq 0 \tag{16}$$

On the other hand, the time-like displacement operators can be expressed in general as

$$A = \int_0^\infty 2\lambda d\lambda \int\cdots\int (dl_\mu)^4 a(\lambda^2)\, \delta(l_\mu l^\mu + \lambda^2) \exp(i l^\mu p_\mu/\hbar) \tag{17}$$

with the matrix elements, which are different from zero only for r_μ's satisfying the condition

$$r_\mu r^\mu \leq 0. \tag{18}$$

Now we see that an arbitrary operator of the first class A can be decomposed into the space-like operator and the time-like operator as follows:

$$A = \int_{-\infty}^{+\infty} d\Lambda \int\cdots\int (dl_\mu)^4 a(\Lambda)\, \delta(l_\mu l^\mu - \Lambda) \exp(i l^\mu p_\mu/\hbar). \tag{19}$$

II. Classification of Elementary Systems

According to the general arguments on nonlocal operators in the preceding section, the most general nonlocal scalar field can be written in the form

(20)

which can be considered as a superposition of irreducible fields.

I. General Properties of Nonlocal Operators

In Part I of the paper under the same title, particular types of nonlocal fields were considered in detail. In order to cope with the problem of the interaction between nonlocal fields or that between a local field and a nonlocal field, however, we have to investigate more general properties common to various types of nonlocal operators. An arbitrary ~~operat~~ nonlocal operator A can be represented by a matrix $(x' \ A \ x'')$ with rows and columns, which are characterized by the eigenvalues of the space-time operators in the representation in x -space. In general, the individual matrix element $(x' \ A \ x'')$ is not a pure number, but a sub-matrix with rows and columns, which are characterized by eigenvalues of numbers of particles in various possible states. In other words, $(x' \ A \ x'')$ can be regarded as an ensemble of a family of functions of x' and x'', or a family of functions $A(X , r)$ of X and r , where

(1)

Now it is clear that any function or a whole family of functions $A(X , r)$ can be written in the integral form

(2)

where k and l are arbitrary four vectors, which are independent of X and r . This is equivalent to the relation

(3)

which, in turn, can be regarded as the matrix representation of the relation

(4)

between the operators A, x and $\bar{p}$.

I. General Properties of Nonlocal Operators

In Part I of the paper under the same title, particular types of nonlocal fields were considered in detail. In order to cope with the problem of the interaction between nonlocal fields or that between a local field and a nonlocal field, however, we have to investigate more general properties common to various types of nonlocal operators. An arbitrary ~~operat~~ nonlocal operator A can be represented by a matrix (x' A x'') with rows and columns, which are characterized by the eigenvalues of the space-time operators in the representation in x -space. In general, the individual matrix element (x' A x'') is not a pure number, but a sub-matrix with rows and columns, which are characterized by eigenvalues of numbers of particles in various possible states. In other words, (x' A x'') can be regarded as an ensemble of a family of functions of x' and x'', or a family of functions A(X , r) of X and r , where

(1)

Now it is clear that any function or a whole family of functions A(X , r) can be written in the integral form

(2)

where k and l are arbitrary four vectors, which are independent of X and r . This is equivalent to the relation

(3)

which, in turn, can be regarded as the matrix representation of the relation

(4)

between the operators A, x and p .

As the vectors k and l on the right hand side of (4) are, in general, independent of each other, two factors and are not commutative, but they satisfy the commutation relation

(5)

Thus, (4) can be written alternatively in the form

(6)

or

(7)

where and are related with each other and with by the simple relations

(8)

In this way any nonlocal operator A can be expanded into Fourier series or expressed as Fourier integral in 8-dimensional x-p-space formally, in spite of the fact that x and p are noncommutative operators. The Fourier transform (or ∅ or) of A is an ensemble of operators characterized by real parameters k and l .

Now there are various types of operators among the nonlocal operators in the most general sense. Among others, the following three classes of operators are of particular interest in connection with the formulation of the problem of interaction between nonlocal fields. The first class of operators consists of those which are relativistically invariant and are independent of the field quantities. More precisely, an operator A_1 belongs to the first class, if the folowing two conditions are satisfied:

(i) The matrix element must be invariant with respect to inhomogeneous Lorentz group including the translation of the origin of coordinate system.

As the vectors k and l on the right hand side of (4) are, in general, independent of each other, two factors and are not commutative, but they satisfy the commutation relation

(5)

Thus, (4) can be written alternatively in the form

or (6)

where and are related with each other and with by the simple relations (7)

(8)

In this way any nonlocal operator A can be expanded into Fourier series or expressed as Fourier integral in 8-dimensional x-p-space formally, in spite of the fact that x and p are noncommutative operators. The Fourier transform (or ∅ or) of A is an ensemble of operators characterized by real parameters k and l .

Now there are various types of operators among the nonlocal operators in the most general sense. Among others, the following three classes of operators are of particular interest in connection with the formulation of the problem of interaction between nonlocal fields. The first class of operators consists of those which are relativistically invariant and are independent of the field quantities. More precisely, an operator A belongs to the first class, if the folowing two conditions are satisfied:

(i) The matrix element must be invariant with respect to <u>inhomogeneous</u> Lorentz group including the translation of the origin of coordinate system.

(ii) Each matrix element, which is by itself a submatrix with rows and columns characterized by the number of particles in various possible states, must be ~~a certain~~ the unit submatrix multiplied by a certain function of x' and x''.

The condition (i) implies that (x' A x'') = A(X , r) must be an invariant function of r alone, because the parameters X and r are transformed into

(9)

respectively, by a general inhomogeneous Lorentz transformation

(10)

of the space-time operators. The condition (ii) implies further that this function A(r) is an ordinary complex function of r r multiplied by the unit submatrix.

The second and more general class of operators, in which the operators of the first class are all included, consists of those which satisfy the condition (i), but do not necessarily satisfy the condition (ii). The nonlocal field operators, which were considered in Part I, can be regarded as those which belong to the second class. For example, the scalar field U and U^*, which are defined by (32)' and (34) respectively, become invariant with respect to the inhomogeneous Lorentz transformation (10), if we assume that the operators u(k , l) and u^*(k , l) are transformed by the rule

(11)

This gives rise to no change in the form of commutation relations (4), (5), (12) and (37) in Part I. Thus the fundamental ~~equations~~ laws for the nonlocal field are indeed invariant with respect to the whole group of inhomogeneous Lorentz transformations as it shouls be. Furthermore, the operator such as n(k, l, m), which represents the number of elementary particles in the quantum state characterized by the quantum numbers k, l, m, is invariant in the same sense.

(ii) Each matrix element, which is by itself a submatrix with rows and columns characterized by the number of particles in various possible states, must be ~~a~~ ~~~~ the unit submatrix multiplied by a certain function of x' and x''.

The condition (i) implies that (x' A x'') = A(X , r) must be an invariant ~~~~ function of r alone, because the parameters X and r are transformed into

(9)

respectively ~~,~~ by a general inhomogeneous Lorentz transformation

(10)

of the space-time operators. The condition (ii) implies further that this function A(r) is an ordinary complex function of r r multiplied by the unit submatrix.

The second and more general class of operators, in which the operators of the first class are all included, consists of those which satisfy the condition (i), but do not necessarily satisfy the condition (ii). The nonlocal field operators, which were considered in Part I, can be regarded as those which belong to the second class. For example, the scalar field U and U^*, which are defined by (32)' and (34) respectively, become invariant with respect to the inhomogeneous Lorentz transformation (10), if we assume that the operators u(k , l) and u^*(k , l) are transformed by the rule

(11)

This gives rise to no change in the form of commutation relations (4), (5), (12) and (37) in Part I. Thus the fundamental ~~~~ laws for the nonlocal field are indeed invariant with respect to the whole group of inhomogeneous Lorentz transformations as it shouls be. Furthermore, the operator such as n(k, l, m), which represents the number of elementary particles in the quantum state characterized by the quantum numbers k, l, m, is invariant in the same sense.

Now, any operator A of the first class can be written in the form

(12)

or

(13)

where $\bar{A}($ $)$ is a certain function of alone. More precisely, $\bar{A}$ is a unit submatrix multiplied by a numerical function of . In particular, if $\bar{A}$ has the form

(14)

A itself is a unit matrix multiplied by a certain number. Among more general types of the operator of the first class, we can discriminate the space-like displacement operators from the time-like displacement operators. The former can be written in the form

(15)

where is an arbitrary function of . Such an operator A as given by (15) has ~~ingen~~ general matrix elements (x' A x''), which are different from zero only for those values of x'-x'' satisfying the condition

(16)

On the other hand, the time-like displacement operators can be expressed in general as

(17)

with the matrix elements, which are different from zero only for x'-X'' satisfying the condition

(18)

Now we see that arbitrary operator of the first class A can be decomposed into the space-like operator and the time-like operator as follows:

(19) (

Now, any operator A of the first class can be written in the form

(12)

or

(13)

where $\bar{A}($) is a certain function of alone. More precisely, $\bar{A}$ is a unit submatrix multiplied by a numerical function of . In particular, if $\bar{A}$ has the form

(14)

A itself is a unit matrix multiplied by a certain number. Among more general types of the operator of the first class, we can discriminate the space-like displacement operators from the time-like displacement operators. The former can be written in the form

(15)

where is an arbitrary function of . Such an operator A as given by (15) has in~~gen~~ general matrix elements (x' A x''), which are different from zero only for those values of x'-x'' satisfying the condition

(16)

On the other hand, the time-like displacement operators can be expressed in general as

(17)

with the matrix elements, which are different from zero only for x'-X'' satisfying the condition

(18)

Now we see that arbitrary operator of the first class A can be decomposed into the space-like operator and the time-like operator as follows:

(19) (

II. Classification of Elementary Systems

According to the general arguments on nonlocal operators in the preceding section, the most general nonlocal scalar field can be written in the form

(20)

which can be considered as a superposition of irreducible fields.