$$\frac{d^2u}{dr^2} - \frac{l(l+1)u}{r^2} + k^2 u = 0$$

$kr = x$ $$\frac{d^2u}{dx^2} - \frac{l(l+1)u}{x^2} + u = 0$$

$$u = x^{\frac{1}{2}} J_{l+\frac{1}{2}}(kr)$$

$$\frac{du}{dx} = x^{\frac{1}{2}} \frac{dJ_{l+\frac{1}{2}}}{dx} + \frac{1}{2} x^{-\frac{1}{2}} J$$

$$\frac{d^2u}{dx^2} = x^{\frac{1}{2}} \frac{d^2J}{dx^2} + x^{-\frac{1}{2}} \frac{dJ}{dx} - \frac{1}{4} x^{-\frac{3}{2}} J$$

$$\frac{d^2J}{dx^2} + \frac{1}{x}\frac{dJ}{dx} + \left(1 - \frac{(l+\frac{1}{2})^2}{x^2}\right) J = 0$$

$$\frac{d^2u}{dr^2} - \frac{l(l+1)u}{r^2} - \kappa^4 r^2 u + \mu u = 0$$

$$u = e^{-\lambda r^2} v$$

$$\frac{du}{dr} = e^{-\lambda r^2} \frac{dv}{dr} - 2\kappa r e^{-\kappa r^2} v$$

$$\frac{d^2u}{dr^2} = e^{-\lambda r^2} \frac{d^2v}{dr^2} - 4\lambda r e^{-\kappa r^2} \frac{dv}{dr}$$
$$+ 4\lambda^2 r^2 e^{-\kappa r^2} v$$
$$- 2\lambda e^{-\kappa r^2} v$$

$$4\lambda^2 = \kappa^4 \qquad \lambda = \frac{\kappa^2}{2}$$

$$x = \frac{\kappa^2 r^2}{2} \qquad \kappa^2 r^2 = \frac{2x}{\cancel{\kappa^2}}$$

$$\frac{du}{dr} = \kappa^2 r \frac{du}{dx} \qquad \kappa^4 r^2 = 2\kappa^2 x$$

$$\frac{d^2u}{dr^2} = \kappa^4 r^2 \frac{d^2u}{dx^2} + \kappa^2 \frac{du}{dx}$$

$$\kappa^4 r^2 \frac{d^2u}{dx^2} + \kappa^2 \frac{du}{dx} - \frac{l(l+1)}{r^2} u$$

$$- \kappa^4 r^2 u + \mu u = 0$$

$$\frac{d^2u}{dx^2} + \frac{1}{2x} \frac{du}{dx} - \frac{l(l+1)}{4x^2} u$$

$$- u + \frac{\mu}{2\kappa^2 x} u = 0$$

$$\frac{d^2u}{dx^2} + \frac{1}{2x} \frac{du}{dx} - \left(\frac{\kappa^2 l(l+1)}{4x^2} + 1 - \frac{\mu}{2\kappa^2 x} \right) u$$

$$= 0$$

$$u = e^{-x} v$$

$$\frac{du}{dx} = e^{-x} \frac{dv}{dx} - e^{-x} v$$

$$\frac{d^2u}{dx^2} = e^{-x} \frac{d^2v}{dx^2} - 2 e^{-x} \frac{dv}{dx} + e^{-x} v$$

$$\frac{d^2v}{dx^2} - 2\frac{dv}{dx} + \frac{1}{2x}\frac{dv}{dx} - \frac{1}{2x}v$$

$$- \left(\frac{l(l+1)}{4x^2} - \frac{\mu}{2\kappa^2 x}\right)v = 0$$

$$\frac{d^2v}{dx^2} - \left(2 - \frac{1}{2x}\right)\frac{dv}{dx}$$

$$- \left(\frac{l(l+1)}{4x^2} - \left(\frac{\mu}{2\kappa^2} - \frac{1}{2}\right)\frac{1}{x}\right)v = 0$$

$$v = x^{l+1} w$$

$$\frac{dv}{dx} = x^{l+1}\frac{dw}{dx} + (l+1)x^{l} w$$

$$\frac{d^2v}{dx^2} = x^{l+1}\frac{d^2w}{dx^2} + 2(l+1)x^{l}\frac{dw}{dx} + l(l+1)x^{l-1}w$$

$$\frac{d^2w}{dx^2} + \frac{2(l+1)}{x}\frac{dw}{dx} + 0$$

$$v = x^{\alpha} w$$

$$\frac{dv}{dx} = x^{\alpha}\frac{dw}{dx} + \alpha x^{\alpha-1} w$$

$$\frac{d^2v}{dx^2} = x^{\alpha}\frac{d^2w}{dx^2} + 2\alpha x^{\alpha-1}\frac{dw}{dx}$$

$$+ \alpha(\alpha-1)x^{\alpha-2} w$$

$$\frac{d^2w}{dx^2} + \frac{2\alpha}{x}\frac{dw}{dx} + \frac{\alpha(\alpha-1)}{x^2}w$$

$$\cancel{+\left(2-\frac{1}{2x}\right)\left(1+\frac{\alpha}{x}\right)}$$

$$+\left(2-\frac{1}{2x}\right)\frac{dw}{dx} - \frac{\alpha}{x}\left(2-\frac{1}{2x}\right)w$$

$$-\left(\frac{l(l+1)}{4x^2} - \left(\frac{\mu}{2x^2}-\frac{1}{2}\right)\frac{1}{x}\right)w = 0$$

$$\alpha(\alpha-1) = \frac{l(l+1)}{4} - \frac{\alpha}{2}$$

$$\cancel{\left(\alpha - \frac{\quad}{2}\right)\left(\quad\right)}$$

$$\alpha^2 - \frac{\alpha}{2} - \frac{l(l+1)}{4} = 0$$

$$\left(\alpha + \frac{l}{2}\right)\left(\alpha - \frac{l+1}{2}\right) = 0$$

$$\frac{l+1}{x}\frac{dw}{dx}$$

$$+\,2 - \frac{1}{2x}$$

$$\alpha = \frac{l+1}{2}; \qquad 2\ \cancel{l+1}\ +$$

$$\frac{d^2w}{dx^2} + \left(\frac{l+1+\frac{1}{2}}{x} - 2\right)\frac{dw}{dx}$$

$$-\left(\frac{l+1}{x} + \left(\frac{\mu}{2x^2}-\frac{1}{2}\right)\frac{1}{x}\right)w = 0$$

$$x = \frac{y}{2}: \quad \frac{d^2w}{dy^2} + \left(\frac{l+1+\frac{1}{2}}{2y} - 1\right)\frac{dw}{dx} - \left(\frac{l+1}{2y} + (\ ?\ )\frac{1}{2}\right)w = 0$$

$$\alpha = \frac{l+1}{2} + \frac{1}{4}$$

$$\frac{l+1}{2} + \left(\frac{\mu}{2\kappa^2} - \frac{1}{2}\right)\frac{1}{2} = n$$

$$\frac{\mu}{2\kappa^2} = 2n - l - 1 - \frac{1}{2}$$

$$= 2(n - \alpha)$$

$$L_n^{(\alpha)}(y)$$

$$y = \frac{\kappa^2 r^2}{4} \qquad \alpha = \frac{l+1}{2} + \frac{1}{4}$$

$$H_{2n}(x) = (-2)^n n! \, L_n^{(-1/2)}\left(\frac{x^2}{2}\right)$$

$$H_{2n+1}(x) = (-2)^n n! \, L_n^{1/2}\left(\frac{x^2}{2}\right)$$

$$L_n^{(\alpha_1+\alpha_2+\cdots\alpha_k+k-1)}(x_1+x_2+\cdots+x_k)$$

$$=\sum_{i_1+i_2+\cdots+i_k=n} L_{i_1}^{(\alpha_1)}(x_1)L_{i_2}^{(\alpha_2)}(x_2)\cdots L_{i_k}^{(\alpha_k)}(x_k)$$

$k \Rightarrow 4$: $n \to n+1$ $\alpha_1+\alpha_2+\cdots+\alpha_k+3 \Rightarrow k+1$

$$L_{n+1}^{(k+1)}(+x_1+x_2+x_3+x_4)$$

$$=\sum_{i_1+i_2+i_3+i_4=n} L_{i_1}^{(\alpha_1)}(x_1)L_{i_2}^{(\alpha_2)}(x_2)L_{i_3}^{(\alpha_3)}(x_3)L_{i_4}^{(\alpha_4)}(x_4)$$

$\alpha_1+\alpha_2+\alpha_3+\alpha_4=k-2$

$$H_{2n}(x)=(-2)^n n!\,L_n^{(-1/2)}\left(\frac{x^2}{2}\right)$$

$L_{n+1}^{(k+1)}(R^2)$: polynomials of

$(x_1^2, x_2^2, x_3^2, x_4^2)$

can be expanded into series of

$H_{2n_1}(x_1), H_{2n_2}(x_2), H_{2n_3}(x_3), H_{2n_0}(x_0)$

$e^{+x_0^2}$? (extra factor in $\cancel{X}(x_n)$)

finite degrees with respect to n_1, n_2, n_3

infinite degrees " " n_0

Not a single expansion over the whole space!!!

~~A~~ ~~F~~ functions which can be written as a product form in $(R, \chi, \theta, \varphi)$

versus those

in $(x_1 x_2 x_3 x_0)$