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N30

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Quantum Mechanics
I. Physics 247(~248)
M. W. 4:10~5:30 Pupin
329

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Advanced Quantum Mechanics

1952 ~ 1953

Winter Session Physics 247.

Reference

W. Pauli, Meson Theory
G. Wentzel, Q.T. of Fields
E. Fermi, Elementary Particles

~~Part One: Relativistic Field Theory~~

I. Review of Relativistic Field Theory

II. ~~Modern~~ Further Development of Field Theory

III. Application to Q.E.D.
Lamb Shift
Anomalous Mg. Moment of the Electron

IV. Application to Mesons and Nucleons

V. Application to β-decay

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Part Two : Application
of Group Theory
to Quantum Mechanics

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Review of Field Theory 1
as discussed in 237&238
Quantum Mechanics
last year

(i) Field Equations
scalar field
(pseudoscalar field)
vector field
(pseudovector field)
spinor field

(ii) (Second Quantization) { B. E.
Commutation Relations { F. D.

(iii) Schrödinger Equation
one-time formalism
many-time formalism.

Modern method of quantization
~~Complet~~ of Fields

Dyson, Advanced Q. M.

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$$T_{\mu\nu} = L\,\delta_{\mu\nu} - \frac{\partial L}{\partial \frac{\partial \varphi}{\partial x_\nu}} \frac{\partial \varphi}{\partial x_\mu}$$

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I. Recapitulation of

General procedure of formulating quantum theory of fields is as follows: (see Wentzel, Q. T. of Fields)

classical field quantity φ, real scalar field

Lagrangian density (invariant)

$$L\left(\varphi, \frac{\partial \varphi}{\partial x_\mu}\right)$$

$x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$.

$$\delta \int L \, dx_1 dx_2 dx_3 dx_4 = 0.$$

$$\frac{\partial L}{\partial \varphi} - \sum \frac{\partial}{\partial x_\mu}\left(\frac{\partial L}{\partial \frac{\partial \varphi}{\partial x_\mu}}\right) = 0$$

$\delta\varphi = 0$ on the boundary surface

Example: Klein-Gordon equation

$$L = -\frac{1}{2}\left(\frac{\partial \varphi}{\partial x_\mu}\frac{\partial \varphi}{\partial x_\mu} + \varkappa^2 \varphi^2\right)$$

$$\frac{\partial^2 \varphi}{\partial x_\mu \partial x_\mu} - \varkappa^2 \varphi = (\Box - \varkappa^2)\varphi = 0$$

Energy-Momentum Tensor

$$T_{\mu\nu} = \frac{\partial \varphi}{\partial x_\mu}\frac{\partial L}{\partial \frac{\partial \varphi}{\partial x_\mu}} + L\,\delta_{\mu\nu}$$

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$$\frac{\partial T_{\mu\nu}}{\partial x_\mu} = \frac{\partial^2\varphi}{\partial x_\mu \partial x_\mu}\frac{\partial L}{\partial \frac{\partial\varphi}{\partial x_\nu}} + \frac{\partial\varphi}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\left(\frac{\partial L}{\partial \frac{\partial\varphi}{\partial x_\mu}}\right)$$

$$+ \frac{\partial\varphi}{\partial x_\mu}\frac{\partial L}{\partial\varphi}\delta_{\mu\nu}$$

$$+ \frac{\partial^2\varphi}{\partial x_\mu \partial x_\nu}\frac{\partial L}{\partial \frac{\partial\varphi}{\partial x_\mu}}\delta_{\mu\nu} + \frac{\partial L}{\partial x_\mu}$$

$$= \frac{\partial^2\varphi}{\partial x_\mu \partial x_\nu}\left(\frac{\partial L}{\partial \frac{\partial\varphi}{\partial x_\mu}} - \frac{\partial L}{\partial \frac{\partial\varphi}{\partial x_\mu}}\right)$$

$$+ \frac{\partial\varphi}{\partial x_\mu}\left(\frac{\partial}{\partial x_\mu}\left(\frac{\partial L}{\partial \frac{\partial\varphi}{\partial x_\mu}}\right) - \frac{\partial L}{\partial\varphi}\right)$$

(provided that L does not contain x_μ explicitly)

In particular, $= 0$ if

$$L = -\frac{1}{2}\left(\frac{\partial\varphi}{\partial x_\mu}\frac{\partial\varphi}{\partial x_\mu} + \kappa^2\varphi^2\right)$$

$$T_{\mu\nu} = \frac{\partial\varphi}{\partial x_\mu}\frac{\partial\varphi}{\partial x_\nu} + L\,\delta_{\mu\nu}$$

$T_{\mu\nu} = T_{\nu\mu}$
(symmetric tensor)

$T_{44} = -H$ energy density

$$= -\frac{1}{2}\left(\frac{\partial\varphi}{\partial x_k}\frac{\partial\varphi}{\partial x_k} - \frac{\partial\varphi}{\partial x_4}\frac{\partial\varphi}{\partial x_4} + \kappa^2\varphi^2\right)$$

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$$\frac{\partial T_{41}}{\partial x_1} + \frac{\partial T_{42}}{\partial x_2} + \frac{\partial T_{43}}{\partial x_3} + \frac{\partial T_{44}}{\partial x_4} = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t} + \operatorname{div} \mathbf{G} = 0$$

$\mathbf{G} = (ic\,T_{41}, ic\,T_{42}, ic\,T_{43})$

momentum density

$G_k = \frac{i}{c} T_{4k}$

$$= \int H \, dx_1 dx_2 dx_3$$

total energy is conserved because

$H(t) = H(x_0)$ = surface integral at infinity $= 0$

if φ is sufficiently small for at large distances from the origin.

Momentum density

$\mathbf{G} = (\frac{i}{c} T_{41}, \frac{i}{c} T_{42}, \frac{i}{c} T_{43})$

$$\frac{\partial T_{1k}}{\partial x_1} + \frac{\partial T_{2k}}{\partial x_2} + \frac{\partial T_{3k}}{\partial x_3} + \frac{\partial T_{4k}}{\partial x_4} = 0$$

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total momentum

$$\overline{G_T} = \int G_{T\mu} \, dx_1 dx_2 dx_3$$

is conserved.

Angular momentum density:

$$M_{\lambda\mu\nu} = x_\mu T_{\lambda\nu} - x_\nu T_{\lambda\mu}$$

$$\frac{\partial M_{\lambda\mu\nu}}{\partial x_\lambda} = T_{\lambda\mu} x_\nu + x_\mu \frac{\partial T_{\lambda\nu}}{\partial x_\lambda} - T_{\lambda\mu\nu} - x_\nu \frac{\partial T_{\lambda\mu}}{\partial x_\lambda}$$

$$= T_{\lambda\mu} x_\nu - T_{\lambda\mu} x_\nu$$

$$\frac{\partial M_{\lambda\mu\nu}}{\partial x_\nu} = 0$$

if $T_{\mu\lambda} = T_{\lambda\mu}$.

~~In that case~~

$$\overline{M_{\lambda\mu\nu}} = -\overline{M_{\lambda\nu\mu}}$$

(antisymmetric ~~tensor~~ with respect to μ and ν)

If $T_{\mu\lambda} = T_{\lambda\mu}$,

$$\overline{M_{\lambda\mu\nu}} = \int M_{\lambda\mu\nu} \, dx_1 dx_2 dx_3$$

is conserved.

~~$\overline{M_{\lambda\mu\nu}} = \int$~~

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$$+\frac{\dot{r}}{c}\overline{M_{kl4}} = \int (x_k G_l - x_l G_k)\, dx_1 \times dx_2 dx_3$$

Thus,

$\frac{\dot{r}}{c} M_{kl4}$

can be interpreted as angular momentum density.

In general, the canonical e.m. tensor $T_{\mu\nu}$ may not be symmetrical, but we can always construct a symmetrical tensor in the following way $\Theta_{\mu\nu}$ (See Wentzel, Appendix)

which satisfies ~~$\Theta_{\mu\nu} = T_{\mu\nu} + T'_{\mu\nu}$~~

$$\Theta_{\mu\nu} = \Theta_{\nu\mu}, \quad \frac{\partial \Theta_{\mu\nu}}{\partial x_\nu} = 0, \quad \int \Theta_{\mu 4}(dx)^3 = \int T_{\mu 4}(dx)^3$$

$\Theta_{\mu\nu} = T_{\mu\nu} + T'_{\mu\nu}$, where

$$\left.\begin{aligned} \frac{\partial T'_{\mu\nu}}{\partial x_\nu} &= 0 \\ \int T'_{\mu 4}(dx)^3 &= 0 \end{aligned}\right\} *$$

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Let us consider ~~~~ classical fields!

Rigid rotation in homogeneous

infinitesimal Lorentz Transf.

$$\delta x_\mu = \sum_{\nu=1}^{4} \omega_{\mu\nu} x_\nu \qquad \omega_{jk}: \text{real} \quad \omega_{4k}: \text{imag}$$

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

(!! $x'_\mu = x_\mu + \delta x_\mu = a_{\mu\nu} x_\nu$

$x'_\mu x'_\mu = x_\mu x_\mu + 2 x_\mu \delta x_\mu$

$= x_\mu x_\mu + 2\omega_{\mu\nu} x_\mu x_\nu$

$= x_\mu x_\mu$ $\qquad (\omega_{\mu\nu} + \omega_{\nu\mu}) x_\mu x_\nu$

~~For~~ In general

$a_{\mu\nu} a_{\mu\lambda} x_\nu x_\lambda = x_\nu x_\nu$

$a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda}$

$a_{\mu\nu} = \delta_{\mu\nu} + \omega_{\mu\nu}$.

$(\delta_{\mu\nu} + \omega_{\mu\nu})(\delta_{\mu\lambda} + \omega_{\mu\lambda}) = \delta_{\nu\lambda}$

$\delta_{\nu\lambda} + \omega_{\lambda\nu} + \omega_{\mu\nu} \omega_{\mu\lambda} = 0$

$\omega_{\lambda\nu} = -\omega_{\nu\lambda}$

Variation of the field quantities $\varphi^\sigma(x)$ at a fixed space-time point

$(\varphi^\sigma \to (\varphi')^\sigma = \varphi^\sigma + \delta\varphi^\sigma)$ $\quad \delta\varphi^\sigma = \sum_\rho \Lambda^\sigma_\rho \varphi^\rho$ $\quad \Lambda_{\sigma\rho}(\omega_{\lambda\mu})$

$\Lambda_{\sigma\rho} = \sum_{\lambda,\mu=1}^{4} \frac{\partial \Lambda_{\sigma\rho}}{\partial \omega_{\lambda\mu}} \omega_{\lambda\mu}$

$\Lambda^{\lambda\mu}_{\sigma\rho} \left(= \frac{\partial \Lambda_{\sigma\rho}}{\partial \omega_{\lambda\mu}} \right) = -\Lambda^{\mu\lambda}_{\sigma\rho}$

$= -\Lambda^{\mu\lambda}_{\sigma\rho}$ since symmetric part has no contribution to $\Lambda_{\sigma\rho}$.

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$$\delta \frac{\partial \varphi^\sigma}{\partial x_\mu} = \sum_\rho \Lambda_{\sigma\rho} \frac{\partial \varphi^\rho}{\partial x_\mu} + \sum_{\nu=1}^{4} \omega_{\mu\nu} \frac{\partial \varphi^\sigma}{\partial x_\nu}$$

~~[Variation]~~

$$\left(\because \quad \frac{\partial \varphi^{\sigma\prime}}{\partial x'_\mu} = \frac{\partial}{\partial x'_\mu}\left(\varphi^\sigma + \sum_\rho \Lambda_{\sigma\rho} \varphi^\rho\right) \right.$$

$$= \frac{\partial x_\nu}{\partial x'_\mu} \frac{\partial \varphi^\sigma}{\partial x_\nu} + \sum_\rho \Lambda_{\sigma\rho} \frac{\partial \varphi^\rho}{\partial x_\mu}$$

$$x_\nu = x'_\nu - \sum \omega_{\nu\lambda} x_\lambda' = x'_\nu + \sum \omega_{\lambda\nu} x'_\lambda$$

$$= \frac{\partial \varphi^\sigma}{\partial x_\mu} - \sum_\nu \omega_{\nu\mu} \frac{\partial \varphi^\sigma}{\partial x_\nu}$$

$$+ \sum_\rho \Lambda_{\sigma\rho} \frac{\partial \varphi^\rho}{\partial x_\mu} + \sum_\nu \omega_{\mu\nu} \frac{\partial \varphi^\sigma}{\partial x_\nu}$$

$$\delta \frac{\partial \varphi^\sigma}{\partial x_\mu} = \sum_\nu \omega_{\mu\nu} \frac{\partial \varphi^\sigma}{\partial x_\nu} + \sum_\rho \Lambda_{\sigma\rho} \frac{\partial \varphi^\rho}{\partial x_\mu}$$

$$L\left(\varphi^\sigma, \frac{\partial \varphi^\sigma}{\partial x_\mu}\right) \to L' = L\left(\varphi^{\sigma\prime}, \frac{\partial \varphi^{\sigma\prime}}{\partial x'_\mu}\right)$$

$$\delta L = L' - L = 0$$

or

$$\sum_\rho \frac{\partial L}{\partial \varphi^\rho_{x_1}} \delta \varphi^\rho_{x_1} + \sum_\rho \sum_\mu \frac{\partial L}{\partial \frac{\partial \varphi^\rho_{x_2}}{\partial x_\mu}} \delta\left(\frac{\partial \varphi^\rho_{x_2}}{\partial x_\mu}\right) = 0$$

or

$$\sum_{\lambda,\mu} \left\{ \sum_{\rho,\sigma} \frac{\partial L}{\partial \varphi^\rho_{x_3}} \Lambda^{\lambda\mu}_{\rho\sigma} \varphi_\sigma + \sum_{\nu,\rho,\sigma} \frac{\partial L}{\partial \frac{\partial \varphi^\rho_{x_4}}{\partial x_\nu}} \left(\Lambda^{\lambda\mu}_{\rho\sigma} \frac{\partial \varphi_\sigma}{\partial x_\mu} + \frac{\partial \varphi^\rho}{\partial x_\mu} \delta_{\lambda\nu} \right) \right\} \omega_{\lambda\mu} = 0$$

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$$\Sigma_{\lambda\mu} = \sum_{\rho,\sigma} \left(\frac{\partial L}{\partial \varphi_\rho} \varphi_\sigma + \sum_{\nu=1}^{4} \frac{\partial L}{\partial \left(\frac{\partial \varphi_\rho}{\partial x_\nu}\right)} \frac{\partial \varphi_\sigma}{\partial x_\nu} \right) \Lambda^{\nu\mu}_{\rho\sigma}$$

$$+ \sum_{\rho} \frac{\partial L}{\partial \frac{\partial \varphi_\rho}{\partial x_\lambda}} \frac{\partial \varphi_\rho}{\partial x_\mu}$$

is symmetrical $\Sigma_{\lambda\mu} = \Sigma_{\mu\lambda}$ (because $\Sigma_{\lambda\mu}$ is determined coordinate system and be chosen arbitrarily

$$F_{\nu\lambda\mu} = \sum_{\rho,\sigma} \frac{\partial L}{\partial \frac{\partial \varphi_\rho}{\partial x_\nu}} \Lambda^{\lambda\mu}_{\rho\sigma} \varphi_\sigma$$

$$= - \sum F_{\nu\mu\lambda}$$

$$\sum_\nu \frac{\partial F_{\nu\lambda\mu}}{\partial x_\nu} = \sum_{\rho,\sigma} \left(\frac{\partial L}{\partial \varphi_\rho} \varphi_\sigma \right.$$

$$\left. + \sum_\nu \frac{\partial L}{\partial \frac{\partial \varphi_\rho}{\partial x_\nu}} \frac{\partial \varphi_\sigma}{\partial x_\nu} \right) \Lambda^{\lambda\mu}_{\rho\sigma}$$

or

$$\Sigma_{\lambda\mu} = \sum_{\nu=1}^{4} \frac{\partial F_{\nu\lambda\mu}}{\partial x_\nu} + T_{\lambda\mu} + L\delta_{\lambda\mu}$$

The conditions

$$\begin{cases} G_{\nu\lambda\mu} = - G_{\lambda\nu\mu} \\ \frac{1}{2}(G_{\nu\lambda\mu} - G_{\nu\mu\lambda}) = F_{\nu\lambda\mu} \end{cases}$$

are satisfied by

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$$G_{\nu\lambda\mu} = F_{\mu\lambda\nu} + F_{\lambda\mu\nu} + F_{\nu\lambda\mu}$$

(∵ $G_{\lambda\nu\mu} = F_{\mu\nu\lambda} + F_{\nu\mu\lambda} + F_{\lambda\nu\mu}$)

$$= -F_{\mu\lambda\nu} - F_{\nu\lambda\mu} - F_{\lambda\mu\nu}$$

$$= -G_{\nu\lambda\mu}$$

(fixed / ines for an arbitrary $\omega_{\mu\nu} = -\omega_{\nu\mu}$ can)

$$G_{\nu\lambda\mu} - G_{\nu\mu\lambda} = F_{\mu\lambda\nu} + F_{\lambda\mu\nu} + F_{\nu\lambda\mu}$$

$$- F_{\lambda\mu\nu} - F_{\mu\lambda\nu} - F_{\nu\mu\lambda}$$

$$= F_{\nu\lambda\mu} - F_{\nu\mu\lambda} = 2F_{\nu\lambda\mu}$$

Conversely, from

$$\tfrac{1}{2}(G_{\nu\lambda\mu} - G_{\nu\mu\lambda}) = F_{\nu\lambda\mu}$$

$$G_{\nu\lambda\mu} = -G_{\nu\mu\lambda} \quad (= F_{\lambda\nu\mu})$$

$$F_{\mu\lambda\nu} + F_{\lambda\mu\nu} + F_{\nu\lambda\mu}$$

$$= \tfrac{1}{2}\{ G_{\mu\lambda\nu} - G_{\mu\nu\lambda}$$

$$+ G_{\lambda\mu\nu} - G_{\lambda\nu\mu}$$

$$+ G_{\nu\lambda\mu} - G_{\nu\mu\lambda} \}$$

$$= G_{\lambda\nu\mu}$$

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$$\Theta_{\lambda\mu} = T_{\lambda\mu} + T'_{\lambda\mu}$$

$$T'_{\lambda\mu} = -\sum_{\nu=1}^{4} \frac{\partial G_{\nu\lambda\mu}}{\partial x_\nu}$$

$$\Theta_{\lambda\mu} - \Theta_{\mu\lambda} = T_{\lambda\mu} - T_{\mu\lambda} - \frac{\partial G_{\nu\lambda\mu}}{\partial x_\nu} + \frac{\partial G_{\nu\mu\lambda}}{\partial x_\nu}$$

$$= T_{\lambda\mu} - T_{\mu\lambda} + \frac{\partial}{\partial x_\nu}\left(F_{\mu\lambda\nu} + F_{\lambda\mu\nu} + F_{\nu\lambda\mu} - F_{\lambda\mu\nu} - F_{\mu\lambda\nu} - F_{\nu\mu\lambda} \right)$$

$$= T_{\lambda\mu} - \frac{\partial F_{\nu\lambda\mu}}{\partial x_\nu} - L\delta_{\lambda\mu} - \left(T_{\mu\lambda} - \frac{\partial F_{\nu\mu\lambda}}{\partial x_\nu} - L\delta_{\mu\lambda} \right)$$

$$= \Sigma_{\lambda\mu} - \Sigma_{\mu\lambda} = 0$$

$$\sum_{\lambda=1}^{4} \frac{\partial T'_{\lambda\mu}}{\partial x_\lambda} = \sum_{\nu,\lambda=1}^{4} \frac{\partial^2 G_{\nu\lambda\mu}}{\partial x_\nu \partial x_\lambda}$$

$$= \frac{1}{2} \sum \left(\frac{\partial^2 G_{\nu\lambda\mu}}{\partial x_\nu \partial x_\lambda} + \frac{\partial^2 G_{\lambda\nu\mu}}{\partial x_\lambda \partial x_\nu} \right)$$

$$= 0$$

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$$\int T'_{4\mu}(dx)^3 = + \int \sum_\nu \frac{\partial G_{\nu 4\mu}}{\partial x_\nu} (dx)^3$$

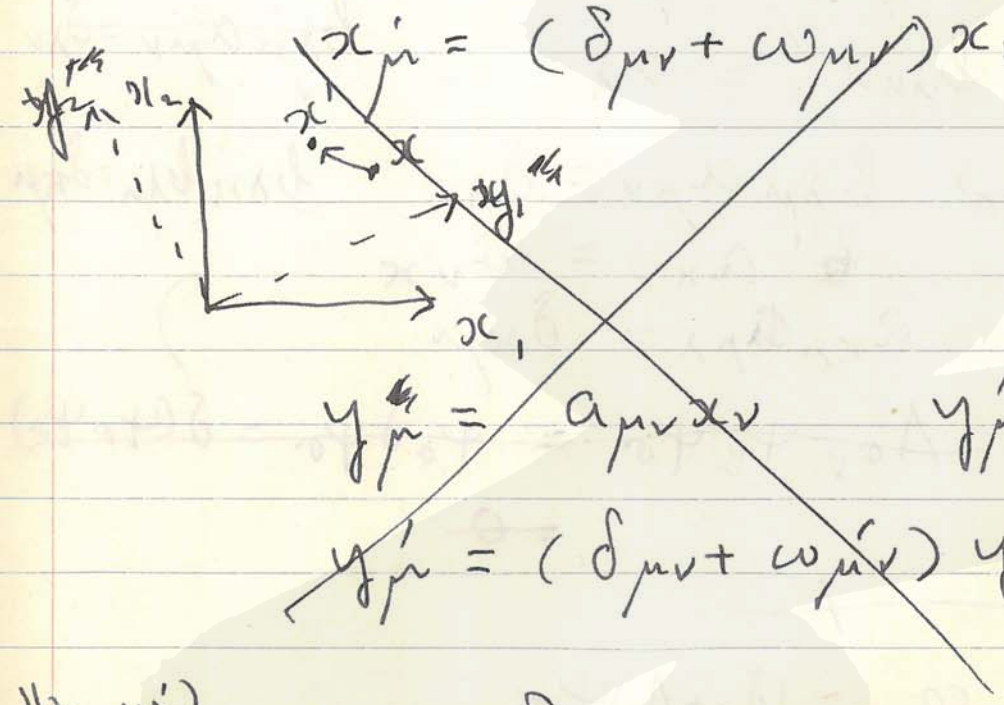
$$= \int \sum_k \frac{\partial G_{k4\mu}}{\partial x_k} (dx)^3 = 0$$

$$\therefore G_{44\mu} = 0$$

Problem: Prove that $\omega_{\mu\nu}, \Sigma_{\lambda\mu}$ are tensors.

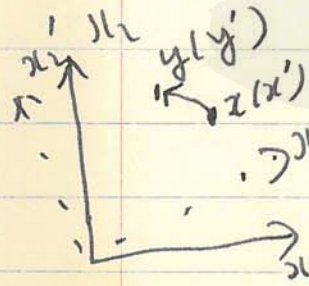
Note: Proof that $\omega_{\mu\nu}, \Sigma_{\lambda\mu}$ are tensors.

$$x'_\mu = (\delta_{\mu\nu} + \omega_{\mu\nu}) x_\nu$$



$$y'_\mu = a_{\mu\nu} x_\nu \qquad y'_\mu = a_{\mu\nu} x'_\nu$$

$$y'_\mu = (\delta_{\mu\nu} + \omega_{\mu'\nu}) y_{\mu\nu}$$



$$y_\mu = (\delta_{\mu\nu} + \omega_{\mu\nu}) x_\nu,$$

$$x'_\mu = a_{\mu\nu} x_\nu, \qquad y'_\mu = a_{\mu\nu} y_\nu$$

$$y'_\mu = (\delta_{\mu\nu} + \omega_{\mu'\nu}) x'_\nu$$

$$a_{\mu\nu} y_\nu = (\delta_{\mu\nu} + \omega_{\mu'\nu}) a_{\nu\lambda} x_\lambda$$

$$a_{\mu\nu} (\delta_{\nu\lambda} + \omega_{\nu\lambda}) x_\lambda = (\delta_{\mu\nu} + \omega_{\mu'\nu}) a_{\nu\lambda} x_\lambda$$

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$$a_{\mu\nu}(\delta_{\nu\lambda} + \omega_{\nu\lambda}) = (\delta_{\mu\nu} + \omega_{\mu'\nu}) a_{\nu\lambda}$$

or

$$a_{\mu\nu}\omega_{\nu\lambda} = \omega_{\mu'\nu} a_{\nu\lambda} \quad \times a_{\kappa\lambda}$$

$$a_{\mu\nu} a_{\kappa\lambda} \omega_{\nu\lambda} = \omega_{\mu'\kappa}$$

because

$$a_{\nu\lambda} a_{\kappa\lambda} = \delta_{\nu\kappa}$$

(∵ $b_{\lambda\mu} x'_\mu = a_{\mu\nu} x_\nu$

$b_{\lambda\mu} x'_\mu = x_\lambda$ ~~$b_{\lambda\mu} a_{\mu\nu} = \delta_{\lambda\nu}$~~

$b_{\lambda\kappa} b_{\lambda\mu} a_{\mu\nu} = \delta_{\lambda\nu}$ $b_{\lambda\kappa} b_{\lambda\mu} = \delta_{\kappa\mu}$

$a_{\kappa\nu} = b_{\nu\kappa}$

$a_{\kappa\lambda} a_{\mu\lambda} = \delta_{\kappa\mu}$)

~~$\Lambda_{\sigma\rho} \psi_\rho \psi_\sigma = \psi_\sigma \delta\psi_\sigma = \delta(\psi_\sigma \psi_\sigma)$~~

~~$= 0$~~

or

$$\varphi_\sigma = (\delta_{\sigma\rho} + \Lambda_{\sigma\rho}) \psi_\rho$$

$$\Lambda_{\sigma\rho} = \Lambda^{\lambda\mu}_{\sigma\rho} \omega_{\lambda\mu}$$

$$\Lambda_{\sigma\rho} \psi_\rho \psi_\sigma = 0$$

$$\sum_{\rho\lambda\mu} \Lambda^{\lambda\mu}_{\sigma\rho} \omega_{\lambda\mu} \psi_\rho \psi_\sigma = 0$$

$$B^{\lambda\mu} \omega_{\lambda\mu} = 0$$

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$$\psi'_\rho = A_{\rho\sigma}\psi_\sigma \quad \left.\right\}$$
$$\varphi'_\rho = A_{\rho\sigma}\varphi_\sigma$$

$$A_{\rho\sigma}A_{\sigma\tau} = A'_{\rho\sigma}A_{\sigma\tau}$$

~~$\Lambda'_{\rho\sigma} = B_{\rho\sigma,\tau\omega}A_{\tau\omega}$~~

$\Lambda_{\sigma\tau}$: behaves like a tensor ~~$A_{\rho\sigma}\psi_\rho$~~ in ψ-space

Thus $\Lambda'_{\rho\sigma}\psi'_\rho\varphi'_\sigma$
$= \Lambda_{\rho\sigma}\psi_\rho\varphi_\sigma$

$\mp$ is invariant,

$$* \quad T_{\mu\nu} = \mathcal{L}\delta_{\mu\nu} - \sum_\sigma \frac{\partial \mathcal{L}}{\partial \frac{\partial \varphi_\sigma}{\partial x_\mu}} \frac{\partial \varphi_\sigma}{\partial x_\nu}$$

$$\textcircled{T}_{\mu\nu} = T_{\mu\nu} - \sum_{\nu=1}^{4} \frac{\partial G_{\nu\lambda\mu}}{\partial x_\nu}$$

$$G_{\nu\lambda\mu} = \Gamma_{\mu\lambda\nu} + \Gamma_{\lambda\mu\nu} + \Gamma_{\nu\lambda\mu}$$

$$\Gamma_{\nu\lambda\mu} = \sum_{\rho,\sigma} \frac{\partial \mathcal{L}}{\partial \frac{\partial \varphi_\rho}{\partial x_\nu}} \Lambda^{\lambda\mu}_{\rho\sigma}\varphi_\sigma$$

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Method of quantization of field originally made use of the canonical formalism as follows:

$$\varphi(x) \to \pi(x) = \frac{\partial h}{\partial\left(\frac{\partial \varphi(x)}{\partial t}\right)} = \frac{1}{ic}\frac{\partial L}{\partial \frac{\partial \varphi}{\partial x_4}}$$

commutation relations:

$$\left.\begin{aligned} &[\varphi(x,t), \pi(x',t)] = i\hbar\,\delta(x,x') \\ &[\varphi(x,t), \varphi(x',t)] = 0 \\ &[\pi(x,t), \pi(x',t)] = 0 \end{aligned}\right\}$$

field equations: (Heisenberg repr.)

$$\left\{\begin{aligned} i\hbar\frac{\partial \varphi}{\partial t} &= [\varphi, \bar{H}] \\ i\hbar\frac{\partial \pi}{\partial t} &= [\pi, \bar{H}] \end{aligned}\right.$$

Schrödinger equation $i\hbar\frac{\partial \Psi}{\partial t} = \bar{H}\Psi$ (Schrödinger repr.)

However, this method ~~did~~ was apparently not relativistically, because the time t ~~took~~ played a peculiar role as distinct from the space variables x, y, z. In modern field theories as developed by

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Tomonaga, Schwinger and others, we make use of the interaction representation:

~~Schrödinger representation~~

* (i) Heisenberg representation

$$i\hbar \frac{\partial \varphi^{(H)}(x,t)}{\partial t} = [\varphi^{(H)}(x,t), \bar{H}]$$
$$i\hbar \frac{\partial \pi^{(H)}(x,t)}{\partial t} = [\pi^{(H)}(x,t), \bar{H}]$$

$$[\varphi^{(H)}(x,t), \pi^{(H)}(x',t)] = i\hbar\delta(x,x')$$

$$i\hbar \frac{\partial \Psi^{(H)}(\varphi(x),t)}{\partial t} = 0$$

(ii) Schrödinger representation

$$\varphi^{(S)} = e^{-i\bar{H}t/\hbar} \varphi^{(H)} e^{i\bar{H}t/\hbar}$$

$$\pi^{(S)} = e^{-i\bar{H}t/\hbar} \pi^{(H)} e^{i\bar{H}t/\hbar}$$

$$i\hbar \frac{\partial \varphi^{(S)}}{\partial t} = 0$$
$$i\hbar \frac{\partial \pi^{(S)}}{\partial t} = 0$$

$$\Psi^{(S)}(\varphi(x), t) = e^{-i\bar{H}t/\hbar} \Psi^{(H)}(\varphi, t)$$

$$i\hbar \frac{\partial \Psi^{(S)}}{\partial t} = \bar{H} \Psi^{(S)}$$

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8

(iii) Interaction Representation

$$\bar{H} = \int H\,dv = \bar{H}_0 + \bar{H}'$$

$$H = H_0 + H' \qquad \bar{H}_0 = \int H_0\,dv$$

$$\bar{H}' = \int H'\,dv$$

$$\left.\begin{aligned} \varphi^{(I)} &= e^{i\bar{H}_0 t/\hbar}\,\varphi^{(S)}\,e^{-i\bar{H}_0 t/\hbar} \\ \Pi^{(I)} &= e^{i\bar{H}_0 t/\hbar}\,\varphi^{(S)}\,e^{-i\bar{H}_0 t/\hbar} \end{aligned}\right\}$$

$$\left.\begin{aligned} i\hbar\frac{\partial\varphi^{(I)}}{\partial t} &\simeq [\varphi^{(I)}, \bar{H}_0] \\ i\hbar\frac{\partial\Pi^{(I)}}{\partial t} &\simeq [\Pi^{(I)}, \bar{H}_0] \end{aligned}\right\}$$

gives rise to the free field equations which are relativistically invariant:

Example:

$$H_0 = \frac{1}{2}\left(\frac{\partial\varphi}{\partial x_k}\frac{\partial\varphi}{\partial x_k} + \frac{\partial\varphi}{\partial x_4}\frac{\partial\varphi}{\partial x_4} + \kappa^2\varphi^2\right)$$

$$\Pi = \frac{1}{ic}\frac{\partial L}{\partial\frac{\partial\varphi}{\partial x_4}} = \frac{-1}{ic}\frac{\partial\varphi}{\partial x_4} = \frac{i}{c}\frac{\partial\varphi}{\partial x_4}$$

$$H_0 = \frac{1}{2}\left(\frac{\partial\varphi}{\partial x_k}\frac{\partial\varphi}{\partial x_k} + c^2\Pi^2 + \kappa^2\varphi^2\right)$$

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$$[\varphi^{(x)}, \overline{H_0}] = i\hbar c^2 \pi^{(x)}$$

or $$i\hbar \frac{\partial \varphi^{(x)}}{\partial t} = i\hbar c^2 \pi^{(x)}$$

or $$\pi^{(x)} = \frac{1}{c^2}\frac{\partial \varphi^{(x)}}{\partial t} = \frac{\hat{\imath}}{c}\frac{\partial \varphi^{(x)}}{\partial x_4}$$

$$i\hbar \frac{\partial \pi^{(x)}}{\partial t} = [\pi^{(x)}, \overline{H_0}]$$

$$\rightarrow (\Box - \varkappa^2)\varphi^{(x)} = 0.$$

commutation relations

$$[\varphi^{(x)}(x,t), \pi^{(x)}(x',t')] = i\hbar c\,\delta(x,x')$$

etc.

are entitled to a relativistic form:

$$[\varphi(x,t), \varphi(x',t')]$$

$$= i\hbar c\,\Delta(x-x', t-t')$$

where

$$\Delta(x,t) = \frac{1}{(2\pi)^3}\iiint \frac{e^{i(kx - k_0 ct)} - e^{-i(kx - k_0 ct)}}{2ik_0}(dk)^3$$

$$k_0 = +\sqrt{k^2 + \varkappa^2}$$

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4

or

$$A(x_\mu) = \frac{1}{(2\pi)^3} \int \frac{e^{ik_\mu x_\mu} - e^{-ik_\mu x_\mu}}{2\, k_4} dk$$

$$A(r, t) = \frac{1}{(2\pi)^3 i} \int_{-\infty}^{t\infty} \cdots \int e^{i(kr - k_s ct)}$$

$$\times \frac{k_0}{|k_s|} \delta(k^2 - k_s^2 + \varkappa^2)\, dk_0\, dk_x\, dk_y\, dk_z$$

$$k_4 = i k_0$$

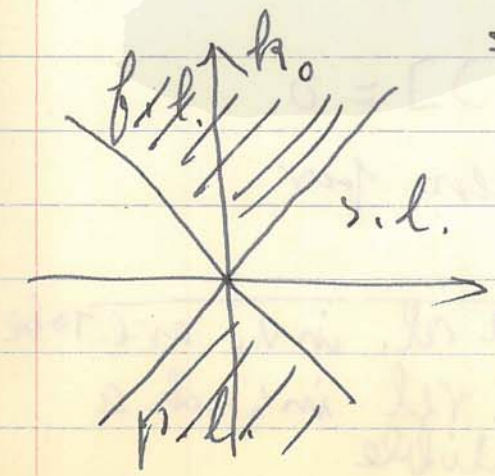
or

$$A(x_\mu) = \frac{-1}{(2\pi)^3} \int \cdots \int e^{i k_\mu x_\mu} \varepsilon(k_\mu)$$

$$\times \delta(k_n k_n + \varkappa^2)\, (dk)^4$$

where

$\varepsilon(k_n) = 1$ for k_n: future-like
$= 0$ for k_n: space-like
$= -1$ for k_n: past-like



future-like : $k_0 > k$
space-like : $|k_0| < k$
past-like : $k_0 < -k$
$-k < k_0 < k$

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$$[\varphi(x_\mu), \varphi(x'_\mu)] = i\hbar c\, \Delta(x_\mu - x'_\mu)$$

Proof: $\frac{1}{c}\frac{\partial \Delta(r,t)}{\partial t}\Big|_{t=0} = -\delta(r)$

or $\frac{1}{c}\frac{\partial \Delta(r-r', t-t')}{\partial t'}\Big|_{t'=t} = \delta(r-r')$

$[\varphi(r,t), \pi(r',t')]$

$= \left[\varphi(r,t), \frac{1}{c^2}\frac{\partial \varphi(r',t')}{\partial t'}\right]$

$= \left[\varphi(r,t), \frac{\partial \varphi(r',t')}{\partial t'}\right]$

$= \frac{\partial}{\partial t'}\left[\varphi(r,t), \varphi(r',t')\right]$

$= i\hbar c \frac{1}{c^2}\frac{\partial \Delta(r-r', t-t')}{\partial t'}\Big|_{t'=t}$

$= i\hbar\, \delta(r-r')$

$[\varphi(r,t), \varphi(r',t)] = 0$

because $\Delta(x_\mu)$ is zero for space-like x_μ.

free field equations which are rel. inv. are to be combined with rel. com. rel in such a way that they are compatible

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Schrödinger equation ⇒ T.S. Equations

$$i\hbar \frac{\partial \Psi^{(S)}(\varphi, t)}{\partial t} = (\bar{H}_0^{(S)} + \bar{H}^{(S)})\Psi^{(S)}$$

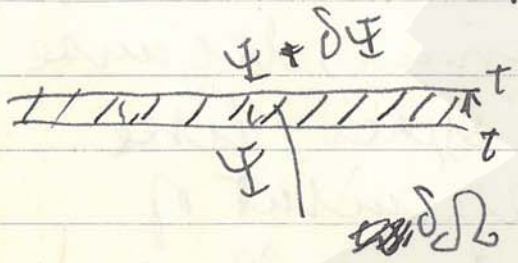
$$\Psi^{(I)} = e^{i\bar{H}_0 t/\hbar}\Psi^{(S)} \qquad \bar{H}_0^{(S)} = \bar{H}_0^{(I)} = \bar{H}_0$$

$$i\hbar \frac{\partial \Psi^{(I)}}{\partial t} = \cancel{e^{i\bar{H}_0 t/\hbar}\bar{H}_0\Psi^{(S)}} + \cancel{\bar{H}_0\Psi^{(S)}}$$

$$+ e^{i\bar{H}_0 t/\hbar}(\bar{H}_0^{(S)} + \bar{H}'^{(S)})\Psi^{(S)}$$

or $$i\hbar \frac{\partial \Psi^{(I)}}{\partial t} \approx \bar{H}'^{(I)}\Psi^{(I)}$$

where $\bar{H}'^{(I)} = e^{i\bar{H}_0 t/\hbar}\bar{H}'^{(S)}e^{-i\bar{H}_0 t/\hbar}$.



t : independent of x, y, z

$$\delta\Psi(\{\varphi(x_n)\}, t) = \bar{H}'\delta t\cdot\Psi$$

$$= \int\cdots\int_{\delta\Omega} H'(dx)^4 \cdot \Psi$$

ζ' : eigenvalues of ζ, ζ : complete set of commuting field operators at time t

$t(x,y,z)$ connected with surface σ

$$\delta\Psi(\{\varphi(x_n)\}, \sigma)$$

$$= \int\cdots\int_{\delta\Omega_\sigma} H'(dx)^4 \cdot \Psi(\{\varphi(x_n)\}, \sigma)$$

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In particular, when $\delta\sigma^{1}$ ($=\delta\sigma_P$) is concentrate about a small region around the point $P(x_\mu)$ we have

$$i\hbar \frac{\delta \Psi(x'_\mu, \sigma)}{\delta \sigma} = H'(x_\mu) \cdot \Psi(x'_\mu, \sigma).$$

Simultaneous equations

Integrability Condition

$$[H'(x_\mu), H'(x'_\mu)] = 0$$

for $x_\mu - x'_\mu$: space-like

T-S equation is invariant, if H' is invariant, because the concept of space-like surface is independent of Lorentz transformation and the volume of the infinitesimal region is also invariant.

Now, H' is invariant, if H' does not contain the time-derivatives of field quantities, because, in that case, $H' = -L'$

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where L' is the interaction part in the hagrangian.

On the contrary, when ~~L' or~~ L' contains the time derivatives of field quantities, H' on the right hand side of T. S. ~~must be replaced by~~ is, in general, depends ~~on~~ not only x_μ, but also on the direction of n_μ of the surface normal at the point x_μ.

The example of such a complicated case will be given later on.

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$(\alpha''|) = \overline{\Psi}(\alpha'')\overline{\Psi}(\alpha')\cdot\Psi$

$\Psi| = \int \Psi(\alpha')(\alpha'|$

$= \int \Psi(\beta')(\beta'|$

$= \int \Psi(\alpha')(\alpha'|\beta')(\beta'|$

$(\alpha''| = (\alpha''|\beta')(\beta'|$

$(\alpha''|\beta') = \overline{\Psi}(\alpha'')\Psi(\beta')$

$\overline{\alpha} = U\alpha U^{-1}$

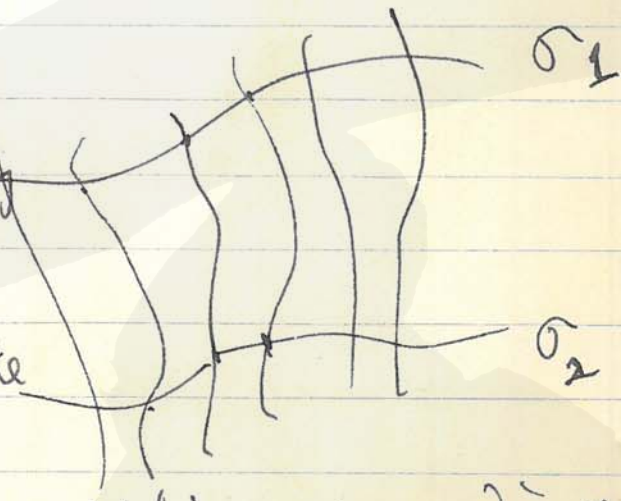
$\overline{\Psi}(\alpha'')\cdot\overline{\Psi}(\overline{\alpha}') = U\Psi(\alpha')$

$(\alpha''|\overline{\alpha}') = (\alpha''|U|\alpha')$

$(\overline{\alpha}'|\alpha'') = (\alpha'|U^{\dagger}|\alpha'')$

$= (\alpha|U^{-1}|\alpha'')$

* By "similarly constructed" we mean, the corresponding points on σ_1 and σ_2 are related by a definite way, for instance, by the help of a curvilinear coordinates u_μ, where σ_1, σ_2 are two surfaces, such u_4 = const.



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II. Schwinger's Variation (II, 1)
Principle in Q. T. of Fields.

Schwinger, P. R. 82 (1951), 914
Dyson, A. Q. M.
Dirac, P. Q. M. § 32.

The ~~so~~ eigen~~state~~ vector for an eigenstate of with a set of eigenvalues ζ' of a commuting operators ζ, which were complete set of constructed from the field quantities ϕ^α and their derivatives $\frac{\partial \phi^\alpha}{\partial x_\mu}$ on the surface σ; be

$$\Psi(\zeta', \sigma)$$

$\zeta_2 \sigma_2 \rightarrow \zeta_1 \sigma_1$ *

ζ_1, ζ_2 are similarly constructed operators sets which possess identical eigenvalue spectrum. Then ζ_1 and ζ_2 are related by a unitary transformation:

$$\zeta_1 = U_{12} \zeta_2 U_{12}^{-1}$$

$$\Psi(\zeta_1', \sigma_1) = U_{12} \Psi(\zeta_2', \sigma_2)$$

where $\zeta_1' \equiv \zeta_2'$ (same sets of eigenvalues)

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where U_{12} is a unitary operator with the matrix element

$$(\zeta_1', \sigma_1 | \zeta_2'', \sigma_2) = (\zeta_2' \sigma_2 | U_{12}^{-1} | \zeta_2'', \sigma_2)$$

Proof:

$\zeta_2(\sigma_2) \to \alpha$ $\qquad$ $\zeta_2' \to \alpha'$

$\zeta_1(\sigma_1) \to \beta$ $\qquad$ $\zeta_1'' \to \beta''$

$$\Psi = \int \Psi(\alpha') d\alpha' (\alpha' |$$

$$\Psi = \int \Psi(\beta'') d\beta'' (\beta'' |$$

$$= \int \Psi(\alpha') d\alpha' (\alpha' | \beta'') d\beta'' (\beta'' |$$

$$\text{or } (\alpha' | \simeq \int (\alpha' | \beta'') d\beta'' (\beta'' |$$

Thus we ~~take an active regard~~ construct form the ~~rotation~~ transformation function $(\alpha' | \beta'')$, the operator U, whose matrix element is ~~represented~~ given by

$$(\alpha' | U | \alpha'') = (\alpha' | \beta'')$$

where $\alpha'' \equiv \beta''$, then

$$(\alpha' | U^\dagger | \alpha'') = \overline{(\alpha'' | U | \alpha')}$$

$$= (\alpha'' | \beta') = (\beta' | \alpha'')$$

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(I, 2)

$$(\alpha'|UU^\dagger|\alpha'') = \int (\alpha'|U|\alpha''')\, d\alpha'''\, (\alpha'''|U^\dagger|\alpha'')$$

$$= \int (\alpha'|\beta''')\, d\beta'''\, (\beta'''|\alpha'') = \delta(\alpha', \alpha'')$$

$$(\alpha'|U^\dagger U|\alpha'') = \delta(\alpha', \alpha'')$$

or $UU^\dagger = U^\dagger U = 1$.

~~$(\alpha'|1) = (\alpha'|U|\alpha'')\, d\alpha''$~~

$$\left.\begin{aligned} \Psi(\beta'') &= \int \Psi(\alpha')\, d\alpha'\, (\alpha'|\beta'') \\ U\Psi(\alpha'') &= \int \Psi(\alpha')\, d\alpha'\, (\alpha'|U|\alpha'') \end{aligned}\right\}$$

$\therefore \Psi^\dagger(\alpha') U \Psi(\alpha'') = (\alpha'|U|\alpha'')$

or

$\Psi(\beta'') = U\Psi(\alpha'')$ with $\alpha'' \equiv \beta''$

$\Psi(\alpha'') = U^{-1}\Psi(\beta'')$

Q. E. D. ⌋

$$\left.\begin{aligned} \zeta_1 &= U_{12}\, \zeta_2\, U_{12}^{-1} \\ \Psi(\zeta_1', \sigma_1) &= U_{12}\, \Psi(\zeta_2', \sigma_2) \end{aligned}\right\} \quad (1)$$

$(\zeta_1', \sigma_1 | \zeta_2'' \sigma_2) = (\zeta_2' \sigma_2 | U_{12}^{-1} | \zeta_2'', \sigma_2)$

$(\because\ (\beta'|\alpha'') = (\alpha'|U^\dagger|\alpha'') = (\alpha'|U^{-1}|\alpha''))$

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The temporal development of a system is determined by the relationships between eigenvectors associated with different space-like surfaces or by the transformation function (1). The operator U_{12}^{-1} depends on the choice of commuting operators ζ_1 and ζ_2 on the surfaces σ_1 and σ_2 respectively.

Any infinitesimal change in the quantities on which (1) depends induces a corresponding change in U_{12}^{-1}.

$$\delta(\zeta_1', \sigma_1 | \zeta_2'', \sigma_2) = (\zeta_2', \sigma_2 | \delta U_{12}^{-1} | \zeta_2'', \sigma_2) \quad (2)$$

Since U_{12} is unitary, we have

$$U_{12} U_{12}^{\dagger} = 1$$

$$\delta U_{12} \cdot U_{12}^{\dagger} + U_{12} \cdot \delta U_{12}^{\dagger} = 0$$

so that

$$(i U_{12} \delta U_{12}^{\dagger})^{\dagger} = -i \delta U_{12} \cdot U_{12}^{\dagger}$$

$$= +i U_{12} \delta U_{12}^{\dagger}$$

or $i U_{12} \delta U_{12}^{\dagger}$ is an hermitian operator and we may write

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(II.3)

$$i U_{12} \delta U_{12}^{+} = i U_{12} \delta U_{12}^{-1} = \hbar \delta W_{12}$$

~~where~~ or

$$\delta U_{12}^{-1} = (i/\hbar)\, U_{12}^{-1} \delta W_{12}$$

where δW_{12} is an infinitesimal hermitian operator.

Thus, (2) becomes

$$\delta(\zeta_1', \sigma_1 | \zeta_2'' \sigma_2) = (i/\hbar)(\zeta_2' \sigma_2 | U_{12}^{-1} \delta W_{12} | \zeta_2'' \sigma_2)$$

$$= (\tfrac{i}{\hbar}) \int (\zeta_2' \sigma_2 | U_{12}^{-1} | \zeta_2''' \sigma_2) d\zeta_2''' (\zeta_2''' \sigma_2 | \delta W_{12} | \zeta_2'' \sigma_2)$$

$$= (\tfrac{i}{\hbar}) \int (\zeta_1' \sigma_1 | \zeta_2''' \sigma_2)\, d\zeta_2''' (\zeta_2''' \sigma_2 | \delta W_{12} | \zeta_2'' \sigma_2)$$

$$= (\tfrac{i}{\hbar}) (\zeta_1' \sigma_1 | \delta W_{12} | \zeta_2'' \sigma_2)$$

which is the mixed representation of δW_{12}.

$$(\beta' | A | \alpha'') = \int (\beta' | \alpha''') d\alpha''' (\alpha''' | A | \alpha'')$$

$\Psi^{\dagger}(\beta)$: $A \Psi(\alpha'') = \int \Psi(\alpha''') d\alpha''' (\alpha''' | A | \alpha'')$

$\Psi^{\dagger}(\beta') A \Psi(\alpha'') = \int (\beta' | \alpha''') d\alpha''' (\alpha''' | A | \alpha'')$

$\therefore$ $(\beta' | A | \alpha'') = \Psi^{\dagger}(\beta') A \Psi(\alpha'')$

or $(\zeta_1', \sigma_1 | \delta W_{12} | \zeta_2'', \sigma_2) = \Psi^{\dagger}(\zeta_1') \delta W_{12} \Psi(\zeta_2'')$

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transformation functions

Now the law of composition of

$$(\zeta_1' \sigma_1 | \zeta_3''' \sigma_3) = \int (\zeta_1', \sigma_1 | \zeta_2'' \sigma_2)\, d\zeta_2'' \times (\zeta_2'', \sigma_2 | \zeta_3''', \sigma_3)$$

gives rise to

$$\delta(\zeta_1', \sigma_1 | \zeta_3''', \sigma_3) = \int \delta(\zeta_1', \sigma_1 | \zeta_2'' \sigma_2)\, d\zeta_2'' (\zeta_2'', \sigma_2 | \zeta_3''', \sigma_3) + \int (\zeta_1', \sigma_1 | \zeta_2'' \sigma_2)\, d\zeta_2''\, \delta(\zeta_2'', \sigma_2 | \zeta_3''' \sigma_3)$$

or

$$\left(\frac{i}{\hbar}\right)(\zeta_1' \sigma_1 | \delta W_{13} | \zeta_3''' \sigma_3) = \left(\frac{i}{\hbar}\right)\int (\zeta_1' \sigma_1 | \delta W_{12} | \zeta_2'' \sigma_2)\, d\zeta_2'' (\zeta_2'', \sigma_2 | \zeta_3''', \sigma_3) + \left(\frac{i}{\hbar}\right)\int (\zeta_1', \sigma_1 | \zeta_2'' \sigma_2)\, d\zeta_2'' (\zeta_2'' \sigma_2 | \delta W_{23} | \zeta_3''' \sigma_3)$$

or

$$= \left(\frac{i}{\hbar}\right)(\zeta_1' \sigma_1 | \delta W_{12} | \zeta_3'' \sigma_3) + \left(\frac{i}{\hbar}\right)(\zeta_1', \sigma_1 | \delta W_{23} | \zeta_3'' \sigma_3)$$

Hence, we obtain

$$\delta W_{13} = \delta W_{12} + \delta W_{23}.$$

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(II. 4)

Further, let us consider one more important property, which the operators ~~is to~~ $\delta\Psi_n$ must satisfy, if q. ~~mechanical~~ principles of q.m. are to be applied to our system consisting of ~~quantized~~ fields.

$\alpha \to \alpha' = \alpha + \delta\alpha$ (supposing that eigenvalues of α are $(-\infty, +\infty)$)

$\Psi(\alpha') \to \Psi(\alpha') = \Psi((\alpha+\delta\alpha)')$

$= \Psi(\alpha') + \delta\Psi(\alpha') = U\Psi(\alpha')$

$U = 1 + (i/\hbar) F$

$U^{-1} = 1 - (i/\hbar) F$ } F: infinitesimal hermitian operator

$\delta\Psi(\alpha') = U\Psi(\alpha') - \Psi(\alpha')$

F is hermitian because

$U = 1 + (i/\hbar) F.$

$U^\dagger = 1 - (i/\hbar) F^\dagger$) $F = F^\dagger$.

$U^{-1} = 1 - (i/\hbar) F$

$\delta\Psi(\alpha') = +\left(\frac{i}{\hbar}\right) F \Psi(\alpha')$

$(\alpha'|) = \Psi^\dagger(\alpha')$

$\delta(\alpha'|) = -\frac{i}{\hbar} \Psi^\dagger(\alpha') F \Psi$

$= -\frac{i}{\hbar}(\alpha'|F|\ )$

or

$(\hbar/i)\,\delta(\alpha'|) = -\int (\alpha'|F|\alpha'')(\alpha''|\ )$

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which is ~~the~~ a differential equation for the representation $(\alpha'|\beta'')$ of arbitrary state vector Ψ.

$$\alpha \to \alpha + \delta\alpha$$
$$\beta \to \beta + \delta\beta$$
$$\delta(\alpha'|\beta'') = \delta\Psi^{\dagger}(\alpha')\cdot\Psi(\beta'') + \Psi^{\dagger}(\alpha')\delta\Psi(\beta'')$$
$$= -\frac{i}{\hbar}(\alpha'|F_\alpha - F_\beta|\beta'')$$
$$-(\hbar/i)\delta(\alpha'|\beta'') = \int(\alpha'|F_\alpha|\alpha'')d\alpha''(\alpha''|\beta'') - \hbar\int(\alpha'|\beta''')d\beta'''(\beta'''|F_\beta|\beta'')$$

Thus the infinitesimal change in the transformation function $(\alpha'|\beta'')$ is given by the difference of ~~two~~ mixed representations of $(\hbar/i)F_\alpha$, $(\hbar/i)F_\beta$.

In field theory, we may assume that

$$\delta(\zeta_1'\sigma_1|\zeta_2''\sigma_2) = \left(\frac{\hbar}{i}\right)(\zeta_1'\sigma_1|F(\sigma_1) - F(\sigma_2)|\zeta_2''\sigma_2)$$

$F(\sigma)$ can be called the infinitesimal generating operator on σ.

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(II, 5')

or we start from a fundamental assumption

$$\delta W_{12} = -(F(\sigma_1) - F(\sigma_2)) \quad (I)$$

The second fundamental assumption, which is necessary for deriving all the ~~rela~~ basic relations in the q.t. of field is

$$W_{12} = \frac{1}{c} \int_{\sigma_2}^{\sigma_1} (dx)^4 L[x] \quad (II)$$

where $L[x]$ is an invariant function of fields and their derivatives at the point $x_\mu \equiv x$

$$L[x] = L(\varphi^\alpha(x), \varphi_\mu^\alpha(x))$$

$$\varphi_\mu^\alpha(x) \equiv \partial_\mu \varphi^\alpha(x)$$

$$\partial_\mu = \frac{\partial}{\partial x_\mu}$$

and $iL[x]$ is an hermitian operator.

From the second assumption, we conclude that the fundamental equation for the transformation function

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$$\delta(\zeta_1', \sigma_1 | \zeta_2'', \sigma_2) = \left(\frac{+i}{\hbar}\right)(\zeta_1', \sigma_1 | \delta W_{12} | \zeta_2'', \sigma_2)$$

$$= \frac{i}{\hbar c}(\zeta_1', \sigma_1 | \delta \int_{\sigma_2}^{\sigma_1} (dx)^4 L \, | \zeta_2'', \sigma_2)$$

is unaltered in form by a change in the coordinate system.

~~From~~ The first assumption implies that the variation

$$\delta \int_{\sigma_2}^{\sigma_1} (dx)^4 L = +\{F(\sigma_1) - F(\sigma_2)\}$$

of the action integral operator is unaffected by the infinitesimal variation of the field quantities in the interior of the region bounded by σ_1 and σ_2 and depends only on operators attached to the boundary surfaces.

From this variation principle or the principle of Stationary action (Schwinger) ~~we can~~ derived all the required relations in q. t. of fields.

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(II, E)

(i) Equations of motion

Let us fix the points (without performing any deformation of the surfaces σ_1 and σ_2) and perform only infinitesimal variations of field quantities $\phi^{\alpha}(x)$ at each point. Then we have ($\delta_0\phi^{\alpha}(x)$)

$$+\delta_0 W_{12} = \left(\frac{1}{c}\right)\int_{\sigma_2}^{\sigma_1} (dx)^4\, \delta_0 L$$

$$\delta_0 L = \left(\frac{\partial L}{\partial \phi^{\alpha}}\right)\delta_0\phi^{\alpha} + \left(\frac{\partial L}{\partial \phi^{\alpha}_{\mu}}\right)\partial_\mu \delta_0\phi^{\alpha}$$

$$= \left\{\left(\frac{\partial L}{\partial \phi^{\alpha}}\right) - \partial_\mu\left(\frac{\partial L}{\partial \phi^{\alpha}_{\mu}}\right)\right\}\delta_0\phi^{\alpha}$$

$$+ \partial_\mu\left\{\left(\frac{\partial L}{\partial \phi^{\alpha}_{\mu}}\right)\delta_0\phi^{\alpha}\right\} \qquad (2)$$

where the expression for $\delta_0 L$ is to be understood symbolically, since the order of the operators in L must not be altered in course of effecting variation.

~~factual~~

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In order that the right hand side ~~is is~~ depend~~s~~ only on the change in φ^α on the boundaries σ_1, σ_2, well-known, the equations of motion

$$\partial_\mu \left(\frac{\partial L}{\partial \varphi^\alpha_\mu} \right) = \frac{\partial L}{\partial \varphi^\alpha} \qquad (3)$$

must be satisfied everywhere in the region between σ_1 and σ_2. Since σ_1, σ_2 is arbitrary, (e.m) must be satisfied at any point in space-time world.

It should be kept in mind, however, that in deriving formally (e.m), we had to assume ~~that~~ $\delta_0\varphi^\alpha$ to be arbitrary small numbers, not operators, so that $\delta_0\varphi^\alpha$ can be shifted freely to right or left in the r.h.s. of (2).

To what extent this is justified will be discussed later on.

We have, in this case, the generating inf. operator.

$$F(\sigma) = +\frac{1}{c} \int d\sigma_\mu \cdot \frac{\partial L}{\partial \varphi^\alpha_\mu} \delta_0 \varphi^\alpha \sim$$

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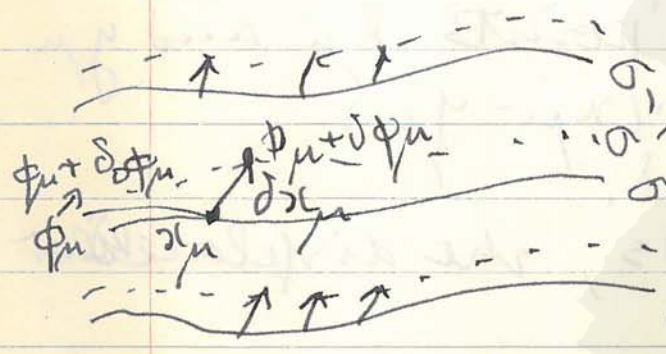
Next, let us consider a more general variation in which both the field quantities and the space-time points

$$\delta W_{12} = \left(\frac{1}{c}\right)\int_{\sigma_2}^{\sigma_1} (dx)^4\, \delta_0 \mathcal{L}$$

$$+\frac{1}{c}\left(\int_{\sigma_1} - \int_{\sigma_2}\right) d\sigma_\mu\, \delta x_\mu\, \mathcal{L}$$



where δx_μ is the displacement of the point x_μ and $\delta_0\phi^\alpha_\mu$ is the change in ϕ^α at a fixed point x_μ. The total change $\delta\phi^\alpha$ in $\phi^\alpha(x)$ is the sum of $\delta_0\phi^\alpha$ and the change in $\phi^\alpha(x)$ produced by moving from the point x on σ to $x' = x + \delta x$ on σ'.

In this case, we have

$$F(\sigma) = \pm\frac{1}{c}\int d\sigma_\mu \left\{\left(\frac{\partial \mathcal{L}}{\partial \phi^\alpha_\mu}\right)\delta_0\phi^\alpha + \mathcal{L}\,\delta x_\mu\right\}$$

$$\rightarrow = \pm\int d\sigma_\mu\, \Pi^\alpha_\mu\, \delta_0\varphi^\alpha \quad \text{where } \Pi^\alpha_\mu = \frac{1}{c}\frac{\partial L}{\partial \varphi^\alpha_\mu}$$

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In order to ~~obtain~~ derive ~~the~~ definite conclusions ~~of physical~~ in which are physically important, we must confine our attention to the displacement δx which corresponds to the ~~rigid~~ infinitesimal displacement of a four-dimensional rigid body, for which the four dimensional distance of two points x_μ and y_μ

$$(x_\mu - y_\mu)\cdot(x_\mu - y_\mu)$$

are not changed.

In other words, the displacement

$$\delta x_\mu = x'_\mu - x_\mu$$

must be of the form

$$\delta x_\mu = \varepsilon_\mu + \omega_{\mu\nu}\delta x_\nu$$
$$\omega_{\mu\nu} = -\omega_{\nu\mu},$$

which can be regarded as an infinitesimal inhomogeneous Lorentz transformation.

Then, we have

$$\partial_\nu \delta x_\mu = \omega_{\mu\nu}$$
$$= -\partial_\mu \delta x_\nu$$

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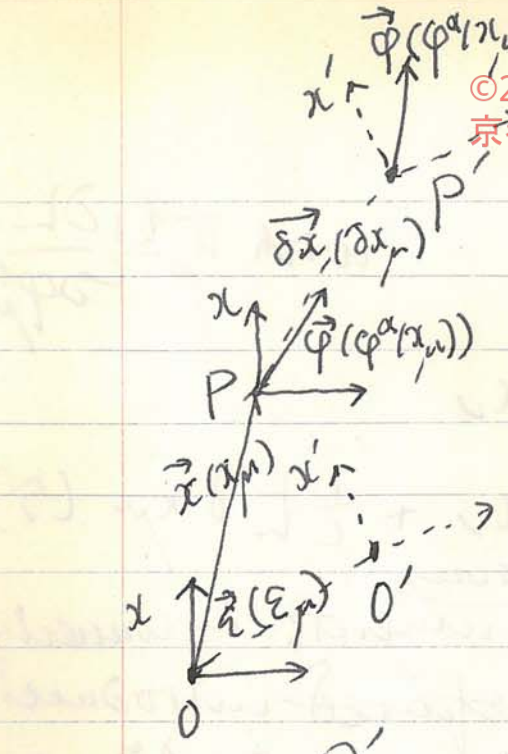

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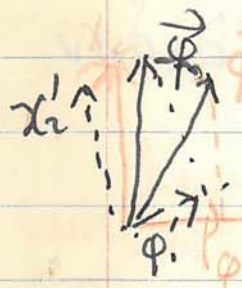
$$x_\mu' = x_\mu + \delta x_\mu$$
$$= \xi_\mu + \omega_{\mu\nu}\delta x_\nu \quad (4)$$

$$\delta\varphi^\alpha(x_\mu) = \varphi^{\alpha\prime}(x_\mu') - \varphi^\alpha(x_\mu)$$
$$\delta_0\varphi^\alpha(x_\mu)$$

where (φ^α) is the α-th component of $\vec{\varphi}$ at the point P' with respect to the new coordinate system.

Thus, $\varphi^\alpha(x_\mu')$ is obtained from $\varphi^\alpha(x_\mu+\delta x_\mu)$ by carrying out as a linear transformation which is inverse to the (4) or



$$\varphi^{\alpha\prime}(x_\mu') = \varphi^\alpha(x_\mu+\delta x_\mu) - \sum \Lambda_{\alpha\beta}\varphi^\beta(x_\mu+\delta x_\mu)$$
$$= \varphi^\alpha(x_\mu+\delta x_\mu) - \sum_{\alpha\beta,\lambda,\mu} \Lambda_{\alpha\beta}^{\lambda\mu}\omega_{\lambda\mu}\varphi^\beta(x_\mu)$$

Thus,

$$\delta\varphi^\alpha(x) = \delta_0\varphi^\alpha(x) + \frac{\partial\varphi^\alpha}{\partial x_\mu}\delta x_\mu - \Lambda_{\alpha\beta}^{\lambda\mu}\omega_{\lambda\mu}\varphi^\beta$$
$$= \delta_0\varphi^\alpha + \varphi_\mu^\alpha\delta x_\mu + \Lambda_{\alpha\beta}^{\lambda\mu}\varphi^\beta\partial_\lambda\delta x_\nu$$

Hence

$$\frac{1}{c}\left\{\left(\frac{\partial L}{\partial\varphi_\mu^\alpha}\right)\delta_0\varphi^\alpha + L\,\delta x_\mu\right\} =$$

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$$= \pi_\mu^\alpha \delta_0 \varphi^\alpha + \frac{1}{c} L \delta x_\mu \qquad \text{with } \pi_\mu^q = \frac{1}{c}\frac{\partial L}{\partial \varphi_\mu^q}$$

$$= \pi_\mu^\alpha \delta\varphi^\alpha - \pi_\mu^\alpha \varphi_\nu^\alpha \delta x_\nu$$

$$- \pi_\mu^q \Lambda_{\alpha\beta}^{\lambda\nu} \varphi^\beta \partial_\lambda \delta x_\nu + \frac{1}{c} L \delta x_\mu \qquad (5)$$

According to the previous arguments in connection with $\Theta_{\mu\nu}$, we ~~defined~~ introduced

$$\frac{1}{c} G_{\mu\lambda\nu} = \pi_\mu^\alpha \Lambda_{\alpha\beta}^{\lambda\nu} \varphi^\beta + \pi_\nu^q \Lambda_{\alpha\beta}^{\lambda\mu} \varphi^\beta + \pi_\lambda^q \Lambda_{\alpha\beta}^{\nu\mu} \varphi^\beta$$

$$= -\frac{1}{c} G_{\lambda\mu\nu}$$

Hence, we have

$$\frac{1}{c} G_{\mu\lambda\nu} \partial_\lambda \delta x_\nu = \pi_\mu^q \Lambda_{\alpha\beta}^{\lambda\nu} \varphi^\beta \partial_\lambda \delta x_\nu + \pi_\nu^q \Lambda_{\alpha\beta}^{\lambda\mu} \varphi^\beta \partial_\lambda \delta x_\nu + \pi_\lambda^q \Lambda_{\alpha\beta}^{\nu\mu} \varphi^\beta \partial_\lambda \delta x_\nu$$

(cancel with the other: $\partial_\lambda \delta x_\nu = -\partial_\nu \delta x_\lambda$)

$$= \pi_\mu^q \Lambda_{\alpha\beta}^{\lambda\nu} \varphi^\beta \partial_\lambda \delta x_\nu .$$

Thus, (5) becomes

$$\pi_\mu^q \delta\varphi^q + \frac{1}{c}\left\{ T_{\mu\nu} \delta x_\nu - G_{\mu\lambda\nu} \partial_\lambda \delta x_\nu \right\}$$

~~$\pi_\mu^q \delta\varphi^q$~~

because $T_{\mu\nu} = L \delta_{\mu\nu} - c\, \pi_\mu^q \varphi_\nu^q$

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$n_\mu \frac{\partial}{\partial x_\lambda}(G_{\lambda\mu\nu}\delta x_\nu)$ $d\sigma_\mu$ → tangential derivatives.

$n_1 = n_2 = n_3 = 0$

$= n_4\left(\frac{\partial G_{1,4\nu}}{\partial x_1} + \frac{\partial G_{2,4\nu}}{\partial x_2} + \frac{\partial G_{3,4\nu}}{\partial x_3}\right)\frac{\partial}{\partial x_4}(G_{\lambda\mu\nu}\delta x_\nu)$ (II, 9)

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$$\frac{1}{c}\left\{\left(\frac{\partial L}{\partial \varphi^a_\mu}\right)\delta_0\varphi^a + L\,\delta x_\mu\right\}$$

$$= \pi^a_\mu \delta\varphi^a + \frac{1}{c}\left\{T_{\mu\nu} + \frac{\partial G_{\mu\lambda\nu}}{\partial x_\lambda}\right\}\delta x_\nu - \frac{\partial}{\partial x_\lambda}\left\{G_{\mu\lambda\nu}\delta x_\nu\right\}$$

$$= \pi^a_\mu \delta\varphi^a + \frac{1}{c}\Theta^{\delta x_\nu}_{\mu\nu} + \frac{\partial}{\partial x_\lambda}(G_{\lambda\mu\nu}\delta x_\nu)$$

$$\Theta_{\mu\nu} = T_{\mu\nu} + \frac{\partial G_{\mu\lambda\nu}}{\partial x_\lambda}$$

$$= T_{\mu\nu} - \frac{\partial G_{\lambda\mu\nu}}{\partial x_\lambda}$$

(because of $G_{\lambda\mu\nu} = -G_{\mu\lambda\nu}$)

$\int d\sigma_\mu \frac{\partial}{\partial x_\lambda}(G_{\lambda\mu\nu}\delta x_\nu) = 0$ for a closed system

In this way, we could show that the ~~symmetric~~ tensor $\Theta_{\mu\nu}$ was introduced naturally in connection with variation of the action operator W_{12}. The relation

$$\Theta_{\mu\nu} = \Theta_{\nu\mu}$$

will be proved immediately in connection with the conservation of angular momentum.

$$F(\sigma) = \frac{1}{c}\int d\sigma_\mu \left\{\mathcal{L}\,\delta x_\mu + \pi^a_\mu \delta\varphi^a + \frac{1}{c}\Theta_{\mu\nu}\delta x_\nu + \frac{\partial}{\partial x_\lambda}(G_{\lambda\mu\nu}\delta x_\nu)\right\}$$

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;;) Conservation laws

For the variations which leave the action operator W_{12} unchanged, we have

$$\delta W_{12} = +F(\sigma_1) + F(\sigma_2) \approx 0$$

and hence the corresponding generating operators (10) are associated with the constants of motion.

For the variation

$$\delta x_\mu = \epsilon_\mu + \omega_{\mu\nu}\delta x_\nu \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

$$\delta\phi^q(x_\mu) = 0$$

W_{12} will not change, ~~because, this is nothing but the change in $F(x)$ due to the Lorentz transformation of coordinate system~~,

~~$x'_\mu = x_\mu + \epsilon_\mu + \omega_{\mu\nu}\delta x_\nu$~~

~~and the corresponding change in ϕ^q~~

~~$\phi^{q'} =$~~

by such a common rigid displacement ~~of the~~ and rotation of positive space-time region together with the

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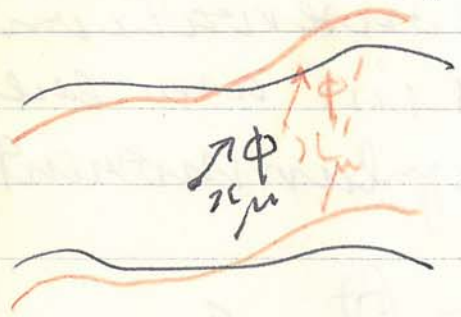
boundary surfaces σ_1, σ_2 and with the changed ~~shape~~ ~~of the entire~~ corresponding field.



$$\phi'^{\alpha}(x'_\mu) = \phi^{\alpha}(x_\mu)$$

We have entirely the same field and same domain as before, if you ~~use the new coordinate system~~ for expressing the points and the components of fields in terms of the new coordinate system (~~x~~ with $'$)

The corresponding inf. gen. oper

$$F_{\delta\alpha}(\sigma) = \frac{1}{c}\int_\sigma d\sigma_\mu \Theta_{\mu\nu}$$

$$\varepsilon_\nu P_\nu(\sigma) + \frac{1}{2}\omega_{\mu\nu} J_{\mu\nu}(\sigma)$$

with

$$P_\nu = \frac{1}{c}\int_\sigma d\sigma_\mu \Theta_{\mu\nu}$$

$$J_{\mu\nu} = \frac{1}{c}\int_\sigma d\sigma_\lambda M_{\lambda\mu\nu}$$

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$$M_{\lambda\mu\nu} = x_\nu \Theta_{\lambda\mu} - x_\mu \Theta_{\lambda\nu}$$

$$= \Theta_{\lambda\nu} x_\nu - \Theta_{\lambda\nu} x_\mu$$

gives rise to the conservation of linear momentum and energy vector P_ν and the angular momentum tensor $J_{\mu\nu}$

$$\therefore \Theta_{\mu\nu} \delta x_\nu = \Theta_{\mu\nu} \varepsilon_\nu + \Theta_{\mu\nu} \omega_{\nu\lambda} \delta x_\lambda$$

$$= \Theta_{\mu\nu} \varepsilon_\nu + \frac{1}{2} \left(\Theta_{\mu\lambda} \omega_{\lambda\nu} x_\nu + \Theta_{\mu\nu} \omega_{\nu\lambda} x_\lambda \right)$$

$$= \Theta_{\mu\nu} \varepsilon_\nu + \frac{1}{2} \omega_{\lambda\nu} \left(x_\nu \Theta_{\mu\lambda} - x_\lambda \Theta_{\mu\nu} \right)$$

$$x_\nu \Theta_{\lambda\mu} - x_\mu \Theta_{\lambda\nu} = x_\nu T_{\lambda\mu} - x_\mu T_{\lambda\nu} - \left(x_\nu \frac{\partial G_{\lambda\mu\nu}}{\partial x_\lambda} - x_\mu \frac{\partial G_{\lambda\nu\mu}}{\partial x_\lambda} \right)$$

orbital and spin angular momenta.

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~~$\approx T_{\mu\nu}\varepsilon_\nu + \frac{1}{2}\omega_{\lambda\nu}(x_\nu T_{\mu\lambda} - x_\lambda T_{\mu\nu})$~~

$$\int d\sigma_\mu T_{\mu\nu}\delta x_\nu$$

$$\simeq \varepsilon_\nu \int d\sigma_\mu T_{\mu\nu} \quad \top \to (H)$$

$$+\frac{1}{2}\omega_{\mu\nu}\int (x_\nu T_{\lambda\mu} - x_\mu T_{\lambda\nu})\, d\sigma_\lambda$$

Accordingly,

$$\left.\begin{aligned} P_\nu(\sigma_1) - P_\nu(\sigma_2) &= 0 \\ J_{\mu\nu}(\sigma_1) - J_{\mu\nu}(\sigma_2) &= 0 \end{aligned}\right\}$$

Corresponding differential conservation laws ~~$\partial_\mu T$~~ are

$$\partial_\mu T_{\mu\nu} = 0$$

$$\partial_\lambda M_{\lambda\mu\nu} = 0 \qquad T \to (H)$$

because

$$\frac{1}{c}\int_{\sigma_2}^{\sigma_1} (\partial_\mu T_{\mu\nu})\, dx_\mu = P_\nu(\sigma_1) - P_\nu(\sigma_2) = 0$$

for arbitrary surfaces σ_1, σ_2

etc.

The latter relation results in

$$\partial_\lambda M_{\lambda\mu\nu} = T_{\mu\nu} - T_{\nu\mu} = 0$$

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Conservation of charge

If the Lagrangian density is invariant with respect to the transformation

$$\left.\begin{aligned}\phi^{\alpha} &\to \phi^{\alpha} e^{i\gamma_\alpha} \\ \phi^{\alpha\dagger} &\to \phi^{\alpha\dagger} e^{-i\gamma_\alpha}\end{aligned}\right\} \quad \gamma_\alpha \text{ real}$$

where γ_α is a constant, the corresponding infinitesimal transformation is

$$\delta\phi^{\alpha} = i\gamma_\alpha \phi^{\alpha} = -\frac{ie_\alpha}{\hbar c}\delta\lambda$$

$$\gamma_\alpha = -\frac{1}{\hbar c} e_\alpha \delta\lambda$$

$e_\alpha = \pm e$ or 0.

generating operator

$$F_{\delta\lambda}(\sigma) = -\frac{ie_\alpha}{\hbar c}\int d\sigma_\mu e_\alpha \Pi_\mu^{\alpha} \phi^{\alpha} \delta\lambda$$

$$= \frac{1}{c} Q(\sigma)\delta\lambda$$

$$Q(\sigma) = \frac{1}{e}\int d\sigma_\mu \cdot j_\mu$$

$$j_\mu = -\frac{i}{\hbar}\Pi_\mu^{\alpha} e_\alpha \phi^{\alpha}$$

Q is the total charge of the system

$$Q(\sigma_1) = Q(\sigma_2)$$

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iii) Commutation Relations (B.E. Stat)

So far, we did not consider seriously the non-commutativity of operators appearing in the Lagrangian L. If they were all commutative with each other, we have just the classical field theory, in which the transformation function is always a δ-function. In quantum theory, on the contrary, some of the field quantities and their derivatives must be non-commutative. Therefore, the variation $d\varphi^a(x)$ may themselves operators, in general, which do not necessarily commute with the field quantities and their derivatives. The restrictions which are to imposed on $d\varphi^a(x)$ are that $\varphi^a(x, \sigma_1) \to \varphi^{a'}(x', \sigma_1')$, $\varphi^a(x, \sigma_2) \to \varphi^{a'}(x', \sigma_2')$ must be an infinitesimal unitary transformation. Let us take the case $\delta x_\mu = 0$. Then, the change in the transformation

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function can be written symbolically

$$\delta(\zeta_1', \sigma_1 | \zeta_2'', \sigma_2)$$

$$= \left(\frac{-i}{\hbar}\right)(\zeta_1', \sigma_1 | \delta W_{12} | \zeta_2'', \sigma_2)$$

$$\delta W_{12} = + \left(F_{\delta\varphi}(\sigma_1) - F_{\delta\varphi}(\sigma_2)\right)$$

$$F_{\delta\varphi}(\sigma) = +\frac{1}{c}\int d\sigma_\mu \left(\frac{\partial L}{\partial \varphi^a_\mu}\right)\cdot \delta\varphi^a$$

where we must keep in mind that $\delta\varphi^a$ are operators so that the positions of $\delta\varphi^a$ in the symbolical expression $\left(\frac{\partial L}{\partial \varphi^a_\mu}\right)\delta\varphi^a$ are not necessarily on the right, but may be on the extreme somewhere else.

~~From the definition of $(\zeta_1', \sigma_1 | \zeta_2'', \sigma_2)$,~~ ~~we have~~

~~$$\delta(\zeta_1', \sigma_1 | \zeta_2'', \sigma_2) = \Psi^\dagger(\zeta_1', \sigma_1$$~~

Now the variation $\delta\varphi^a(x, \sigma)$ gives rise to ~~the change~~ a certain change $\delta\Psi(\zeta', \sigma)$ in the eigenvector $\Psi(\zeta', \sigma)$, hence, for any operator G

$$\delta(\zeta', \sigma | G | \zeta'', \sigma)$$
$$= \Psi^\dagger(\zeta', \sigma) G\, \delta\Psi(\zeta'', \sigma) + \delta\Psi^\dagger(\zeta', \sigma) G\, \Psi(\zeta'', \sigma)$$

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where

$$\delta\Psi(\zeta',\sigma) = \left(\frac{i}{\hbar}\right) F_{\delta\varphi}\, \Psi(\zeta',\sigma)$$

or

$$\delta(\zeta',\sigma|G|\zeta'',\sigma)$$

$$= \frac{i}{\hbar}(\zeta',\sigma|[G, F_{\delta\varphi}]|\zeta'',\sigma)$$

which must be equal to the element (= matrix) corresponding change $\delta_\varphi G$

$$-(\zeta',\sigma|\delta_\varphi G|\zeta'',\sigma)$$

Thus, we obtain

$$+ i\hbar\, \delta_\varphi G = [G, F_{\delta\varphi}]$$

If we take $G \equiv \varphi^a$

$$+ i\hbar\, \delta\varphi^a = [\varphi^a(x),\ F_{\delta\varphi}]$$

Thus, if we restrict the variations $\delta\varphi^a$ which are commutative with all other operators in L, we have (still symbolically)

$$+ i\hbar\, \delta\varphi^a_{(x,\sigma)} = \frac{1}{c}\int d\sigma'_\mu \left[\frac{\partial L(x',\sigma)}{\partial \varphi^a_\mu}\right] \delta\varphi^a_{(x',\sigma)}$$

x : fixed.

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$d\sigma_\mu = n_\mu d\sigma$ $\quad n_\mu n_\mu = -1$ $\quad n_1, n_2, n_3$: pure imaginary, n_4: real

$dx_\mu d\sigma_\mu$: $dx_4 \cdot dx_1 dx_2 dx_3$: pure imaginary

In order that this is true for arbitrary ~~distribution~~ function $\delta\varphi^a(x)$ on σ, the relation

$$[\varphi^\alpha(x), \Pi^\beta(x')] = i\hbar\, t_{\alpha\beta}\, \delta_\sigma(x-x')$$

$$\int \delta_\sigma(x-x')\cdot d\sigma' f(x') = f(x)$$

must hold, where

$n_\mu = (in_x, in_y, in_z, n_4)$ $\quad \Pi^a(x) = \pm\frac{1}{c} n_\mu \frac{\partial \mathcal{L}}{\partial \frac{\partial \varphi^a}{\partial x_\mu}} = \pm n_\mu \Pi^a_\mu$

$n_\mu = (0,0,0,1)$

$n_4^2 - n_x^2 - n_y^2 - n_z^2 = 1$

Thus our case reduces to the case of Bose-statistics for the particles associated with the field φ^α. This implies that the eigenvalues of a real (hermitian) field $\varphi^a(x)$ run from $(-\infty, +\infty)$, so that φ^α and $\varphi^\alpha + \delta\varphi^\alpha$ have the identical spectrum.

This situation gives rise to the

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further consequence that the canonical momentum $\pi^\alpha(x)$ could be changed by an infinitesimal amount $\delta\pi^\alpha(x)$ without affecting $\varphi^\alpha(x)$ and without changing the commutation relation between φ^α and π^α on any s. l. surface σ. Such a change ~~and~~ can be ~~generated~~ represented by a generating g function

$$F_{\delta\pi} = -\int d\sigma \cdot \delta\pi^\alpha \cdot \varphi^\alpha$$

because

$$i\hbar\,\delta_{\delta\pi}\pi^\alpha = [\pi^\alpha, \int d\sigma \cdot \delta\pi^\alpha \cdot \varphi^\alpha]$$

holds identically, ~~if we assume~~ due the commutation relation between π^α and φ^α.

~~In order further that the va~~ In addition to this, ~~we have to~~ we assume ~~that~~ from the beginning that

$$[\varphi^\alpha(x,\sigma), \varphi^\beta(x',\sigma)] = 0$$

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~~in order that $\varphi^\alpha(x)$ etc. can be~~

$$[\pi^\alpha(x,0),\ \pi^\beta(x',0)] = 0,$$

~~we have~~ every thing is consistent because, then

$$i\hbar\delta_\pi\varphi = [\varphi^\alpha,\ F_{\delta\pi}] = 0$$

$$i\hbar\delta_\varphi\pi = [\pi^\alpha,\ F_{\delta\varphi}] = 0$$

are satisfied identically as it should be. Conversely, in order that $F_{\delta\varphi}$, $F_{\delta\pi}$ are the generating operators for the changes $(\delta\varphi^\alpha, 0)$ and $(0, \delta\pi^\alpha)$ respectively, the above commutation relations must hold.

It should be noticed that ~~the~~ $F_{\delta\pi}$ is intimately connected with the ambiguity of the hamiltonian h. Namely, if we change h into

$$h'(\varphi^\alpha, \varphi^\alpha_\mu) = h(\varphi^\alpha, \varphi^\alpha_\mu) + c\,\partial_\nu f_\nu(\varphi^\alpha, \varphi^\alpha_\mu)$$

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the corresponding W_{12} changes into

$$W_{12}' = W_{12} - \left(\int_{\sigma_1} - \int_{\sigma_2}\right) d\sigma$$

which does not change the equations of motion.
If we take, in particular

$$f_\nu = -\pi_\nu^\alpha \varphi^\alpha \qquad \pi_\nu^\alpha = \frac{1}{c}\frac{\partial L}{\partial \dot{\varphi}_\nu^\alpha}$$

we have

$$\delta \int d\sigma_\nu f_\nu = -\delta \int d\sigma\, \pi^\alpha \varphi^\alpha$$

$$= -\int d\sigma \left(\pi^\alpha \delta\varphi^\alpha + \delta\pi^\alpha \cdot \varphi^\alpha\right)$$

Then the corresponding generating operation is

$$F' \equiv F_{\delta\pi} = -\int d\sigma\, \delta\pi^\alpha \cdot \varphi^\alpha .$$

More generally, by taking

$$f_\nu = -\lambda \pi_\nu^\alpha \varphi^\alpha$$

we have

$$F' = \int d\sigma \left\{(1-\lambda)\,\pi_\nu^\alpha \delta\varphi^\alpha - \lambda\, \delta\pi^\alpha \cdot \varphi^\alpha\right\}$$

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(iv) Commutation Relations (cont'd)

Fermi-Dirac Statistics

According to the above discussion, it is clear that if we assume that the field quantities φ^α have continuous eigenvalues $(-\infty, +\infty)$, then the commutation relations between φ^α, $\pi^\alpha = \pm \frac{1}{c} n_\mu \frac{\partial L}{\partial \frac{\partial \varphi_\alpha}{\partial x_\mu}} = \pm n_\mu \pi^\alpha_{,\mu}$

are necessarily of B.E. type.

In order to have, F.D. type commut. relations, ~~Therefore~~ on the contrary, the variations $\delta\varphi^\alpha$ of φ^α must be such that they do not change the form of the anticipated commutation relations of the type

$$[\varphi^\alpha(x,\sigma), \varphi^\beta(x',\sigma)]_+ = 0$$

$$[\varphi^\alpha(x,\sigma), \varphi^\beta(x',\sigma)]_+ = i\hbar \delta_{\alpha\beta} \delta(x-x')$$

etc.

Namely, we ~~expect to~~ have to restrict the variations to those which satisfy

$$[\varphi^\alpha(x,\sigma) + \delta\varphi^\alpha(x,\sigma), \varphi^\beta(x',\sigma) + \delta\varphi^\beta(x',\sigma)]_+ = 0$$

etc

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(II.16)

These relations are obviously satisfied by ~~take~~ assuming $\delta\varphi^\alpha$ to be all anticommutative with all field quantities and their derivatives on σ.

$$[\delta\varphi^\alpha(x,\sigma),\ \varphi^\beta(x',\sigma)]_+ = 0$$
$$[\delta\varphi^\alpha(x,\sigma),\ \pi^\beta(x',\sigma)]_+ = 0$$

In that case, we obtain *

$$-i\hbar\,\delta\varphi^\alpha(x,\sigma) = \frac{1}{\hbar}\int d\sigma'\,[\underbrace{\pi^\beta_{(x',\sigma)}\delta\varphi^\beta(x',\sigma)}_{\varphi^\alpha(x,\sigma),}]$$

$$= \frac{1}{\hbar}\int d\sigma'\,[\varphi^\alpha(x,\sigma),\ \pi^\beta(x',\sigma)]_+\,\delta\varphi^\beta(x',\sigma)$$

or

$$[\varphi^\alpha(x,\sigma),\ \pi^\beta(x',\sigma)]_+ = i\hbar\,\delta_{\alpha\beta}\,\delta_\sigma(x-x')$$

Similarly,

$$[\varphi^\alpha(x,\sigma),\ \varphi^\beta(x',\sigma)]_+ = 0$$

etc.

Conversely, if we ~~assume~~ assume these commutation relations, the above variations $\delta\varphi^\alpha$ (or $\delta\pi^\alpha$) which commute with all φ^α, π^α on the surface σ are consistent with these commut.

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of q^α and π^α)

relations and the eigenvalues will not be ~~dis~~ altered by such ~~an~~ variations. Thus, the whole procedure is self-consistent.

We know from the arguments by Pauli and Fierz that the commutation relations are intimately connected with transformation properties of the field quantities q^α. Schwinger tried to derive the relationships between spin and statistics by taking into account the time reversal.

---

* π^α is defined in the following way:

$$\delta L = \frac{\partial L}{\partial q^\alpha}\delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha}\delta \dot{q}^\alpha$$

$$\pi^\alpha \delta \dot{q}^\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha$$

where $\frac{\partial L}{\partial \dot{q}^\alpha_\mu} n_\mu \delta q^\alpha$ denotes an operator which can be obtained by replacing $\dot{q}^\alpha_\mu$ by δq^α and taking the sum. Then, if we shift δq^α to the extreme right by taking into account the anti-commutativity, we obtain the expression for δ of π^α.

$$\delta\{\bar{\Psi}(\zeta',\sigma_1|\Psi(\zeta'',\sigma_2)\} = \left(-\frac{i}{\hbar}\right)\bar{\Psi}\{F(\sigma_1)-F(\sigma_2)\}\Psi \qquad (\text{II}, 17)$$

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遂に Schrödinger Equation

Schrödinger equation ~~is contained in~~ can be derived from the variation principle as a special case. From the fundamental relation

$$\delta(\zeta_1',\sigma_1|\zeta_2'',\sigma_2) = \left(\frac{i}{\hbar}\right)(\zeta_1',\sigma_1|\delta W_{12}|\zeta_2''\sigma_2)$$

with $\quad \delta W_{12} = +(F(\sigma_1)-F(\sigma_2))$

we can drive *

$$\Psi((\zeta+\delta\zeta)',\sigma+\delta\sigma) = \Psi'(\zeta,\sigma)$$
$$= [1+(i/\hbar)F]\Psi(\zeta',\sigma)$$

where

$$F = \bar{F}\delta\varphi + F\delta x$$

Now

$$\Psi((\zeta+\delta\zeta)',\sigma) = [1+(\tfrac{i}{\hbar})F\delta\varphi]\Psi(\zeta',\sigma)$$

while

$$\bar{F}\delta x = \frac{1}{c}\int d\sigma_\mu \Theta_{\mu\nu}\delta x_\nu$$

generates

$$\Psi(\zeta',\sigma+\delta\sigma) = [1+(\tfrac{i}{\hbar})F\delta x]\Psi(\zeta',\sigma)$$

We restrict ourselves to plane surfaces and consider only rigid displacement of σ:

$$\delta_x(\zeta',\sigma|\ ) = \delta_x\bar{\Psi}(\zeta',\sigma)\Psi$$
$$= (i/\hbar)(\zeta',\sigma|F|\ )$$
$$\partial_x(\zeta',\sigma|\ ) = \varepsilon_\mu\partial_\mu(\zeta',\sigma) + \omega_{\mu\nu}\delta_{\mu\nu}$$

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generalized Schrödinger eq.:

$$i\hbar\,\delta_n(\zeta',\sigma|\,) = (\zeta',\sigma|P_\mu(\sigma)|\,)$$
$$= \int(\zeta',\sigma|P_\mu(\sigma)|\zeta'',\sigma)\,d\zeta''(\zeta'',\sigma|\,)$$

for translation and

$$i\hbar\,\delta_{\mu\nu}(\zeta',\sigma|\,) = (\zeta',\sigma|J_{\mu\nu}(\sigma)|\,)$$
$$= \int(\zeta',\sigma|J_{\mu\nu}|\zeta'',\sigma)\,d\zeta''(\zeta'',\sigma|\,)$$

for rotations.

In particular, we have

$$+\quad i\hbar\frac{\partial(\zeta',t|\,)}{\partial t} = \int(\zeta',t|\,icP_4(\sigma)|\zeta'',t)\,d\zeta''(\zeta'',t|\,)$$

$$icP_4(\sigma) = ic\int\mathcal{H}_{44}(dx)^3$$
$$= -H \qquad *$$

Note that what we have been considering is equivalent to Schrödinger representation in that, with the change of σ, field operators were displaced without any alteration.

v) Operator Eq. (Heisenberg eq. of motion)

On the other hand, any operator $G(\sigma)$

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generalized Schrödinger eq.:

$$i\hbar\,\delta_n(\zeta',\sigma|\,) = -(\zeta',\sigma|P_\mu(\sigma)|\,)$$
$$= -\int(\zeta',\sigma|P_\mu(\sigma)|\zeta'',\sigma)\,d\zeta''(\zeta'',\sigma|\,)$$

for translation and

$$i\hbar\,\delta_{\mu\nu}(\zeta',\sigma|\,) = (\zeta',\sigma|J_{\mu\nu}(\sigma)|\,)$$
$$= -\int(\zeta',\sigma|J_{\mu\nu}|\zeta'',\sigma)\,d\zeta''(\zeta'',\sigma|\,)$$

for rotations.

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(II. 18)

which was constructed from field quantities on σ has a matrix

$$(\zeta', \sigma | G(\sigma) | \zeta'', \sigma)$$

that is independent of σ, or

$$(\zeta', \sigma | G(\sigma) | \zeta'', \sigma + \delta\sigma) = (\zeta', \sigma | G(\sigma) | \zeta'', \sigma)$$

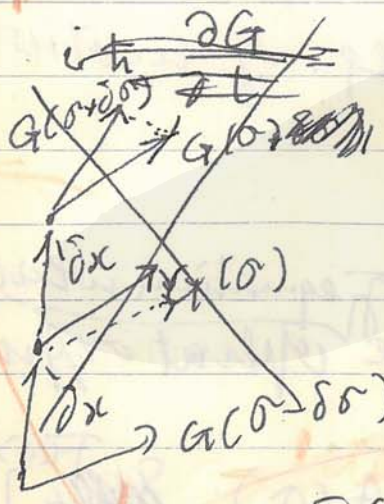
or

$$\Psi^{\dagger}(\zeta', \sigma) G(\sigma) \Psi(\zeta'', \sigma + \delta\sigma) = \Psi(\zeta', \sigma) G(\sigma) \Psi(\zeta', \sigma)$$

or

$$\delta_x G(\sigma) + \left(\frac{i}{\hbar}\right)\left[G(\sigma), F_{\delta x} \right] = 0$$

$$+ i\hbar\, \delta_x G = -\left[G, F_{\delta x} \right]$$



$$+ i\hbar \frac{\partial G}{\partial t} = \left[G, \overline{H} \right]$$

which is the change in the operator $G(t)$ (or $G(\sigma)$) due to the change in Φ, t (or σ).

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The equations of motion thus obtained are the same as those which we ~~had~~ derived directly from variation principle.

It seems rather strange that we derived the same equations twice (in different ways), but one can think that this guarrantees the consistency of the whole argument on the quantum mechanical level, because at the beginning, ~~we~~ commutation relations were not explicitly taken into account, so that we did not know the compatibility of eq. of motion and com. rel.

~~ii) Integrability~~ equation which ~~they~~ General form for the operator A must satisfy is

$$i\hbar\,\delta A(\sigma) = [A(\sigma), \overset{F(\sigma)}{\delta W_2}]$$

where ~~δW_2~~ $F(\sigma)$ is the generating operator relating to the simultaneous variation ~~of~~ $\delta\varphi^a, \delta x_\mu$ on σ.

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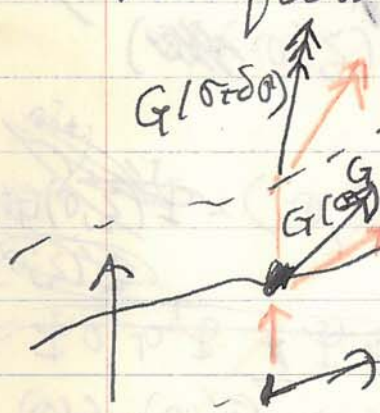
(II, 19)

* ~~(***)~~ We must keep in mind the physical meaning of the displacement in the space-time world ~~without the change~~ is associated with the shifting of the field distribution.



By this, the eigenvector $\Psi(\zeta', \sigma)$ is replaced by $\Psi'_{(\zeta';\sigma)} = \Psi(\zeta', \sigma+\delta\sigma)$, so that

$$\delta_x\Psi(\zeta', \sigma) = \Psi(\zeta', \sigma+\delta\sigma) - \Psi(\zeta', \sigma) = \left(\frac{i}{\hbar}\right) F_{\delta x}\cdot\Psi$$

In particular $t \to t+\delta t$.

$$\delta_t\Psi(\zeta', t) = \Psi(\zeta', t-\delta t) - \Psi(\zeta', t) = -\frac{\partial\Psi}{\partial t}\cdot\delta t$$

correspondingly, $\delta_t\{\Psi^\dagger(\zeta', t)\Psi\} = \Psi^\dagger(\zeta', t-\delta t)\Psi - \Psi^\dagger(\zeta', t)\Psi$

$$\delta_t(\zeta', t|\ ) = -\frac{\partial(\zeta', t|\ )}{\partial t}\delta t.$$

Thus, we have the correct eq.

$$i\hbar\frac{\partial(\zeta', t|\ )}{\partial t} = \int(\zeta t|H|\zeta' t)\,d\zeta' \times(\zeta', t|\ )$$

with $\delta_x\Psi^\dagger(\zeta', \sigma)\Psi = \Psi^\dagger(\zeta', \sigma)F_{\delta x}\Psi$

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$\Psi'(\zeta', \sigma) = \Psi(\zeta', \sigma - \delta\sigma)$

$G'(\sigma) = G(\sigma - \delta\sigma)$

Similarly, the matrix element

~~$\langle \zeta', \sigma + | G(\sigma + \delta\sigma) |$~~

~~$\Psi^{\dagger}(\zeta', \sigma + \delta\sigma) G(\sigma + \delta\sigma)$~~

$G(\sigma + \delta\sigma)$ as looked upon from the displaced system.

$\Psi^{\dagger}(\zeta', \sigma) G'(\sigma + \delta\sigma) \Psi(\zeta'', \sigma + \delta\sigma)$

must be the same as

$\Psi^{\dagger}(\zeta', \sigma) G(\sigma) \Psi(\zeta'', \sigma) =$

or

$\delta\Psi^{\dagger} G \Psi + \Psi^{\dagger} G \delta\Psi = 0$

or

where $\delta_x G(\sigma) = G'(\sigma) - G(\sigma)$.

$\delta_x G + \frac{i}{\hbar}[G, F\delta x] = 0$

or $+ i\hbar\, \partial_x G = [G, F\delta x]$ $\quad \left(= \frac{\partial G}{\partial x_\mu} \delta x_\mu \right)$

$\delta_x G = G'(\sigma) - G(\sigma) = G(\sigma - \delta\sigma) - G(\sigma)$

$+ i\hbar \frac{\partial G}{\partial t} \delta t = [G, -\bar{H}\delta t]$ *

~~Remark~~ * $L = -iL'$

$T_{44} = L \cdots = -iL' \cdots$

$= iH'$

$P_4 = \frac{1}{c} \int T_{44}\, d\sigma = \frac{i}{c} \bar{H}'$

$\bar{F}_{\partial x_4} \delta x_4 = \frac{i}{c} \bar{H}' i c \delta t = -\bar{H}' \delta t$

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(Ⅳ.25)

Problem 1.)

Example. 1. Neutral scalar field.

φ there's no question about the order of operators in L.

$$L = +\frac{i}{2}\left(\frac{\partial\varphi}{\partial x_\mu}\frac{\partial\varphi}{\partial x_\mu} + \varkappa^2\varphi^2\right)$$

$$T_{\mu\nu} = L\delta_{\mu\nu} - \Pi_\mu^{(n)}\varphi_\nu^{(n)}$$

$$\Pi_\mu^{(n)} = \frac{1}{c}\frac{\partial L}{\partial \varphi_\mu^{(n)}} = -\frac{i}{c}\frac{\partial\varphi}{\partial x_\mu}$$

$$\Pi_4 = +\frac{1}{c^2}\frac{\partial\varphi}{\partial t}$$

~~$\Pi_4 = \frac{1}{c}\frac{\partial\varphi}{\partial t}$~~

$$T_{\mu\nu} = -\frac{i}{2}\left(\frac{\partial\varphi}{\partial x_\mu}\frac{\partial\varphi}{\partial x_\mu} + \varkappa^2\varphi^2\right)\delta_{\mu\nu} + i\frac{\partial\varphi}{\partial x_\mu}\frac{\partial\varphi}{\partial x_\nu}$$

$$T_{44} = -\frac{i}{2}\left(\frac{\partial\varphi}{\partial x_j}\frac{\partial\varphi}{\partial x_j} + \varkappa^2\varphi^2 - \frac{\partial\varphi}{\partial x_4}\frac{\partial\varphi}{\partial x_4}\right)$$

$$= +iH$$

$\Theta_{\mu\nu} = T_{\mu\nu}$

because ~~$\Lambda_{\mu\rho} = 0$~~ $\Lambda = 0$

$G_{\mu\rho;\lambda\nu} = 0$

$$F(\sigma) = \frac{1}{4}\int d\sigma_\mu\left(\Pi_\mu\delta\varphi + \frac{1}{c}T_{\mu\nu}\delta x_\nu\right)$$

provided that $\delta\varphi$ are numbers.

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If we consider the particular variation

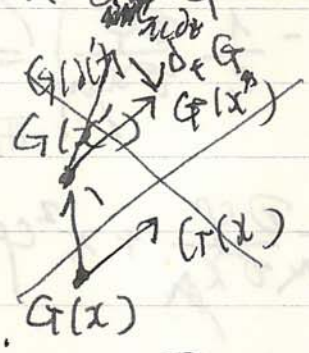
$$\delta x_1 = \delta x_2 = \delta x_3 = 0$$

$$\delta x_4 = ic\,\delta t,$$

$$F = \int \mathcal{H}(dx) > \int \bar{\mathcal{H}}\,\delta t = \delta t\,\bar{H}$$

$$\frac{1}{ic}\delta t$$

with $\delta_\varepsilon G = \frac{1}{i\hbar}[G, \bar{H}]\,\delta t$



$$\delta_\varepsilon G = \left(+\frac{\partial G}{\partial t}\cdot\delta t\right) + \frac{\partial G}{\partial (ict)}\,ic\,\delta t =$$

$$i\hbar\frac{\partial G}{\partial t} = [G, \bar{H}]$$

$d\sigma_1 = dx_2 dx_3 dx_4$: pure imaginary

$d\sigma_4 = dx_1 dx_2 dx_3$: real

$d\sigma_\mu = n_\mu\, d\sigma$

$d\sigma_\mu d\sigma_\mu = n_\mu n_\mu \cdot (d\sigma)^2$

$n_\mu n_\mu = 1$! $\quad d\sigma_\mu d\sigma_\mu = (d\sigma)^2$

$n_\mu = (0, 0, 0, 1)$ for example.

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$\frac{\partial L}{\partial \frac{\partial \psi}{\partial x_4}} = i\hbar\psi^\dagger = \bar{\psi}\gamma_4$ (II, 21)

Problem 2.

Example 2: Spinor Field

ψ with four components

$L = \bar{i}c\{\bar{\psi}\gamma_\mu \partial_\mu \psi + \chi mc\bar{\psi}\psi\}$

$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$

$\{+i\hbar\gamma_\mu\partial_\mu + imc\}\psi = 0$

$\frac{\partial L}{\partial \frac{\partial \psi}{\partial x_4}} = \cdot ic\frac{\partial L}{\partial \frac{\partial \varphi}{\partial t}}$

$\gamma_4 = \rho_3 \qquad \rho_3\gamma_k = i\rho_1\sigma_k$

$\gamma_4^2 \cdot i\hbar\{\psi(x), \psi^\dagger(x')\}_+ = \delta(x-x')$ $\Big| \frac{\partial L}{\partial \frac{\partial \varphi}{\partial t}} = -\frac{i}{c}L'$

$\{i\hbar \cdot -i\gamma_\mu\partial_\mu + mc\}\psi = 0$

$\{-i\hbar(i\gamma_4\gamma_k)\partial_k + \gamma_4 mc\}\psi = i\hbar\, i\gamma_4\frac{1}{ic}\frac{\partial\psi}{\partial t}$

$\gamma_4^2 = 1. \qquad \gamma_4 = \rho_3 \qquad i\gamma_4\gamma_k = i\rho_3\gamma_k = \rho_1\sigma_k$

$\gamma_k = -i\rho_3\rho_1\sigma_k = -\rho_2\sigma_k$

In order that L is hermitian,

$\psi^\dagger\bar{\psi}^\dagger = \bar{\psi}\psi$

$\partial_\mu\psi^\dagger \cdot \gamma_\mu^\dagger\bar{\psi}^\dagger = \bar{\psi}\gamma_\mu\tilde{\partial}_\mu\psi$ $\}(\gamma_\mu^\dagger = \gamma_\mu)$

we can choose $\bar{\psi}$ such that

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$$\bar{\psi} = \psi^\dagger \gamma_4 , \quad \bar{\psi}^\dagger = \gamma_4 \psi$$

Then, we have

$$\psi^\dagger \bar{\psi}^\dagger = \psi^\dagger \gamma_4 \psi = \bar{\psi}\psi ,$$

and but $\bar{\psi}\gamma_\mu \partial_\mu \psi$ is not hermitian,

because $\partial_\mu \psi^\dagger \cdot \gamma_\mu \gamma_4 \psi = \psi^\dagger \gamma_4 \gamma_\mu \tilde{\partial}_\mu \psi$

$= -\partial_\mu \psi^\dagger \cdot \gamma_4 \gamma_\mu \psi$

does not hold, in general. However,

the expression

$$\psi^\dagger \gamma_4 \gamma_\mu \partial_\mu \psi + \tilde{\partial}_\mu \psi^\dagger \gamma_\mu \gamma_4 \psi$$

$$= \bar{\psi}\gamma_\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma_\mu \psi$$

is hermitian, because its hermitian conjugate is

$$\tilde{\partial}_\mu \psi^\dagger \cdot \gamma_\mu \gamma_4 \psi + \psi^\dagger \gamma_4 \gamma_\mu \partial_\mu \psi$$

~~or $\bar{\psi}\gamma_\mu \partial_\mu \psi$~~

Thus, one may take

$$\mathcal{L} = i\hbar c \left\{ \frac{1}{2} \left(\bar{\psi}\gamma_\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \cdot \gamma_\mu \psi \right) + \varkappa \bar{\psi}\psi \right\} \qquad \varkappa = \frac{mc}{\hbar}$$

with $\bar{\psi} = \psi^\dagger \gamma_4$

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(IV.22)

Invariance with respect to
~~time~~ time reflection
(time reversal)

~~In order to extend the~~ apply the general ~~procedure~~ The transformation of the state vector due to the change in the order of time is ~~some~~ represented by the corresponding change in the ~~state~~ corresponding Schrödinger function in one-time formalism

$t' = -t$ $\quad \Psi(\zeta', t) \to \Psi'(\zeta', t') = (\Psi^\dagger(\zeta', t))^T$

$(-) i\hbar \frac{\partial \Psi(\zeta', t)}{\partial t} = \bar{H} \Psi$ $\quad \bar{H}$: Hermitian.

~~$i\hbar \frac{\partial \tilde{\Psi}(\zeta', t)}{\partial t} = \tilde{\bar{H}} \tilde{\Psi}$~~ $\quad (+) i\hbar \frac{\partial \Psi^\dagger}{\partial t} = \Psi^\dagger \bar{H}$

$\to (-) i\hbar \frac{\partial \Psi(\zeta', t')}{\partial t'} = (\bar{H} \Psi^\dagger(\zeta', -t))^T = \Psi'(\zeta', t') \bar{H}$

$= (+) -i\hbar \frac{\partial (\Psi^\dagger(\zeta', -t))^T}{\partial t'}$ $\quad (=) \Psi'(\zeta', t') \bar{H}$

~~$\Psi'^\dagger(\zeta', t') = \Psi^\dagger(\zeta', -t')$~~

or $\quad -i\hbar \frac{\partial \Psi'(\zeta', t')}{\partial t'} = \bar{H} \Psi'(\zeta', t')$

etc.

Symbolically, one may write

$\Psi \to \tilde{\Psi} = (\Psi^\dagger)^T$
$\Psi^\dagger \to \tilde{\Psi}^\dagger = \Psi^T$

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$= \sum_j \alpha_{0j} \beta_{j0}$

$= \sum_j \beta_{0j}^T \alpha_{j0}^T$

$\sum_j \alpha_{ij} \beta_{jk} = \left(\sum_j \beta_{kj}^T \alpha_{ji}^T\right)^T$ ↓ $P(iR)$

For any operator A, we have

$$A\Psi \simeq \Psi^T A^T.$$

$$\Psi^\dagger A = A^T (\Psi^\dagger)^T = A^T \tilde{\Psi}$$

$$(a|A|b) \simeq \Psi_a^\dagger A \Psi_b \simeq \Psi_b^T A^T \tilde{\Psi}_a$$

Ψ^*, $\tilde{\Psi}$: one column, n rows; $\Psi^\dagger$: one row, n-colum

More generally, if ~~the complex operators~~ Ψ ~~themselves undergo a linear transformation, which can be also due to the time reflection~~

$\tilde{x}_a \simeq x_a$, $\tilde{x}_b \simeq x_b$ $\quad a, b = 1, 2, 3$

$\Psi_{\tilde{a}} = R \tilde{\Psi}_a$

$\tilde{A} = R A R^{-1}$

$\Psi_{\tilde{a}} = R \Psi_a$

$\Psi_{\tilde{a}} = (R \Psi_a)^T$ ← $\Psi_{\tilde{a}}^\dagger = \Psi_a^\dagger R^T$

$\tilde{R}^T = R^\dagger$

$R^T = \tilde{R}$

where R is a unitary operator, we have

$$(a|b) = \Psi_a^\dagger \Psi_b$$

$$(\tilde{b}|\tilde{a}) = \Psi_b^\dagger R^{-1} \cdot R \Psi_a = \Psi_b^\dagger \Psi_a = \Psi_{\tilde{a}}^\dagger \Psi_b = (a|b)$$

$$= (\Psi_a^\dagger R^\dagger)^T (R \Psi_b)^T = \Psi_a^T \Psi_b.$$

$$(\tilde{b}|\tilde{a}) = \Psi_b^T \cdot \tilde{R}_a = \Psi_{\tilde{a}}^T \cdot \tilde{\Psi}_b$$

$$= \Psi_b^T R^T \cdot \tilde{R} \tilde{\Psi}_a = (\Psi_a^\dagger R^\dagger R \Psi_b)^T$$

$$= \Psi_a^\dagger \Psi_b$$

$\Psi_b^T R^T \tilde{R} \tilde{\Psi}_a$

$= \Psi_b^T (\Psi_a^\dagger)^T$

$= (\Psi_a^\dagger \Psi_b)^T$

$= \Psi_a^\dagger \Psi_b$

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(II, 23)

$$(a|A|b) = (\bar{b}|\bar{A}|\bar{a})$$

with $\bar{A} = (RAR^{-1})^T$

$$\because \quad \Psi_b^\dagger R^T (RAR^{-1})^T \tilde{R} \Psi_a$$
$$= \Psi_a^\dagger A \Psi_b$$

$$\bar{x}_k = x_k, \quad \bar{x}_4 = -x_4$$
$$\bar{\zeta} = R\zeta R^{-1}$$
$$\bar{\Psi}(\zeta', \sigma) = R\Psi(\zeta', \sigma)$$

$$\delta(\bar{\zeta}_2'', \sigma_2 | \bar{\zeta}_1', \sigma_1) \longrightarrow \delta(\bar{b}|\bar{a})\,(\bar{b}|\bar{A}|\bar{a})$$
$$= (i/\hbar c)(\bar{\zeta}_2'', \sigma_2 | \cdot \delta \int_{\sigma_1}^{\sigma_2} \bar{L}(d\bar{x})^4 | \bar{\zeta}_1', \sigma_1)$$

$$\bar{L} = (RLR^{-1})^T$$
$$= L^{(\mathrm{row})}((R\phi^a R^{-1})^T, \pm \bar{\partial}_\mu((R\varphi^a R^{-1})^T))$$

where $\pm$ sign indicates the effect of the coord. transf. $\bar{x}_\mu' = \pm x_\mu$.
$L^{(\mathrm{row})}$ symbolizes the reversal in the order of all factors induced by transposition.
The linear transformation

$$R\varphi^a R^{-1} = R^{a\beta}\varphi^\beta$$

is compensated by the transformation of

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gradient vector and the net result is

$$L = (\pm) L^{(\text{Rep})}(\varphi^{aT}, \partial_\mu \varphi^{aT})$$

where $\pm$ refers to integer and half-inter spin fields respectively,

Reason: In the case of 0-spin field, Lagrangian is certainly invariant with respect to time reversal.

In the case of $\frac{1}{2}$-spin field, L changes sign.

(i) $x_k' = x_k$, $x_4' = -x_4$

$\psi' = \gamma_1\gamma_2\gamma_3\psi$ $\quad \psi^{\dagger\prime} = \psi^\dagger \gamma_3\gamma_2\gamma_1$

$\rho_2\sigma_1\ \rho_2\sigma_2\ \rho_2\sigma_3$ $\quad \bar\psi'\gamma_\mu\psi' = \psi^{\dagger\prime}\gamma_4\gamma_\mu\psi' = \psi^\dagger\gamma_3\gamma_2\gamma_1\gamma_4\gamma_\mu\gamma_1\gamma_2\gamma_3\psi$ $\quad \sim i\rho_2$

$i\rho_2$

$\mu=4$: $\bar\psi'\gamma_4\psi' = \psi^\dagger\psi = \bar\psi\gamma_4\psi$

$\mu=1,2,3$: $\bar\psi'\gamma_k\psi' = -\psi^\dagger\gamma_4\gamma_k\psi = -\bar\psi\gamma_k\psi$

(ii) $x_k' = -x_k$, $x_4' = x_4$

$\psi' = \gamma_4\psi$ $\quad \psi^{\dagger\prime} = \psi^\dagger\gamma_4$

$\bar\psi'\gamma_\mu\psi' = \psi^{\dagger\prime}\gamma_4\gamma_\mu\psi' = \psi^\dagger\gamma_\mu\gamma_4\psi$

$\bar\psi'\gamma_4\psi' = +\bar\psi\gamma_4\psi$

$\bar\psi'\gamma_k\psi' = -\bar\psi\gamma_k\psi$

$\bar\psi'\psi' = \psi^{\dagger\prime}\gamma_4\psi' = \psi^\dagger\gamma_3\gamma_2\gamma_1\gamma_4\gamma_1\gamma_2\gamma_3\psi = -\bar\psi\psi$

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Notion of pseudo-scalar etc. is ambiguous, if we include the time reflection. (II, 29)

Thus, $\bar{\psi}\psi$ and $\bar{\psi}\gamma_\mu\partial_\mu\psi$ change sign by time reversal.

In order that the above equation for the probability amplitude is invariant,

$$L^{Rev}(\varphi^{\alpha T}, \partial_\mu\varphi^{\alpha T}) = \pm L(\varphi^{\alpha T}, \frac{\partial\varphi^{\alpha T}}{\partial x_\mu})$$

according as the field particles have integer or half-integer spin. This can be achieved by ~~taken~~ symmetrizing or anti-symmetrizing the Lagrangian. For instance, in the case of spinor field

$$L \rightarrow \frac{1}{2}\{\bar{\psi}^{**}\psi - \psi\bar{\psi}\} + \cdots$$

$$L^{(Rev)}(\psi^T, \frac{\partial\psi^T}{\partial x_\mu}) = \{\frac{1}{2}(\bar{\psi}^T\psi^T - \psi^T\bar{\psi}^T)\}^{Rev} + \cdots$$

$$= -\frac{1}{2}\{\psi^T\bar{\psi}^T - \bar{\psi}^T\psi^T\} = -L(\psi^T, \frac{\partial\psi^T}{\partial x_\mu})$$

Also, for $\psi\gamma_\mu\partial_\mu\psi$

$\frac{1}{2}(\psi\gamma_\mu\partial_\mu\psi - \partial_\mu\psi\gamma_\mu\bar{\psi})$ **

hermitian conj.

$\frac{1}{2}(-\partial_\mu\bar{\psi}\gamma_\mu\psi + \psi\gamma_\mu\partial_\mu\bar{\psi})$

** These two terms are almost the same because of the com. rel. which is derived ~~from~~ ~~the~~ afterwards

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Then, $\delta\varphi^\alpha$ must ~~be~~ commute~~abbee~~ or anti-commute with field quantities in order to have the equations of motion which ~~I~~ is similar to ~~we expect from~~ the classical field equation because those terms in δL which differ from each other only by the location of $\delta\varphi^\alpha$ must add up when $\delta\varphi^\alpha$ shifted to the extreme ~~left~~ right.

Thus, we obtain the comm. rel. which are, in turn, consistent with the ~~properties of the~~ conditions imposed on $\delta\varphi^\alpha$.

The above proof of the ~~connection~~ relationship between spin and statistics is simple but there remains the doubt as to the physical significance of the transformation of the type (i). Namely the question arises whether there is more suitable transf. of field quantities due to the time reversal.* (such as that as is associated with the charge conjugation.)

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* See also
Watanabe Phys. Rev 84 (1951), 1008
Notion of Revised State (in general) (II. 28)
Pauli & Frez have shown that
(i) B.E. quantization of half integer spin field is untenable because the total energy is not positive definite.
(ii) F.D. quantization of integer spin field is not possible because the anti-commutation relations of infinitesimal character ~~are~~ so result in the trivial case of zero field.
Feynman argued on the basis of charge symmetry concept.

$\gamma_1, \gamma_3, \gamma_4$; real elements ; $\sigma_2\gamma_2$: imag. el.

$x_k' = x_k$ $\quad x_4' = -x_4$ $\quad [\sigma_2\gamma_2]_+ = 0$

$\psi' = \sigma_2\gamma_1\gamma_2\gamma_3\psi$

$\psi^{\dagger\prime} = \psi\gamma_3\gamma_2\gamma_1\sigma_2$ $\quad \gamma_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$\bar\psi'\gamma_4\psi' = \psi^{\dagger\prime}\psi' = +\psi\psi^\dagger = \psi\gamma_4\bar\psi \to -\bar\psi\gamma_4\psi$ + const.

$\bar\psi'\gamma_k\psi' = -\psi\gamma_k\bar\psi \to \bar\psi\gamma_k\psi$

$\bar\psi'\gamma_2\psi' = \psi\gamma_2\bar\psi \to \bar\psi\gamma_2\psi$ $\quad \gamma_2$: imag.

$\bar\psi'\gamma_3\psi' = -\psi\gamma_3\bar\psi \to \bar\psi\gamma_3\psi$

$\bar\psi'\psi' = -\psi\bar\psi \to \bar\psi\psi$.

or $\bar\psi\psi - \psi\bar\psi \to \bar\psi\psi - \psi\bar\psi$

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* iv) Operator form for the Variation Principle (Heisenberg eq. of motion)

$$\delta(\zeta_1', \sigma_1 | \zeta_2'' \sigma_2) = \left(\frac{i}{\hbar}\right)(\zeta_1', \sigma_1 | \delta W_{12} | \zeta_2'', \sigma_2)$$

$$(\zeta', \sigma | A | \zeta'' \sigma)$$

$$= \iint (\zeta', \sigma | \zeta''', \sigma') \overset{\vee}{(} \zeta''', \sigma' | A | \zeta^{IV}, \sigma') (\zeta^{IV}, \sigma' | \zeta'', \sigma)$$

$d\zeta'''$ $d\zeta^{IV}$

* If we perform a variation only on σ, but not on σ', we obtain

$$\delta(\zeta', \sigma | A | \zeta'', \sigma)$$

$$= \left(+\frac{i}{\hbar}\right) \delta W_{12} (\zeta', \sigma | [\delta W_{12}, A] | \zeta'', \sigma)$$

On the other hand, if A is a function of field operators on σ: $A(\sigma)$, then, variation on σ gives rise to a certain intrinsic change δA in A, but $(\zeta' \sigma | \delta A | \zeta'' \sigma)$ is unchanged by simultaneous change of A and the eigenvectors $\Psi(\zeta', \sigma)$, so that

$$(\zeta', \sigma | \delta A | \zeta'', \sigma) = -\delta(\zeta' \sigma | A | \zeta'' \sigma)$$

or

$$i\hbar \delta A(\sigma) = [\delta W_{12}, A(\sigma)]$$

or

$$\rightarrow i\hbar \delta A(\sigma) = [A(\sigma), \delta W_{12}]$$

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III. Formal Integration of Equations of Q.T. of Field (IV.1)

Yang, Feldman, P.R. 79 (1950), 972
G. Källen, Arkiv för Fysik 2 (1950) 187; 371.

Free spinor field:

$$(\gamma_\mu \frac{\partial}{\partial x_\mu} + M)\psi^{(0)}(x) = 0$$

Free real (neutral) scalar field

$$(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - \varkappa^2)\varphi^{(0)}(x) = 0$$

$M, \varkappa$: $\frac{(\text{mass})\,c}{\hbar}$

$(dx)^4 = dx_1 dx_2 dx_3 dx_0$

Free Lagrangian

$$\mathcal{L}_0 = -\frac{\hbar}{2}\left(\frac{\partial\varphi}{\partial x_\mu}\frac{\partial\varphi}{\partial x_\mu} + \varkappa^2\varphi^2\right)$$

$$\pm \frac{i\hbar c}{2}\left(\bar\psi\gamma_\mu\frac{\partial\psi}{\partial x_\mu} - \frac{\partial\bar\psi}{\partial x_\mu}\gamma_\mu\psi + 2M\bar\psi\psi\right)$$

Interaction $\mathcal{L}' = -ig\bar\psi\psi\varphi$ $(= -H')$

Total Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}'$$

$$(\gamma_\mu\frac{\partial}{\partial x_\mu} + M)\psi = \frac{g}{\hbar c}\varphi\psi$$

$$(\Box - \varkappa^2)\varphi = -g\bar\psi\psi \equiv -\rho.$$

$$(\frac{\partial\bar\psi}{\partial x_\mu}\gamma_\mu - M)\psi = -\frac{g}{\hbar c}\varphi\psi$$

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It is more convenient to use the notations

$$\varphi \to \varphi' = \frac{1}{\sqrt{\hbar c}}\varphi$$

$$q \to q' = \frac{q}{\sqrt{\hbar c}} \text{ (dimensionless)}$$

and then drop ' everywhere.

$$L = L_0 + L'$$

$$L_0 = -\frac{i\hbar c}{2}\left\{\frac{\partial\varphi}{\partial x_\mu}\frac{\partial\varphi}{\partial x_\mu} + \varkappa^2\varphi^2\right.$$

$$\left. + \bar\psi\gamma_\mu\frac{\partial\psi}{\partial x_\mu} - \frac{\partial\bar\psi}{\partial x_\mu}\gamma_\mu\psi \right.$$

$$\left. + 2M\bar\psi\psi\right\}$$

$$L' = -i\hbar c\, g\,\bar\psi\psi\varphi = -H'$$

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu} + M\right)\psi = g\varphi\psi$$

$$\left(\Box - \varkappa^2\right)\varphi = -g\bar\psi\psi$$

$$\varphi^2\bar\psi\gamma_\mu\psi M\psi = -g\bar\psi\varphi$$

The commutation relations for free fields were originally

$$[\varphi^{(0)}(x), \varphi^{(0)}(x')] = i\hbar c\,\Delta(x-x'),$$

but now

$$[\varphi^{(0)}(x), \varphi^{(0)}(x')] = i\Delta(x-x')$$

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$$[\psi^{(0)}(x), \bar{\psi}^{(0)}(x')]_t = -iS(x-x')$$

$$S(x) = \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - M\right) \Delta_M(x)$$

$$\Delta(x) = -\frac{1}{(2\pi)^3} \int\cdots\int e^{ik_\mu x_\mu} \varepsilon(k_\mu)\, \delta(k_\mu k_\mu + \varkappa^2) \times (dk)^4$$

$dk_4 = i\,dk_0$

$$= \frac{1}{(2\pi)^3 i} \int\cdots\int e^{i(\boldsymbol{k}\boldsymbol{r} - k_0 x_0)}\, \varepsilon(k_0)\, \delta(\boldsymbol{k}^2 - k_0^2 + \varkappa^2) \times (d\boldsymbol{k})^3\, dk_0$$

$$\int_{-\infty}^{+\infty} e^{-ik_0 x_0}\, \varepsilon(k_0)\, \delta(\boldsymbol{k}^2 - k_0^2 + \varkappa^2)\, dk_0$$

$$= \int_{-\infty}^{\infty} \frac{e^{-ik_0 x_0}}{2k_0} \varepsilon(k_0)\, d(k_0)^2\, \delta(\boldsymbol{k}^2 - k_0^2 + \varkappa^2)$$

$$= \int_{\infty}^{0} \frac{e^{-ik_0 x}}{2k_0}\, d(k_0)^2\, \delta(\boldsymbol{k}^2 - k_0^2 + \varkappa^2)$$

$$= \frac{e^{-ik_0 x_0}}{2k_0}\Bigg|_{k_0 = \sqrt{\boldsymbol{k}^2 + \varkappa^2}} - \frac{e^{ik_0 x_0}}{2k_0}\Bigg|_{k_0 = \sqrt{\boldsymbol{k}^2 + \varkappa^2}}$$

$$\Delta(x) = \frac{1}{(2\pi)^3} \iiint \frac{e^{i(\boldsymbol{k}\boldsymbol{r} - k_0 x_0)} - e^{-i(\boldsymbol{k}\boldsymbol{r} - k_0 x_0)}}{2ik_0} (d\boldsymbol{k})^3$$

$$-\frac{1}{c} \frac{\partial \Delta(r,t)}{\partial t}\Bigg|_{t=0} = \delta(r)$$

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$$[\psi_\alpha^{(0)}(x), \bar{\psi}_\beta^{(0)}(x')]_+ = -i S_{\alpha\beta}(x-x')$$

$$[\psi_\alpha^{(0)}(x), \psi_\beta^{(0)\dagger}(x')]_+ = -i S_{\alpha\delta}(\gamma_4)_{\delta\beta}$$

$$= -i\left(\frac{\partial}{\partial x_4} + \gamma_k\gamma_4\frac{\partial}{\partial x_k} - M\gamma_4\right)_{\alpha\beta}\Delta_M$$

$$= -\frac{i}{ic}\frac{\partial \Delta_M}{\partial t}\Big|_{t=t'}\delta_{\alpha\beta} = \delta(x-x')\,\delta_{\alpha\beta}$$

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(III, 3)

Integral equations:

Green function

$$(\Box - \varkappa^2)\, G(x) = -\delta(x)$$

$$\delta(x) \equiv +\delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4)$$
$$= -i\,\delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4)$$

or it is more convenient to take

~~$(\Box - \varkappa^2)\,\Delta(x$~~

$$\delta(x) \equiv \delta(x_1)\,\delta(x_2)\delta(x_3)\delta(x_0)$$

$$\int_{-\infty}^{+\infty} \delta(x_0)\,dx_0 = \int_{-i\infty}^{+i\infty} \delta(x_4)\,d\,ix_4 = 1$$
$$= \int \delta(x_4)\, i\, dx_0$$

$$i\,\delta(x_4) = \delta(x_0) \quad \text{or} \quad \delta(x_4) = -i\,\delta(x_0)$$

According to the latter definition

$$\delta(x) = \frac{1}{(2\pi)^4} \int\!\!\cdots\!\!\int_{-\infty}^{+\infty} e^{i(\mathbf{k}\mathbf{r} - k_0 x_0)}\,(d\mathbf{k})^3 (dk_0)$$

$k_4 = i k_0$

$$= \frac{-i}{(2\pi)^4} \int\!\cdots\!\int e^{i k_\mu x_\mu}\,(dk)^4$$

$$G(x) = \frac{-i}{(2\pi)^4} \int\!\cdots\!\int \frac{e^{i k_\mu x_\mu}}{k_\mu k_\mu + \varkappa^2}\,(dk)^4$$

$$= \frac{1}{(2\pi)^4} \int\!\cdots\!\int \frac{e^{i(\mathbf{k}\mathbf{r} - k_0 x_0)}}{\mathbf{k} - k_0^2 + \varkappa^2}\,(d\mathbf{k})^3 (dk_0)$$

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This integral is indeterminate (and, in general, divergent) due to the singularity at $k_0 = \pm\sqrt{k^2+\kappa^2}$. In order to have unambiguous and convergent result, let us start from the integration with respect to k_0 and

$$\Delta(x) = \frac{1}{(2\pi)^4} P\int\!\!\int \frac{e^{i(\mathbf{k}\mathbf{r} - k_0 x_0)}}{k^2 - k_0^2 + \kappa^2} \times d\mathbf{k}\, dk_0$$

$$= \frac{1}{(2\pi)^4} \iiint (d\mathbf{k})^3 e^{i\mathbf{k}\mathbf{r}} P\int_{-\infty}^{+\infty} \frac{e^{-ik_0 x_0}}{(k^2 - k_0^2 + \kappa^2)} dk_0$$

$$P\int_{-\infty}^{+\infty} \frac{e^{-ik_0 x_0}}{k^2 - k_0^2 + \kappa^2} dk_0 = \lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{-\sqrt{k^2+\kappa^2}-\varepsilon} + \int_{-\sqrt{k^2+\kappa^2}+\varepsilon}^{\sqrt{k^2+\kappa^2}-\varepsilon} + \int_{\sqrt{k^2+\kappa^2}+\varepsilon}^{+\infty}\right\} \times \frac{e^{-ik_0 x}}{k^2 - k_0^2 + \kappa^2} dk_0$$

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$$P\int_b^c \frac{f(x)}{x-a}dx = \lim_{\varepsilon\to 0}\left\{\int_b^{a-\varepsilon}\frac{f(x)}{x-a}dx + \int_{a+\varepsilon}^{c}\frac{f(x)}{x-a}dx\right\}$$

$b<a<c$

$$= \lim_{\varepsilon\to 0}\left\{\int_{b-a}^{-\varepsilon}\frac{f(y+a)}{y}dy + \int_{+\varepsilon}^{c-a}\frac{f(y+a)}{y}dy\right\}$$

$$= \lim_{\varepsilon\to 0}\left\{\int_{b-a}^{-\varepsilon}\frac{f(y+a)}{y}dy - \int_{a-c}^{-\varepsilon}\frac{f(a-y)}{y}dy\right\} \qquad y\to -y$$

$b-a < a-c$

$$= \lim_{\varepsilon\to 0}\left\{\int_{a-c}^{-\varepsilon}\frac{f(a+y)-f(a-y)}{y}dy\right\} - \int_{b-a}^{a-c}\frac{f(y+a)}{y}dy$$

is finite and definite, if $f(x)$ is regular at $x=a$.

Now, instead of taking principal value, we take the complex integral of the following type:

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$$\int \frac{e^{-ik_0x_0}\,dk_0}{k^2-k_0^2+\kappa^2} = P\int_{-\infty}^{+\infty} - \pi i \left[\frac{e^{-ik_0x_0}}{+\sqrt{k^2+\kappa^2}}\right]_{k_0=-\sqrt{\;}}$$

$$C_1 \quad -(k_0+\sqrt{\;})(k_0-\sqrt{\;})$$

$$- \pi i \left[\frac{e^{ik_0x_0}}{-k_0-\sqrt{k^2+\kappa^2}}\right]_{k_0=+\sqrt{\;}}$$

$$= P\int_{-\infty}^{+\infty} + (\pi i)\left[\frac{e^{-ik_0x_0}}{2k_0} - \frac{e^{+ik_0x_0}}{2k_0}\right]_{k_0=+\sqrt{\;}}$$

Now, if we define

$$\Delta^{(ret)}(x) = \frac{1}{(2\pi)^4}\iiint\!\!\int_{C_1} \frac{e^{i(\vec{k}\vec{r}-k_0x_0)}}{k^2-k_0^2+\kappa^2}(dk)^3(dk_0)$$

$$= \overline{\Delta}(x) - \frac{1}{2}\Delta(x)$$

Similarly

$$\Delta^{(adv)}(x) = \frac{1}{(2\pi)^4}\iiint\!\!\int_{C_2} \;,,$$

$$= \overline{\Delta}(x) + \frac{1}{2}\Delta(x)\,.$$

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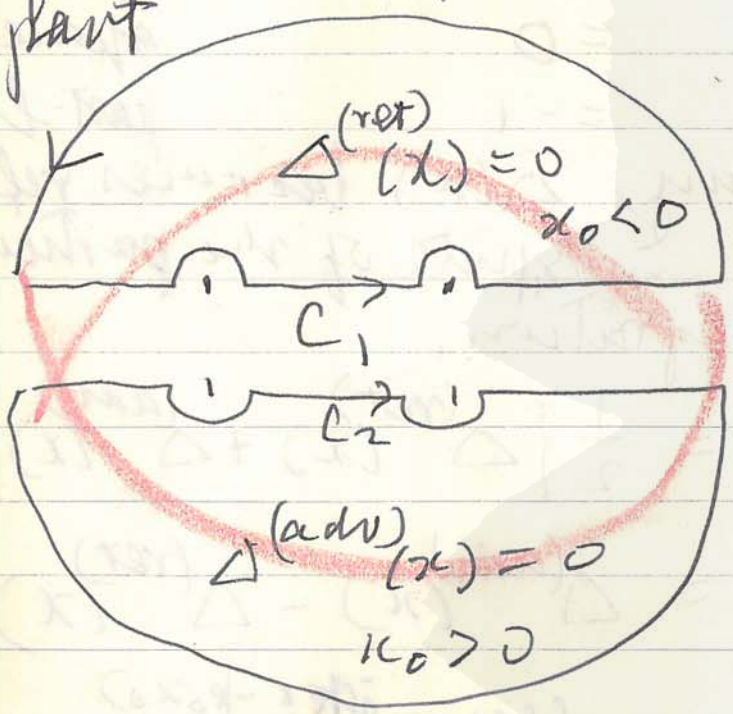
(II,5)

$\Delta^{(ret)}(x) = 0$ for $x_0 < 0$

because $-ik_0 x_0$ is real negative for k_0 has positive imaginary part



Likewise,
$\Delta^{(adv)}(x) = 0$ for $x_0 > 0$.

Thus, $\overline{\Delta}(x) = \frac{1}{2}\Delta(x)$ for $x_0 < 0$
$= -\frac{1}{2}\Delta(x)$ for $x_0 > 0$

Since $\Delta(x)$ is zero for space-like x, one may write

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$$\bar{\Delta}(x) = -\frac{1}{2}\varepsilon(x)\Delta(x).$$

where

$\varepsilon(x) = 1$ $\quad$ x: future-like
$= 0$ $\quad$ space-like
$= -1$ $\quad$ past-like

In this way, $\bar{\Delta}(x)$ becomes relativistic invariant in spite of the particular way of integration.

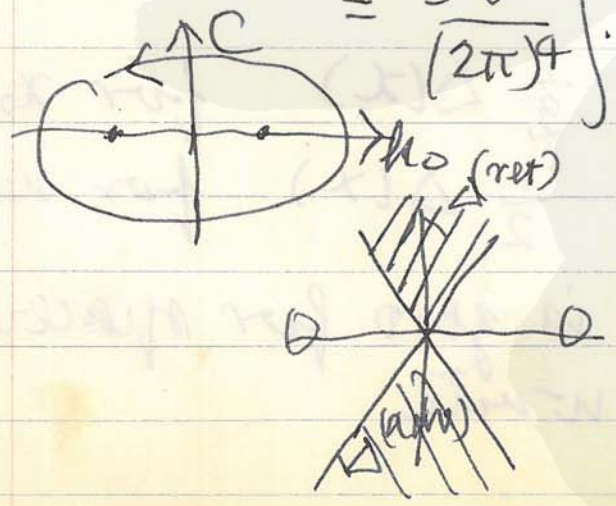
$$\bar{\Delta}(x) = \frac{1}{2}\{\Delta^{(ret)}(x) + \Delta^{(adv)}(x)\}$$

$$\Delta(x) = \Delta^{(adv)}(x) - \Delta^{(ret)}(x)$$

$$= \frac{1}{(2\pi)^4}\int\int\int\int_C \frac{e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - k_0 x_0)}}{k^2 - k_0^2 + \varkappa^2}(dk)^3(dk_0)$$

$k_4 = ik_0$
$-ik_4 = k_0$

$$= -\frac{i}{(2\pi)^4}\int\cdots\int_C \frac{e^{ik_\mu x_\mu}}{k_\mu k_\mu + \varkappa^2}(dk)^4$$



$\Delta(x_k, -x_4) = -\Delta(x_k, x_4)$
$\therefore \bar{\Delta}(x_k, -x_4) = \Delta(x_k, x_4)$
$\Delta(-x_k, x_4) = +\Delta(x_k, x_4)$

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$$\overline{\Delta}(-x_4, x_4) = \overline{\Delta}(x_4, x_4)$$
$$\overline{\Delta}(-x_4, -x_4) = \overline{\Delta}(x_4, x_4).$$

$$(\Box - x^2)\Delta(x) = 0$$

$$(\Box - x^2) G_r(x,x') = -\delta(x-x')$$

where

$$G_r(x,x') = \left(\overline{\Delta}(x-x') - \Delta^{(ret)}(x-x')\right)\theta^{(adv)}(x-x')$$

Thus, if φ is an arbitrary solution of

$$(\Box - x^2)\varphi(x) = -\rho(x),$$

then

$$\varphi'(x) \equiv \varphi(x) - \int G_r(x,x')\rho(x')(dx)^4$$

satisfies the free field eq,

$$(\Box - x^2)\varphi'(x) = 0$$

where

$$(dx)^4 = dx_1\, dx_2\, dx_3\, dx_0$$
$$= (-i)(dx_1)(dx_2)(dx_3)(dx_4)$$

In particular, if $G_r(x,x') \equiv \Delta^{(ret)}(x-x')$

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and if

$$\lim_{x_0 \to -\infty} \int\cdots\int \Delta^{(ret)}(x-x')\,\rho(x')\,(dx')^4 = 0$$

remote past

or $\rho(x) \to 0$ for $x_0 \to -\infty$.

then

$$\varphi(x) - \int\int \Delta^{(ret)}(x-x')\,\rho(x')\,(dx')^4 \equiv \varphi^{(in)}(x)$$

coincide with the actual field in the remote past but behaves like a free field, which we call incoming field.

$$\varphi(x) = \varphi^{(in)}(x) + \int\cdots\int \Delta^{(ret)}(x-x')\,\rho(x') \times (dx')^4$$

$$\cdot\,[\varphi^{(in)}(x), \varphi^{(in)}(x')] = i\Delta(x-x')$$

Similar, one may write

$$\varphi(x) = \varphi^{(out)}(x) + \int\cdots\int \Delta^{(adv)}(x-x')\,\rho(x') \times (dx')^4$$

$$[\varphi^{(out)}(x), \varphi^{(out)}(x')] = i\Delta(x-x')$$

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(IV. 7)

where $\varphi^{(out)}$ is the outgoing (free) field, which coincides with the actual field φ at remote future*.

$$\varphi^{(out)}(x) = \varphi^{(in)}(x) + \int\cdots\int \Delta(x-x')\rho(x')(dx')^4$$

$$= \varphi^{(in)}(x) + \int\cdots\int \Delta(x-x')\, g\,\bar{\psi}(x')\psi(x') \times (dx')^4$$

Now, there is a relation between

$$\varphi^{(out)}(x) = S^{-1}\varphi^{(in)}(x)S$$

where S is a unitary operator, because $\varphi^{(in)}(x)$ and $\varphi^{(out)}$ satisfy the same commutation relations, so that the energy spectrum must be identical. ~~Correspondingly, we have~~

~~$$\Psi((\zeta^{(out)})', \quad) = \Psi((\zeta^{(in)})', \quad)$$~~

---

* Φ: reinserting operator
$\varphi^{(in)}$, $\varphi^{(out)}$: interaction regions.

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The equations for $\psi(x)$, $\bar{\psi}(x)$ can be integrated in the similar way. The equation

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + M\right)_{\alpha\beta} \bar{S}_{\beta\gamma}(x) = -\delta(x)$$

is satisfied by

$$\bar{S}_{\alpha\beta}(x) = \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - M\right)_{\alpha\beta} \bar{\Delta}_M(x)$$

$$S^{(ret)}(x) = \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - M\right) \Delta_M^{(ret)}(x)$$

$$S^{(adv)}(x) = \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - M\right) \Delta_M^{(adv)}(x)$$

$$\psi(x) = \psi^{(in)}(x) - \int\cdots\int S^{(ret)}(x-x') g \varphi \psi(x') \times (dx')^4$$

$$\bar{\psi}(x) = \psi^{(out)}(x) - \int\cdots\int S^{(adv)}(x-x') g \bar{\varphi}(x') \psi(x') \times (dx')^4$$

$$\left(\frac{\partial \bar{\psi}}{\partial x_\mu} \gamma_\mu - \bar{\psi} M\right) = -g \bar{\psi} \varphi$$

$$\frac{\partial \bar{S}_{\alpha\beta}(x)}{\partial x_\mu} (\gamma_\mu)_{\gamma\beta} - \bar{S}_{\alpha\beta} M$$

$$= \frac{\partial}{\partial x_\mu} \left\{ (\gamma_\nu)_{\alpha\gamma} \frac{\partial}{\partial x_\mu} - M \delta_{\alpha\beta} \right\} \bar{\Delta}_M(x) - \cdots M$$

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$$\frac{\partial}{\partial x_\mu}\left(\int \rho(x')\bar{S}(x'-x)(dx)^4\right)\gamma_\mu - (\quad)M$$

$$= \int \rho(x')\left(-\frac{\partial \bar{S}}{\partial x'_\mu}\gamma_\mu - M\bar{S}\right)$$

$$-\frac{\partial \bar{S}_{\alpha\beta}(x'-x)}{\partial x'_\mu}(\gamma_\mu)_{\gamma\beta} - M\bar{S}_{\alpha\beta}(x'-x)$$

$$= -\left(\gamma_\nu \frac{\partial}{\partial x'_\nu} + M\right)\left(\gamma_\mu \frac{\partial}{\partial x'_\mu} - M\right)\bar{\Delta}_M(x'-x)$$

$$= \delta(x).$$

$$= \rho(x).$$

$$\bar{\psi}(x) = \bar{\psi}^{(in)} - \int_{-\infty}^{t_0}\int g\bar{\psi}(x')\varphi(x')S^{(adv)}(x'-x) \times (dx')^4$$

because $S^{(adv)}(x'-x)$ is zero for $x'_0 > x_0$ and so for $x_0 \to +\infty$, the second term tends to zero.

$$\bar{\psi}(x) = \bar{\psi}^{(out)} - \int\int g\bar{\psi}(x')\varphi(x')S^{(ret)}(x'-x) \times (dx')^4$$

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two sets of operators

Since, $\{\varphi^{(in)}(x), \psi^{(in)}(x), \bar{\psi}^{(in)}(x)\}$
and $\{\varphi^{(out)}(x), \psi^{(out)}(x), \bar{\psi}^{(out)}(x)\}$

satisfy the same set of c.r.,
there must be a unitary transfor.

$$\left.\begin{aligned}
\varphi^{(out)}(x) &= S^{-1}\cdot\varphi^{(in)}(x)S \\
\psi^{(out)}(x) &= S^{-1}\psi^{(in)}(x)S \\
\bar{\psi}^{(out)}(x) &= S^{-1}\bar{\psi}^{(in)}(x)S
\end{aligned}\right\}$$

The physical meaning of the operator S is ~~obvious~~ as follows:

$$\Psi(\{\zeta^{(out)}\}', \sigma) = S^{-1}\Psi(\{\zeta^{(in)}\}', \sigma)$$

$$\Psi(\{\zeta^{(in)}\}'', \sigma) = S\,\Psi(\{\zeta^{(out)}\}'', \sigma)$$

$$\Psi^{+}(\{\zeta^{(out)}\}', \sigma)\Psi(\{\zeta^{(in)}\}'', \sigma)$$
$$= \Psi^{+}(\{\zeta^{(out)}\}', \sigma)S\,\Psi(\{\zeta^{(out)}\}'', \sigma)$$

$$(\{\zeta^{(out)}\}', \sigma \mid \{\zeta^{(in)}\}'', \sigma)$$
$$= (\{\zeta^{(out)}\}', \sigma \mid S \mid \{\zeta^{(out)}\}'', \sigma)$$

$\sigma \to +\infty$,

$$(\{\zeta^{(out)}\}', +\infty \mid \{\zeta^{(in)}\}'', +\infty)$$
$$= (\{\zeta^{(out)}\}', +\infty \mid S \mid \{\zeta^{(out)}\}'', +\infty)$$

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$$= (\zeta', +\infty | S | \zeta'', +\infty)$$

where ζ denotes the set of Heisenberg operators which coincide with $\zeta^{(out)}$ at $x_0 = +\infty$.

Thus the matrix elements of S represents the probability amplitude that $\zeta^{(out)}(+\infty) = \zeta(+\infty)'$ has ~~the~~ a set of values $(\zeta^{(out)}_{(+\infty)})'' = (\zeta(+\infty))'$ in the state in which $\zeta(+\infty)$ was to have a set of values $(\zeta^{(in)}_{(+\infty)})''$, if ~~there were no interac~~ the fields were kept free from interaction between them. This is nothing but the S-matrix as ~~in the case~~ introduced by Heisenberg and extensively used in r.f.t. recently.

~~Actu~~ As a matter of fact, one can show that the above S is identical as the matrix S which was obtained as the result of integration of T.S. eq. in many-time formalism

$$S = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar c}\right)^n \frac{1}{n!} \int \cdots \int P\{H'(x') H'(x'') \cdots H'(x^{(n)})\} \times (dx')^4 \cdots (dx^{(n)})^4$$

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where H' is expressed in terms of the interaction representation or incoming operators field operators.

Proof; $S^\dagger = S^{-1}$

$$\Psi((\zeta^{(out)})', \sigma) = S^{-1}\Psi((\zeta^{(in)})', \sigma)$$

or

$$\Psi^\dagger((\zeta^{(out)})', \sigma) = \Psi^\dagger((\zeta^{(in)})', \sigma)(S^{-1})^\dagger$$
$$= \Psi^\dagger((\zeta^{(in)})', \sigma)S$$

$$((\zeta^{(out)})', +\infty \,|\, (\zeta^{(in)})'', \sigma)$$
$$= ((\zeta^{(in)})', \sigma|S|(\zeta^{(in)})'', \sigma)$$

$$((\zeta^{(out)})', +\infty \,|\, (\zeta^{(in)})'', +\infty)$$
$$= ((\zeta^{(in)})', +\infty|S|(\zeta^{(in)})'', +\infty)$$
$$(= ((\zeta^{(out)})', +\infty|S|(\zeta^{(out)})'', +\infty))$$

Now in order to prove the above statement, let us ~~consider~~ introduce the following function

$$\Delta^\sigma(x, x') = \Delta^{(ret)}(x-x') \quad \text{if } x' \text{ is later than } \sigma$$
$$= \Delta^{(adv)}(x-x') \quad \text{if } x' \text{ is earlier than } \sigma,$$

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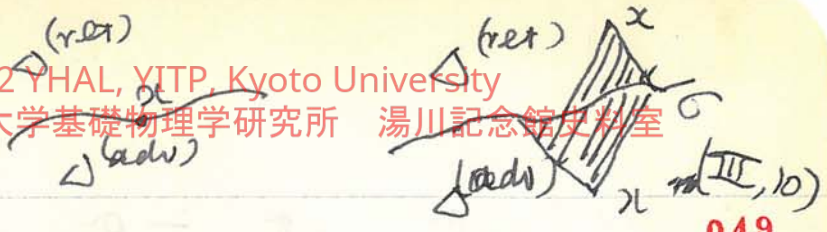

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~~x belongs to σ~~ or

$$\Delta^{\sigma}(x,x') = \frac{1-\varepsilon(\sigma,x')}{2}\Delta^{(ret)}(x-x') + \frac{1+\varepsilon(\sigma,x')}{2}\Delta^{(adv)}(x-x')$$

$\varepsilon(\sigma,x') = +1$ σ later than x'

$= -1$ σ earlier than x'.

x ⋯ or $\Delta^{\sigma}(x,x') = -\frac{1}{2}[\varepsilon(x-x') - \varepsilon(\sigma,x')]\Delta(x-x')$

σ —

x' ⋯ $S^{\sigma} = (\gamma_\mu \frac{\partial}{\partial x_\mu} - M)\Delta_M^{\sigma}$

$$\varphi(x) \simeq \varphi^{\sigma}(x) + \int\cdots\int \Delta^{\sigma}(x-x') g \bar{\psi}(x')\psi(x') (dx')^4$$

$$\psi(x) \simeq \psi^{\sigma}(x) - \int\cdots\int S^{\sigma}(x-x') g \varphi(x')\psi(x') \times (dx')^4$$

$$\bar{\psi}(x) \simeq \bar{\psi}^{\sigma}(x) - \int\cdots\int g\bar{\psi}(x')\varphi(x') S^{\sigma}(x'-x) \times (dx')^4$$

$$S^{\sigma}(x-x') = (\gamma_\mu \frac{\partial}{\partial x_\mu} - M)\Delta_M^{\sigma}(x-x')$$

$$S^{\sigma}(x'-x) = +(\gamma_\mu \frac{\partial}{\partial x'_\mu} - M)\Delta_M^{\sigma}(x'-x)$$

$$\left(\frac{\partial S^{\sigma}(x'-x)}{\partial x_\mu}\gamma_\mu - S^{\sigma}(x'-x)M\right) = \delta(x'-x)$$

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φ^σ, ψ^σ, $\bar{\psi}^\sigma$ are free fields which coincide with actual fields on σ, because Δ^σ and S^σ vanish for x on σ. ($\varepsilon(x-x') - \varepsilon(\sigma; x')$ is different from zero only if x and x' are opposite sides of σ)

$$\left.\begin{aligned} [\varphi^\sigma(x), \varphi^\sigma(x')] &= i\Delta(x-x') \\ [\psi^\sigma_\alpha(x), \bar{\psi}^\sigma_\beta(x')]_+ &= i S_{\alpha\beta}(x-x') \end{aligned}\right\}$$

$$\left.\begin{aligned} \varphi^{\sigma'}(x) &= U^{-1}(\sigma', \sigma'')\varphi^{\sigma''}(x) U(\sigma', \sigma'') \\ \psi^{\sigma'}(x) &= U^{-1}(\sigma', \sigma'')\psi^{\sigma''}(x) U(\sigma', \sigma'') \\ \bar{\psi}^{\sigma'}(x) &= U^{-1}(\sigma', \sigma'')\bar{\psi}^{\sigma''}(x) U(\sigma', \sigma'') \end{aligned}\right\}$$

U: unitary. $U(\sigma'', \sigma') = U(\sigma'', \sigma) U(\sigma, \sigma')$

$U(+\infty, -\infty) = S$

$S = U(+\infty, -\infty) = \cdots U(\sigma_0, \sigma_{-1}) U(\sigma_0, \sigma_0) U(\sigma_2, \sigma_1) \cdots$

$U(\sigma, -\infty) = U(\sigma, -\infty) U(\sigma', \sigma)$ ← $+\infty$

If $\sigma' = \sigma + \delta\sigma$,

$U(\sigma+\delta\sigma, \sigma) = 1 - i R(\sigma, \sigma+\delta\sigma, \sigma)$

$\times \int\int\int_{\sigma}^{\sigma+\delta\sigma} \mathcal{L}(x', \sigma)(dx')^4$

where R is an infinitesimal hermitian operator.

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$$U(\sigma+\delta\sigma, -\infty) = U(\sigma, -\infty)\, U(\sigma+\delta\sigma, \sigma)$$

$$U(\sigma+\delta\sigma, \sigma) = 1 - iR(\sigma+\delta\sigma, \sigma)$$

$$\varphi^{\sigma+\delta\sigma}(x) = U^{-1}(\sigma+\delta\sigma, \sigma)\,\varphi^{\sigma}(x)\,U(\sigma+\delta\sigma, \sigma)$$

$$= (1+iR)\,\varphi^{\sigma}(1-iR)$$

$$\varphi^{\sigma+\delta\sigma}(x) - \varphi^{\sigma}(x) = -i\,[\varphi^{(\sigma)}, R(\sigma+\delta\sigma, \sigma)]$$

On the other hand,

$$\varphi^{\sigma+\delta\sigma}(x) + \int\cdots\int \Delta^{\sigma+\delta\sigma}(x-x')\, g\, \bar{\psi}(x')\,\psi(x')\,(dx')^4$$

$$= \varphi^{\sigma}(x) + \int\cdots\int \Delta^{\sigma}(x-x')\, g\, \bar{\psi}(x')\,\psi(x')\,(dx')^4$$

$$\varphi^{\sigma+\delta\sigma}(x) - \varphi^{\sigma}(x) = \int_{\sigma}^{\sigma+\delta\sigma}\cdots\int \{\Delta^{(ret)}(x-x') - \Delta^{(adv)}(x-x')\}$$

$$\times\, g\,\bar{\psi}(x')\,\psi(x')\,(dx')^4$$

$$= -\int_{\sigma}^{\sigma+\delta\sigma}\cdots\int \Delta(x-x')\, g\, \bar{\psi}^{\sigma}(x')\,\psi^{\sigma}(x')\,(dx')^4$$

$$= i\int_{\sigma}^{\sigma+\delta\sigma}\cdots\int [\varphi^{\sigma}(x), \varphi^{\sigma}(x')]\, g\, \bar{\psi}^{\sigma}(x')\,\psi^{\sigma}(x')$$

$$\times\, (dx')^4$$

$$= -i\,[\varphi^{\sigma}(x), \int_{\sigma}^{\sigma+\delta\sigma}\cdots\int -g\,\underbrace{\bar{\psi}^{\sigma}(x')}_{\bar{\varphi}^{\sigma}(x')}\,\psi^{\sigma}(x')\,(dx')^4]$$

Hence, we have

$$R(\sigma+\delta\sigma, \sigma) = \frac{1}{\hbar c}\int_{\sigma}^{\sigma+\delta\sigma}\int H'(\varphi^{\sigma}(x'), \psi^{\sigma}(x'), \bar{\psi}^{\sigma}(x'))\,(dx')^4$$

$$H'^{(\sigma)} = -g\,\varphi^{\sigma}\bar{\psi}^{\sigma}\psi^{\sigma}$$

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$$U(\sigma+\delta\sigma,-\infty) = U(\sigma,-\infty)\left(1-\frac{i}{\hbar c}\int_{\sigma}^{\sigma+\delta\sigma} H'^{(x)}(dx')^4\right)$$

$$= U(\sigma,-\infty)\left(1-\frac{i}{\hbar c}\int_{\sigma}^{\sigma+\delta\sigma} H'^{(x)}(dx')^4\right) U^{-1}(\sigma,-\infty)\cdot U(\sigma,-\infty)$$

$$= \left(1-\frac{i}{\hbar c}\int_{\sigma}^{\sigma+\delta\sigma} H'^{(in)}(dx')^4\right) U(\sigma,-\infty)$$

because

$$U(\sigma',\sigma)\,\varphi^{\sigma'}\,U^{-1}(\sigma',\sigma) = \varphi^{\sigma}$$

or

$$U(\sigma,-\infty)\,\varphi^{\sigma}\,U^{-1}(\sigma,-\infty) = \varphi^{(in)}$$

$$\delta U(\sigma,-\infty) = \left(-\frac{i}{\hbar c}\right)\int_{\sigma}^{\sigma+\delta\sigma} H^{(in)}(dx')^4 \cdot 1 \times U(\sigma,-\infty)$$

In particular, if $\delta\sigma$ is concentrated around one point $P(x)$ on σ (a small region $\delta\omega_P$)

$$\frac{\delta U(\sigma,-\infty)}{\delta\omega_P} = -\frac{i}{\hbar c} H'^{(in)} U(\sigma,-\infty)$$

This is the same equation as that which obtain for the Schrödinger function $(\zeta', \sigma[\,)$ in many-time formalism*.

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$$S=\sum_{n=0}^{\infty}\left(\frac{-i}{\hbar c}\right)^n\int\cdots\int P\{H'^{(in)}(x')\,\theta(x'-x'')\,H'^{(in)}(x'')\,\theta(x''-x''')\cdots\theta(x^{(n-1)}-x^{(n)})\times H'^{(in)}(x^{(n)})\}\,(dx')^4\cdots(dx^{(n)})^4$$

We obtain, thus, the formal expression for $U(+\infty,-\infty)=S$

$$S=\sum_{n=0}^{\infty}\left(\frac{-i}{\hbar c}\right)^n\frac{1}{n!}\int\cdots\int P\{H'^{(in)}(x'),H'^{(in)}(x''),\cdots H'^{(in)}(x^{(n)})\}\,(dx')^4\cdots(dx^{(n)})^4$$

P product $x'>x''>\cdots$

Thus, the interaction representations of fields, $\varphi^{(int)}$ etc are to be identified with $\varphi^{(in)}$, etc.

~~Problem~~ One can obtain similar expression for S in terms of $\varphi^{(out)}$, etc.

$$*\quad \Psi((\zeta^{(\sigma')})',\sigma')=U(\sigma',\sigma)\Psi((\zeta^{(\sigma)})',\sigma)$$

or

~~$\Psi((\zeta^{(\sigma)})',\sigma')=U(\sigma',\sigma)\Psi((\zeta^{(\sigma)})',\sigma')$~~

$$\times\Psi\quad \Psi^{\dagger}((\zeta^{(\sigma')})',\sigma')=\Psi^{\dagger}((\zeta^{(\sigma)})',\sigma)\,U(\sigma',\sigma)$$

$$((\zeta^{(\sigma')})',\sigma'\,|\,)=((\zeta^{(\sigma)})',\sigma\,|\,U(\sigma',\sigma)\,|\,)$$

$\sigma'=\sigma+\delta\sigma$

$$=\int(\zeta^{(\sigma)})',\sigma|\,U(\sigma',\sigma)\,|(\zeta^{(\sigma)})'',\sigma)\,d(\zeta^{(\sigma)})''(\zeta^{(\sigma)})'',\sigma|)$$

~~$i\hbar\frac{\partial((\zeta^{(\sigma)})',t'|)}{\partial t'}=$~~

$$\frac{\delta((\zeta^{(\sigma)})',\sigma|\,)}{\delta\sigma_p}=-\frac{i}{\hbar c}\int(\zeta^{(\sigma)})',\sigma|H'^{(\sigma)}|(\zeta^{(\sigma)})'',\sigma)\,d(\zeta^{(\sigma)})''((\zeta^{(\sigma)})'',\sigma|)$$

$(\sigma)\to(in)$ just as before.

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One-time formalism:

$$\frac{\partial U(x_0,-\infty)}{\partial x_0} = \frac{-i}{\hbar c}\int H'^{(in)}\,dx\,dy\,dz\,dt\cdot U(x_0,-\infty)$$

1st approx.

$$U(x_0,-\infty) = \left(\frac{i}{\hbar c}\right)\int_{-\infty}^{x_0}\!\!\int\!\!\int\!\!\int H'^{(in)}(x)(dx)^4$$

$$1 -$$

because $U(-\infty,-\infty) = 1$

2nd approx.

$$U(x_0,-\infty) \simeq 1 - \left(\frac{i}{\hbar c}\right)\int_{-\infty}^{x_0}\!\!\int\!\!\int\!\!\int H'^{(in)}(x')(dx')^4$$

$$+\left(\frac{-i}{\hbar c}\right)^2\int_{-\infty}^{x_0}\!\!\int\!\!\int\!\!\int_{-\infty}^{x_0'}\!\!\int\!\!\int\!\!\int H'^{(in)}(x')H'^{(in)}(x'')(dx')^4(dx'')^4$$

$$+ \cdots\cdots$$

In our case, the result is independent of the choice of coordinate system because

$$[H'^{(in)}(x'),\ H'^{(in)}(x'')] = 0$$

for space-like $x'-x''$.

$$[\varphi(x')\bar{\psi}(x')\psi(x'),\ \varphi(x'')\bar{\psi}(x'')\psi(x'')]$$
$$= \varphi(x')\varphi(x'')\cdot\bar{\psi}_\alpha(x')\psi_\alpha(x')\bar{\psi}_\beta(x'')\psi_\beta(x'')$$
$$- \varphi(x'')\varphi(x')\cdot\bar{\psi}_\beta(x'')\psi_\beta(x'')\bar{\psi}_\alpha(x')\psi_\alpha(x')$$

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$$= \varphi(x')\varphi(x'')\{\bar{\psi}_\alpha(x')(-iS_{\alpha\beta}(x'-x''))\psi_\beta(x'') - \bar{\psi}_\alpha(x')\bar{\psi}_\beta(x'')\psi_\alpha(x')\psi_\beta(x'') - \bar{\psi}_\beta(x'')(-iS_{\beta\alpha}(x''-x'))\psi_\alpha(x')\} + \bar{\psi}_\beta(x'')\bar{\psi}_\alpha(x')\bar{\psi}_\beta(x'')\psi_\alpha(x')]$$

$$S_{\alpha\beta}(x'-x'') = \left(\gamma_\mu \frac{\partial}{\partial x'_\mu} - M\right)_{\alpha\beta} \Delta_M(x'-x'')$$

$$S_{\beta\alpha}(x''-x') = \left(\gamma_\mu \frac{\partial}{\partial x''_\mu} - M\right)_{\beta\alpha} \Delta_M(x''-x')$$

$$= \cdots = \frac{i}{c}(\gamma_4)_{\alpha\beta}\delta(x'-x'')$$

For $x_0' \simeq x_0''$:

$$= \varphi(x')\varphi(x'')\frac{i}{c}\{\bar{\psi}_\alpha(x')\gamma_4\psi(x')\delta(x'-x'') - \bar{\psi}(x')\gamma_4\psi(x')\delta(x'-x'')\}$$

$$= 0$$

For $x_0' \neq x_0''$: one can choose the axes such that $x_0' = x_0''$ for $x'-x''$ space-like.

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IV. Structure of S-Matrix

IV.1

F.J. Dyson, P.R. 25 (1949), 486
25 (1949), 1736

We have obtained the formal expression for S-matrix. It has been shown by a great number of physicists; S-matrix furnishes a very convenient method of finding probabilities of various collision processes. Before going into the application of S-matrix to ~~actual~~ specific collision processes, let us discuss briefly its general properties. and how the inherent divergence difficulties could be obviated.

We start from the vacuum state $(0|S|0)$ denote the matrix element, for which the ~~initial~~ row and column both refer to the vacuum. By the vacuum here we mean ~~there's no~~ the eigenvector for no incoming ~~free~~ free particles of any kind in a very large but finite volume. space domain with a

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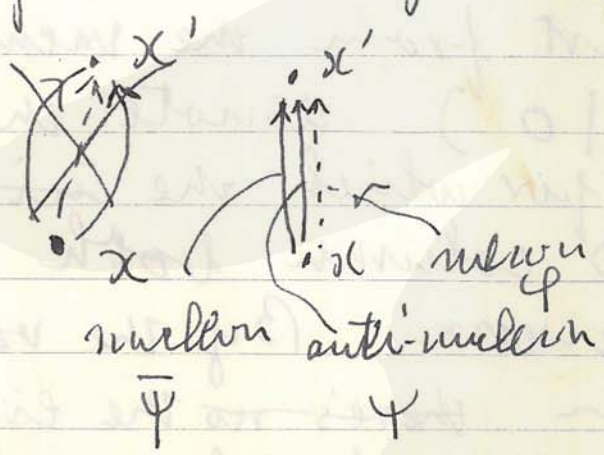
$$q^2(0|\int\cdots\int\varphi(x)\bar{\psi}(x)\psi(x)A(x,x')\varphi(x')\bar{\psi}(x')\psi(x')A(x',x'')\varphi(x'')\bar{\psi}(x'')\psi(x'')\cdots|0)$$

$$(0|S|0) = 1 + i\int q\,(0|\int q\,\varphi(x)\bar{\psi}(x)\psi(x)\,x(dx)^4|0)$$

$$-\frac{q^2}{2}(0|\int\cdots\int P(\varphi(x)\bar{\psi}(x)\psi(x),\varphi(x')\bar{\psi}(x')\psi(x'))(dx)^4(dx')^4|0)$$

$$+\cdots$$

The first order term ~~can be corresponds to Feynman-Dyson diagram~~ to the graph: gives no contribution vanishes, of course, because ~~energy-momentum~~ there's no matrix elements corresponding to $(0,0)$. The second order term ~~can be~~ corresponds to the following Feynman-Dyson graph:



It order make the ~~mean~~ correspondence between the graph and the formula. expression more precise, let us write:

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$$\varphi(x) = \varphi^{(-)}(x) + \varphi^{(+)}(x)$$

$$\varphi^{(-)}(x) = \sum_{\boldsymbol{k}} \sqrt{\frac{1}{2k_0 V}}\, a_{\boldsymbol{k}}\, e^{i(\boldsymbol{k}\boldsymbol{x} - k_0 x_0)}$$

$k_0 = +\sqrt{\boldsymbol{k}^2 + \varkappa^2}$

$$\varphi^{(+)}(x) = \sum_{\boldsymbol{k}} \sqrt{\frac{1}{2k_0 V}}\, a_{\boldsymbol{k}}^{*}\, e^{-i(\boldsymbol{k}\boldsymbol{x} - k_0 x_0)}$$

negative frequency part $\varphi^{(-)}$
→ annihilation operator
positive f.p. $\varphi^{(+)}$
→ creation operator
Only $\varphi^{(+)}$ gives non-zero results when applied to the vacuum state.

$$\frac{1}{\sqrt{2p_0 V}} \cdot \frac{1}{\sqrt{p_0 + M}} \qquad p_+^2 + p^2 = p_0^2 + 2p_0 M + M^2 + p^2 = 2p_0(p_0 + M)$$

$$\psi(x) = \sum_{\boldsymbol{p}} \frac{1}{\sqrt{V(p_+^2 + p^2)}} \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - M\right)\left(\psi_{\boldsymbol{p}}^{(-)}(x) + \psi_{\boldsymbol{p}}^{(+)}(x)\right)$$

$p_0 = +\sqrt{p^2 + M^2}$, $P = p_0 + M$

$$\psi_{\boldsymbol{p}}^{(-)}(x) = \begin{pmatrix} b_{\boldsymbol{p}}^{(1)} \\ b_{\boldsymbol{p}}^{(2)} \\ 0 \\ 0 \end{pmatrix} e^{i(\boldsymbol{p}\boldsymbol{x} - p_0 x_0)}$$

$$\psi_{\boldsymbol{p}}^{(+)}(x) = \begin{pmatrix} 0 \\ 0 \\ c_{\boldsymbol{p}}^{(1)*} \\ c_{\boldsymbol{p}}^{(2)*} \end{pmatrix} e^{-i(\boldsymbol{p}\boldsymbol{x} - p_0 x_0)}$$

$$[b_{\boldsymbol{p}}^{(j)}, b_{\boldsymbol{p}'}^{(k)}]_+ = \delta_{\boldsymbol{p}\boldsymbol{p}'} \delta_{jk}$$
$$[b_{\boldsymbol{p}}^{(j)}, c_{\boldsymbol{p}'}^{(k)}]_+ = 0$$
etc

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or $\psi(x) = \sum_{p_\mu} \frac{1}{\sqrt{V(p^2+p^2)}} (\gamma_\mu p_\mu + M) \psi_{p_\mu} e^{i p_\mu x_\mu}$

$p_\mu p_\mu + M^2 = 0$

where

$$\psi_p = \begin{pmatrix} b^{(1)}_p \\ b^{(2)}_p \\ 0 \\ 0 \end{pmatrix} \quad \text{for } -ip_4 = p_0 > 0$$

$$\psi_p = \begin{pmatrix} 0 \\ 0 \\ c^{(1)*}_p \\ c^{(2)*}_p \end{pmatrix} \quad \text{for } -ip_4 = p_0 < 0$$

$$\psi(x) = \psi^{(-)(p)}(x) + \psi^{(+)(p)}(x)$$

$$\psi^{(-)(p)}(x) = \sum_{p_0} \frac{1}{\sqrt{V(p^2+p^2)}} (i\gamma_k p_k - \gamma_4 p_0 - M)(i\gamma_\mu p_\mu - M) \psi^{(-)(p)}_{p_0} \times e^{i(p_0 x - p_0 x_0)}$$

$$\psi^{(+)(q)}(x) = \sum_{p_0} \frac{1}{\sqrt{V(p^2+p^2)}} (-i\gamma_k p_k + \gamma_4 p_0 - M)(-i\gamma_\mu p_\mu - M) \psi^{(+)(q)}_{p_0} \times e^{-i(p_0 x - p_0 x_0)}$$

$\psi^{(-)(p)}(x)$ designates the annihilation operator for the particle.

$\psi^{(+)(q)}(x)$: creation operator for the anti-particle.

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IV. 3

$$\psi^{\dagger}(x) = \psi^{(-)(a)}(x) + \psi^{(+)(p)}(x)$$

$$\psi^{(-)(a)} \approx \sum_{p} \frac{1}{\sqrt{V(p^2+p^2)}} \psi_{p}^{(-)(a)} (i\gamma_k \tilde{p}_k + i\gamma_4 p_0 - M) \times e^{i(pr - p_0 x_0)}$$

$\tilde{p}_4 = -p_4$

$$\psi^{(+)(p)} \approx \sum_{p} \frac{1}{\sqrt{\quad}} \psi_{p}^{(+)(p)} (-i\gamma_m \tilde{p}_m - M) \times e^{-i(pr - p_0 x_0)}$$

$$(-i\gamma_k p_k + i\gamma_4 p_0 - M)$$

$$\psi_{p}^{(-)(p)} = \begin{pmatrix} b_{p}^{(1)} \\ b_{p}^{(2)} \\ 0 \\ 0 \end{pmatrix} \qquad \psi_{p}^{(+)(a)} \approx \begin{pmatrix} 0 \\ 0 \\ c_{p}^{(1)*} \\ c_{p}^{(2)*} \end{pmatrix}$$

$$\psi_{p}^{(+)(p)} = (b_{p}^{(1)*}, b_{p}^{(2)*}, 0, 0)$$

$$\psi_{p}^{(-)(a)} = (0, 0, c_{p}^{(1)}, c_{p}^{(2)})$$

$$\bar{\psi}(x) = \bar{\psi}^{(-)(a)}(x) + \bar{\psi}^{(+)(p)}(x)$$

$$\bar{\psi}^{(-)(a)} = \psi^{(-)(a)} \gamma_4$$

$$\bar{\psi}^{(+)(p)} = \psi^{(+)(p)} \gamma_4$$

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$$-\frac{g^2}{2}\langle 0|\int\int T(\varphi\bar{\psi}\psi(x)\,\psi(x),\ \varphi(x')\bar{\psi}(x')\psi(x'))|0\rangle$$
$$(dx)^4(dx')^4$$

$$\simeq -g^2\langle 0|\int_{-\infty}^{+\infty}dx_0\int_{-\infty}^{+\infty}dx_0'\int\cdots\int(dx)^3(dx')^3$$
$$\times\varphi(x)\bar{\psi}(x)\psi(x)\,\theta(x-x')$$
$$\times\varphi(x')\bar{\psi}(x')\psi(x')\,|\,0\rangle$$

$$\theta(x) = 1 \quad \text{for } x_0>0$$
$$= 0 \quad \text{for } x_0<0$$

Non-zero ~~matrix element~~ contribution to the matrix element $\langle 0|S_2|0\rangle$ comes from the terms

$$\langle 0|\,\varphi^{(-)}(x)\bar{\psi}_\alpha^{(-)(a)}(x)\,\psi_\alpha^{(-)(p)}(x)\,\varphi^{(+)}(x')\,\bar{\psi}_\beta^{(+)(p)}(x')\,\psi_\beta^{(+)(a)}(x')\,|0\rangle$$

$\theta(x-x')$

x

$$=\theta(x-x')\{\langle 0|\varphi^{(-)}(x)\varphi^{(+)}(x')|0\rangle$$
$$\times\langle 0|\bar{\psi}_\alpha^{(-)(a)}(x)\,\psi_\beta^{(+)(a)}(x')|0\rangle$$
$$\times\langle 0|\psi_\alpha^{(-)(p)}(x)\,\bar{\psi}_\beta^{(+)(p)}(x')|0\rangle\}$$

x'

"There is another ~~term~~ strange term of the form

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$$\theta(x-x')\{(0|\varphi^{(-)}(x)\varphi^{(+)}(x')|0)$$
$$\times(0|\bar{\psi}^{(-)(a)}(x)\psi^{(+)a}(x)|0)\times(0|\psi^{(-)(a)}(x')\psi^{(+)(a)}(x')|0)\},$$

but the latter will compensate with a ~~Minus~~ similar term, if we take anti-symmetrized interaction

$$\frac{1}{2}g\varphi(x)\{\bar{\psi}(x)\psi(x)-\psi(x)\bar{\psi}(x)\}$$

Now the vacuum expectation value

$$(0|\varphi^{(-)}(x)\varphi^{(+)}(x')|0)$$
$$=(0|\sum_k\sqrt{\frac{1}{2k_0V}}a_k e^{i(kr-k_0x_0)}\cdot\sum_{k'}\sqrt{\frac{1}{2k_0'V}}a_{k'}^{*}$$
$$e^{-i(k'r'-k_0'x_0')}|0)$$
$$=\sum_k\frac{1}{2k_0V}e^{i\{k(r-r')-k_0(x_0-x_0')\}}$$
$$=\frac{1}{(2\pi)^3}\iiint\frac{e^{i\{k(r-r')-k_0(x_0-x_0')\}}}{2k_0}(dk)^3$$

We introduce

$$\Delta^{(1)}(x)=\frac{1}{(2\pi)^3}\iiint\frac{e^{i(kr-k_0x_0)}+e^{-i(kr-k_0x_0)}}{2k_0}\times(dk)^3$$

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$$= \frac{1}{(2\pi)^3} \int\cdots\int_{-\infty}^{+\infty} e^{ik_\mu x_\mu} \delta(k_\mu k_\mu + \kappa^2)\, (dk)^3 (dk_0)$$

whereas, we define Δ

$$\Delta(x) = \frac{1}{(2\pi)^3 i} \iiint \frac{e^{i(\mathbf{kr} - k_0 x_0)} - e^{-i(\mathbf{kr} - k_0 x_0)}}{2k_0} \times (dk)^3$$

$$= \frac{1}{(2\pi)^3 i} \int\cdots\int_{-\infty}^{+\infty} e^{ik_\mu x_\mu} \varepsilon(k)\, \delta(k_\mu k_\mu + \kappa^2) \times (dk)^3 (dk_0)$$

thus

$$\Delta^{(\pm)}(x) = \frac{1}{(2\pi)^3} \iiint \frac{e^{i(\mathbf{kr} \mp k_0 x_0)}}{2k_0} (dk)^3$$

$$= \frac{1}{2}\left(\Delta^{(1)}(x) \pm i\,\Delta(x)\right)$$

or $\langle 0 | \varphi^{(-)}(x)\, \varphi^{(+)}(x') | 0 \rangle = \Delta^{(+)}(x - x')$

or $\langle 0 | \varphi(x)\, \varphi(x') | 0 \rangle = \Delta^{(+)}(x - x')$

incidentally, we have

$\langle 0 | P(\varphi(x), \varphi(x')) | 0 \rangle$

$= \Delta^{(+)}(x - x') \quad x_0 > x_0'$

$= \Delta^{(+)}(x' - x) \quad x_0 < x_0'$

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(N.4)'

$$\langle 0|P(\varphi(x),\varphi(x'))|0\rangle = \frac{1}{2}\Delta_F(x-x')$$

$$= \Delta^{(-)}(x-x') \quad \text{for } x > x'$$

$$= \Delta^{(-)}(x'-x) \quad \text{for } x < x'$$

$$= \frac{1}{2}\{1+\varepsilon(x,x')\}\Delta^{(-)}(x-x')$$

$$+\frac{1}{2}\{1-\varepsilon(x,x')\}\Delta^{(-)}(x'-x)$$

$$= \frac{1}{2}\left(\Delta^{(-)}(x-x')+\Delta^{(-)}(x'-x)\right)$$

$$+\frac{1}{2}\varepsilon(x,x')\{\Delta^{(-)}(x-x')-\Delta^{(-)}(x'-x)\}$$

$$= \frac{1}{2}\{\Delta^{(1)}(x-x')$$

$$+i\varepsilon(x-x')\Delta(x-x')\}$$

$$= \frac{1}{2}(\Delta^{(1)}-2i\overline{\Delta})$$

$$\Delta_F(x) = \Delta^{(1)}-2i\overline{\Delta}$$

$$= \Delta^{(1)} - i(\Delta^{(ret)}+\Delta^{(adv)})$$

~~Δ_F is the ~~

Δ_F is the function, which ~~derivat~~ reproduces the propagation of scalar wave associated with the virtual scalar (neutral) particle.

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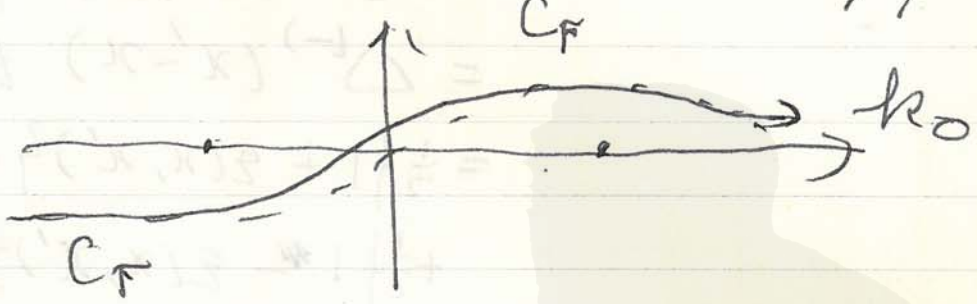
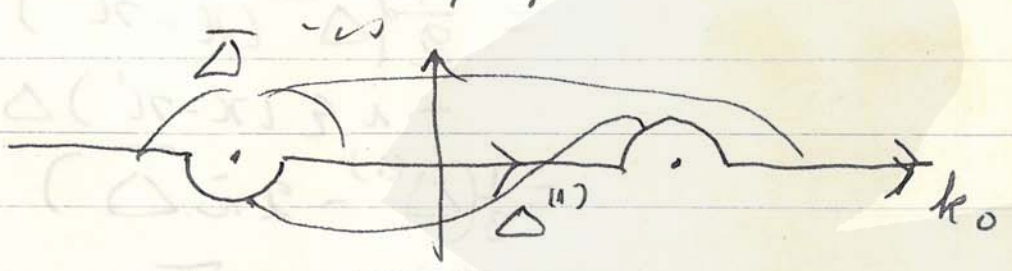
$$D_F(x) = \frac{1}{(2\pi)^3} \frac{1}{\pi i} \int e^{i k_\mu x^\mu} \frac{(dk)^4}{k_\mu k_\mu + \varkappa^2}$$



$$\int \frac{dk_0\, e^{i k_\mu x_\mu}}{k_\mu k_\mu + \varkappa^2 - i\varepsilon}$$



$$k_0 = \pm\sqrt{k^2 + \varkappa^2 - i\varepsilon}$$

$$\omega = \begin{cases} +\sqrt{k^2+\varkappa^2}\left(1 - \dfrac{i\varepsilon}{2\sqrt{k^2+\varkappa^2}}\right) \\ -\sqrt{k^2+\varkappa^2}\left(1 - \dfrac{i\varepsilon}{2\sqrt{k^2+\varkappa^2}}\right) \end{cases}$$

$$(0 \mid P(\varphi(x), \varphi(x')) \mid 0) = \frac{1}{2} \Delta_F(x - x')$$

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(IV, 5)

Similarly

$$\langle 0 | \bar{\psi}_\alpha^{(-)(a)}(x) \psi_\beta^{(+)(a)}(x') | 0 \rangle$$

$$= \langle 0 | \sum_{p} \frac{1}{\sqrt{V(p^2+p^2)}} (\psi_p^{(-)(a)})_\sigma (i\gamma_\mu p_\mu - M)_{\sigma\rho} (\gamma_4)_{\rho\alpha}$$

$$\times e^{i(px - p_0 x_0)} \times \sum_{p'} \frac{1}{\sqrt{V(p'^2+p'^2)}} (-i\gamma_\mu p'_\mu - M)_{\beta\tau} (\psi_{p'}^{(+)(a)})_\tau$$

$$\times e^{-i(p'x' - p'_0 x'_0)} | 0 \rangle$$

$$\psi_p^{(+)(a)} = \begin{pmatrix} 0 \\ 0 \\ c_p^{(1)*} \\ c_p^{(2)*} \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\psi_p^{(-)(a)} = (0, 0, c_p^{(1)}, c_p^{(2)})$$

$$= \sum_{p} \frac{1}{V(p^2+p^2)} \sum_{\sigma=3,4} (i\gamma_\mu p_\mu - M)_{\sigma\rho} (\gamma_4)_{\rho\alpha}$$

$$\times (-i\gamma_\mu p_\mu - M)_{\beta\sigma} e^{i p_\mu (x_\mu - x'_\mu)}$$

$$\sum_{\sigma=3,4} (i\gamma_\mu p_\mu - M)_{\sigma\rho} (\gamma_4)_{\rho\alpha} (-i\gamma_\mu p_\mu - M)_{\beta\sigma}$$

$$= \left\{ (-i\gamma_\mu p_\mu - M) \frac{1-\gamma_4}{2} (i\gamma_\mu p_\mu - M) \gamma_4 \right\}_{\beta\alpha}$$

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$$= -(-i\gamma_\mu \partial_\mu - M)_{\beta\beta} (M + \gamma_4 p_0)_{\beta\alpha}$$

In any case,
$(0 | \psi^{(-)(a)}_\alpha(x) \cdot \psi^{(+)(a)}_\beta(x') | 0)$
has only negative frequency part with respect to $x_0 - x_0'$ or positive freq. part with respect to $x_0' - x_0$.
If we construct, on the other hand,
$(0 | \psi^{(-)(p)}_\beta(x') \bar{\psi}^{(+)(p)}_\alpha(x) | 0)$,
it contains only negative frequency part with respect to $x_0' - x_0$, or only positive frequency part with respect to $x_0 - x_0'$.

Now, from the commutation relation
$$\psi_\alpha(x) \cdot \bar{\psi}_\beta(x') + \bar{\psi}_\beta(x') \psi_\alpha(x) = -i S_{\alpha\beta}(x - x')$$
we obtain
$$(0 | \psi_\alpha(x) \bar{\psi}_\beta(x') + \bar{\psi}_\beta(x') \psi_\alpha(x) | 0) = -i S_{\alpha\beta}(x - x')$$
since the r.h.s. is a ordinary number.

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IV. 6

$$\langle 0 | \psi_\alpha^{(+)(p)}(x)\,\bar{\psi}_\beta^{(+)(p)}(x') + \bar{\psi}_\beta^{(-)(a)}(x')\,\psi_\alpha^{(+)(a)}(x) | 0 \rangle$$
$$= -i S_{\alpha\beta}(x - x')$$

or interchanging α, β and x, x',

$$\langle 0 | \psi_\beta^{(+)(p)}(x')\,\bar{\psi}_\alpha^{(+)(p)}(x) + \psi_\alpha^{(-)(a)}(x)\,\psi_\beta^{(+)(a)}(x') | 0 \rangle$$
$$= -i S_{\beta\alpha}(x' - x).$$

$$S_{\beta\alpha}(x'-x) = \left(\gamma_\mu \frac{\partial}{\partial x'_\mu} - M\right)_{\beta\alpha} \Delta_M(x'-x)$$

$$\Delta_{M(x)}(x'-x) = \frac{1}{(2\pi)^3 i} \iiint \frac{e^{i(\mathbf{p}\mathbf{r} - p_0 x_0)} - e^{-i(\mathbf{p}\mathbf{r} - p_0 x_0)}}{2 p_0} (d\mathbf{p})^3$$

$$= \frac{-i}{} \left(\Delta_M^{(-)}(x) - \Delta_M^{(+)}(x)\right)$$

$$S_{\beta\alpha}(x'-x) = -i\left(S_{\beta\alpha}^{(-)}(x'-x) - S_{\beta\alpha}^{(+)}(x'-x)\right)$$

$$S^{(\pm)}(x) = \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - M\right) \Delta_M^{(\pm)}(x)$$

$$\langle 0 | \psi_\beta^{(-)(p)}(x')\,\bar{\psi}_\alpha^{(+)(p)}(x) + \psi_\alpha^{(-)(a)}(x)\,\psi_\beta^{(+)(a)}(x') | 0 \rangle$$
$$= -S_{\beta\alpha}^{(-)}(x'-x) + S_{\beta\alpha}^{(+)}(x'-x)$$

First term on l. h. s. contains only neg. freq. part with respect to $x_0' - x_0$,

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which must be equal to $-S^{(-)}_{\beta\alpha}(x'-x)$,
while the second term on R.h.s.
contains only pos. ν's $\psi^{(+)}$ which
must be equal to $S^{(+)}_{\beta\alpha}(x'-x)$:

$$(0|\psi^{(-)(p)}_\beta(x')\overline{\psi^{(+)(p)}_\alpha}(x)|0) = -S^{(-)}_{\beta\alpha}(x'-x)$$

or

$$(0|\psi^{(-)(p)}_\alpha(x)\overline{\psi}^{(+)(p)}_\beta(x')|0) = -S^{(-)}_{\alpha\beta}(x-x')$$

$$(0|\overline{\psi}^{(-)(a)}_\alpha(x)\psi^{(+)(a)}_\beta(x')|0) = S^{(+)}_{\beta\alpha}(x'-x)$$

$$= -S_{\beta\alpha}(x+x')$$

or

$$(0|\psi_\alpha(x)\overline{\psi}_\beta(x')|0) = -S^{(-)}_{\alpha\beta}(x-x') \quad **$$

Now, $\Delta^{(-)}$ and $\Delta^{(+)}$ functions are singular on the light cone. In order to see what kind of singularities they have on the light cone, let us investigate more in detail the behavior of $\Delta^{(+)}$ and $\Delta^{(-)}$ functions. We obtained the result already for Δ-fct (see Q.M. course):

** $(0|\overline{\psi}_\alpha(x)\psi_\beta(x')|0) = -S^{(-)}_{\beta\alpha}(x-x')$

$(0|\psi_\beta(x')\psi_\alpha(x)|0) = -S'^{(-)}_{\alpha\beta}(x'-x)$

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(IV.6)

$$(0|\Psi_\alpha(x)\bar{\Psi}_\beta(x')|0)$$

$$=(0|\psi_\alpha^{(-)(p)}(x)\psi_\beta^{(+)(p)}(x')|0)$$

$$=-S^{(-)}_{\alpha\beta}(x-x')$$

$$(0|P(\psi_\alpha(x),\bar{\psi}_\beta(x'))|0)$$

$$=-S^{(-)}_{\alpha\beta}(x-x') \quad \text{for } x>x'$$

$$-S^{(-)}_{\alpha\beta}(x'-x) \quad \text{for } x<x'$$

$$(0|\varepsilon(x,x')P(\psi_\alpha(x),\bar{\psi}_\beta(x'))|0)$$

$$=-\frac{1}{2}(1+\varepsilon(x-x'))S^{(-)}_{\alpha\beta}(x-x')$$

$$+\frac{1}{2}(1-\varepsilon(x-x'))S^{(-)}_{\alpha\beta}(x'-x)$$

$$=\cancel{-\frac{1}{2}(1+\varepsilon(x-x'))}$$

$$\frac{1}{2}(S^{(-)}_{\alpha\beta}(x'-x)-S^{(-)}_{\alpha\beta}(x-x'))$$

$$-\frac{1}{2}(S^{(-)}_{\alpha\beta}(x'-x)+S^{(-)}_{\alpha\beta}(x-x'))$$

$$=-\frac{1}{2}\{S^{(1)}_{\alpha\beta}(x-x')-2i\bar{S}_{\alpha\beta}(x-x')\}$$

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$$= \frac{1}{2} \left(S_{F\alpha\beta}(x - x') \right)_{\alpha\beta}$$

$$\omega \left(0 \mid P\left(\psi_\alpha(x), \overline{\psi}_\beta(x') \right) \mid 0 \right)$$

$$= -\frac{1}{2} \varepsilon(x, x') \left(S_F(x - x') \right)_{\alpha\beta}$$

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T. H.

$$\Delta(x) = \frac{1}{s}\frac{dF(s)}{ds} \quad \text{inside the lightcone}$$
$$= 0 \quad \text{outside}$$

$s = \sqrt{-x_\mu x_\mu}$ for time-like x

~~$s = \sqrt{+x_\mu x_\mu}$ for space-like x~~

$F(s) = -\frac{1}{4\pi} J_0(\kappa s)$ for future-like x.

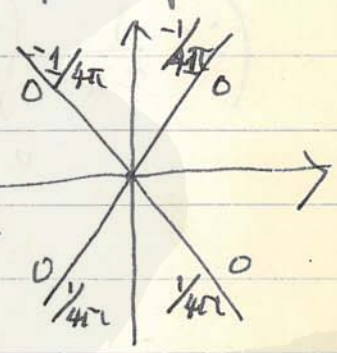
$F(s) = 0$

$F(s) = \frac{1}{4\pi} J_0(\kappa s)$ for past-like x.

$\left(J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \cdots \text{ for small } x \right)$

$\int \Delta(x)\, s\, ds = -\frac{1}{4\pi}\delta(-x_\mu x_\mu)$

across the lightcone $= \frac{1}{4\pi} - 1/4\pi + \cdots$



or

$\frac{1}{2}\int \Delta(x)\, ds^2 = -1/4\pi + \cdots$

or $\Delta(x) = -\frac{1}{4\pi}\delta(-x_\mu x_\mu) + \cdots = -\frac{1}{4\pi}\delta(x_\mu x_\mu) + \cdots$

for future-like x.

$\Delta(x) = +\frac{1}{4\pi}\delta(x_\mu x_\mu) + \cdots$

for past-like x.

$(\ \Delta(0) = 0\ )$

$\Delta^{(1)}(x) = \frac{1}{2\pi^2 x^2} + \frac{\pi^2}{4\pi^2}\left(\log \frac{\gamma}{2}\kappa |x^2|^{1/2} - \frac{1}{2}\right)$

+ finite terms.

$\gamma = 1.781\cdots$

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Schwinger, 25 (1949), 651.
Pauli, R. M. P.

At the origin $\Delta^{(1)}(0)$ is indeterminate

$$\Delta^{(1)}(0) = \frac{1}{(2\pi)^3}\int d\boldsymbol{k}\int \delta(k_0^2 - k^2 - \varkappa^2)\,dk_0$$

$$= \frac{1}{(2\pi)^3}\int \frac{d\boldsymbol{k}}{\sqrt{k^2+\varkappa^2}} = \frac{1}{4\pi^2}\lim_{k\to\infty}\left\{k^2 - \varkappa^2\left(\log\frac{2k}{\varkappa} - \frac{1}{2}\right)\right\}$$

$$\left(\because\ \frac{1}{2\pi^2}\int_0^\infty \frac{k^2\,dk}{\sqrt{k^2+\varkappa^2}} = \frac{1}{2\pi^2}\left\{\int_0^\infty k\,dk - \int_0^\infty \frac{(\sqrt{k^2+\varkappa^2}-k)\,k\,dk}{\sqrt{k^2+\varkappa^2}}\right.\right.$$

$$\left.\left.\int_0^\infty \frac{\sqrt{k^2+\varkappa^2}-k}{\sqrt{k^2+\varkappa^2}}k\,dk = \int_\varkappa^\infty \frac{\varkappa^2}{2\varkappa}\,d\varkappa + \text{finite term}\right.\right.$$

$$\sqrt{k^2+\varkappa^2} = \varkappa \qquad 2k\,dk = 2\varkappa\,d\varkappa$$

$$k = \sqrt{\varkappa^2-\varkappa^2} = \varkappa\left(1-\frac{\varkappa^2}{2\varkappa^2}\right) - O\left(\frac{1}{\varkappa^3}\right)$$

$$\left.= \frac{\varkappa^2}{2}\log\frac{k}{\varkappa} + \text{finite term}\right)$$

In general,

$$\Delta^{(1)}(0) = \lim_{k\to\infty}\left(c_1 k^2 + c_2\varkappa^2\log\frac{k}{\varkappa} + c_3\varkappa^2\right)$$

c_1, c_2, c_3 : constants which depend on the procedure of integration, but independent of $\varkappa^2$.

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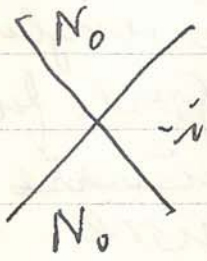
$$N_0(x) = \frac{2}{\pi}\log\frac{x}{2} - \gamma H_0^{(1)}(ix) - iN_0(x)$$

$$\Delta^{(1)}(x) = \frac{1}{s}\frac{dF^{(1)}(x)}{ds}$$

$$F^{(1)}(x) = \begin{cases} \frac{1}{4\pi}N_0(\kappa s) & s = \sqrt{x_0^2 - r^2} \\ -\frac{i}{4\pi}H_0^{(1)}(i\kappa s) & s = \sqrt{r^2 - x_0^2} \end{cases} \quad \begin{matrix} \text{for } |x_0| > r \\ \text{for } r > |x_0| \end{matrix}$$



where N_0 is Neumann function, $H_0^{(1)}$ is first Hankel function, which have both log-singularity on light cone, so that $\Delta^{(1)}(x)$ has singularity proportional to $1/x^2$.

Now, ~~what we~~ back to the expression

$$\langle 0|\int\!\!\int \theta(x-x')\,\varphi(x)\bar{\psi}(x)\psi(x)\cdot\varphi(x')\bar{\psi}(x')\psi(x')|0\rangle \frac{(dx)^4(dx')^4}{S}$$

$$= \int\!\!\int \theta(x-x')\langle 0|\varphi(x)\varphi(x')|0\rangle\langle 0|$$
$$\times \langle 0|\psi_\alpha^{(-)(a)}(x)\psi_\beta^{(+)(a)}(x')|0\rangle$$
$$\times \langle 0|\psi_\alpha^{(-)(\beta)}(x)\psi_\beta^{(+)(\beta)}(x')|0\rangle (dx)^4(dx')^4$$

$$= \int\!\!\int \theta(x-x')\cdot\Delta^{(-)}(x-x')\,S_{\beta\alpha}^{(-)}(x-x')\,S_{\alpha\beta}^{(-)}(x-x')(dx)^4(dx')^4$$

$$= \int\!\!\int \theta(\xi)\Delta^{(-)}(\xi)S_{\beta\alpha}^{(-)}(\xi)S_{\alpha\beta}^{(-)}(\xi)(dx)(d\xi)^4$$

finite region or δ-function

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where

$$\left.\begin{aligned} X &= \tfrac{1}{2}(x+x') \\ \xi &= x-x' \end{aligned}\right\}$$

$$(dX)^4(d\xi)^4 = (dx)^4(dx')^4$$

This is a strongly divergent integral. However, from the following ~~procedure~~ version, we can ~~eliminate~~ ignore such a divergence.

$(j|S|k)$ $\quad j=k=0$ vacuum

$$(0|S|0) = 1 + (0|S_2|0)t \cdots$$

$$(j|S|0) = (0|S|k) = 0$$

for $j\neq 0$ and $k\neq 0$.

$$\int\!\!\int \varphi(x)\bar{\psi}(x')\psi(x') \cdots \varphi(x^{(n)})\bar{\psi}(x^{(n)})\psi(x^{(n)})$$

$$\times (dx)^4 \cdots (dx^{(n)})^4$$

~~each factor~~.

$$\left.\begin{aligned} X &= \frac{1}{n}(x'+x''+\cdots+x^{(n)}) \\ \xi^{(1)} &= x'-x'' \\ \xi^{(2)} &= x''-x''' \\ &\vdots \\ \xi^{(n-1)} &= x^{(n-1)}-x^{(n)} \end{aligned}\right\}$$

$$x' = X + \frac{1}{n}\{(x''-x'')+(x'''-x''')+\cdots(x^{(n)}-x^{(n)})\}$$

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$$x' = X + \frac{1}{n}\{\xi^{(1)} + (\xi^{(1)} + \xi^{(2)}) + (\xi^{(1)} + \xi^{(2)} + \xi^{(3)}) + \cdots + (\xi^{(1)} + \cdots + \xi^{(n-1)})\}$$

$$x'' = X + \frac{1}{n}\{(x'' - x') + (x''' - x'') + \cdots (x^{(n)} - x^{(n)})\}$$

$$= X + \frac{1}{n}\{-\xi^{(1)} + \xi^{(2)} + (\xi^{(2)} + \xi^{(3)}) + \cdots + (\xi^{(2)} + \cdots + \xi^{(n-1)})\}$$

$$\vdots$$

$$x^{(n)} = X + \frac{1}{n}\{(x^{(n)} - x') + (x^{(n)} - x'') + \cdots + (x^{(n)} - x^{(n-1)})\}$$

$$= X + \frac{1}{n}\{-(\xi^{(1)} + \cdots \xi^{(n-1)}) + \cdots - (\xi^{(2)} + \cdots + \xi^{(n-1)}) + \cdots + (-\xi^{(n-1)})\}$$

$$g^n \int\cdots\int \varphi(x')\,\bar{\psi}(x')\,\psi(x') \cdots \varphi(x^{(n)})\,\bar{\psi}(x'')\,\psi(x^{(n)}) \times (\text{const})(dX)^4 (d\xi'')^4 \cdots (d\xi^{(n-1)})^4$$

$$e^{\pm i k x} e^{\pm i p' x} e^{\pm i q' x} \cdots$$

$$e^{i\{\sum_{\alpha=1}^{n}(\pm k^{(\alpha)}) + \sum_j (\pm p^{(\alpha)}) + \sum_j (\pm q^{(\alpha)})\} X}$$

$$\int e^{i\{\sum_{\alpha=1}^{n}(\pm k_0^{(\alpha)}) + \sum_j (\pm p^{(\alpha)}) + \sum (\pm q^{(\alpha)})\} X_0} dX_0$$

$\neq 0$ only for $\sum_{\alpha=1}^{n}(\quad) = 0$

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Now ~~the state~~ the total energy of the state j is equal to

$$\text{const.} \sum_{\alpha=1}^{n} (\cdot\cdot)$$

and if $j \neq 0$, total energy is positive and greater than 0. Thus, it follow immediately that

$$(j \neq S | 0) = 0,$$

and similarly

$$(0 | S | k) = 0$$

	0 , 1 , 2 , . . .
0	$(0\|S\|0)$ 0 0 0 —
1	0 ⋮
2	0 ⋮
⋮	0 $(p\|S\|p)$ 0 0 ⋯
⋮	0 ⋮
1	0 ⋮
⋮	⋮ ⋱

~~As we expected~~ This shows that ~~the vacuum~~ there's no transition between vacuum and the states, in which there are one or more particles. The only trouble is that $(0|S|0)$ is not 1, but infinity or

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(vacuum fluctuation)
, at least, indeterminate, owing to our expectation.

However, let us forget this for the moment and proceed further. Among states other than vacuum, the simplest ones are those in which there is only one particle of a certain type. Let us take a state in which there is only one ~~particle~~ spinless particle with the momentum p etc (and energy $p_0 c = \sqrt{p^2 + M^2}\cdot c$ etc.

$(p|S|p) \neq 0$ and all other elements $(p'|S|p) = 0$ etc, because total energy and momentum cannot be conserved simultaneously between the state p and any other state in our case. (We exclude the case, in which a particle can decay spontaneously into two or three particles, such as π-meson decay or neutron β-decay.) Reason is (similar to/~~the same~~ as) the cancellation of $(j|S|0)$.

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From physical standpoint, we expect ~~that~~ again that

$$(p|S|p) = 1,$$

but this is not ~~true~~ only false, true. More than that it turns out to be infinite due to the appearance of infinite self-energy.

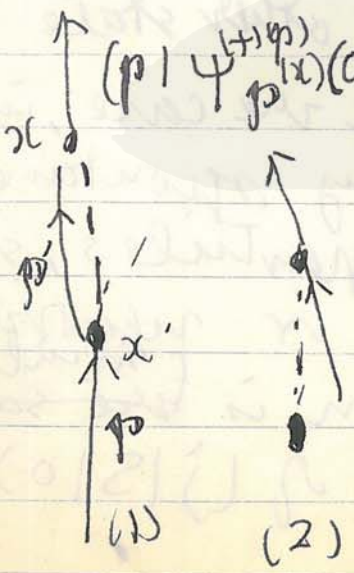
$$(p|\int \theta(x-x')\varphi(x)\bar{\psi}(x)\psi(x)\varphi(x')\bar{\psi}(x')\psi(x') \times (dx)^4(dx')^4|p)$$

$$= \frac{1}{2}\iint \theta(x-x')(0|\varphi(x)\varphi(x')|0) \times (p|\bar{\psi}(x)\psi(x)\bar{\psi}(x')\psi(x')|p)(dx)^4 \times (dx')^4$$

$$(p|\cdot\begin{pmatrix}\overset{(+)(p)}{\psi}(x)\\ +\overset{(-)(a)}{\psi}(x)\end{pmatrix}\begin{pmatrix}\overset{(-)(p)}{\psi}(x)\\ +\overset{(+)(a)}{\psi}(x)\end{pmatrix}\begin{pmatrix}\overset{(+)(p)}{\psi}(x')\\ +\overset{(-)(a)}{\psi}(x')\end{pmatrix}\begin{pmatrix}\overset{(-)(p)}{\psi}(x')\\ +\overset{(+)(a)}{\psi}(x')\end{pmatrix}|p)$$

$$(p|\overset{(+)(p)}{\psi_p}(x)(0|\sum_{p'}\overset{(-)(p')}{\psi_{p'}}(x)\overset{(+)(p')}{\psi_{p'}}|0)\overset{(-)(p)}{\psi_p}(x')|p) \quad (1)$$

$$\overset{(+)(p)}{\psi_p}(x)\overset{(-)(p)}{\psi_p}(x')\overset{(-)(a)}{\psi_{p'}}(x')\overset{(+)(a)}{\psi_{p'}}(x') \quad (2)$$



~~because of conservation of momentum or it is to be compensated by $\bar{\psi}$ is to be compensated~~

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†

$$= \sum_{p,p'} \psi^{(-)(a)}_{p'}(x)\psi^{(+)(a)}_{p'}(x')\,|p,0\rangle + |0,0\rangle\,\psi^{(-)(p)}_{p}(x)\psi^{(+)(p)}_{p}(x')\,|p,0\rangle$$

IV. 11.

by a-w

$$\sum_{p,p'} \psi^{(+)(a)}_{p'}(x)\,\psi^{(+)(a)}_{p'}(x')\,\psi^{(+)(p)}_{p}(x')\,\psi^{(-)p}_{p}(x') \qquad (3)$$

will disappear, if we take anti-symm.

$$\langle p|\sum_{p''=p} \psi^{(-)(a)}_{p''}(x)\,\psi^{(-)(p)}_{p'}(x)\,\psi^{(+)(p)}_{p'}(x')\,\psi^{(+)(a)}_{p''}(x')\,|p\rangle \qquad (4)$$

$$= \langle 0|\sum_{p',p''} \qquad\qquad |0\rangle$$

$$= \langle p|\sum_{p',p''} \psi^{(-)(a)}_{p''}(x)\psi^{(-)(p)}_{p''}(x)\,\psi^{(+)(p)}_{p'}(x')\,\psi^{(+)(a)}_{p'}(x')\,|p\rangle$$

$$= \quad ,,$$

$$= \langle 0|\psi^{(-)(a)}_{\alpha}(x)\,\psi^{(+)(a')}_{\alpha'}(x')|0\rangle\langle\alpha|\psi^{(-)(p)}_{\alpha}(x)\,\psi^{(+)(p)}_{\alpha}(x')|\beta\rangle$$

(1) + (4)

(1):

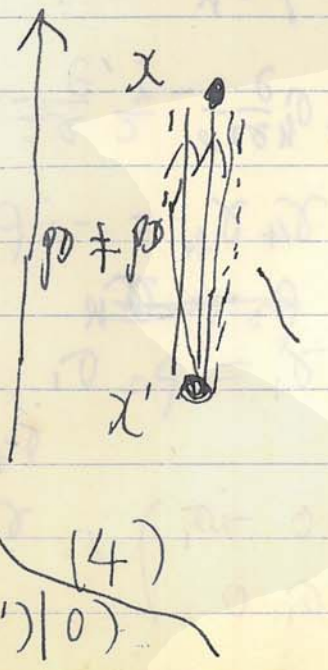
$$\rightarrow \langle\alpha|\sum_{\alpha'}\psi^{(+)(a)}_{\alpha'}(x)\,\psi^{(-)(b)}_{\alpha}(x)\,\psi^{(+)(b)}_{\alpha'}(x')\,|\alpha\rangle$$

$$\times \langle\alpha|(x')\psi_{\alpha}\,|\alpha\rangle$$

$$= \langle\alpha|\sum_{\alpha'}\psi^{(+)(a)}_{\alpha'}(x)\,\psi^{(+)(b)}_{\alpha'}(x')\,|\alpha\rangle$$

$$= \langle\alpha|(x')\psi_{\alpha}\,\psi^{(+)(a)}_{\alpha}(x)\,|\alpha\rangle$$

$$\times \langle 0|\sum_{\alpha'}\psi^{(+)(a)}_{\alpha'}(x)\,\psi^{(+)(a')}_{\alpha'}(x')|0\rangle$$



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$$p^2 + p^2 = (p_0 + M)^2 + p^2$$

$$= p_0^2 + 2p_0M + M^2 + p^2 = 2p_0(M + p_0)$$

$$\psi_p^{(-)(p)}(x) = \frac{1}{\sqrt{V \cdot 2p_0(M+p_0)}} (i\gamma_\mu p_\mu - M)\, \psi_{p0}^{(-)(p)} \times e^{i p_\mu x_\mu}$$

$$\psi_{p_0}^{(+)(p)}(x) = \frac{1}{\sqrt{V \cdot 2p_0(M+p_0)}} \psi_{p}^{(+)(p)} (-i\gamma_\mu \tilde{p}_\mu - M) \times \gamma_4\, e^{-i p_\mu x_\mu}$$

(4):

$$(0 \mid \psi_{p\alpha}^{(-)(p)}(x)\, \psi_{p\beta}^{(+)(p)}(x') \mid 0)$$

$$= \frac{1}{V\, 2p_0(M+p_0)} (i\gamma_\mu p_\mu - M)_{\alpha\alpha_1} (\gamma_4)_{\alpha_1\beta} e^{i p_\mu (x_\mu - x'_\mu)}$$

$$(\gamma_\mu \frac{\partial}{\partial x_\mu} + M)\psi = 0$$

$$(\gamma_4 \gamma_k \frac{\partial}{\partial x_k} + \frac{\partial}{\partial x_4} + \gamma_4 M)\psi = 0$$

$$(-i\rho_1 \sigma_k \frac{\partial}{\partial x_k} - \frac{i}{c}\frac{\partial}{\partial t} + \gamma_4 M)\psi = 0$$

$$\gamma_4 \gamma_k = -i\rho_1 \sigma_k = \rho_3 \rho_2 \sigma_k$$

$$\gamma_1 = \rho_2 \sigma_1,\ \gamma_2 = \rho_2 \sigma_2,\ \gamma_3 = \rho_2 \sigma_3,\ \gamma_4 = \rho_3$$

$$\gamma_1 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix},\quad \gamma_2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}\quad \gamma_3 = \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}$$

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$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$i\gamma_\mu p_\mu - M = i\gamma_k p_k - \gamma_4 p_0 - M$$

$$= \begin{pmatrix} -M-p_0 & 0 & p_z & p_x - ip_y \\ -p_0 - M & 0 & p_z & p_x - ip_y \\ 0 & -p_0 - M & p_x + ip_y & -p_z \\ p_z & p_x - ip_y & p_0 - M & 0 \\ -p_x - ip_y & p_z & 0 & p_0 - M \end{pmatrix}$$

$$(i\gamma_\mu p_\mu - M)_{\alpha 1} (-i\gamma_\mu \tilde{p}_\mu - M)_{1\beta} (\gamma_4)_{\alpha\beta}$$

$$= \begin{pmatrix} -p_0 - M \\ 0 \\ p_z \\ -p_x - ip_y \end{pmatrix} \begin{pmatrix} -p_0 - M & 0 & p_z & p_x - ip_y \end{pmatrix}$$

$$= \begin{pmatrix} (p_0+M)^2 & 0 & -(p_0+M)p_z & -(p_0+M)(p_x - ip_y) \\ 0 & 0 & 0 & 0 \\ p_z(-p_0-M) & 0 & p_z^2 & p_z(p_x - ip_y) \\ (-p_x - ip_y)(-p_0-M) & 0 & (p_x - ip_y)p_z & (p_x - ip_y)^2 \end{pmatrix}$$

$$-i\gamma_\mu \tilde{p}_\mu - M = -i\gamma_k p_k \gamma_4 - \gamma_4 p_0 - M$$

$$= \begin{pmatrix} -p_0 - M & 0 & -p_z & -p_x + ip_y \\ 0 & -p_0 - M & -p_x - ip_y & p_z \\ p_z & p_x + ip_y & p_0 - M & 0 \\ p_x - ip_y & -p_z & 0 & p_0 - M \end{pmatrix}$$

$$= \begin{pmatrix} p_0+M & 0 & -p_z & -p_x + ip_y \\ 0 & p_0+M & -p_x - ip_y & p_z \\ p_z & p_x - ip_y & \frac{p_z^2}{p_0+M} & \frac{p_z(p_x - ip_y)}{p_0+M} \\ p_x + ip_y & -p_z & \frac{p_z(p_x + ip_y)}{p_0+M} & \frac{p_x^2 + p_y^2}{p_0+M} \end{pmatrix}$$

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Let us the average over spins:

$$\frac{1}{4V p_0 (M+|p_0|)} \{(i\gamma_\mu p_\mu - M)\frac{1+\gamma_4}{2}(-i\gamma_\mu \tilde{p}_\mu - M)\gamma_4\}_{\alpha\beta} e^{ip_\mu(x_\mu - x'_\mu)}$$

$$\text{Tr}\{\}_{\alpha\beta} = \{(i\gamma_k p_k + \gamma_4 p_0 - M)\left(\frac{1+\gamma_4}{2}\right) \times (-i\gamma_k p_k - \gamma_4 p_0 - M)\gamma_4\}_{\alpha\beta}$$

$$= \frac{1}{2}\{(i\gamma_k p_k - \gamma_4 p_0 - M)(-i\gamma_k p_k - \gamma_4 p_0 - M)\gamma_4 + (i\gamma_k p_k - \gamma_4 p_0 - M)\gamma_4(-i\gamma_k p_k - \gamma_4 p_0 - M)\gamma_4\}_{\alpha\beta}$$

$$= \frac{1}{2}\{[-(i\gamma_k p_k)^2 + (\gamma_4 p_0)^2 + M^2 - 2(i\gamma_k p_k)(\gamma_4 p_0) + 2\gamma_4 p_0 M]\gamma_4 + (i\gamma_k p_k - \gamma_4 p_0 - M)(i\gamma_k p_k - \gamma_4 p_0 - M)\}_{\alpha\beta}$$

$$= -(p_0 + M)(i\gamma_k p_k - \gamma_4 p_0 - M)_{\alpha\beta}$$

$$(p|\psi^{(-)(p)}_{p\alpha}(x)\psi^{(+)(p)\dagger}_{p\beta}(x')|p0) = \frac{-1}{4p_0 V}(i\gamma_k p_k - \gamma_4 p_0 - M)_{\alpha\beta} \times e^{ip_\mu(x_\mu - x'_\mu)}$$

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IV. 13

~~$(p_0 | \psi^{(+)(p)}(x)(0) \psi^{(-)}_{p_0}$~~

Let us define

$$\Psi(p,j) = b^{(j)\dagger}_{p_0} \Psi(p_0)$$

or $|p\rangle = b^{(j)\dagger}_{p_0} |0\rangle$

i.e. or $\Psi^\dagger(p,j)\Psi(p,j) = \langle 1 | b^{(j)}_{p_0} |0\rangle$

$\Psi^\dagger(p,j) b^{(j)}_{p_0} \Psi(0)$)

~~$\Psi^{(+)(p)}$~~ also $\Psi^\dagger(0) \cdot b^{(j)\dagger}_{p_0} = \Psi^\dagger(p,j)$

~~$p_0 | b^j ($~~ Ψ ~~$)$~~

$$(p_0 | \left(\psi^{(+)(p)}(x) + \psi^{(-)(p)}(x)\right)\left(\psi^{(+)(p)}(x) + \psi^{(-)(p)}(x)\right)\left(\psi^{(+)(p)}(x') + \psi^{(-)(p)}(x')\right)\left(\psi^{(+)(p)}(x') + \psi^{(-)(p)}(x')\right)| p_0)$$

$$= (0 | b^{(j)}_{p_0} (\quad)(\quad)(\quad)(\psi^{(-)(p)}(x')) b^{(j)\dagger}_{p_0} | 0)$$

$$= (p_0 | b^{(j)}_{p_0} \cdot \psi^{(+)(p)}(x)\psi^{(-)(p)}(x) \cdot \psi^{(+)(p)}(x') \cdot \psi^{(-)(p)}(x') b^{(j)\dagger}_{p_0} | 0)$$
$$+ (0 | b^{(j)}_{p_0} \cdot \psi^{(-)(p)}(x)\psi^{(+)(p)}(x) \cdot \psi^{(+)(p)}(x')\psi^{(+)(p)}(x') b^{(j)\dagger}_{p_0} | 0)$$

Vacuum terms:

$$-\langle 0 | \psi^{(-)(p)}(x)\psi^{(-)(p)}(x)\psi^{(+)(p)}(x')\psi^{(+)(p)}(x') | 0\rangle$$

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$$= (0|(b_{p}^{(j)} \psi^{(+)(p)}(x))|0)(0|\psi^{(-)(p)}(x)\psi^{(+)(p)}(x')|0)$$
$$\times (0|\psi^{(-)(p)}(x') b_{p}^{(j)\dagger}|0)$$
$$+ (0|b_{p}^{(j)} \psi^{(-)(p)}(x)\psi^{(+)(p)}(x') b_{p}^{(j)\dagger}|0)$$
$$\times (0|\psi^{(-)(a)}(x)\psi^{(+)(a)}(x')|0)$$
$$- (0|\psi^{(-)(p)}(x)\psi^{(+)(p)}(x')|0)$$
$$\times (0|\psi^{(-)(a)}(x)\psi^{(+)(a)}(x')|0)$$

Now

$$(0|b_{p}^{(j)}\psi^{(-)(p)}(x)\psi^{(+)(p)}(x')b_{p}^{(j)\dagger} - \psi^{(-)(p)}(x)\psi^{(+)(p)}(x')|0)$$
$$= (0|b_{p}^{(j)}[\psi^{(-)(p)}(x), \psi^{(+)(p)}(x')]_{+} b_{p}^{(j)\dagger}|0)$$
$$- (0|b_{p}^{(j)}\psi^{(+)(p)}(x')|0)(0|\psi^{(-)(p)}(x) b_{p}^{(j)\dagger}|0)$$
$$- (0|[\quad]_{+}|0) - (0|\psi^{(+)(p)}(x')\psi^{(-)(p)}(x)|0)$$

1 $= (0|b_{p}^{(j)} b_{p}^{(j)\dagger}|0)(0|[\quad]_{+}|0)$
$$- (0|b_{p}^{(j)}\psi^{\dagger(p)}(x')|0)(0|\psi^{(-)(p)}(x) b_{p}^{(j)\dagger}|0)$$
$$- (0|[\quad]_{+}|0)$$
$$= -(0|b_{p}^{(j)}\psi^{\dagger(p)}(x')|0)(0|\psi^{(-)(p)}(x) b_{p}^{(j)\dagger}|0)$$

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$$(p\alpha|\bar{\psi}_\alpha(x)\psi_\alpha(x)\bar{\psi}_\beta(x')\psi_\beta(x')|p\alpha)$$

$$=(0|b_{p\alpha}^{j\alpha}\psi_\alpha^{(+)(p)}(x)|0)(0|\psi_\beta^{(-)(p)}(x')b_{p\alpha}^{j\alpha\dagger}|0)$$

$$\times\{(0|\psi_\alpha^{(-)(p)}(x)\psi_\beta^{(+)(p)}(x')|0)$$

$$-(0|\psi_a^{(-)(a)}(x)\psi_\beta^{(+)(a)}(x')|0)\}$$

$$(0|b_{p\alpha}^{j\alpha}\psi_\alpha^{(+)(p)}(x)|0)=\frac{1}{\sqrt{2p_0(M+p_0)V}}$$

$$\times\{(-i\gamma_\mu\tilde{p}_\mu-M)\gamma_4\}_{j\alpha}e^{-ip_\mu x_\mu}$$

$$(0|\psi_\beta^{(-)(p)}(x')b_{p\alpha}^{j\alpha\dagger}|0)=\frac{1}{\sqrt{2p_0(M+p_0)V}}$$

$$\times(i\gamma_\mu p_\mu-M)_{\beta j}e^{ip_\mu x_\mu}$$

$$\frac{1}{2}\sum_{j=1,2}(0|b_{p\alpha}^{j\alpha}\psi_\alpha^{(+)(p)}(x)|0)(0|\psi_\beta^{(-)(p)}(x')b_{p\beta}^{j\alpha\dagger}|0)$$

$$=\frac{-1}{4p_0V}(p_0+M)(i\gamma_\mu p_\mu-M)_{\beta\alpha}e^{-ip_\mu(x_\mu-x'_\mu)}$$

$$\left(\frac{1}{2}\left\{(i\gamma_\mu p_\mu-M)\frac{1+\gamma_4}{2}(-i\gamma_\mu\tilde{p}_\mu-M)\gamma_4\right\}_{\beta\alpha}\right.$$

$$\left.=-\frac{1}{2}(p_0+M)(i\gamma_\mu p_\mu-M)_{\beta\alpha}\right)$$

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$$\overline{(p|\,|p)}^{sp} = -\frac{1}{4p_0 V}(i\gamma_\mu p_\mu - M)$$

$$\times e^{-ip_\mu(x_\mu - x'_\mu)}\{(-S^{(-)}_{\alpha\beta}(x-x')) + S^{(-)}_{\beta\alpha}(x-x')\}$$

$$S^{(-)}_{\alpha\beta}(x) = \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - M\right)\Delta^{(-)}_M(x)$$

$$\Delta^{(-)}_M = \frac{1}{2}\left(\Delta^{(1)}_M + i\Delta_M\right)$$

$$\Delta^{(-)}_M(x) = \frac{1}{(2\pi)^3}\iiint \frac{e^{ip'_\mu x_\mu}}{2p_0}(dp')^3$$

$$\Delta^{(1)}_M = \frac{1}{(2\pi)^3}\int\cdots\int e^{ip'_\mu x_\mu}\delta(p'_\mu p'_\mu + M^2)(dp')^4$$

$$i\Delta_M = \frac{1}{(2\pi)^3}\int\cdots\int e^{ip'_\mu x_\mu}\varepsilon(p'_\mu)\delta(\cdots)(dp')^4$$

$$\Delta^{(-)}_M = \frac{1}{(2\pi)^3}\int\cdots\int \frac{1+\varepsilon(p'_\mu)}{2} e^{ip'_\mu x_\mu}\delta(\cdots)(dp')^4$$

$$S^{(-)}_{\alpha\beta}(x) = \frac{1}{(2\pi)^3}\int\cdots\int \frac{1+\varepsilon(p'_\mu)}{2}(i\gamma_\mu p'_\mu - M)_{\alpha\beta} e^{ip'_\mu x_\mu}\delta(p'_\mu p'_\mu + M^2)(dp')^4$$

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(IV, 体)

~~$S^{(-)}_{\alpha\beta}(x) \simeq$~~

$$\langle p | \bar{\psi}_\alpha(x) \psi_\alpha(x) \bar{\psi}_\beta(x') \psi_\beta(x') | p \rangle$$

$$= \langle 0 | b_p^{(j)} \psi_\alpha^{(+)(p)}(x) | 0 \rangle \langle 0 | \psi_\beta^{(-)(p)}(x') b_p^{(j)+} | 0 \rangle$$

$$\times \langle 0 | \psi_\alpha^{(-)(p)}(x) \psi_\beta^{(+)(p)}(x') | 0 \rangle$$

$$- \langle 0 | b_p^{(j)} \psi_\beta^{(+)(p)}(x') | 0 \rangle \langle 0 | \psi_\alpha^{(-)(p)}(x) b_p^{(j)+} | 0 \rangle$$

$$\times \langle 0 | \psi_\alpha^{(-)(a)}(x) \psi_\beta^{(+)(a)}(x') | 0 \rangle$$

$$\langle 0 | b_p^{(j)} \psi_\alpha^{(+)(p)}(x) | 0 \rangle = \frac{1}{\sqrt{2p_0(M+p_0)V}} \times (-i\gamma_\mu \tilde{p}_\mu - M) \gamma_4 \chi_{j\alpha} e^{-i p_\mu x_\mu}$$

$$\langle 0 | \psi_\beta^{(-)(p)}(x') b_p^{(j)+} | 0 \rangle = \frac{1}{\sqrt{\quad}} \times (i\gamma_\mu p_\mu - M) \cdot e^{i p_\mu x'_\mu}$$

$$\frac{(j)}{\langle 0 | b_{p_1}^{(j)} \psi_\alpha(x) | 0 \rangle \langle 0 | \psi_\beta(x') b_p^{(j)+} | 0 \rangle} dp$$

$$= -\frac{1}{4p_0 V} (i\gamma_\mu p_\mu - M)_{\beta\alpha} e^{-i p_\mu (x_\mu - x'_\mu)}$$

$$\frac{}{\langle 0 | b_p^{(j)} \psi_\beta(x') | 0 \rangle \langle 0 | \psi_\alpha(x) b_p^{(j)+} | 0 \rangle} dp$$

$$= -\frac{1}{4p_0 V} (i\gamma_\mu p_\mu - M)_{\alpha\beta} e^{i p_\mu (x_\mu - x'_\mu)}$$

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$$\overline{(p|\qquad|p)^{\gamma\beta}} = +\frac{1}{4p_0V}\Big\{(i\gamma_\mu p_\mu - M)_{\beta\alpha} e^{-ip_\mu(x_\mu - x'_\mu)} S^{(-)}_{\alpha\beta}(x-x') - (i\gamma_\mu p_\mu - M)_{\alpha\beta} e^{ip_\mu(x_\mu - x'_\mu)} S^{(-)}_{\beta\alpha}(x-x')\Big\}$$

$$S^{(-)}_{\alpha\beta}(x) = (\gamma_\mu \frac{\partial}{\partial x_\mu} - M)\Delta^{(-)}_M(x)$$

$$\Delta^{(-)}_M(x) = \frac{1}{(2\pi)^3}\int\cdots\int \frac{1+\varepsilon(p'_\mu)}{2} e^{ip'_\mu x_\mu} \times \delta(p'_\mu p'_\mu + M^2)(dp')^4$$

$$S^{(-)}_{\alpha\beta}(x) = \frac{1}{(2\pi)^3}\int\cdots\int \frac{1+\varepsilon(p'_\mu)}{2}(i\gamma_\mu p'_\mu - M)_{\alpha\beta} \times e^{ip'_\mu x_\mu}\delta(p'_\mu p'_\mu + M^2)(dp')^4$$

$$\frac{1}{4p_0V}(p|S|p^0) = \frac{e^2}{}\int\cdots\int \varepsilon(x-x')p_\mu \Delta^{(-)}(x-x')\frac{1}{4p_0V}$$
$$\Big\{(i\gamma_\mu p_\mu - M)_{\beta\alpha} e^{-ip_\mu(x_\mu - x'_\mu)} S^{(-)}_{\alpha\beta}(x-x') - (i\gamma_\mu p_\mu - M)_{\alpha\beta} e^{ip_\mu(x_\mu - x'_\mu)} S^{(-)}_{\beta\alpha}(x-x')\Big\} \times (dk)^4(dk')^4$$

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More precisely, the matrix element for one, ~~electron~~ (sp. particles) state has the form

$$(pj|-g^2\iint\theta(x-x')\varphi(x)\bar{\psi}(x)\psi(x)\cdot\varphi(x')\bar{\psi}(x')\psi(x')\times(dx)^4(dx')^4|p'j')$$

if we take the limit $V\to\infty$, so that p are continuous. $j=1,2$ denote two directions of the spin:

$$\psi_\cdot(x) = \psi^{(-)(p)}(x) + \psi^{(+)(a)}(x)$$

$$\psi^{(-)(p)}_\alpha(x) = \frac{1}{\sqrt{(2\pi)^3}}\sum_{j=1,2}\iiint(u_{pj})_\alpha b^{(j)}_p e^{i(pr-p_0x_0)}(dp)^3$$

$$p_0 = +\sqrt{p^2+M^2} = \iiint(\psi^{(+)}_{pj})_\alpha b^{(j)}_p$$

$$\psi^{(+)(a)}_a(x) = \frac{1}{\sqrt{(2\pi)^3}}\sum_{j=1,2}\iiint(\tilde{v}_{pj})_\alpha c^{(j)*}_p e^{-i(pr-p_0x_0)}(dp)^3$$

$$= \iiint(\psi_{pj})\, c^{(j)*}_p$$

$$\bar{\psi}(x) = \psi^{(+)(p)}(x) + \psi^{(-)(a)}(x)$$

$$\psi^{(+)(p)}_a(x) = \frac{1}{\sqrt{(2\pi)^3}}\sum_j\iiint(\tilde{u}_{pj})_\beta(\gamma_4)_{\beta\alpha}\, b^{(j)*}_p \times e^{-i(pr-p_0x_0)}(dp)^3$$

$$\psi^{(-)(a)}_a(x) = \frac{1}{\sqrt{(2\pi)^3}}\sum_j\iiint(v_{pj})_\beta(\gamma_4)_{\beta\alpha}\, c^{(j)}_p \overbrace{e^{i(pr-p_0x_0)}(dp)^3}$$

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where u's and v's are constant spinors satisfying

$$\left.\begin{aligned}
&\sum_\alpha (\hat{u}_{p,j})_\alpha (u_{p,j'})_\alpha = \delta_{jj'} \\
&\sum_\alpha (\tilde{v}_{p,j})_\alpha (v_{p,j'})_\alpha = \delta_{jj'} \\
&\sum_\alpha (v_{-p,j})_\alpha (u_{p,j'})_\alpha = 0 \\
&\sum_\alpha (\hat{v}_{-p,j})_\alpha (\tilde{u}_{p,j'})_\alpha = 0
\end{aligned}\right\}$$

and b's and c's are operators satisfying

$$[b_p^{(j)}, b_{p'}^{(j')*}]_+ = \delta_{jj'} \delta(p-p')$$
$$[b_p^{(j)}, b_{p'}^{(j')}]_+ = 0 \quad \text{etc}$$
$$[c_p^{(j)}, c_{p'}^{(j')*}]_+ = \delta_{jj'} \delta(p-p')$$
$$[b_p^{(j)}, c_{p'}^{(j')}]_+ = 0 \quad \text{etc}$$

In order to define one particle (and many particle) states unambiguously, it is appropriate to proceed as follows:

$$\Psi(p,j) = b_p^{(j)*} \Psi(0)$$
$$\Psi^*(p,j) = \Psi^*(0)\, b_p^{(j)}$$

Normalization:

$$\Psi^*(p,j)\,\Psi(p',j') = \delta(p-p')\,\delta_{jj'}$$

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$$\Psi(p,j;p',j') = b_p^{j\dagger} b_{p'}^{j'\dagger} \Psi(0)$$
$$\Psi(p',j';p,j) = b_{p'}^{j'\dagger} b_p^{j\dagger} \Psi(0)$$
$$= -\Psi(p,j;p',j')$$

- - -

Similarly, for anti particles, and can further be extended to include ~~but~~ states in which both particles and anti-particles exist.

$$\varphi(x) = \varphi^{(-)}(x) + \varphi^{(+)}(x)$$
$$\varphi^{(-)}(x) = \frac{1}{\sqrt{(2\pi)^3}} \iiint \frac{1}{\sqrt{2k_0}} a_k e^{i(kr - k_0 x_0)} (dk)^3$$
$$\varphi^{(+)}(x) = \frac{1}{\sqrt{(2\pi)^3}} \iiint \frac{1}{\sqrt{2k_0}} a_k^* e^{-i(kr - k_0 x_0)} (dk)^3$$

$$[a_k, a_{k'}^*] = \delta(k - k')$$
$$[a_k, a_{k'}] = 0 \quad \text{etc.}$$

$$\Psi(k) = a_k^* \Psi(0)$$
$$\Psi^*(k) = \Psi^*(0) a_k$$
$$\Psi(k,k') = a_k^* a_{k'}^* \Psi(0)$$
$$= \Psi(k',k)$$

etc.

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$$\langle p j | -g^2 \int\cdots\int \theta(x-x')\varphi(x)\bar{\psi}(x)\psi(x)\cdot\varphi(x')\bar{\psi}(x')\psi(x') \times (dx)^4(dx')^4 | p' j' \rangle$$

$$= g^2 \int\cdots\int \theta(x-x') \langle 0_{sp}| \varphi(x)\varphi(x') |0_{sp}\rangle \times \langle 0_{sp}| b^{(j)}_{p} \bar{\psi}(x)\psi(x)\bar{\psi}(x')\psi(x') b^{(j')*}_{p'} |0_{sp}\rangle \times (dx)^4(dx')^4$$

We must subtract the vacuum matrix element

$$\langle 0 | -g^2 \int\cdots\int \cdots | 0\rangle \delta(p-p')\delta_{jj'}$$

$$= -g^2 \int\cdots\int \theta(x-x') \langle 0|\varphi(x)\varphi(x')|0\rangle \times \langle 0|\bar{\psi}(x)\psi(x)\bar{\psi}(x')\psi(x')|0\rangle \cdot (dx)^4(dx')^4 \delta(p-p')\delta_{jj'}$$

$$\langle 0| b^{(j)}_{p} \bar{\psi}_\alpha(x)\psi_\alpha(x)\bar{\psi}_\beta(x')\psi_\beta(x') b^{(j')*}_{p'} |0\rangle$$

$$= \langle 0| b^{(j)}_{p} \begin{pmatrix}\psi^{(+)(p)}_\alpha(x)\\ \psi^{(-)(a)}_\alpha(x)\end{pmatrix}\begin{pmatrix}\psi^{(-)(p)}_\alpha(x)\\ \psi^{(+)(a)}_\alpha(x)\end{pmatrix}\begin{pmatrix}\psi^{(+)(p)}_\beta(x')\\ \psi^{(-)(a)}_\beta(x')\end{pmatrix}\begin{pmatrix}\psi^{(-)(p)}_\beta(x')\\ \psi^{(+)(a)}_\beta(x')\end{pmatrix} b^{(j')}_{p'} |0\rangle$$

$$= \langle 0| b^{(j)}_{p} \psi^{(+)(p)}_\alpha(x)|0\rangle\langle 0|\psi^{(-)(p)}_\alpha(x)\psi^{(+)(p)}_\beta(x')|0\rangle \times \langle 0|\psi^{(-)(p)}_\beta(x') b^{(j')}_{p'}|0\rangle \quad (1)$$

$$+ \langle 0| b^{(j)}_{p} \psi^{(+)(p)}_\beta(x)\psi^{(+)(p)}_\beta(x') b^{(j')}_{p'}|0\rangle \times \langle 0|\psi^{(-)(a)}_\alpha(x)\psi^{(+)(a)}_\beta(x')|0\rangle \quad (2)$$

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vacuum term:

$$\propto \langle 0|\psi^{(-)(p)}(x)\psi^{(+)(p)}(x')|0\rangle\langle 0|\psi^{(-)(a)}(x)\psi^{(+)(a)}(x')|0\rangle(3)$$
$$\times \delta(p-p')\delta_{jj'}$$

(2) − (3):

$$\langle 0|b_p^{(j)}\psi_\alpha^{(-)(p)}(x)\psi_\beta^{(+)(p)}(x')b_{p'}^{(j')*} - \psi_\alpha^{(-)(p)}(x)\psi_\beta^{(+)(p)}(x')|0\rangle$$

$$= \langle 0|b_p^{(j)}[\psi_\alpha^{(-)(p)}(x), \psi_\beta^{(+)(p)}(x')]_+ b_{p'}^{(j')*}|0\rangle$$
$$- \langle 0|b_p^{(j)}\psi_\beta^{(+)(p)}(x')|0\rangle\langle 0|\psi_\alpha^{(-)(p)}(x)b_{p'}^{(j')*}|0\rangle$$
$$- \langle 0|[\psi_\alpha^{(-)(p)}(x), \psi_\beta^{(+)(p)}(x')]_+|0\rangle \frac{\delta(p-p')}{\delta_{jj'}}$$
$$+ \langle 0|\psi_\beta^{(+)(p)}(x')\psi_\alpha^{(-)(p)}(x)|0\rangle\delta(p-p')\delta_{jj'}$$

$$= \langle 0|[\quad]_+\delta(p-p')\delta_{jj'}|0\rangle$$
$$- \langle 0|b_p^{(j)}\psi_\beta^{(+)(p)}(x')|0\rangle\langle 0|b_{p'}^{(j')*}\psi_\alpha^{(-)(p)}(x)|0\rangle$$
$$- \langle 0|[\quad]_+\delta(p-p')\delta_{jj'}|0\rangle$$
$$- 0$$
$$= -\langle 0|b_p^{(j)}\psi_\beta^{(+)(p)}(x')|0\rangle\langle 0|\psi_\alpha^{(-)(p)}(x)b_{p'}^{(j')*}|0\rangle$$

Thus the net result is

$$\langle 0|b_p^{(j)}\psi_\alpha^{(+)(p)}(x)|0\rangle\langle 0|\psi_\beta^{(-)(p)}(x)b_{p'}^{(j')*}|0\rangle$$
$$\times\langle 0|\psi_\alpha^{(-)(p)}(x)\psi_\beta^{(+)(p)}(x')|0\rangle$$

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$$-(0|b_p^{ij}\psi_\beta^{(+)(p)}(x')|0)(0|\psi_\alpha^{(-)(p)}(x)b_{p'}^{i'j'*}|0)$$
$$\times(0|\psi_\alpha^{(-)(a)}(x)\psi_\beta^{(+)(a)}(x')|0)$$

$$(0|b_p^{ij}\psi_\alpha^{(+)(p)}(x)|0)=\frac{1}{\sqrt{(2\pi)^3}}\sum_\sigma(\tilde{u}_{pj})_\sigma(\gamma_4)_{\sigma\alpha}e^{-i(px-p_0x_0)}$$

$$(0|\psi_\beta^{(-)(p)}(x')b_{p'}^{i'j'*}|0)=\frac{1}{\sqrt{(2\pi)^3}}(u_{p'j'})_\beta e^{i(p'x'-p_0'x_0')}$$

$$(0|\psi_\alpha^{(-)(p)}(x)\psi_\beta^{(+)(p)}(x')|0)=-S_{\alpha\beta}^{(-)}(x-x')$$

$$(0|b_p^{ij}\psi_{p_\beta}^{(+)(p)}(x)|0)=\frac{1}{\sqrt{(2\pi)^3}}\sum_\rho(\tilde{u}_{pj})_\rho(\gamma_4)_{\rho\beta}e^{-i(px'-p_0x_0')}$$

$$(0|\psi_\alpha^{(-)(p)}(x)b_{p'}^{i'j'*}|0)=\frac{1}{\sqrt{(2\pi)^3}}(u_{p'j'})_\alpha e^{i(p'x-p_0'x_0)}$$

$$(0|\psi_\alpha^{(-)(a)}(x)\psi_\beta^{(+)(a)}(x')|0)=-S_{\beta\alpha}^{(-)}(x-x')$$

$x=X+\frac{1}{2}\xi$, $\quad x'=X-\frac{1}{2}\xi$

Now the whole expression becomes

$$-g^2\int\cdots\int(dX)^4(d\xi)^4\theta(\xi)\,\delta^{(-)}(\xi)\times\frac{1}{(2\pi)^3}$$
$$\{\sum(\tilde{u}_{pj})_\alpha(u_{p'j'})_\sigma e^{-i(p_\mu-p_\mu')X_\mu}e^{-\frac{i}{2}(p_\mu+p_\mu')\xi_\mu}$$
$$\times(-S_{\sigma\beta}^{(-)}(\xi))-\Sigma$$

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Once

(IV.19)

Comm. Rel.:

$$\{b^{(j)}_{p\sigma}, \bar{\psi}_\alpha(x)\}_+ = \frac{1}{\sqrt{(2\pi)^3}} \sum_\sigma (\tilde{u}_{p\sigma j})_\beta (\gamma_4)_{\beta\alpha} \times e^{-ip_\mu x_\mu}$$

$$\{b^{(j)}_{p\sigma}, \psi_\alpha(x)\}_+ = 0$$

$$\{b^{(j)*}_{p\sigma}, \bar{\psi}_\alpha(x)\}_+ = 0$$

$$\{b^{(j)*}_{p\sigma}, \psi_\alpha(x)\}_+ = \frac{1}{\sqrt{(2\pi)^3}} (u_{p\sigma j})_\alpha e^{ip_\mu x_\mu}$$

$$b^{(j)}_{p\sigma} \bar{\psi}_\alpha(x) \psi_\alpha(x) \bar{\psi}_\beta(x') \psi_\beta(x') b^{(j')*}_{p\sigma'}$$

$$= \{b^{(j)}_{p\sigma}, \bar{\psi}_\alpha(x)\}_+ \{b^{(j')*}_{p\sigma'}, \psi_\beta(x')\}_+ \psi_\alpha(x) \bar{\psi}_\beta(x')$$

$$- \{b^{(j)}_{p\sigma}, \bar{\psi}_\beta(x')\}_+ \{b^{(j')*}_{p\sigma'}, \psi_\alpha(x)\}_+ \bar{\psi}_\alpha(x) \psi_\beta(x')$$

$$+ b^{(j)}_{p\sigma} b^{(j')}_{p\sigma'} \bar{\psi}_\alpha(x) \psi_\alpha(x) \bar{\psi}_\beta(x') \psi_\beta(x')$$

$$+ \{b^{(j)}_{p\sigma}, \bar{\psi}_\alpha(x)\} \{b^{(j')*}_{p\sigma'}, \psi_\alpha(x)\} \bar{\psi}_\beta(x') \psi_\beta(x')$$

$$+ \{b^{(j)}_{p\sigma}, \bar{\psi}_\beta(x')\} \{b^{(j')*}_{p\sigma'}, \psi_\beta(x)\} \bar{\psi}_\alpha(x) \psi_\alpha(x)$$

$$\theta(x-x') \tfrac{1}{2} P\{\varphi(x)\bar{\psi}(x)\psi(x), \varphi(x')\bar{\psi}(x')\psi(x')\}$$

$$= \frac{1}{2}\Big\{ (\theta(x-x') - \theta(x'-x)) \varphi(x)\bar{\psi}(x)\psi(x)\varphi(x')\bar{\psi}(x')\psi(x') + \theta(x'-x)\,\varphi(x')\bar{\psi}(x')\psi(x')\varphi(x)\bar{\psi}(x)\psi(x) \Big\}$$

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~~$-q^2\int\cdots\int (dx)^4(dz)^4\,\theta(z)\Delta$~~

$$= \frac{1}{2}\{\theta(x-x')\varphi(x)\varphi(x') + \theta(x'-x)\varphi(x')\varphi(x)\}$$

$$\times\{\theta(x-x')\bar{\psi}(x)\psi(x)\bar{\psi}(x')\psi(x') + \theta(x'-x)\bar{\psi}(x')\psi(x')\bar{\psi}(x)\psi(x)\}$$

$$= \frac{1}{2}\{P(\varphi(x),\varphi(x'))\,P(\bar{\psi}(x)\psi(x),\bar{\psi}(x')\psi(x'))\}$$

~~$AB=C$~~ $\quad\beta(x,x')$

$$\theta(x,x')A(x,x') + \theta(x',x)A(x',x)\beta(x',x)$$

$$= \{\theta(x,x')A(x,x') + \theta(x',x)A(x',x)\}$$

$$\times\{\theta(x,x')\beta(x,x') + \theta(x',x)\beta(x',x)\}$$

$$\theta(x,x')(A(x,x')+\beta(x,x'))$$

$$+\theta(x,x')(A(x',x')+\beta(x',x'))$$

$$= \{\theta(x-x')A(x,x') + \theta(x,x')\beta(x,x')\}$$

$$+\{\theta(x,x')A(x',x) + \theta(x',x)\beta(x',x)\}$$

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$$\frac{1}{2} P\left(b_{p\sigma}^{(j)} \bar{\psi}_\alpha(x) \psi_\alpha(x), \bar{\psi}_\beta(x') \psi_\beta(x') b_{p\sigma'}^{(j')*} \right)$$

$$= \frac{1}{2} \Big\{ \theta(x-x') \left[b_{p\sigma}^{(j)}, \bar{\psi}_\alpha(x) \right]_+ \left[b_{p\sigma'}^{(j')*}, \psi_\beta(x') \right]_+ \psi_\alpha(x) \bar{\psi}_\beta(x')$$

$$- \theta(x-x') \left[b_{p\sigma}^{(j)}, \bar{\psi}_\beta(x') \right]_+ \left[b_{p\sigma'}^{(j')*}, \psi_\alpha(x) \right]_+ \bar{\psi}_\alpha(x) \psi_\beta(x')$$

$$+ \theta(x'-x) \left[b_{p\sigma}^{(j)}, \bar{\psi}_\beta(x') \right]_+ \left[b_{p\sigma'}^{(j')*}, \psi_\alpha(x') \right]_+ \psi_\beta(x') \bar{\psi}_\alpha(x)$$

$$- \theta(x'-x'') \left[b_{p\sigma}^{(j')}, \bar{\psi}_\alpha(x) \right]_+ \left[b_{p\sigma'}^{(j'')*}, \psi_\beta(x') \right]_+ \bar{\psi}_\beta(x') \psi_\alpha(x) \Big\}$$

$$= \frac{1}{2} \left[b_{p\sigma}^{(j)}, \bar{\psi}_\alpha(x) \right]_+ \left[b_{p\sigma'}^{(j')*}, \psi_\beta(x') \right]_+$$

$$\times \left\{ \theta(x-x') \psi_\alpha(x) \bar{\psi}_\beta(x') - \theta(x'-x) \bar{\psi}_\beta(x') \psi_\alpha(x) \right\}$$

$$- \frac{1}{2} \left[b_{p\sigma}^{(j')}, \bar{\psi}_\beta(x') \right]_+ \left[b_{p\sigma'}^{(j'')*}, \psi_\alpha(x) \right]_+$$

$$\times \left\{ \theta(x-x') \bar{\psi}_\alpha(x) \psi_\beta(x') - \bar{\psi}_\beta(x') \psi_\alpha(x) \right\}$$

$$\theta(x-x') \psi_\alpha(x) \bar{\psi}_\beta(x') - \theta(x'-x) \bar{\psi}_\beta(x') \psi_\alpha(x)$$

$$= \frac{1}{2} \left[\psi_\alpha(x), \bar{\psi}_\beta(x') \right]_- + \frac{\varepsilon(x,x')}{2} \left[\psi_\alpha(x), \bar{\psi}_\beta(x') \right]_+$$

$$= \frac{-i}{2} S_{\alpha\beta}(x-x') + \frac{\varepsilon(x,x')}{2} \left\{ \psi_\alpha(x), \bar{\psi}_\beta(x') \right\}$$

$$= \frac{1}{2} \left(\psi_\alpha(x) \bar{\psi}_\beta(x') - \bar{\psi}_\beta(x') \psi_\alpha(x) \right) - \frac{i}{2} \varepsilon(x,x') S_{\alpha\beta}(x-x')$$

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$$\langle 0 | \tfrac{1}{2} P(\quad) | 0 \rangle$$

$$= \frac{1}{2} \{ [b^{ij)}_{p\sigma}, \bar{\psi}_\alpha(x)]_+ [b^{ij)*}_{p\sigma}, \psi_\beta(x')]_+$$

$$\times \Big[\Big(-\frac{1}{2} S^{(+)}_{\alpha\beta}(x-x') + \frac{1}{2} S^{(-)}_{\alpha\beta}(x-x') \Big)$$

$$- \frac{i}{2} \varepsilon(x,x') S_{\alpha\beta}(x-x') \Big]$$

$$+ [b^{ij)}_{p\sigma}, \bar{\psi}_\beta(x')]_+ [b^{ij)*}_{p\sigma'}, \psi_\alpha(x)]_+$$

$$\left[\alpha \rightleftarrows \beta, \; x \rightleftarrows x' \right] \}$$

$$\to -\frac{1}{2} S^{ij)}_{\alpha\beta}(x-x') + -\frac{i}{2} \varepsilon(x,x') S_{\alpha\beta}(x-x')$$

$$= -\frac{1}{2} \left(S_F(x-x') \right)_{\alpha\beta}$$

$$\langle 0 | \tfrac{1}{2} P(\quad) | 0 \rangle \qquad \to (-\tfrac{1}{2} S_F(x-x'))$$

$$= \frac{1}{2} \{ [b^{ij)}_{p\sigma}, \bar{\psi}_\alpha(x)]_+ [b^{ij)*}_{p\sigma'}, \psi_\beta(x')]_+$$

$$\times \left(-\tfrac{1}{2} S_F(x-x') \right)_{\alpha\beta}$$

$$+ [b^{ij)}_{p\sigma}, \bar{\psi}_\beta(x')]_+ [b^{ij)*}_{p\sigma'}, \psi_\alpha(x)]_+$$

$$\times \left(-\tfrac{1}{2} S_F(x'-x) \right)_{\beta\alpha} \}$$

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$$-\frac{q^2}{2}\frac{1}{(2\pi)^2}\int\int\{(\tilde{u}_{p,j})_\rho(\gamma_4)_{\rho\alpha}(u_{p'j'})_\beta e^{-ip_\mu x_\mu + ip'_\mu x'_\mu}\times\left(-\tfrac{1}{2}S_F(x-x')\right)_{\alpha\rho} + (\tilde{u}_{p,j})_\rho(\gamma_4)_{\rho\beta}(u_{p'j'})_\alpha \times e^{-ip_\mu x'_\mu + ip'_\mu x_\mu}\left(-\tfrac{1}{2}S_F(x'-x)\right)_{\beta\alpha}\}(dx)^4(dx')^4$$

(IV.26) $+ip'_\mu x'_\mu - ip_\mu x_\mu$

$\beta\alpha$ ↓↓ $\alpha\beta$

$$x = X + \tfrac{1}{2}\xi \qquad x + x' = 2X$$
$$x' = X - \tfrac{1}{2}\xi \qquad x - x' = \xi$$

$$e^{-ip_\mu x_\mu + ip'_\mu x'_\mu} = e^{-i(p_\mu - p'_\mu)X_\mu - \frac{1}{2}(p_\mu + p'_\mu)\xi_\mu}$$

$$\int e^{-i(p_\mu - p'_\mu)X_\mu}(dX)^4 = \frac{(2\pi)^4}{(2\pi)^4}\delta(p_\mu - p'_\mu)$$

$$= (2\pi)^4\delta(p-p')\delta(p_0 - p'_0)$$

$$-g^2\cdot\pi\int\int(\tilde{u}_{p,j})_\rho(\gamma_4)_{\rho\alpha}(u_{p'j'})_\beta\,\delta(p-p')\delta(p_0-p'_0)$$
$$\times\int\int\{-\tfrac{1}{2}(S_F(x-x')_{\alpha\beta}e^{-ip_\mu\xi_\mu} - \tfrac{1}{2}(S_F(x'-x)_{\alpha\rho}e^{+ip_\mu\xi_\mu}\}(d\xi)^4$$

$\frac{1}{2}O_F(x-x')$

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$$\Delta_F(x) \approx \frac{1}{(2\pi)^3}\frac{1}{\pi i}\int\!\!\int_{C_R} \frac{e^{ik_\mu x_\mu}(dk)^4}{k_\mu k_\mu + \lambda^2 - i\varepsilon}$$

$$S_F(x) = \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - M\right)(\Delta^{(x)}_M)_F$$

$$(p_0 j|S|p'j') = \frac{g^2}{(2\pi)^3}(\bar{u}_{pj})_\rho (\gamma_4)_{\rho\alpha}(u_{p'j'})_\beta$$
$$(2\pi)^4\delta(p_0-p')\delta(p_0-p'_0)\int \Delta_F(x)\{(S_F(x))_{\alpha\beta}e^{ip_\mu x_\mu}$$
$$+(S_F(-x))_{\alpha\beta}e^{ip_\mu x_\mu}\}(dx)^4$$

$$\frac{-1}{(2\pi)^6\pi^2}\int\!\!\int\!\!\int\!\!\int \frac{e^{ik_\mu x_\mu}}{k_\mu k_\mu + \lambda^2 - i\varepsilon}(iq_\mu\gamma_\mu - M)_{\alpha\beta}\frac{e^{iq_\mu x_\mu}}{q_\mu q_\mu + M^2 - i\varepsilon'}e^{-ip_\mu x_\mu}$$
$$+\frac{e^{ik_\mu x_\mu}}{k_\mu k_\mu + \lambda^2 - i\varepsilon}(-iq_\mu\gamma_\mu - M)_{\alpha\beta}\frac{e^{iq_\mu x_\mu}}{q_\mu q_\mu + M^2 - i\varepsilon'}e^{ip_\mu x_\mu}$$
$$(dk)^4$$

$$= \frac{(2\pi)^4}{(2\pi)^6\pi^2}\frac{1}{k_\mu k_\mu + \lambda^2 - i\varepsilon}(iq_\mu\gamma_\mu - M)_{\alpha\beta}\frac{1}{q_\mu q_\mu + M^2 - i\varepsilon'}$$
$$\times\delta(k_\mu + q_\mu - p_\mu)$$
$$+\frac{(2\pi)^4}{(2\pi)^6\pi^2}\frac{1}{k_\mu k_\mu + \lambda^2 - i\varepsilon}(-iq_\mu\gamma_\mu - M)_{\alpha\beta}\frac{1}{q_\mu q_\mu + M^2 - i\varepsilon'}$$
$$\times\delta(k_\mu + q_\mu + p_\mu)$$

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$$= -\frac{1}{(2\pi)^2\pi^2}\int\cdots\int \frac{(dk)^4 \{ i(p_\mu - k_\mu)\gamma_\mu - M \}_{\alpha\beta}}{(k_\mu k_\mu + \kappa^2 - i\epsilon)((p_\mu - k_\mu)^2 + M^2 - i\epsilon')}$$

$$k_\mu''' \approx k'_\mu + \frac{1}{2}p_\mu ;$$

$$\simeq \frac{-1}{(2\pi)^2\pi^2}\int\cdots\int \frac{(dk')^4 \{ \frac{1}{2}ip_\mu\gamma_\mu - ik'_\mu\gamma_\mu - M \}_{\alpha\beta}}{((k'_\mu + \frac{1}{2}p_\mu)^2 + \kappa^2 - i\epsilon)(k'_\mu - \frac{1}{2}p_\mu)^2 + M^2 - i\epsilon')}$$

$$= -\frac{1}{4\pi^4}\int\cdots\int \frac{\{ -ik'_\mu\gamma_\mu + \frac{1}{2}(ip_\mu\gamma_\mu + M) - \frac{3}{2}M \}_{\alpha\beta}}{(k'_\mu k'_\mu)^2 \left(1 + \frac{k'_\mu p_\mu + \frac{1}{4}p_\mu^2 + \kappa^2 - i\epsilon}{k'_\mu k'_\mu}\right)\left(1 + \frac{-k'_\mu p_\mu + \frac{1}{4}p_\mu^2 + M^2 - i\epsilon'}{k'_\mu k'_\mu}\right)}$$

$$= -\frac{1}{4\pi^4}\int\cdots\int \frac{-ik'_\mu\gamma_\mu}{(k'_\mu k'_\mu)^2}\left(1 - O\left(\frac{1}{k'_\mu k'_\mu}\right)\right)$$

$$-\frac{3}{2}\cdot\frac{M}{4\pi^4}\int\cdots\int \frac{(dk)^4}{(k'_\mu k'_\mu)^2}\,\delta_{\alpha\beta}$$

$$\int_0^K\cdots\int_{C_F} \frac{(dk')^4}{k'_\mu k'_\mu} = 2\pi^2 i \log K .$$

$$\lim_{p\to\infty}\int S_F(x) S_F(x) e^{-ip_\mu x_\mu}(dx)^4 = \frac{+3M}{4\pi^2} i \log\Lambda \,\delta_{\alpha\beta}$$

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$$(p,j\,|S|\,p',j') = \frac{+3g^2\cdot i}{(4\pi)^2} M \log\Lambda$$

$$(2\pi)^4\delta(p-p')\,\delta(p_0-p_0')\sum_{p,\alpha}(\bar{u}_{p_0 j})_\beta(\gamma_4)_{\beta\alpha}(u_{p'j'})_\alpha$$

$$\frac{1}{(2\pi)^3}$$

Feynman's method of integration

$$\frac{1}{ab} = \int_0^1 du\,\frac{1}{(au+b(1-u))^2}$$

$$\int\cdot\int(dk)^4\int_0^1 du\,\frac{\{i(p_\mu-k_\mu)\gamma_\mu - M\}_{\alpha\beta}}{\{(k_\mu k_\mu+\kappa^2-i\varepsilon-2u\,p_\mu k_\mu + u\,p_\mu p_\mu + u(M^2-\kappa^2-i\varepsilon'+i\varepsilon)\}^2}$$

$$k_\mu = k'_\mu + u\,p_\mu$$

$$\int\cdot\int(dk')^4\int_0^1 du\,\frac{\{i(1-u)p_\mu\gamma_\mu - i k'_\mu\gamma_\mu - M\}_{\alpha\beta}}{\{k'_\mu k'_\mu+\kappa^2-i\varepsilon+u(1+u)p_\mu p_\mu + u(M^2-\kappa^2-i\varepsilon'+i\varepsilon)\}^2}$$

$$= \int_0^1 du\,\{i(1-u)p_\mu\gamma_\mu - M\}_{\alpha\beta}\int\cdot\int\frac{(dk')^4}{(k'_\mu k'_\mu)^2}$$

$$+\cdots$$

$$= \{\left[u-\frac{u^2}{2}\right]_0^1 i p_\mu\gamma_\mu - M\}_{\alpha\beta}\,2\pi^2 i\log\Lambda$$

$$= -\frac{3}{2}M\delta_{\alpha\beta}\cdot 2\pi^2 i\log\Lambda$$

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Procedure of Renormalization
(Mass renormalization)

The above logarithmic divergence can be compensated by adding an ~~additional~~ invariant term in the Hamiltonian

$$H'(x) = -\hbar c\, \delta M \cdot \bar{\psi}(x)\psi(x)$$

or in the Lagrangian:

$$L'(x) = \hbar c\, \delta M\, \bar{\psi}(x)\psi(x)$$

$$-\frac{i}{\hbar c}\int\cdots\int H''(x)(dx)^4 = i\frac{1}{(2\pi)^3}(\bar{u}_\rho)(\Gamma_4)(u_\beta)$$

$$\times \delta M \times (2\pi)^4 \delta(\quad)\delta(\quad)\delta(\quad)\delta(\quad),$$

if we assume

$$\delta M = \frac{-3g^2 M}{(4\pi)^2}(\log\Lambda + \cdots)$$

Such a term has the same form as the mass term

$$-\hbar c\, M\, \bar{\psi}(x)\psi(x)$$

in the original Lagrangian, so that we can argue that the experimentally measured mass of the spinor particle

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were not the mass M from which we started, but ~~$M+\delta M$~~ could be $M+\delta M$, because there is no ~~distinction~~ way to experimentally discriminating between them and, if we take the latter viewpoint, the compensating term $H''(x)$ is to be added. This is the so-called renormalization of mass, which is a formal way of removing the divergence ~~δM~~ due to the infinite self-energy in a relativ. invariant way.

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V. Quantum Electrodynamics

$$L_{sp} = \frac{\hbar c}{2}\left(\bar{\psi}\gamma_\mu \frac{\partial \psi}{\partial x_\mu} - \frac{\partial \bar{\psi}}{\partial x_\mu}\gamma_\mu \psi + 2M\bar{\psi}\psi\right)$$

$$L_{el.mg.} = -\frac{\hbar c}{4}\left\{\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right)\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) + 2\frac{\partial A_\mu}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}\right\}$$

$$= -\frac{\hbar c}{4}\left\{2\frac{\partial A_\nu}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\mu} - 2\frac{\partial A_\mu}{\partial x_\nu}\frac{\partial A_\nu}{\partial x_\mu} + 2\frac{\partial A_\mu}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}\right\}$$

$$= -\frac{\hbar c}{2}\left(\frac{\partial A_\nu}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial}{\partial x_\nu}\left(A_\mu \frac{\partial A_\nu}{\partial x_\mu}\right) + \frac{\partial}{\partial x_\mu}\left(A_\mu \frac{\partial A_\nu}{\partial x_\nu}\right)\right)$$

$$= -\frac{\hbar c}{2}\left\{\frac{\partial A_\nu}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial}{\partial x_\mu}\left(A_\nu \frac{\partial A_\mu}{\partial x_\nu} - A_\mu \frac{\partial A_\nu}{\partial x_\nu}\right)\right\}$$

Free el. mg. field

$$\Box A_\nu \equiv \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\mu} = 0$$

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$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

$$F_{12} = H_z$$

$$F_{34} = \frac{\partial A_3}{\partial x_4} - \frac{\partial A_4}{\partial x_3}$$

$$= i\left(-\frac{1}{c}\frac{\partial A_z}{\partial t} - \frac{\partial A_0}{\partial z}\right)$$

$A_4 = iA_0$, $F_{34} = +iE_z$

// etc.

Commutation relations:

$$[A_\mu(x), A_\nu(x')] = i D(x-x') \delta_{\mu\nu}$$

$$D(x) = \{\Delta_0(x)\}_{x=0}.$$

Supplementary condition:

$$A_\mu(x) = A_\mu^{(-)}(x) + A_\mu^{(+1)}(x)$$

$$A_\mu^{(-)}(x) = \frac{1}{\sqrt{(2\pi)^3}} \iiint_{k_0=+k} \frac{1}{\sqrt{2k_0}} a_{k,\mu} e^{ik_\mu x_\mu} (dk)^3$$

$$A_\mu^{(+1)}(x) = \frac{1}{\sqrt{(2\pi)^3}} \iiint \frac{1}{\sqrt{2k_0}} \overline{a}^{*}_{k,\mu} e^{-ik_\mu x_\mu} (dk)^3$$

$$\{a_{k\mu}, \overline{a}^{*}_{k'\nu}\} = \delta(k-k')\delta_{\mu\nu}$$

etc.

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~~However, for $\mu=4$, A_4 is pure imaginary or anti-hermitian, so that $a_{k,4}$ and $a^*_{k,4}$~~

For $\mu=1,2,3$, A_μ is hermitian and

$$\tilde{a}_{k,j} = a^*_{k,j}$$

so that we have

$$[a_{k,j},\ a_{k',j'}] = \delta(k-k')\delta_{jj'}$$

$j \neq j' = 1,2,3$.

$a_{k,j}$: annihilation operators

$a^*_{k,j}$: creation operators.

However, for $\mu=4$, A_4 is anti-herm. and we may write

$$A^*_4 = -A_4$$

$$a_{k,4} = +i b^*_{k,4}$$

$$\tilde{a}_{k,4} = +\bar{i} b_{k,4}$$

$$[b_{k,4},\ b^*_{k',4}] = \delta(k-k')$$

Thus, $-i\tilde{a}_{k,4}$ instead of $a_{k,4}$ is the annihilation operator while $-ia_{k,4}$ is the creation operator for the photon with the negative

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energy $-\hbar k c$, and momentum $\hbar k c$, and the direction of polarization being in time-direction.

The difficulty of negative energy photons is eliminated by the supplementary condition

$$\frac{\partial A_\mu^{(+)}}{\partial x_\mu}\Psi = 0 \qquad \frac{\partial A_\mu}{\partial x_\mu}\Psi = 0$$

or

$$\{(k_\mu a_{k\mu}\Psi) = 0 \quad (\Psi^\dagger(k_\mu \bar{a}_{k\mu}) = 0$$

$$k_\mu \bar{a}_{k\mu}\Psi = 0$$

which are q.m. version of classical Lorentz condition

$$\frac{\partial A_\mu}{\partial x_\mu} = 0$$

Reason:

$$\{k_j a^*_{kj} + i k_0 \cdot i b_{k4}\}\Psi = 0$$

$$\{k_j a^*_{kj} - k_0 b_k\}\Psi = 0$$

$$a_{kj} = b_{k1}e_{1j} + b_{k2}e_{2j} + b_{k3}e_{3j}$$

$$e_{1j} \perp e_{2j} \perp e_{3j}$$

$$e_{3j} = k_j / \sqrt{k_1^2 + k_2^2 + k_3^2}$$

$$(b^*_{k3} - b_{k4})\Psi = 0$$

Thus, if Ψ contains the eigenstate in which ~~one or more~~ n_3, n_4 photons of the

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~~this (positive) and the (two neg. energy)~~
fourth type are present, b_{k4} result in the eigenstate with the photons ~~of the fourth type are present one less, which must be compensated by the eigenstate with photon~~ (n_3, n_4-1), which is to be compensated by (n_3-1, n_4-1) already existent in I. If we proceed in this way, we arrive finally at $(n_3-n_4, 0)$, ~~For example, if $n_3 \geq n_4$~~ if $n_3 - n_4$ may ~~=+~~ thus; we take a appropriate superposition of eigenstates

$(n_3-n_4, 0)$ $(n_3-n_4+1, 1)$ . . .
. . . . . (n_3, n_4) . . . .

On the contrary, if $n_3 < n_4$, $(0, n_4-n_3)$ cannot be compensated. For this reason, ~~the~~ only those eigenstates in which more positive energy photons are ~~positive~~ present than negative energy photons. Thus, there will be only ~~no~~ states of total ~~positive~~ positive (or zero) energy ~~because of~~ zero with $n_3 = n_4$ In spite of the existence of energy energy photons. (From the restriction $(b_{pn} a_{pn}) \Psi = 0$ we have the restriction $n_3 \leq n_4$)

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Gauge-Invariance

$$A'_\mu = A_\mu + \frac{\partial \Lambda}{\partial x_\mu}$$

$\Box \Lambda = 0$:

$$\Lambda = \iiint \{ c_k e^{ik_\mu x_\mu} + \tilde{c}_k e^{-ik_\mu x_\mu} \} \frac{1}{\sqrt{(2\pi)^3 2k_0}} \times (dk)^3$$

$$\frac{\partial \Lambda}{\partial x_\mu} = \lambda \cdot \iiint (ik_\mu c_k e^{ik_\mu x_\mu} - ik_\mu \tilde{c}_k e^{-ik_\mu x_\mu}) \frac{1}{\sqrt{(2\pi)^3 2k_0}} \times (dk)^3$$

$$[a_{k\mu}, k_\nu a^*_{k'\nu}] = \delta(k-k') k_\mu$$

$$[\bar{a}^*_{k'\mu}, k_\nu a_{k'\nu}] = -\delta(k-k') k_\mu$$

$$U = 1 + \lambda (i k_\mu \bar{a}^*_{k\mu} c_k + i k_\mu a_{k\mu} \tilde{c}_k)$$

λ : infinitesimal

$$a'_\mu = U^{-1} a_\mu U = \lambda (a_\mu + i k_\mu c_k)$$

$$\bar{a}'_\mu = U^{-1} \bar{a}^*_\mu U = \lambda (\bar{a}^*_\mu - i k_\mu \tilde{c}_k)$$

$$U = 1 + \lambda \int (i k_\mu \bar{a}_{k\mu} c_k + i k_\mu a_{k\mu} \tilde{c}_k) \times (dk)^3$$

$$[k_\mu \bar{a}_{k\mu}, k'_\nu \bar{a}_{k'\nu}] = 0$$

because $k_\mu k_\mu = 0$.

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$$\Psi' = U^{-1}\Psi$$
$$= (1 - \lambda \int (i k_\mu \bar{a}_{k\mu} c_k + i k_\mu a_{k\mu} \tilde{c}_k)(dk)^3)\Psi$$

$\Psi^\dagger k_\mu a_{k\mu} = 0$

Supplem. cond.
$$k_\mu \bar{a}_{k\mu} \Psi = 0 \quad .$$
$$(k_\mu a_{k\mu} \Psi = 0)$$
$$\Psi_1' \Psi_2'' = \Psi_1 \Psi_2$$

Vacuum State ; the conditions
$$b_{k1} \Psi(0) = 0$$
$$b_{k2} \Psi(0) = 0$$
are to be satisfied in addition to
$$(b_{k3}^* - b_{k4}) \Psi(0) = 0$$
$$(b_{k3} - b_{k4}^*) \Psi(0) = 0$$

This definition is relativ. invariant, since the ~~transverse~~ concept of Transverse wave is relativistic. Namely, the direction of polarization ~~must satisfy~~ is defined by the
$$e_\mu k_\mu = 0$$
$$e_\mu = \xi_1 e_{1\mu} + \xi_2 e_{2\mu} + \xi_3 (e_{3\mu} \quad e.$$

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the restrictions are follows:

~~$a_{k\mu} = b'_{k\nu} e_{\nu\mu}$~~ *

$b'_{k4} = i\, b_{k4}$

$b'_{kj} = b_{kj}$ $\quad j = 1, 2, 3.$

~~$a_{k\mu} e'_\mu$~~ From any direction which is perpendicular to k_μ;

~~$a_{k\mu} e'_\mu$~~ can be determined into the direction which is parallel to k_μ itself *
and that which is genuinely per
an arbitrary vector which is
parallel to k_μ itself * can be
subtracted. The latter
does not make any difference
in $a_{k\mu} e'_\mu$ because of the
supplementary condition.

Thus vacuum can be defined as

$$a_{k\mu} e'_\mu \Psi(0) = 0$$

where e'_μ is any vector $\perp$ ar to k_μ.

* Note that k_μ is at once parallel and perpendicular to itself, because $k_\mu k_\mu = 0$.

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the definition of vacuum is gage invariant. So is the definition of S-matrix. can be written conveniently

Vacuum expection value

$$(0|A_\mu(x)A_\nu(x')|0) = \varepsilon_{\mu\nu} D^{(-)}(x-x')$$

$$a_{k4}\bar{a}_{k4} = -b_{k4}^* b_{k4} \qquad \varepsilon_{\mu\bar{\nu}}(1,1)$$
$$\bar{a}_{k4} a_{k4} = -b_{k4} b_{k4}^*$$

~~together with~~ This relation can be written in a rel. inv. form

$$(0|e'_\mu A_\mu(x)\, e''_\nu A_\nu(x')|0)$$
$$= e'_\mu e''_\nu \delta_{\mu\nu} D^{(-)}(x-x')$$

for any two directions, which are both perpendicular to k_μ, because of suppl. cond. and

$$(0|\tilde{e}'_\mu \bar{a}_{k\mu} \cdot e_\nu a_{k\nu}|0) = 0$$

and if we take into account S.C., this amount to the same thing as to say that there are only transverse photons, so that we may always assume

$$(0|(\gamma_\mu)_{\alpha\beta} A_{k\beta\mu}(x)\, (\gamma_\nu)_{\rho\sigma} A_{\rho\sigma\nu}(x')|0) = \frac{1}{2}(\gamma_\mu)_{\alpha\beta}(\gamma_\mu)_{\rho\sigma}$$
$$\times \Delta^{(-)}(x-x')$$

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Interaction between spinor and elmg. fields.

$$\frac{i}{c} j_\mu(x) = -ie\,\bar\psi\gamma_\mu\psi \quad \left(-\frac{ie}{2}(\bar\psi\gamma_\mu\psi^* - \psi^*\gamma_\mu\bar\psi)\right)$$

electronic charge : $-e$.

$$L'(x) = +\frac{1}{c} j_\mu(x) A_\mu(x)$$

$$H'(x) = -\frac{1}{c} j_\mu(x) A_\mu(x)$$

$$= \frac{ie\cdot\hbar c}{} A_\mu \bar\psi\gamma_\mu\psi$$

$$H''(x) = -\delta m \cdot \hbar c \cdot \bar\psi(x)\psi(x)$$

$$\delta m = \frac{3}{2\pi}\frac{e^2}{4\pi} M(\log K + \cdots)$$

The calculation is essentially the same as that for scalar neutral field, the only difference being the change in the sign and the extra factor 2.

The field equations

$$\left.\begin{aligned} \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\mu} &= ie\,\bar\psi\gamma_\mu\psi \\ \gamma_\mu\frac{\partial\psi}{\partial x_\mu} + M\psi &= -ie\,A_\mu\gamma_\mu\psi \\ \frac{\partial\bar\psi}{\partial x_\mu}\gamma_\mu - M\bar\psi &= ie\,\bar\psi A_\mu\gamma_\mu \end{aligned}\right\}$$

* electron is regarded as the particle. m is the electronic mass. In ordinary units, e^2 is to be replaced by $\hbar\cdot 4\pi\cdot\frac{e^2}{\hbar c}$

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$$A_\mu(x) = A_\mu^{(in)}(x) - ie\int\cdots\int D^{(ret)}(x-x')$$
$$\times \bar\psi(x')\gamma_\mu\psi(x')(dx')^4$$

$$\psi(x) = \psi^{(in)}(x) + ie\int\cdots\int S^{(ret)}(x-x')$$
$$\times \gamma_\mu A_\mu(x')\psi(x')(dx')^4$$

$$\bar\psi(x) = \bar\psi^{(in)}(x) + ie\int \bar\psi(x')\gamma_\mu A_\mu(x')$$
$$\times S^{(adv)}(x'-x)(dx')^4,$$

We can construct the S-matrix as before. The actual form of S-matrix is different from the case of spinor and scalar fields only by

$$\left(\frac{1}{\hbar c}\right)H'(x) = ieA_\mu\bar\psi\gamma_\mu\psi$$
$$\left(\text{or } = \tfrac{1}{2}ieA_\mu(\bar\psi\gamma_\mu\psi - \psi\gamma_\mu\bar\psi)\right)$$

in place of

$$-g\varphi\bar\psi\psi$$

However, there is an additional complication due to the supplementary condition in our case. The S.C. which is customary assumed is

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$$\left(\frac{\partial A_\mu}{\partial x_\mu}\right)\Psi = 0 .$$

for any physically acceptable state Ψ.
This condition is not as simple as it looks, because for practical purposes we have to have a condition in terms of the free field ~~quantities~~ solutions such as incoming fields* instead of the actual (Heisenberg) field operators.

$$\frac{\partial A_\mu}{\partial x_\mu} = \frac{\partial A_\mu^{(in)}}{\partial x_\mu}$$

$$-ie\int\!\!\int \frac{\partial D^{(ret)}(x-x')}{\partial x_\mu}\,\bar{\psi}(x')\gamma_\mu\psi(x')(dx')^4 \neq 0$$

---

* They satisfy the comm. rel.

$$[\psi_\alpha^{(in)}(x), \psi_\beta^{(in)}(x')]_+ = -iS_{\alpha\beta}(x-x')$$

etc.

$$[A_\mu^{(in)}(x), A_\nu^{(in)}(x')] = i\delta_{\mu\nu}D(x-x')$$

etc.

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or $\dfrac{\partial A_\mu^{(in)}}{\partial x_\mu} + ie\int\cdots\int \dfrac{\partial}{\partial x_\mu}\{D^{(ret)}(x-x')\bar{\psi}(x')\gamma_\mu\psi(x')\}dx'$

$= 0$

or $\dfrac{\partial A_\mu^{(in)}(x)}{\partial x_\mu} + ie\int_{\sigma(x\,\mathrm{out}\,\sigma)} D^{(ret)}(x-x')\bar{\psi}(x')\gamma_\mu\psi(x')\times d\sigma_\mu$

σ

$D^{(ret)}$ $\quad - ie\int_{\sigma=-\infty} D^{(ret)}(x-x')\bar{\psi}^{(in)}(x')\gamma_\mu\psi^{(in)}(x')\times d\sigma_\mu$

$\neq 0$

or $\left\{\dfrac{\partial A_\mu^{(in)}(x)}{\partial x_\mu} - ie\int_{\sigma=-\infty} D^{(ret)}(x-x')\bar{\psi}^{(in)}(x')\times\gamma_\mu\psi^{(in)}(x')\times d\sigma_\mu\right\}\psi$

$\times\Psi = 0$

or

$C(x_\mu)\Psi = 0$

$C(x_\mu) = \dfrac{\partial A_\mu^{(in)}(x)}{\partial x_\mu} + ie\int_{\sigma=-\infty} D^{(ret)}(x-x')\bar{\psi}^{(in)}(x')\times\gamma_\mu\psi^{(in)}(x)\,d\sigma_\mu$

because $D^{(ret)}$ can be replaced by

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$$[C(x_\mu), H^{(in)}(x_\mu'')]$$

$$= \frac{ie}{\hbar c}\left\{\left[\frac{\partial A_\mu^{(in)}(x_\mu)}{\partial x_\mu}, A_\nu^{(in)}(x'')\bar{\psi}^{(in)}(x'')\gamma_\nu\psi(x'')\right]\right.$$

$$\left. -ie\int_{\sigma=-\infty} d\sigma_\mu D^{(ret)}(x-x'')\left[\bar{\psi}^{(in)}(x)\gamma_\mu\psi^{(in)}(x'), A_\nu^{(in)}(x'')\bar{\psi}^{(in)}(x'')\gamma_\nu\psi(x'')\right]\right\}$$

$$\left[\frac{\partial A_\mu}{\partial x_\mu}, A_\nu(x'')\bar{\psi}(x'')\gamma_\nu\psi(x'')\right]$$

$$= i\frac{\partial D(x_\mu - x_\mu'')}{\partial x_\mu}\bar{\psi}(x'')\gamma_\mu\psi(x'')$$

$$\left[\bar{\psi}^{(in)}(x')\gamma_\mu\psi(x'), A_\nu(x'')\bar{\psi}(x')\gamma_\nu\psi(x'')\right]$$

$$= -i\left\{A_\nu(x''), \bar{\psi}(x')\gamma_\mu S(x'-x'')\gamma_\nu\psi(x'')\right.$$

$$\left. - A_\nu(x'')\bar{\psi}(x'')\gamma_\nu S(x''-x')\gamma_\mu\psi(x')\right\}$$

$$-ie\int_{\sigma=-\infty} d\sigma_\mu D^{(ret)}(x-x')\{\quad\}$$

$$= -e\int d\sigma_\mu D^{(ret)}(x-x')\{\bar{\psi}(x')\gamma_\mu S(x'-x'')\gamma_\nu\psi(x'')$$

$$- \bar{\psi}(x'')\gamma_\nu S(x''-x')\gamma_\mu\psi(x')\} A_\nu(x'')$$

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Momentum space:

$$A_\mu^{(in)}(x) = \frac{1}{\sqrt{(2\pi)^3}} \iiint \frac{1}{\sqrt{2k_0}} \{ a_{k,\mu} e^{ik_\mu x_\mu} + \bar{a}_{k,\mu} e^{-ik_\mu x_\mu} \} (dk)^3$$

$$\frac{\partial A_\mu^{(in)}(x)}{\partial x_\mu} = \frac{1}{\sqrt{(2\pi)^3}} \iiint \frac{i}{\sqrt{2k_0}} \{ k_\mu a_{k,\mu} e^{ik_\mu x_\mu} - k_\mu \bar{a}_{k,\mu} e^{-ik_\mu x_\mu} \} (dk)^3$$

$$\psi_\alpha^{(in)}(x) = \frac{1}{\sqrt{(2\pi)^3}} \iiint \{ u_{p,j} {}_\alpha b_{p,j} e^{ip_\mu x_\mu} + (\bar{v}_{p,j})_\alpha c^*_{p,j} e^{-ip_\mu x_\mu} \} dp^3$$

$$\bar{\psi}_\beta^{(in)}(x) = \frac{1}{\sqrt{(2\pi)^3}} \iiint \{ (\bar{u}_{p,j})_\beta b^*_{p,j} e^{-ip_\mu x_\mu} + (\bar{v}_{p,j})_\beta c_{p,j} e^{ip_\mu x_\mu} \} dp^3$$

$$D^{(ret)}(x) = \frac{1}{(2\pi)^4} \iiint_{C_1} \frac{e^{ik_\mu x_\mu}}{k^2 - k_0^2} (dk)^3 (dk_0)$$

etc.

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Supplementary Condition in terms of $A^{(0)}$, $\psi^{(0)}$, $\bar{\psi}^{(0)}$:

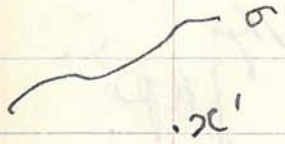
$$A_\mu(x) = A_\mu^{(0)}(x) - ie\int\cdots\int D^{(0)}(x-x') \times \bar{\psi}(x')\gamma_\mu\psi(x')(dx')^4$$

where

$$D^{(0)}(x,x') = \frac{1-\varepsilon(\sigma;x')}{2} D^{(ret)}(x-x') + \frac{1+\varepsilon(\sigma;x')}{2} D^{(adv)}(x-x')$$

$$\varepsilon(\sigma,x') = \begin{cases} +1 & \sigma: \text{later than } x' \\ -1 & \sigma: \text{earlier than } x' \end{cases}$$

$$D^{(0)}(x,x') = -\frac{1}{2}[\varepsilon(x-x') - \varepsilon(\sigma;x')] \times D(x-x')$$



$$\overline{D}(x) = -\frac{1}{2}\varepsilon(x)D(x) = \frac{1}{2}\left(D^{(ret)}(x) + D^{(adv)}(x)\right)$$

$$\Delta(x) = : D^{(adv)} - D^{(ret)}$$

$$\frac{\partial A_\mu}{\partial x_\mu}\Psi = 0 \rightarrow$$

$$\left\{\frac{\partial A_\mu^{(0)}(x)}{\partial x_\mu} + ie\int\cdots\int \frac{\partial}{\partial x'_\mu}\left(D^{(0)}(x-x')\bar{\psi}(x')\gamma_\mu\psi(x')\right)(dx')^4\right\}\Psi = 0$$

$$\left\{\frac{\partial A_\mu^{(0)}(x)}{\partial x_\mu} + ie\int_{\sigma'=+\infty} D^{(0)}(x-x')\bar{\psi}(x')\gamma_\mu\psi(x')\,d\sigma'_\mu\right.$$

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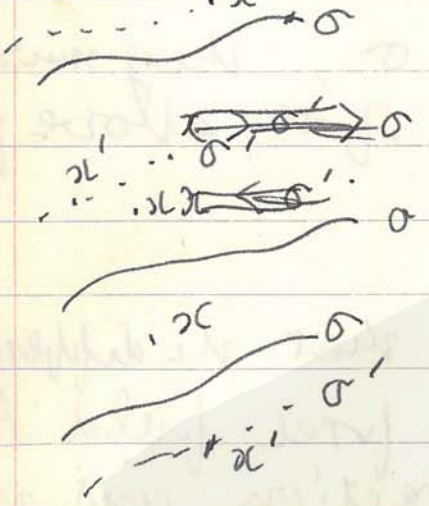
$$-ie\int_{\sigma'=\sigma+0} D^{(\sigma)}(x-x')\bar{\psi}(x')\gamma_\mu\psi(x')d\sigma'_\mu$$

$$+ie\int_{\sigma'=\sigma-0} D^{(\sigma)}(x-x')\bar{\psi}(x')\gamma_\mu\psi(x')d\sigma'_\mu$$

$$-ie\int_{\sigma'=-\omega} D^{(\sigma)}(x-x')\bar{\psi}(x')\gamma_\mu\psi(x')d\sigma'_\mu\}\Psi$$

$$=0$$

case I, $x>\sigma$



$\varepsilon(x-x') - \varepsilon(\sigma,x') = 2$
for $x>\sigma'>\sigma$

$\varepsilon(x-x') - \varepsilon(\sigma,x') = 0$
for $\sigma'>x>\sigma$

$D^{(\sigma)}$ is different from zero only between σ and x

$\varepsilon(x-x') - \varepsilon(\sigma,x') = 0$
for $x>\sigma>\sigma'$

In this case

$$\left\{\frac{\partial A_n^{(\sigma)}(x)}{\partial x_n} + ie\int_\sigma D(x-x')\bar{\psi}^{(\sigma)}(x')\gamma_n\psi^{(\sigma)}(x')\times d\sigma'_n\right\}\Psi=0$$

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Case II, $\sigma > x$

$$\xi(x,x') - \xi(\sigma,x') = -2 \quad \text{for } \sigma > x' > x$$
$$\xi(x,x') - \xi(\sigma,x') = 0 \quad \text{otherwise}$$

Again we have

$$\left\{ \frac{\partial A_\mu^{(0)}(x)}{\partial \sigma(x)} + ie \int_\sigma D(x-x') \bar{\psi}^{(0)}(x') \gamma_\mu \psi^{(0)}(x) \, d\sigma_{x'} \right\} \Psi = 0$$

Thus, if we start from free field operators which coincide with the actual field on σ, they must satisfy the S.C. of the above form at any point x.

It should be noticed that the difference between the entirely free field $A_\mu^{(\text{free})(0)}$ in the absence of interaction and the free field $A_\mu^{(0)}$ above considered is that not only $A_\mu^{(0)}(x) = A_\mu(x)$ on σ with x on σ, but also $n_\mu \frac{\partial A_\mu^{(0)}}{\partial x_\nu} = n_\nu \frac{\partial A_\mu}{\partial x_\nu}$ was required, the latter being governed by the f.o. eq. of motion with interaction.

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In any case, the above discussed S.C. is inconvenient for the actual calculations. There was another difficulty in connection with the above S.C. already in the limit of no interaction ($\lim_{e\to 0}$). Namely, the states Ψ which satisfies

$$\frac{\partial A_\mu}{\partial x_\mu}\Psi = 0$$

in the absence of interaction were necessarily ~~the~~ a definite superposition of states with equal number of longitudinal and scalar (time-like) photons. Such a state can not be normalized in the ordinary sense.

Reason:

$$\left.\begin{aligned}(b_3^* - b_4)\Psi_n &= 0\\ (b_3 - b_4^*)\Psi &= 0\end{aligned}\right\}$$

$$[(b_3^* - b_4),(b_3 - b_4^*)] = 0$$

$$\Psi_n = \sum_{n=0}^{\infty} c_n \frac{(b_3^* b_4^*)^n}{n!}\Psi(0,0)$$

$$(b_3^* - b_4)\Psi = \sum_{n=0}^{\infty} c_n (b_3^*)^{n+1}(b_4^*)^n/n!\;\Psi(0,0)$$

$$- \sum_{n=1}^{\infty} c_n\, n\cdot (b_3^*)^n (b_4^*)^{n-1}/n!\;\Psi(0,0)$$

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$$= \sum_{n=0}^{\infty} (C_n - C_{n+1}) (b_3^*)^{n+1} (b_4^*)^n / n! \quad \times \Psi(0,0)$$

$C_0 = C_1 = C_2 = \cdots$

$$\Psi^T \Psi = (|C_0|^2 + |C_1|^2 + |C_2|^2 + \cdots) \Psi^\dagger(0,0) \Psi(0,0)$$

$$= \infty \times |C_0|^2 \Psi^\dagger(0,0) \Psi(0,0).$$

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Gupta-Bleuler's method of
q. of el. mg. field
S. N. Gupta, Proc. Phys. Soc.
A 63 (1950), 687
K. Bleuler, Helv. Phys. Acta.

$$A_\mu(x) = \sum_k \frac{1}{\sqrt{2kV}} \left(a_{k\mu} e^{ik_\mu x_\mu} + \bar{a}_{k\mu} e^{-ik_\mu x_\mu} \right)$$

$$\bar{a}_{k\mu} = \varepsilon_\mu a^*_{k\mu}$$

$$a_{k\mu} = \sum_{j=1}^{4} e^{(j)}_\mu b_{kj} \qquad \bar{a}_{k\mu} = \sum_{j=1}^{4} e^{(j)}_\mu \bar{b}_{kj}$$

Example

$e^{(1)}_\mu$ $(e^{(1)}_1, e^{(1)}_2, e^{(1)}_3, 0)$
$e^{(2)}_\mu$ $(e^{(2)}_1, e^{(2)}_2, e^{(2)}_3, 0)$
$e^{(3)}_\mu$ $(k_1/k, k_2/k, k_3/k, 0)$
$e^{(4)}_\mu$ $(0, 0, 0, i)$

~~$\sum_{\mu\nu} e_\mu$~~ $\sum_{j,l} \varepsilon_{jl} e^{(j)}_\mu e^{(l)}_\nu = \delta_{\mu\nu}$

$$[A_\mu(x), A_\nu(x')] = i \delta_{\mu\nu} D(x - x')$$

$$[a_{k\mu}, \bar{a}_{k'\nu}] = * \delta_{kk'} \delta_{\mu\nu}$$

$$\{b_{kj}, \bar{b}_{k'l}\} = \varepsilon_{jl} \delta_{kk'}$$

$$k_\mu a_{k\mu} = \sum_{j=3,4} k_\mu e^{(j)}_\mu b_{kj}$$

$$\left(= k (b_{k3} - b_{k4} \right.$$

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$$\frac{1}{ic}H = \sum \frac{\partial A_\mu}{\partial x_4}\frac{\partial L}{\partial \frac{\partial A_\mu}{\partial x_4}} - L$$

$$= -\frac{\partial A_j}{\partial x_4}\left(\frac{\partial A_j}{\partial x_4} - \frac{\partial A_4}{\partial x_j}\right)$$

$$-\left(\frac{\partial A_j}{\partial x_j} + \frac{\partial A_4}{\partial x_4}\right)\frac{\partial A_4}{\partial x_4}$$

$$+\frac{1}{4}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right)\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right)$$

$$+\frac{1}{2}\frac{\partial A_\mu}{\partial x_\mu}\frac{\partial A_\nu}{\partial x_\nu}$$

$$= \mathbf{E}^2 - \frac{\partial A_4}{\partial x_j}\left(\frac{\partial A_j}{\partial x_4} - \frac{\partial A_4}{\partial x_j}\right)$$

$$-\frac{1}{2}\mathbf{E}^2 + \frac{1}{2}\mathbf{H}^2 + \frac{1}{2}\left(\frac{\partial A_j}{\partial x_j} + \frac{\partial A_4}{\partial x_4}\right)$$

$$\times\left(\frac{\partial A_j}{\partial x_j} - \frac{\partial A_4}{\partial x_4}\right)$$

$$= \left(\frac{\partial}{\partial x_0}\right)\left(\frac{\partial A_j}{\partial x_4} + \frac{\partial A_4}{\partial x_j}\right)\left(\frac{\partial A_j}{\partial x_4} - \frac{\partial A_4}{\partial x_j}\right)$$

$$\left(-\frac{1}{2}\right)$$

$$+\frac{1}{2}(\mathrm{curl}\,\mathbf{A})^2$$

$$+\frac{1}{2}\left(\frac{\partial A_j}{\partial x_j} + \frac{\partial A_4}{\partial x_4}\right)\left(\frac{\partial A_j}{\partial x_j} - \frac{\partial A_4}{\partial x_4}\right)$$

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$$= \frac{1}{2}\left\{\left(\frac{\partial A}{\partial x_0}\right)^2 - \left(\frac{\partial A_0}{\partial x_j}\right)^2 + (\mathrm{curl}\,A)^2 + (\mathrm{div}\,A)^2 - \cancel{(\mathrm{grad}\,A_0)}\left(\frac{\partial A_0}{\partial x_0}\right)^2\right\}$$

$$\frac{1}{\hbar c}\overline{H} = \sum_{k,j} k_j \bar{b}_{k,j} b_{k,j}$$

$$\bar{b}_j b_j = b_j^* b_j = n_j \qquad j = 1, 2, 3.$$

$$\Psi(n_3) = (n_3!)^{-1/2} (b_j^*)^{n_1} \Psi(0)$$

$$\left.\begin{aligned}(b_3)_{m,n_1} &= \Psi^+(n_3)\, b_3\, \Psi(n_3) \\ &= n_3^{1/2}\, \delta_{m,n_1-1}\end{aligned}\right\}$$

$$(\overline{b_3^*})_{n_1 m_1 m} = n_3^{1/2}\, \delta_{m,n_1-1}$$

Indefinite metric

$$\overline{\Psi} = \Psi^* \eta$$

$$\overline{\Psi}\Psi = \Psi^* \eta \Psi$$

η ; Hermitian

$$\overline{A} \equiv \eta^{-1} A^* \eta$$

self-adjoint operator

$$A = \overline{A}$$

$$\Psi^* \eta A \Psi = \Psi^* A^* \eta \Psi$$

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(∵ $\bar{\Psi} A \Psi = \bar{\Psi} \bar{A} \Psi$)

the expectation value of self-adjoint operator is <u>real</u>.

$$[b_4, \bar{b}_4] = -1 \qquad \bar{b}_4 = -b_4^*$$
$$[b_4, b_4^*] = 1$$

($a_4 = i b_4$ $\qquad \bar{a}_4 = i \bar{b}_4$

$-a_4^* = i b_4$ $\qquad \bar{a}_4$

$i b_4^* = i \bar{b}_4$

if we assum in order that $A_4 = i A_0$ with A_0: hermitian.

Thus, the assumption

$$\bar{b}_4 = -b_4^* \qquad a_4 = i b_4$$

corresponds to $\qquad \bar{a}_4 = i \bar{b}_4$

$$\bar{a}_4 = a_4^* \qquad a_4^* = -i b_4^*$$

so that A_0 is imag. and A_4 is real, if we adopt ordinary metric.

But in the new metric, A_0 may have real expectation value. if

$$A_0 = \eta^{-1} A_0^* \eta = -\eta^{-1} A_0 \eta$$

or if A_0 anti-commutes with η.

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Namely, we start from

$$-\overline{b_4} b_4 = b_4^* b_4 = n_4$$

$$\Psi(n_4) = (n_4!)^{-1/2} (\overline{b_4})^{n_4} \Psi(0)$$

$$\overline{\Psi}(n_4) = \overline{\Psi}(0) (b_4)^{n_4} (n_4!)^{-1/2}$$

regarding b_4 and $\overline{b_4}$ as annihilation and creation operators formally.

$$(b_4)_{m_4 n_4} = \overline{\Psi}(m_4) b_4 \Psi(n_4)$$

$$= (-1)^{n_4} n_4^{1/2} \delta_{m_4, n_4 - 1}$$

$$(\overline{b_4})_{n_4 m_4} = (-1)^{n_4} n_4^{1/2} \delta_{m_4, n_4 - 1}$$

$$\overline{\Psi}\Psi = \Psi^* \eta \Psi$$

$$\Psi^*(m_1 m_2 m_3 m_4) \eta \Psi(n_1 n_2 n_3 n_4)$$

$$= (-1)^{n_4} \delta_{m_1 n_1} \cdots \delta_{m_4 n_4}$$

$$(m_1 \cdots m_4 | \eta | n_1 \cdots n_4) = (-1)^{n_4} \delta_{m_1 n_1} \cdots \delta_{m_4 n_4}$$

On the other hand, the ordinary algebraic rules for constructing sums and products of matrices are not retained: For example,

$$\left.\begin{aligned} (\overline{b_4} b_4)_{m_4 n_4} &= -(-1)^{n_4} n_4 \, \delta_{m_4 n_4} \\ (b_4 \overline{b_4})_{m_4 n_4} &= -(-1)^{n_4} (n_4 + 1) \, \delta_{m_4 n_4} \end{aligned}\right\}$$

Thus, if we take the vacuum expectation value of $b_4 \overline{b_4}$, we have

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$$(0|b_4\overline{b_4}|0) = -1$$

or $$(0|a_4\overline{a_4}|0) = +1$$

because $a_4 = ib_4$ $\quad \overline{a_4} = i\overline{b_4} = -ib_4^*$

(If we ~~kee~~ stick to the ordinary metric, we would have the same result, but the definition of vacuum was so ~~minus sign~~ complicated for the ~~va~~ perturbation that the verification was ~~value of $a_4\overline{a_4}$.~~ difficult)

In other words, ~~instead~~ of b_4, b_4^*, or $b_4, \overline{b_4}$ are treated just as if they were annih. and creation operators, if we changed the metric. The commutation relation

$$[b_4, \overline{b_4}] = -1$$

was ~~changed into~~ retained

$$[b_4, \overline{b_4}] = 1.$$

However, since η anticommute with $b_4, \overline{b_4}$, A_0 has still real expectation value.

$$\therefore \quad (b_4\eta)_{m_4 n_4} = (-1)^{n_4} n_4^{1/2} \delta_{m_4+1, n_4} (-1)^{n_4}$$

$$(\eta b_4)_{m_4 n_4} = (-1)^{n_4+1} n_4^{1/2} \delta_{m_4-1, n_4}$$

etc.

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Supplementary condition:

$$\left.\begin{aligned} k_\mu e^{(1)}_\mu = k_\mu e^{(2)}_\mu = 0 \\ k_\mu e^{(3)}_\mu \text{ \& } k_\mu e^{(4)}_\mu = 0 \end{aligned}\right\}$$

$e^{(3)} = (k_1/k,\ k_2/k,\ k_3/k,\ 0)$

$e^{(4)} = (0,\ 0,\ 0,\ i)$

$k_4 = ik$

$$\frac{\partial A^{(-)}_\mu}{\partial x_\mu}\Psi = 0$$

$$\overline{\Psi}\ \frac{\partial A^{(+)}_\mu}{\partial x_\mu} = 0$$

$$\left.\begin{aligned} i k_\mu e^{(j)}_\mu b_{kj}\Psi = 0 \\ \overline{\Psi}\, i k_\mu e^{(j)}_\mu b^\dagger_{kj} = 0 \end{aligned}\right\}$$

$$\left\{\begin{aligned} (b_{k3} - b_{k4})\Psi = 0 \\ \overline{\Psi}(b^\dagger_{k3} - b^\dagger_{k4}) = 0 \end{aligned}\right.$$

$$\Psi = \sum_{n_1 \cdots n_4} C_{n_1 n_2 n_3 n_4} \Psi(n_1 n_2 n_3 n_4)$$

(more precisely

$$\Psi = \sum_{kj} C_{\cdots n_{k4} - n_{k4}} \Psi(\cdots n_{k1} \cdots n_{k4} \cdots)$$

$$n_3^{1/2} C_{n_3, n_4 - 1} + n_4^{1/2} C_{n_3 - 1, n_4} = 0$$

$\Psi_0 = \Psi(n_1, n_2, 0, 0)$

$\Psi_1 = \Psi(n_1, n_2, 1, 0) - \Psi(n_1, n_2, 0, 1)$

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$\Psi_2 = \Psi(2,0) - \sqrt{2}\,\Psi(1,1) + \Psi(0,2)$

$\Psi_j = \Psi(j,0) - \sqrt{j}\,\Psi(j-1,1) + \sqrt{\frac{j(j-1)}{1\cdot 2}}\,\Psi(\cdots j-2,2) + (-1)^j \Psi(0,j)$

$\Psi_j \cdot \Psi_j = 0 \qquad j = 1, 2, \cdots$

$\Psi_0 \Psi_0 = 1.$

because $(1-1)^j = 0.$

$\Psi = \Psi_0 + a_1 \Psi_1 + a_2 \Psi_2 + \cdots$

$\bar{\Psi}\Psi = \bar{\Psi}_0 \Psi_0$

$\Psi_1, \Psi_2, \cdots$ are redundant states in many cases.

Main conclusions we obtained:

(i) There is a well-defined vacuum for long. and scalar photons.

(ii) To the expectation value of a diagonal operator such as the Hamiltonian for the radiation for any state satisfying suppl. condition, only the vacuum state Ψ_0 contribute.

(iii) The expection value of any operator ~~can be computed~~ for the vacuum can be computed by assuming

$(0 | a_{k+} \bar{a}_{k+} | 0) = -1, \quad (0 | a_{k3} \bar{a}_{k3} | 0) = 1,$

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~~without causing any contradiction.~~
The great simplification due to these new rules is evident: In the calcul. of matrix elements of S, if there is a contraction of the type

$\overline{\psi(x)}\,\gamma_\mu\psi(x)\,A_\mu(x)$ $\quad$ $\overline{\psi(x')}\,\gamma_\nu\psi(x)\,A_\nu(x')$

we may simply take

$$(0|A_\mu(x)A_\nu(x')|0) = i\delta_{\mu\nu}D(x-x')$$

forgetting all about ~~the strange behavior of the L. and sc. prod.~~ sc. products.
It should be noticed that the above S.C. can be retained even when there is interaction. However, since

$$\Box A_\mu = -j_\mu,$$

~~we have to it is not possible~~ that the Heisenberg operator A_μ can not be decomposed into neg. freq. and pos. freq. part. But $A_\mu^{(in)}$ can still be decomposed and thus we may assume the same S.C. for $A_\mu^{(in(-))}$ as before.

The above argument may not be convincing because of the strange procedure of indefinite metric. However, it was confirmed by

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more orthodox method by T. and S.,
S. Tomonaga, P. T. P. 2(1947), 101,
Z. Koba, T. Tati and S-ichiro Tomonaga, P. T. P. 3(1948), 290,
Schwinger. P. R. 74. 1439(1948)
(1949)

According to Tomonaga, the S. C. is

$$\left\{ \frac{\partial A_\mu^{(ex)}}{\partial x_\mu} + ie \int_\sigma D(x-x') \bar{\psi}(x') \gamma_\mu \psi(x') \times d\sigma_\mu' \right\} \Psi = 0$$

(x on σ

It was proved that this condition is compatible with the T. S. eq.
Namely,

$$\left[i\hbar \frac{\delta}{\delta \sigma_P} - H'(P), \quad \frac{\partial A_\mu^{(ex)}}{\partial x_\mu}\Big|_X + ie \int D(X-\cdots) \right] = 0$$

for any two points P, X on σ.
If we eliminate the l. and sc. photons, we have the familiar formalism.
Covariant elimination of l. and sc. photons was done by Schwinger in a most satisfactory, but very

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(V, 4)

complicated way.

See also

Dyson, P.R. 77 (1950), 420.

$$A_\mu(x) = \sum_{k,j} g(k)\cdot(b_j\, e^{ikx} + b_j^*\, e^{-ikx}) \times e_\mu^{(j)}$$

$$\sum_j e_\lambda^{(j)} e_\mu^{(j)} = \delta_{\mu\nu} \qquad (e_\mu^{(j)} k_\mu = 0 \quad j = 1,2)$$

$$(e_\mu^{(3)} + i e_\mu^{(4)})\, k_\mu = f(k) k_\mu$$

~~$a_4 = -a_4^*$~~ $\quad b_4 = -b_4^*$

~~$(a_\mu^{(3)} + i a_4)\,\Psi = 0$ (S.C.)~~

~~$[a_\mu^{(j)}, a_j]$~~ $k_\mu e_\mu^{(j)} b_j\,\Psi = 0$

$$(b_3 + i b_4)\,\Psi = 0$$

$$[b_j, \bar{b}_\ell] = \delta_{j\ell}$$

(This is different from Gupta's b_4, $\bar{b}_4$ by the factor i)

$$(0|\bar{b}_3^*\, b_3|0) = (e_\mu^{(3)} k_\mu)^2 \phi(k)$$

$$(0|\bar{b}_3\, b_4|0) = -i(e_\mu^{(3)} k_\mu)(e_\mu^{(3)} k_\mu)\phi(k)$$

$$= -i(e_\mu^{(3)} k_\mu)(-i e_\mu^{(4)} k_\mu)\phi(k)$$

$$= -(e_\mu^{(3)} k_\mu)(e_\mu^{(4)} k_\mu)\phi(k)$$

?

$$(0|\bar{b}_j, b_k|0) = \varepsilon_{jk}(e_\mu^{(j)} k_\mu)(e_\nu^{(k)} k_\mu)\phi(k)$$

$$(0|b_j\ \bar{b}_\ell|0) = \delta_{jk} + \varepsilon_{j\ell}\ ,,$$

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Further remark :

$$(b_3 - b_4^*)\,\Psi = 0$$

$$\Psi_0 = \sum_n \frac{(b_3^* b_4^*)^n}{n!}\,\Psi(0,0)$$

$$b_4 b_4^*\,\Psi_0 \approx \sum_n (n+1)\frac{(b_3^* b_4^*)^n}{n!}\,\Psi(0,0)$$

$$b_4^* b_4\,\Psi_0 = \sum_n n\,\frac{(b_3^* b_4^*)^n}{n!}\,\Psi(0,0)$$

$$b_3 b_3^*\,\Psi_0 = \sum_n (n+1)\frac{(b_3^* b_4^*)^n}{n!}\,\Psi(0,0)$$

$$b_3^* b_3\,\Psi_0 = \sum_n n\,\frac{(b_3^* b_4^*)^n}{n!}\,\Psi(0,0)$$

$$\Psi_0^*\, b_4 b_4^*\,\Psi_0 =$$

$$\Psi_0^*(b_3^* b_3 - b_4 b_4^*)\Psi_0 = -\Psi_0^*\Psi_0$$

$$\Psi_0^*(b_3^* b_3^* - b_4^* b_4^*)\Psi_0 = +\Psi_0^*\Psi_0$$

$$\Psi_0^*\big(b_3^* b_3\, e^{-ik(x-x')} + b_3 b_3^*\, e^{+ik(x-x')} - b_4 b_4^*\, e^{-ik(x-x')} - b_4^* b_4\, e^{ik(x-x')}\big)\Psi_0$$

$$\approx \Psi_0^*\big(-e^{-ik(x-x')} + e^{ik(x-x')}\big)\Psi_0$$

$$\to \Psi_0^*\, D\, \Psi_0$$

Schwinger ended up with the eq.

$$i\hbar c\,\frac{\delta\Psi(\sigma)}{\delta\sigma} \approx \Big\{ -\frac{1}{c}\int j^\mu A_\mu^{(+)} - \frac{1}{2}\int n_\nu \frac{\partial D(x-x')}{\partial x_\nu}\frac{1}{c} n_\lambda j_\mu(x)\,\frac{1}{c} j_\lambda(x')\,d\sigma'\Big\}\Psi$$

$$(n_\mu \tfrac{\partial}{\partial x_\mu})^2 D = D. \qquad \Box^2 D = 0$$

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$$A_\mu = n_\mu n_\nu \frac{\partial}{\partial x_\nu}\Lambda(x)$$

$$-\left(\frac{\partial}{\partial x_\mu} + n_\mu n_\nu \frac{\partial}{\partial x_\nu}\right)\Lambda'(x)$$

$$+ A_\mu^{(t)}(x)$$

$$n_\mu A_\mu^{(t)} = 0, \qquad \frac{\partial A_\mu^{(t)}}{\partial x_\mu} = 0$$

$$n_\mu n_\mu = -1.$$

$$\left(n_\mu A_\mu(x) = -n_\mu \frac{\partial \Lambda}{\partial x_\mu} \right.$$

$$\left(\frac{\partial}{\partial x_\mu} + n_\mu n_\nu \frac{\partial}{\partial x_\nu}\right) A_\mu(x)$$

$$= -\left(n_\mu \frac{\partial}{\partial x_\mu}\right)^2 \Lambda'(x) \qquad \left. \right)$$

$$\therefore \frac{\partial A_\mu}{\partial x_\mu} = \left(n_\mu \frac{\partial}{\partial x_\mu}\right)\left(n_\nu \frac{\partial \Lambda}{\partial x_\nu}\right)$$

$$-\left(\frac{\partial^2 \Lambda'}{\partial x_\mu \partial x_\mu} + \left(n_\mu \frac{\partial}{\partial x_\mu}\right)^2 \Lambda'\right)$$

$$\Box\Lambda = \Box\Lambda' = \Box A^{(t)} = 0$$

$$\left\{ n_\mu \frac{\partial \Lambda(x)}{\partial x_\mu}, \; n_\nu \frac{\partial \Lambda(x')}{\partial x'_\nu} \right\} = -i D(x - x')$$

$$\left[\left(n_\mu \frac{\partial}{\partial x_\mu}\right)^2 \Lambda'(x), \; \left(n_\nu \frac{\partial}{\partial x'_\nu}\right)^2 \Lambda'(x') \right]$$

$$= -i \left(n_\mu \frac{\partial}{\partial x_\mu}\right)^2 D(x - x')$$

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$$[\Lambda(x), \Lambda(x')] = i D(x-x')$$
$$[\Lambda'(x), \Lambda'(x')] = -i D(x-x')$$
$$\left(\eta_\mu \frac{\partial}{\partial x_\mu}\right)^2 D = D$$
$$\Box D = 0$$

Satisfy the above e.v. assume also

$$[\Lambda(x), \Lambda'(x')] = 0$$

Further

$$[\Lambda(x), A_\mu^{(t)}(x')] = [\Lambda'(x), A_\mu^{(t')}(x')] = 0$$

as a consequence ~~because~~ of the above relation

$$[A_\mu^{(t)}(x), A_\nu^{(t')}(x')] = i\delta_{\mu\nu} D - i\left\{\frac{\partial^2}{\partial x_\mu \partial x_\nu} + \left(\eta_\mu \frac{\partial}{\partial x_\nu} + \eta_\nu \frac{\partial}{\partial x_\mu}\right)\eta\frac{\partial}{\partial x}\right\} \times D$$

~~S.C.~~ Since we have

$$\frac{\partial A_\mu}{\partial x_\mu}$$

or $= \left(\eta_\mu \frac{\partial}{\partial x_\mu}\right)^2 (\Lambda(x) - \Lambda'(x))$

The S.C.

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(V.19)

$$\left[\frac{\partial A_\mu}{\partial x_\mu} - \int_\sigma D(x-x')\frac{1}{c} j_\mu(x')\, d\sigma'_\mu\right] \times \Psi(\sigma) = 0$$

x : arbitrary point

now ~~becomes~~ is satisfied by $\partial A_\mu / \partial x_\mu$.

$$\left\{\Lambda(x) - \Lambda(x') - \int_\sigma \mathcal{D}(x-x')\frac{1}{c} j_\mu(x')\, d\sigma'_\mu\right\}\Psi(\sigma) = 0$$

Equation of motion:

$$i\hbar\frac{\delta\Psi}{\delta\sigma} = \int -\frac{1}{c} j_\mu(x) A^{(t)}_\mu(x) + \frac{\partial}{\partial x_\mu}\left(\frac{1}{c} j_\mu(x)\Lambda'(x)\right) - \frac{1}{c} n_\mu j_\mu(x)\, n_\nu \frac{\partial}{\partial x_\nu}(\Lambda(x) - \Lambda'(x))\Big\}\Psi$$

$$\Psi(\sigma) \to e^{-iG'(\sigma)}\Psi(\sigma)$$

$$G'(\sigma) = \int_\sigma \frac{1}{c} j_\mu(x)\Lambda'(x)\, d\sigma_\mu$$

$$i\frac{\delta\Psi}{\delta\sigma} = \int -\frac{1}{c} j_\mu A^{(t)}_\mu - \frac{1}{c}\int n_\nu \frac{\partial \mathcal{D}(x,x')}{\partial x_\nu} \times \frac{1}{c} n_\mu j_\mu(x) \frac{1}{c} j_\lambda(x')\, d\sigma'_\lambda \Big\}\Psi$$

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In particular, if

$$n_\mu = (0, 0, 0, i)$$

$$\left(\frac{1}{c}\frac{\partial}{\partial t}\right)^2 \Theta(r,t) = \Delta\Theta(r,t) = D(r,t)$$

and $\quad n_\nu \frac{\partial \Theta}{\partial x_\nu} \to \frac{1}{c}\frac{\partial}{\partial t}\Theta(r-r', 0)$

r, r' on σ or $t = t'$.

$$\frac{1}{c}\frac{\partial}{\partial t}\Theta = \frac{1}{4\pi r}$$

or $\quad n_\mu \frac{\partial}{\partial x_\nu}\Theta(x-x') = \frac{1}{4\pi}\frac{1}{\{(x_\mu - x'_\mu)^2\}^{1/2}}$

$$n_\mu (x_\mu - x'_\mu) = 0.$$

See also

R.P Feynmen, P.R. 76 (1949). 749, 769.

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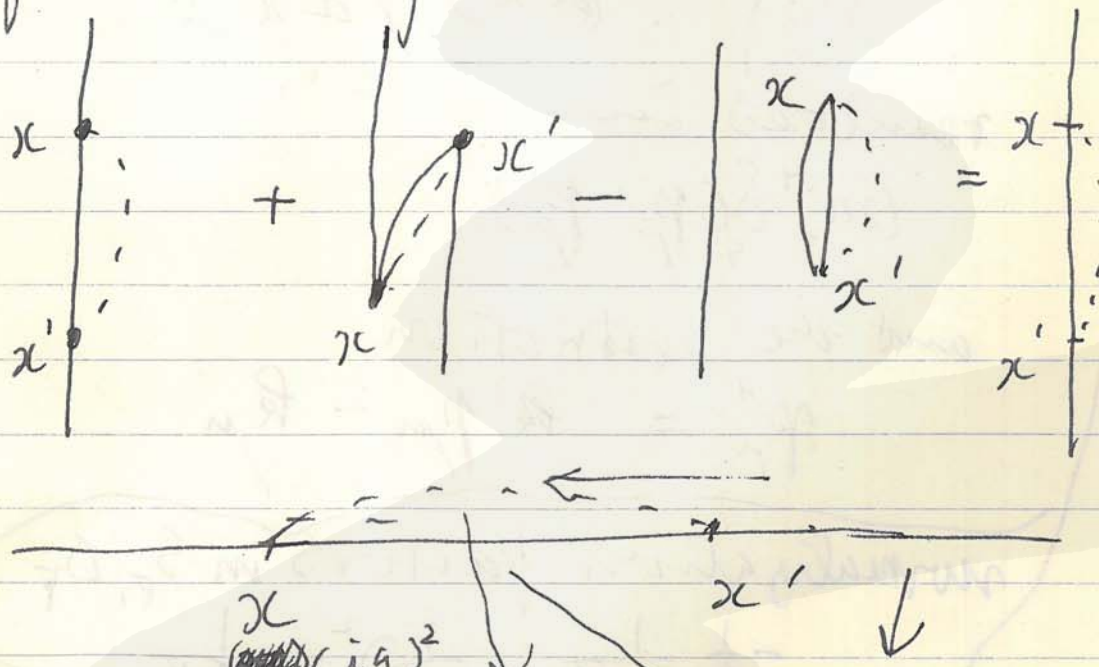
Self-energy of the electron viz

Now we are able go easily through the relativistic formulation of Q.E.d. and apply the above results to S-matrix, in such a way that as if nothing happened in connection with the supplementary condition.

Let us recall the formal procedure of constructing S-matrix



$$(p\,j\,|S_2|\,p'j') = \bar{\psi}_{pj}\,\tfrac{1}{2}\Delta_c(x-x')\,\tfrac{1}{2}S_F(x-x')\,\psi_{p'j'}(x')$$

$$e^{ip_\mu x_\mu}\psi_{p'j'} = \frac{1}{\sqrt{(2\pi)^3}}u_{p'j'}\cdot e^{ip'_\mu x'_\mu}$$

$$e^{-ip_\mu x_\mu}\bar{\psi}_{pj} = \frac{1}{\sqrt{(2\pi)^3}}\bar{u}_{pj}\,\gamma_4\,e^{-ip_\mu x_\mu}$$

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$$-\frac{1}{2} S_F = \int\!\!\int (dp'')^4 \frac{i p''_\mu \gamma_\mu - M}{p''_\mu p''_\mu + M^2 - i\varepsilon} e^{i p''_\mu (x_\mu - x'_\mu)}$$

$$\frac{1}{2} \Delta_F = \int\!\!\int (dk)^4 \frac{1}{k_\mu k_\mu + \varkappa^2 - i\varepsilon'} e^{i k_\mu (x_\mu - x'_\mu)}$$

The integration with respect to

$$\int\!\!\int (dx)^4 (dx')^4$$

resulted in

$$(2\pi)^4 \delta_4 (p_\mu - p'_\mu)$$

and the restriction

$$p''_\mu = p_\mu - k_\mu$$

normalization factors in S_F, Δ_F

$$-\frac{1}{4} \frac{1}{(2\pi)^6} \left(\frac{1}{\pi i}\right)^2 = \frac{1}{(2\pi)^8}$$

$$(p,j|S|p'j') = (2\pi)^4 \delta(p-p') \delta(p_0 - p_0')$$

$$\sum_\alpha (\bar{\psi}_{pj})_\alpha (\psi_{pj'})_\alpha \cdot \frac{3g^2 i}{(4\pi)^2} M \log k.$$

$$\frac{M}{p_0} \delta_{jj'}$$

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Insertion of

$$-\frac{1}{4}\int \Delta_F(x-y)(S_F(x-y))_{\alpha\beta}\, e^{-ip_\mu x_\mu}\, (dx)^4$$

$$= -\frac{1}{(2\pi)^6}\frac{1}{(2\pi)^2}\cdot(2\pi)^4 \int \frac{\{i(p_\mu - k_\mu)\gamma_\mu - M\}_{\alpha\beta}\,(dk)^4}{(k_\mu k_\mu + \chi^2 - i\varepsilon)((p_\mu - k_\mu)^2 + M^2 - i\varepsilon)}$$

$$= \frac{3M}{(4\pi)^2}\, i \log K \cdot \delta_{\alpha\beta} \qquad \frac{3M}{2}\frac{2\pi^2 i}{(2\pi)^4}\log K$$

~~$p_i j | S$~~

$$\delta M = -\frac{3g^2 M}{(4\pi)^2} M \log K$$

the corresponding matrix element in q.e.d. is

$$(p,j|S|p',j') = \cancel{(2\pi)^4 \delta(p-p')\delta(p_0 - p_0')}$$

$$(-i)^2 \int\!\!\int \bar{\psi}_{p'j'}(x')\,(ieA_\mu(x)\bar{\psi}(x)\gamma_\mu\psi(x),\ ieA_\mu(x')\bar{\psi}(x')\gamma_\mu\psi(x'))\,\psi_{pj}(x)\,(dx)^4(dx')^4$$

$$= e^2 \int\!\!\int e^{-i\mu x + i\mu' x'} \bar{\psi}_{p'j'}(x')\,\gamma_\mu\,\psi_{pj}(x)\,\delta(x-x')\,\delta_{\mu\nu}\,\frac{1}{2}D_{\mu F}(x-x')$$

$$\times \frac{1}{2}S_F(x-x')\,\gamma_\nu\,\psi_{p'j'}(x')\, e^{ip_\mu x_\mu}\,(dx_\mu)^4 \times (dx'_\mu)^4$$

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$g^2 \to -e^2$

$$\int \Delta_F(p_E)(S_F(p_E))_{\alpha\beta} e^{-ip_E x_E}(dx_E)^4$$

$$\Rightarrow \int (\gamma_\mu)_{\alpha\rho} D_F(y)(S_F(y))_{\rho\sigma}(\gamma_\mu)^*_{\sigma\beta} \times e^{-ip_E x_E}(dy)^4$$

$$= 2 \int \cdots \text{except for finite terms}$$

$$= \frac{6M}{(4\pi)^2} i \log\kappa \, \delta_{\alpha\beta}$$

$$(p,j|S|p'j') = (2\pi)^4 \delta(\ )\delta(\ )$$

$$\sum_\alpha (\bar{\psi}_{pj})_\alpha (\psi_{p'j'})_\alpha \cdot \frac{-6e^2 i}{(4\pi)^2}(M\log\kappa + \text{finite terms})$$

$$*\langle 0|T(A_\mu(x), A_\nu(x'))|0\rangle = \frac{1}{2}\delta_{\mu\nu} D_F(x-x')$$

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Thus, the divergence in the self energy can be removed by renormalizing mass by δM or subtracting from the Hamiltonian the mass term

$$\text{tic} \cdot \delta M \cdot \bar{\psi}(x)\psi(x)$$

or adding $-\text{tic}\,\delta M\,\bar{\psi}(x)\psi(x)$ with

$$\delta M = \frac{6e^2 M}{(4\pi)^2}(\log K + \text{f.t.}) \quad *$$

Idea of f-Field or C-meson

In connection with the above situation, Pais and Sakata suggested, independently that the infinite self-energy of the electron due to the surrounding electromagnetic field could be

---

* The above rel. cal. agrees with the familiar result which was obtained by conventional perturbation calculations by Weisskopf (See Heitler,)

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complicated by assuming the interaction of the electron with the hypothetical neutral scalar mesons (C-meson) or f-field with sufficiently large mass, if the coupling constant

$$g^2 = 2e^2$$

Thus, the log. singularity disappears and the change in the self-energy δM becomes finite.

However, the self-energy of the spinor particle is not the only divergence in field theories. We have to consider more generally what kind of difficulties we may have if we go further.

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VI. General rules for constructing S-matrix elements.

It was shown that ~~in the case of~~ in constructing the second order matrix elements for one particle states one can apply Feynman's rules:

Each incoming spinor particle line

$$(\psi^{(+)}_{p_j,s}(x'))_\rho = \frac{1}{\sqrt{(2\pi)^3}} (u_{p_j,s})_\rho e^{i p_{j\mu} x'_\mu}$$

(or $(0|\psi\, b^*_{p_j,s}|0)$ or $\{\psi, b^*_{p_j,s}\}_+$)

antiparticle line

$$\psi^{(-)}_{p_j,s}(x_j) = \frac{}{\sqrt{(2\pi)^3}} \bar{v}_{p_j,s} e^{i p_{j\mu} x'_\mu}$$

(or $(0|\bar{\psi}\, c^*_{p_j,s}|0)$ or $\{\bar{\psi}, c^*_{p_j,s}\}_+$)

~~Each~~ scalar particle line

$$\frac{1}{\sqrt{(2\pi)^3 2k_0}} e^{i k_j x'_j}$$

Each outgoing spinor particle line

$$\bar{\psi}_{p_j,s}(x') = \frac{1}{\sqrt{(2\pi)^3}} (\bar{u}_{p_j,s} \gamma_4) e^{-i p_{j\mu} x'_\mu}$$

etc.,

Each interaction Hamiltonian is represented by a vertex with 2 spinor particle lines and one scalar line

$$g\, \bar{\psi}_\alpha(x) \psi_\alpha(x) \varphi(x)$$

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For each virtual particle which are not present initially or finally

scalar particle:

$$\frac{1}{2}\Delta_F(x-x')$$

spinor particle and anti-particle

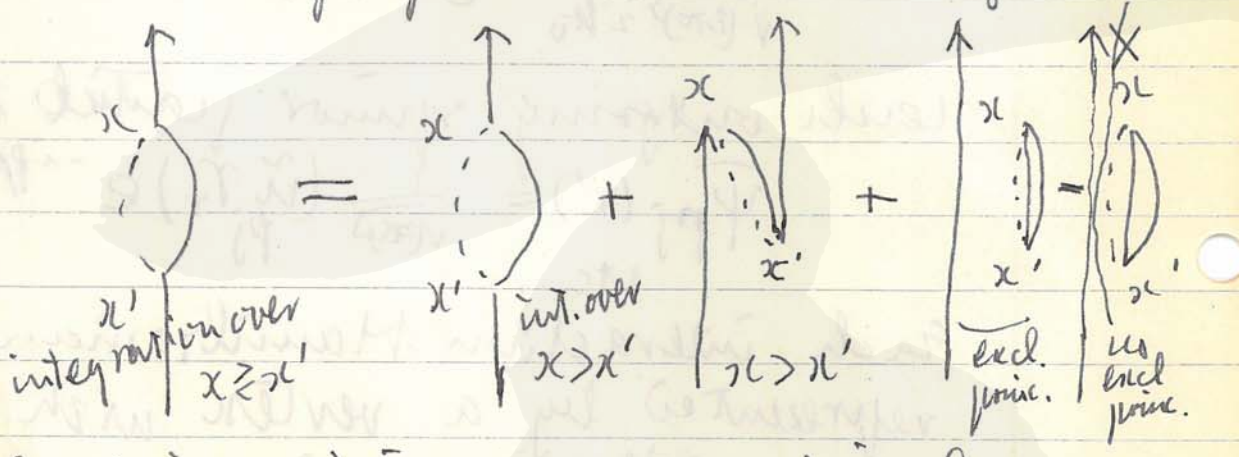
$$-\frac{1}{2}(S_F(x-x'))_{\alpha\beta}$$

$$+\frac{1}{2}(S_F(x'-x))_{\beta\alpha}$$

We can write down the product of these operators corresponding to each graph.

[For example]

In the case of matrix elements for one particle states, one just write down a graph. That's enough.



But according to our original picture it was the sum or difference of the above three graphs.

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In order to see the equivalence more clearly, one can write

$$\langle 0|S^{-1}|0\rangle\langle pj|S|p'j'\rangle$$
$$= \iint \langle 0|S^{-1}|0\rangle \sum_{j''} \langle 0|b_{pj}|pj''\rangle \langle pj''|S|p'j'\rangle \times (dp)^3 j''$$
$$= \langle 0|S^{-1} b_{pj} S|p'j'\rangle$$
$$= \langle 0|S^{-1} b_{pj} S\, b^*_{p'j'}|0\rangle$$
$$= \langle 0|b_{pj} b^*_{p'j'}|0\rangle + \langle 0|S^{-1}[b_{pj}, S]\, b^*_{p'j'}|0\rangle$$
$$\text{or } \langle 0|[S^{-1}, b_{pj}] S\, b^*_{p'j'}|0\rangle$$

$$[b_{pj}, \bar\psi(x)\psi(x)\varphi(x)]$$
$$= [b_{pj}, \bar\psi(x)]_+ \psi(x)\varphi(x)$$
$$+ \bar\psi(x)\, b_{pj}\, \psi(x)\varphi(x) - \bar\psi(x)\psi(x)\varphi(x) b_{pj}$$
$$= \{b_{pj}, \bar\psi(x)\}_+ \psi(x)\varphi(x) + 0$$
$$\text{or } [b_{pj}, \bar\psi(x)\psi(x)\varphi(x)\cdot\bar\psi(x')\psi(x')\varphi(x')]$$
$$= \{b_{pj}, \bar\psi(x)\psi(x)\varphi(x)\}_+ \bar\psi(x')\psi(x')\varphi(x')$$
$$- \bar\psi(x)\psi(x)\varphi(x)[b_{pj}, \bar\psi(x')\psi(x')\varphi(x')]_+$$

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Since $[b_{pj}, S]$ starts from g^2-term*, up to the second order S^{-1} can be identified with the identity operator.

Thus

$$(pj|S_2'|p'j') = (0|[b_{pj}, S_2] b^*_{p'j'}|0)$$

up to the second order

or

$$-(0|\int [b_{pj}, \bar\psi(x)]_+ \psi(x)\varphi(x)\bar\psi(x')\varphi(x') \underbrace{b^*_{p'j'}}_{(-g^2)} + \bar\psi(x)\varphi(x)[b_{pj}, \bar\psi(x')]_+ \psi(x')\varphi(x')\} \theta(x,x')(dx)^4(dx')^4$$

Since $\{b_{pj}, \bar\psi(x)\}_+$ is a c-number, one may write also

$$\int \{[b_{pj}, \bar\psi(x)]_+ \{\psi(x)\varphi(x)\bar\psi(x')\psi(x')\varphi(x') \underbrace{\theta(x-x')}$$
$$- \bar\psi(x')\psi(x')\varphi(x')\psi(x)\varphi(x) \underbrace{\theta(x'-x)}\} \times b^*_{p'j'}$$

---

* Provided that particles are all stable, or no direct decay is possible, of one into two others.

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$[C, AB]_+ = CAB + ABC = [CA + AC]B$

$[C, AB] = [CA]_+ B - ACB = -ABC + ACB$

$= [CA]_+ B - ACB = -A[BC]_+ + ACB$ 4.24

$\langle pj|S'|p'j'\rangle = \langle 0|S^{-1}|0\rangle \langle pj|S|p'j'\rangle$

$= \iint \sum_{pj} \langle 0|S^{-1}|0\rangle \langle 0|b_{pj}|pj\rangle \overset{\wedge}{dp} \langle pj|S|p'j'\rangle$

$= \langle 0|S^{-1} b_{pj} S|p'j'\rangle$

$= \langle 0|S^{-1} b_{pj} S\, b^*_{p'j'}|0\rangle$

$= \langle 0|b_{pj} b^*_{p'j'}|0\rangle$

$+ \langle 0|S^{-1}[b_{pj}, S]\, b^*_{p'j'}|0\rangle$

$(1-S_2)(b_{p_2} S_2 - S_2 b_{p_2})$

$[b_{pj}, \bar\psi(x)\psi(x)\varphi(x)]$

$= \varphi(x)\{[b_{pj}, \bar\psi(x)]_+ \psi(x) - \psi[b_{pj}, \psi(x)]_+\}$

$\{b_{pj}, \bar\psi(x)\psi(x)\varphi(x)\cdot \bar\psi(x')\psi(x')\varphi(x')]$

$= \varphi(x)\varphi(x')\cdot[b_{pj}, \bar\psi(x)\psi(x)\bar\psi(x')\psi(x')]$

$= \varphi(x)\varphi(x')\cdot\{[b_{pj}, \bar\psi(x)\psi(x)]_+ \bar\psi(x')\psi(x')$

$+ \bar\psi(x')\psi(x)[b_{pj}, \bar\psi(x')\psi(x')]_+\}$

$= \varphi(x)\varphi(x')\{[b_{pj}\cdot\bar\psi(x)]_+ \psi(x) - \bar\psi(x)\psi(x')\}$

$\theta(x-x')\simeq \bar\psi b_{pj}\psi$

$+ \psi(x)\psi(x)[b_{pj}, \bar\psi(x')]_+\psi(x)\}$

$x>x'$ $\varphi\cdot$

$= \varphi(x)\varphi(x')\{[b_{pj}, \bar\psi(x)]_+ \psi(x)\bar\psi(x')\psi(x')$

$+ [b_{pj}, \bar\psi(x')]_+ \bar\psi(x)\psi(x')\cdot\psi(x'')\}$

$x'>x$ $\bar\psi(x')\psi(x)$ $\psi(x')$

$b\int b^*\cdot b$

$[b_1, b_2 b_3]$

$= [b_1, b_2]_+ b_3$

$- b_2[b_1, b_3]_+$

$[b_1, b_3 b_2]$

$= [b_1, b_3]b_2$

$- b_3[b_1, b_2]_+$

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$$\theta(x,x')\varphi(x)\varphi(x')\times\{b,\bar\psi_\alpha(x)\}_+\psi_\alpha(x)\bar\psi_\beta(x')\psi_\beta(x')b^*$$
$$+\theta(x',x)\varphi(x')\varphi(x)\times[b,\bar\psi_\alpha(x)]_+\bar\psi_\beta(x')\psi_\alpha(x)\psi_\beta(x')b^*$$
$$=P(\varphi(x),\varphi(x'))\cdot[b\bar\psi(x)]_+P(\psi(x)\bar\psi(x')\psi(x')b^*$$
$$=P(\varphi(x),\varphi(x'))\{[b\bar\psi(x)]_+P(\psi(x)\bar\psi(x'))[\psi(x')b^*]_+$$
$$-P(\,,)\{\,\}+P(\,,)b^*\psi(x')\}$$

The above term is zero for 0-0 element.

$$\varphi(x)\varphi(x')\{[b\bar\psi(x)]_+\{\psi(x)\bar\psi(x')[\psi(x'),b^*]_+$$
$$-\{[b\bar\psi(x)]_+\psi(x)\bar\psi(x')b^*\psi(x')\}$$
$$-[b\bar\psi(x')]_+\{\bar\psi(x)\psi(x')[\psi(x),b^*]_+$$
$$-\bar\psi(x)\psi(x')b^*\psi(x)\}]$$

$$\frac{1}{2}D_F(x-x')\cdot-\frac{1}{2}S_F(x-x')$$

$$\varepsilon(x-x')$$

$$\{\theta(x,x')\varphi(x)\varphi(x')+\theta(x',x)\varphi(x')\varphi(x)\}$$
$$\times\{\theta(x,x')\psi(x)\bar\psi(x')+\theta(x',x)\bar\psi(x')\psi(x)\}$$
$$-\frac{1}{2}\varepsilon(x-x')S_F(x-x')$$

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~~p_i, p'_i~~

$$(p|S'|p') = \cancel{\quad}(0|S^{-1} a_p S a_p^*|0)$$

2nd order:

$$(0|[a_p, S_2)\, a_p^*|0)$$

$$\theta(x-x')[a_p, \varphi(x)\varphi(x')]\bar\psi(x)\psi(x)\bar\psi(x')\psi(x')\, a_p^*$$

$$= \theta(x-x')[a_p, \varphi(x)]\varphi(x') a_p^* \bar\psi(x)\psi(x)\bar\psi(x')\psi(x')$$
$$+ \theta(x-x')[a_p, \varphi(x')]\varphi(x) a_p^* \bar\psi(x)\psi(x)\bar\psi(x')\psi(x')$$

$$= \cancel{\theta(x-}\{a_p, \varphi(x)]\varphi(x') a_p^* \times$$
$$\times \{\theta(x,x')\bar\psi_\alpha(x)\psi_\alpha(x)\bar\psi_\beta(x')\psi_\beta(x')$$
$$+ \theta(x',x)\bar\psi_\beta(x')\psi_\beta(x')\psi_\alpha(x)\psi_\alpha(x)\}$$
$$(0|P(\bar\psi_\alpha(x)\psi_\beta(x'))|0)(0|P(\psi_\alpha(x)\bar\psi_\beta(x'))|0)$$

$$= -\frac{1}{2}\varepsilon(x'-x)(S_F(x'-x))_{\beta\alpha}\frac{1}{2}\varepsilon(x-x')(S_F(x-x'))_{\alpha\beta}$$

$$= \frac{1}{2}(S_F(x'-x))_{\beta\alpha} - \frac{1}{2}(S_F(x-x'))_{\alpha\beta}$$

$$P(\bar\psi_\alpha(x), \psi_\beta(x')) = \theta(x,x')\bar\psi_\alpha(x)\psi_\beta(x') + \theta(x',x)\psi_\beta(x')\bar\psi_\alpha(x)$$

$$= \frac{1}{2}\varepsilon(x'-x)(S_F(x'-x))_{\beta\alpha}$$

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$(2\pi)^4 \cdot \delta(p_\mu - p'_\mu - k_\mu)$ ← $e^{ip_\mu x_\mu}$ $e^{-ip'_\mu x_\mu}$ $e^{-ik_\mu x_\mu}$ 消滅

$$\langle k|S_2'|k\rangle = g^2 \iint \mathrm{Sp}\left(\frac{1}{2} S_F(x-x') \frac{1}{2} S_F(x'-x)\right) \times \frac{1}{\sqrt{(2\pi)^3 2k_0}} e^{-ik_\mu x_\mu} \frac{1}{\sqrt{(2\pi)^3 2k'_0}} e^{ik'_\mu x_\mu} \times (dx)^4 (dx')^4$$

$$= -g^2 (4\pi)^4 \cdot \delta(k-k')\delta(k_0-k'_0) \times \frac{1}{\sqrt{(2\pi)^3 2k_0}} \cdot \frac{1}{\sqrt{(2\pi)^3 2k'_0}} \int\int \frac{1}{(2\pi)^4} \mathrm{Sp}\left[\frac{i\gamma_\mu p_\mu - M}{p_\mu^2 + M^2} \times \frac{i\gamma_\nu (p_\mu - k_\mu) - M}{p'^2_\mu + M^2}\right] (dp)^4$$

p_μ ($p_\mu - k_\mu$) k

$(p_\mu - k_\mu)^2$

quadratic divergence + log. div.

$$\frac{p_\mu p_\mu \cdot (dp_\mu)^4}{(p_\mu p_\mu)^2 + \cdots} \propto \lim_{p\to\infty} \int^p p\,dp$$

$$+ \left\{\frac{\chi^2}{M^2} \lim_{p\to\infty} \log p\right. \cdot \int^p \frac{dp}{p}$$

( ) + ( ) + ( ) − ( )

Ward-Takahashi

no stat. rel.

neutral

This infinite self-energy of scalar meson can be compensated by adding

$$-(\delta\chi)^2 \varphi(x)\varphi(x)$$

to the Hamiltonian. This amount to

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the same thing as to change

$$M \to x + \frac{\delta x}{x} = x\left(1 + \left(\frac{\delta x}{x}\right)^2\right)$$

i.e. $\frac{1}{2}x^2\varphi^2 \to \frac{1}{2}\left(x + \frac{(\delta x)^2}{x}\right)^2\varphi^2$

provided that $\delta x/x \ll 1$.

Self-energy of the photon

~~Vacuum polarization~~

The problem of ~~invariance~~ self-energy of the scalar (or vector) particle can be formally solved by the renormal. above mention in general. However, the case of electromag. field is exceptional, because the renormalized mass as well as the unrenormalized mass of the photon must be zero. Otherwise the additional term which is proportional to $A_\mu A_\mu$ destroys the gage invariance.

The corresponding matrix element is,

$$(k_\mu e_\mu | S_2 | k'_\mu e'_\mu) = e^2 (2\pi)^4 \delta(k - k') \delta(k_0 - k'_0)$$

$$\times \frac{e_\mu}{\sqrt{(2\pi)^3 2k_0}} \cdot \frac{e_\nu}{\sqrt{(2\pi)^3 2k'_0}} \iint \frac{1}{(2\pi)^4} Sp\left[\frac{i\gamma_\kappa p_\kappa - M}{p_\kappa^2 + M^2}\right.$$

$$\left. \times \gamma_\mu \frac{i\gamma_\lambda (p_\lambda - k_\lambda) - M}{(p_\lambda - k_\lambda)^2 + m^2} \gamma_\nu \right] (dp)^4$$

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$$Sp\ \gamma_\mu = 0$$
$$Sp\ \gamma_\mu\gamma_\nu = 4\delta_{\mu\nu}$$
$$Sp\ \gamma_\mu\gamma_\nu\gamma_\kappa = 0$$
$$Sp(\gamma_\mu\gamma_\nu\gamma_\kappa\gamma_\lambda + \gamma_\mu\gamma_\lambda\gamma_\kappa\gamma_\nu) = 8(\delta_{\mu\nu}\delta_{\kappa\lambda} + \delta_{\mu\lambda}\delta_{\nu\kappa} - \delta_{\mu\kappa}\delta_{\nu\lambda})$$

The integral is divergent and contains the term which is proportional to $e_\mu e_\mu$ or A_μ^2. There is no convincing argument that it is zero, although we know that it should be zero.
If we drop this term, the remaining part turns out to be gage invariant and is zero in particular for the free radiation field.

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Vacuum polarization

The problem of self-energy of the photon is very intimately connected with the problem of vacuum polarization. ~~A~~ According to Dirac's original idea, the vacuum is filled with negative energy electrons. When an electric field is applied in a certain direction, these negative energy electrons will be displaced in the opposite direction. The vacuum thus behaves like a dielectric medium, in which the electric polarization is proportional to the external field.

This problem of vacuum polarization can be dealed with relativistically by in the following way:
Suppose that the external potential $A_\mu^{(ext)}$ is due to the ~~external~~ distribution of external current $j_\mu^{(ext)}$. Then the corresponding matrix element of S-matrix in

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vacuum is

$$(0|S|0)$$

where

$$S = \sum_{n=0}^{\infty} \left(\frac{i}{c}\right)^n \int\cdots\int j_\mu(x') A_\mu^{(ext)}(x') D(x'-x'')$$

$$\times j_\mu(x'') A_\mu^{(ext)}(x'') D(x''-x''') \cdots j_\mu(x^{(n)}) A_\mu^{(ext)}(x^{(n)})$$

$$\times (dx')^4 \cdots (dx^{(n)})^4$$

$$\frac{1}{c} j_\mu = -ie\, \bar{\psi} \gamma_\mu \psi$$

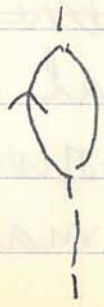
or

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} \int\cdots\int P\left(e\bar{\psi}(x')\gamma_\mu\psi(x')A_\mu^{(ext)}(x'),\right.$$

$$\left. \cdots\cdots\; e\bar{\psi}(x^{(n)})\gamma_\mu\psi(x^{(n)})A_\mu^{(ext)}(x^{(n)})\right)$$

$$(dx')^4 \cdots (dx^{(n)})^4$$

If we take the second order term alone, it corresponds to the graph which is



essentially the same as in the case of self-energy of the

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photon. The only difference being that the ~~quantized el.mag.~~ plane wave $\hat{}$ is replaced by a general (unquantized) external field.

$$\langle 0|S_2|0\rangle = -e^2 \int\!\!\int \mathrm{Sp}\left\{ \tfrac{1}{2} S_F(x-x') \gamma_\mu \tfrac{1}{2} S_F(x'-x) \gamma_\nu \right] A^{(ext)}_\mu(x') A^{(ext)}_\nu(x) (dx)^4 (dx')^4$$

or if we go over to the Fourier transform of S_F, A_μ, A_ν

$$\langle 0|S_2|0\rangle = e^2 \int\!\!\int \mathrm{Sp}\left\{ \frac{i\gamma_\varkappa p_{\varkappa} - M}{p_{\varkappa}^2 + M^2} \gamma_\mu \times \frac{i\gamma_\lambda (p_\lambda - k_\lambda) - M}{(p_\lambda - k_\lambda)^2 + M^2} \gamma_\nu \right] A^{(ext)}_\mu(k) A^{(ext)}_\nu(-k) (dk)^4$$

$\because$

$$\frac{1}{2} S_F(x-x') = i \cdot \frac{1}{(2\pi)^4} \int\!\!\int (dp_\mu)^4 \frac{i \not{p} \gamma_\mu p_\mu - M}{p_\mu p_\mu + M^2 - i\varepsilon} e^{i p_\mu (x_\mu - x'_\mu)}$$

$$A^{(ext)}_{\mu\nu}(x) = \frac{1}{(2\pi)^4} \int A^{(ext)}_\mu(k') e^{i k'_\mu x_\mu} (dk')^4$$

etc,

$$A^{(ext)}_\mu(x') = \int A^{(ext)}_\mu(k) e^{i k_\mu x'_\mu} (dk)^4$$

$e^{i p_\mu (x_\mu - x'_\mu)}$ $e^{-i p'_\mu (x_\mu - x'_\mu)}$ $e^{i k'_\mu x_\mu}$ $e^{i k_\mu x'_\mu}$

$\exp i k'_\mu x \cdot \exp i k_\mu x \cdot \exp\left(i \frac{k'_\mu}{2} (x_\mu - x'_\mu)\right) \exp\left(-i \frac{k_\mu}{2} (\,)\right)$

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$$= \exp i\left(p_\mu - p'_\mu - \frac{k_\mu}{2}\right)(x_\mu - x'_\mu)$$

$$\times \exp i\left(k'_\mu + k_\mu\right)\frac{x_\mu}{2}$$

$$(2\pi)^4 \delta(k_\mu + k'_\mu)$$

$$(2\pi)^4 \delta(p_\mu - p'_\mu - k_\mu)$$

$$\frac{1}{c} j_\mu^{(ext)}(k) = k_\mu k_\nu A_\mu^{(ext)}(k)$$

$$(0|S_2|0) = -\frac{i\pi^2 \cdot e^2}{c} \int dk\, A_\mu(k) \{j_\mu(-k),$$

$$\times \left\{ \frac{8}{3}(\log K) + \frac{4}{3} k_\mu k_\nu \int_0^1 du \frac{u^2(1-2u)(3-2u)}{M^2 + k_\mu k_\nu u(1-u)} \right.$$

$$= -i\frac{e^2}{c}\frac{1}{16\pi^2} \int (dx)^4 A_\mu^{(ext)}(x)$$

$$\times \left\{ \frac{8}{3}(\log K) \cdot j_\mu(x) \right.$$

$$- \frac{4}{3(2\pi)^4} \Box \int dx' \left\{ \int_0^1 du\, u^2(1-2u)(3-2u) \right.$$

$$\left. \left. \times \int (d\mathrm{k})^4 \frac{e^{i\mathrm{k}(x'-x)}}{M^2 + k_\mu k_\mu u(1-u)} \right\} j_\mu(x') \right]$$

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in the first order

Now this has the same form as the ~~the~~ induced we ~~case~~ obtains, if ~~the~~ the current

$$\Delta j_\mu(x) = -\frac{e^2}{16\pi^2}\frac{8}{3}(\log\cdots)j_\mu(x)$$

$$+\frac{ie^2}{16\pi^2}\frac{4}{3(2\pi)^4}\Box\int(dx')^4$$

$$\{\ \ \}\, j_\mu(x')$$

was added in the Hamiltonian. Among them, the first term is proportional to ~~the~~ j in itself, so that a term which is proport. to

$$j_\mu(x)A_\mu^{(ext)}(x)$$

is to be added to the Hamiltonian. This cannot be discriminated from the original Hamiltonian, but rather we may think that the actually observed electric charge is not e, but the one which was already changed by the interaction. Thus the first term can be eliminated easily

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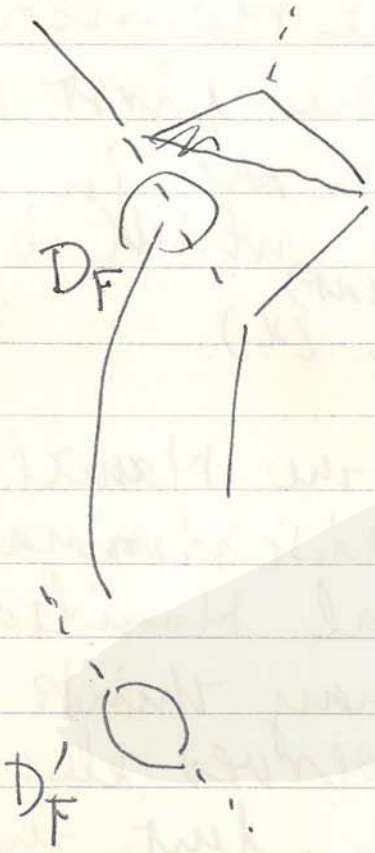
charge renormalization.
The second term give the genuine observable polarization current.

The above argument can be extended to the case when the photon lines are internal lines in a graph.



In that case, D_F is replaced by D', in which same the divergent term as above appears, and the divergence can thus be eliminated by the charge renormalization.

Similarly, in the case of the electron lines, S_F must be replaced by S'_F, in which the divergent term

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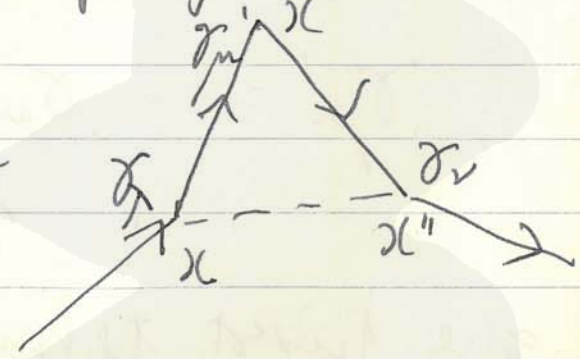
partly

is eliminated by mass renormalization. and partly compensated by the term which appears as follows.

The above charge renormalization appears also as the radiative correction to the vertex in any graph. The corresponding matrix element has the form

$$(-i)^3(+ie)^3\int(dk)^4$$



$$\times \int (dx')^4 (dx'')^4$$

$$\bar{\psi}(x)\,\gamma_\lambda \tfrac{1}{2}\gamma_\mu \tfrac{1}{2} S_F(x-x')\tfrac{1}{2} S_F(x'-x'')$$

$$\gamma_\nu\,\psi(x'')\cdot\tfrac{1}{2} D_F(x-x'')\,\delta_{\lambda\nu}\cdot A_\mu(x')$$

One can see by going over to Fourier transform that the effect of the ~~correction~~ term is equivalent to the radiative correction to γ_μ by

$$\gamma_{\mu'} = -ie^2 \frac{1}{(2\pi)^4}\int(dk)^4\,\chi$$

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$$\times \gamma_\nu \frac{i\gamma_\lambda (p_1 - k)_\lambda - M}{(p_1 - k)^2 + M^2} \gamma_\mu$$

$$\times \frac{i\gamma_\kappa (p_3 - k)_\kappa - M}{(p_3 - k)^2 + M^2} \gamma_\nu \frac{1}{k^2}$$

which reduces to the form

$$\gamma_{\mu'} = \gamma_\mu \times \text{log. div} + \text{finite term.}$$

the first term, fortunately has the same form as that which appears in the normal electron line and compensate with each other.

The second finite term give rise to the observable effects and actually responsible mainly for the hamb shift and abnormalous mag. moment of the electron.

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(1)

General Discussion on the Main Problems of R.F.T.

What we have been discussing in connection with the simplest of the S-matrix elements suggests us a number of important problems:

(i) What kinds of divergences are there in ~~field theories~~ certain types of field theories? Can ~~How~~ they could be removed at least formally by renormalization procedure?

(ii) Is there any possibility of constructing a field theory in which relativistic no divergence appears from the beginning?

(iii) ~~Supposing that~~ How can we justify the adoption of general rules as given by Feynman and Dyson? We ~~have only~~ could so far only verified directly the Feynman's formula for the S-matrix in the simplest cases.

(iv) ~~How~~ What is the connection between the matrix elements of S-matrix and the cross-section of the

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corresponding collision processes?

Let us start from the first problem.
The general term in the S-matrix has the form like

$$g^n(-i)^n\int\cdots\int \bar{\psi}_{\alpha^{(n)}}(x^{(n)})\psi_{\alpha^{(n-1)}}(x^{(n)})\varphi(x^{(n)})\theta(x^{(n)},x^{(n-1)})\bar{\psi}_{\alpha^{(n-1)}}(x^{(n-1)})\psi_{\alpha^{(n-2)}}(x^{(n-1)})\varphi(x^{(n-1)})$$

$$\times\theta(x',x'')\cdots\cdots\theta(x^{(n-1)},x^{(n)})\bar{\psi}_{\alpha^{(n)}}(x^{(n)})\psi_{\alpha^{(n)}}(x^{(n)})$$

$$\times\varphi(x^{(n)})\,(dx')^4\cdots(dx^{(n)})^4$$

If we take the matrix elements between the initial state in which there are ~~this~~ n'_{sp} spinor particles (and anti-particles) and n'_{sc} scalar particles and the final state in which there are n_{sp} spinor particles (and anti-particles) and n_{sc} scalar particles, the corresponding graph will have the corresponding external lines. $n'_{sp}+n_{sp}$ must be an even number.

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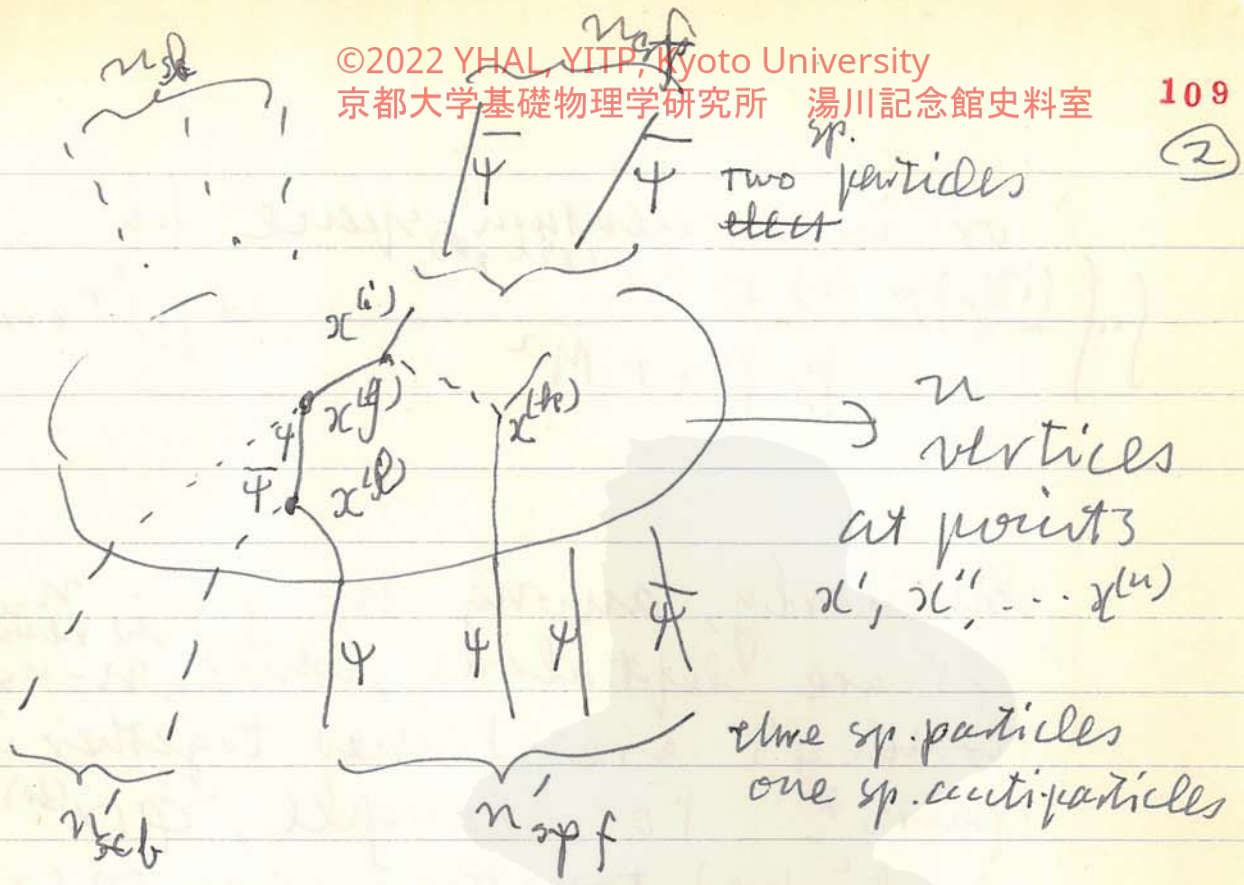

Thus, among $2n$ - ψ's and $\bar{\psi}$'s, $n_{sp} + n'_{sp}$ are left unpaired, * while the remaining $2n - n_{sp} - n'_{sp}$ are linked together in pairs. For example, $\bar{\psi}(x^{(k)})$ which is linked with $\psi(x^{(l)})$, ~~which~~ corresponds to the sp. particle line from $x^{(k)}$ to $x^{(l)}$. According to F.D. rule, this will be represented by

$$-\frac{1}{2} S_F(x^{(l)} - x^{(k)})$$

---

* These are $\frac{n_{sp} + n'_{sp}}{2}$ - ψ's and $\frac{n_{sp} + n'_{sp}}{2}$ - $\bar{\psi}$'s, because the remaining numbers of ψ's and $\bar{\psi}$'s must be equal.

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or in momentum-space by

$$\int\cdots\int \frac{(i\gamma_\mu p_\mu - M)\, e^{ip(x^{(j)} - x^{(k)})}}{p_\mu p_\mu + M^2} (dp)^4 \times \text{const.}$$

Similarly, among n - φ's $n_{sc} + n'_{sc}$ φ's are left alone, while the remaining $n - n_{sc} + n'_{sc}$ ~~which~~ φ's are linked together into pairs*. For example, $\varphi(x^{(k)})$ is linked together with $\varphi(x^{(i)})$ giving rise to

$$\frac{1}{2} \Delta_F (x^{(i)} - x^{(k)})$$

$$\text{const.} \int\cdots\int^{\infty} \frac{e^{ik(x^{(i)} - x^{(k)})}}{k_\mu k_\mu + \varkappa^2} (dk)^4$$

Thus, we have a product of $\frac{1}{2}(2n - n_{sp} - n'_{sp})$ S_F-functions and $\frac{1}{2}(n - n_{sc} - n'_{sc})$ Δ_F-functions. The integration is over $\frac{1}{2} \cdot 4(3n - E_{sp} - E_{sc})$ momentum variables and the integrand

* $n - n_{sc} - n'_{sc}$ must be even

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has the highest power in momentum variables

$$-\frac{1}{2}(2n - E_{sp}) - (n - E_{sc})$$

$$= -2n + \frac{1}{2}E_{sp} + E_{sc}.$$

The integration with respect to $4n$ space-time variables will give rise to $4n$ δ-functions for the momentum variables. Among them 4 δ-fns refer to the conservation of total energy-momentum for initial and final states. Hence, ~~we have~~ after carrying out the integration with respect to $4n-4$ momentum variables, we are left with the integration over

$$2(3n - E_{sp} - E_{sc}) - 4n + 4$$

momentum variables, which means that the integration will be convergent, if

$$2n - \frac{1}{2}E_{sp} - E_{sc} > 2n + 4 - 2E_{sp} - 2E_{sc}$$

or

$$\frac{3}{2}E_{sp} + E_{sc} > 4$$

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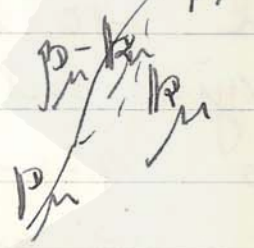
It should be noticed, however, the above argument is not rigorous, because, ~~although~~ even if the over-all picture may be favorable to the convergence, it may well happen that in some part of the graph, there appears a divergent integral which already destroys the whole picture. For instance, if there is in the graph a partial graph of the self-energy type, the integration with respect to k_μ will give the logarithmic divergence.



So let us confine our attention for the moment to the so-called primitive divergence. Namely, if one of the momentum 4-vectors is held fixed, the integration over the remaining variables is convergent. In that case, if ~~only~~ one of the

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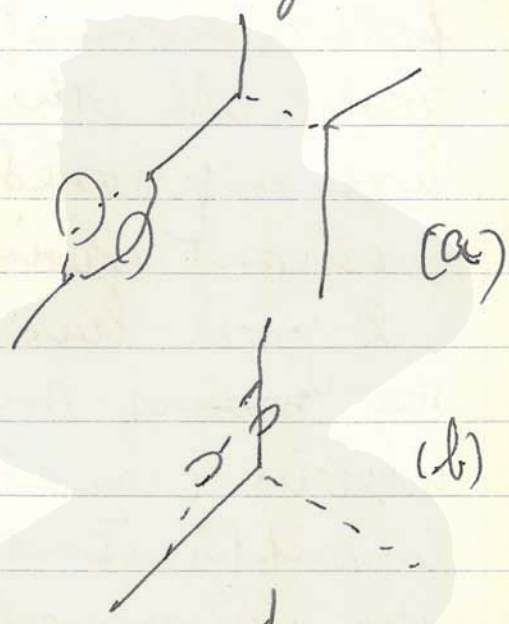
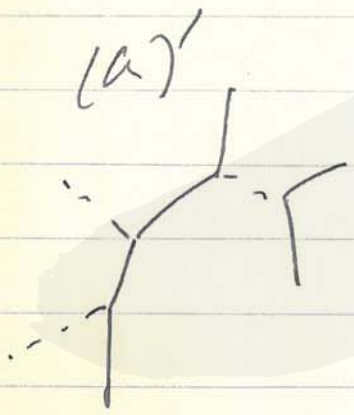
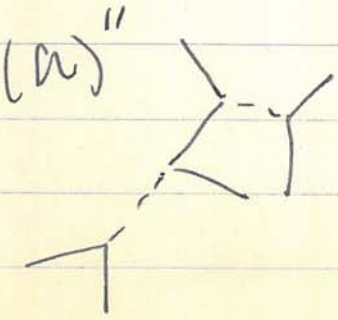
(3)

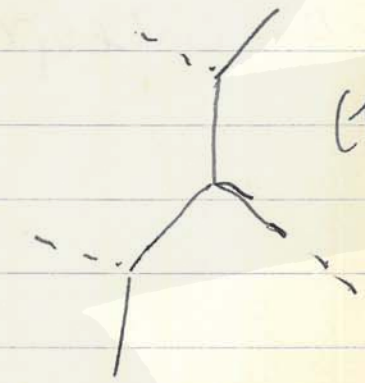
internal lines is removed and is replaced by two external lines, the modified graph will give convergent result. Such a procedure makes sense only for ~~those~~ internal lines ~~which~~ ~~that~~ which do not ~~disconnect the~~ graph separate the graph into two parts.





or





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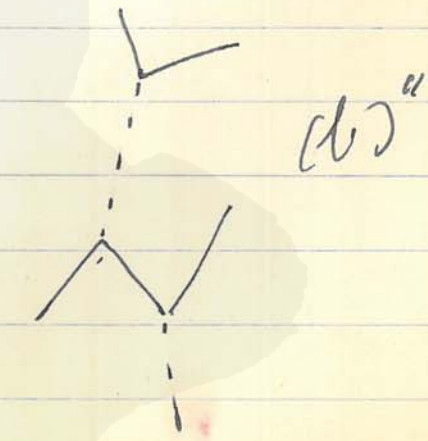

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In the case of the primitive divergence, what matters is the integration with respect to ~~the~~ one set of four momenta, while the integrations over all other internal momenta are supposed to be carried out without giving rise to divergence.

Dyson has proved that in ~~the case of~~ such a case, the above condition E_{sp} and E_{3c} is sufficient for guaranteeing the convergence for the last 4-integrations.

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(4)

Thus, for

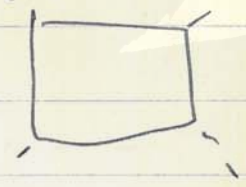
(i) $E_{sp} = 2$ $E_{sc} = 0$ $\frac{3}{2}E_{sp} + E_{sc} = 3$
$(n \geq 2)$ (log. div)

(ii) $E_{sp} = 0$, $E_{sc} = 2$ $\frac{3}{2}E_{sp} + E_{sc} = 2$
$(n \geq 2)$ (quad. div and log.)

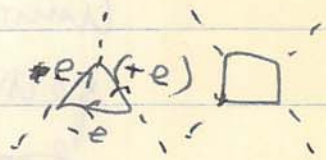
(iii) $E_{sp} \geq 2$: $E_{sc} > 1$ convergent

(iv) $E_{sc} = 3$ $(E_{sp} = 0)$
$(n \geq 3)$ log. div.

(vi) $E_{sc} = 4$ $(E_{sp} = 0)$
$(n \geq 4)$ log. div.

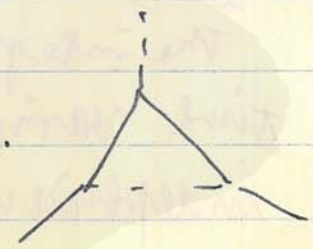


(vii) $E_{sp} > 4$: convergent

In the case of q. e. d., 

vanish due to charge symmetry and gage invariance

(iii) $E_{sp} = 2$, $E_{sc} = 1$.
$(n \geq 3)$ (log. div.)



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(i,i,i) and (v) ~~does~~ not appear ~~for~~ by themselves ~~for~~ in S-matrix, but they may appear as a ~~part~~ factor in the ~~general~~ higher order terms, because in the ~~latter~~ former cases the energy-momentum conservation laws are not satisfied between initial and final states, but in the latter case this does not matter.

N-factors

$$\int \cdots \Big(\underbrace{\qquad\qquad}_{n\text{-factors}} \Big) \cdots (dx^1)^4 \cdots (dx^{(n)})^4$$

Namely, we take out from the general ~~matrix~~ term of Nth order the factor of n-th order.

Among 3n-field quantities $\bar{E} = E_{op} + E_{sc}$ ~~for~~ factors remain unlinked, while ~~for~~ other 3n-$\bar{E}$ factors are linked with each other in pairs. The integration with respect to 4n-space-time variables reduces the number of independent momentum variables

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from $\frac{3}{2}(3n-E)$ to $\frac{3}{2}(3n-E)-4n$ $+4 = 2n-2E+4$, while there is a factor outside the integration of the form of the products of δ-functions corresponding to the conservation of total energy and momentum for particles which are left ~~unlinked~~ unpaired.

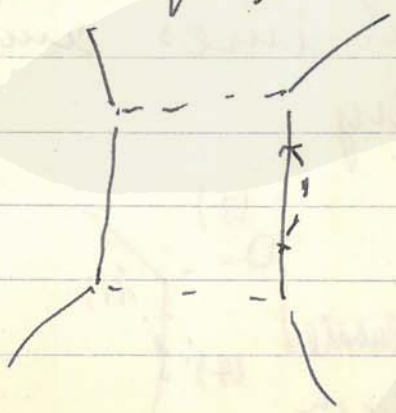
The above considerations were oversimplified in two respects:

(i) The integration may be divergent even when ~~it is~~ the condition

$$\frac{3}{2}E_{sp} + E_{sb} > 4$$

if the graph is not primitive,

is satisfied. Namely, if we take a graph as follows, which is convergent, and add a ~~scalar~~ boson line, then it will be divergent. To each graph, there correspond an infinitely



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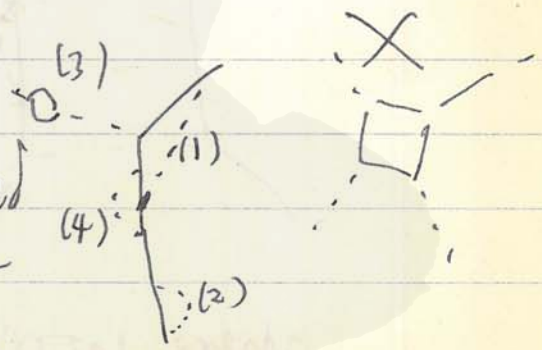
many graphs which can be
constructed from the original one
by adding a number of internal
lines, which ~~will~~ well may give rise to
divergences. However, these divergences
are not new and can be compensated
by adding in advance the
counter term in ~~the orig~~ advance
in the renormalization. In the case
of q.e.d., this was achieved by
the procedure of charge and mass
renormalizations.
More precisely, the net effect of
the totality of
graphs which can be constructed
from one graph is ~~reduced~~ to
modify each of fermion lines,
~~and each~~ boson lines and
~~each~~ vertices, by
adding divergent
terms. ~~This~~ Each
of the can be compensated
by the counter term
due to mass and charge
renormalizations.



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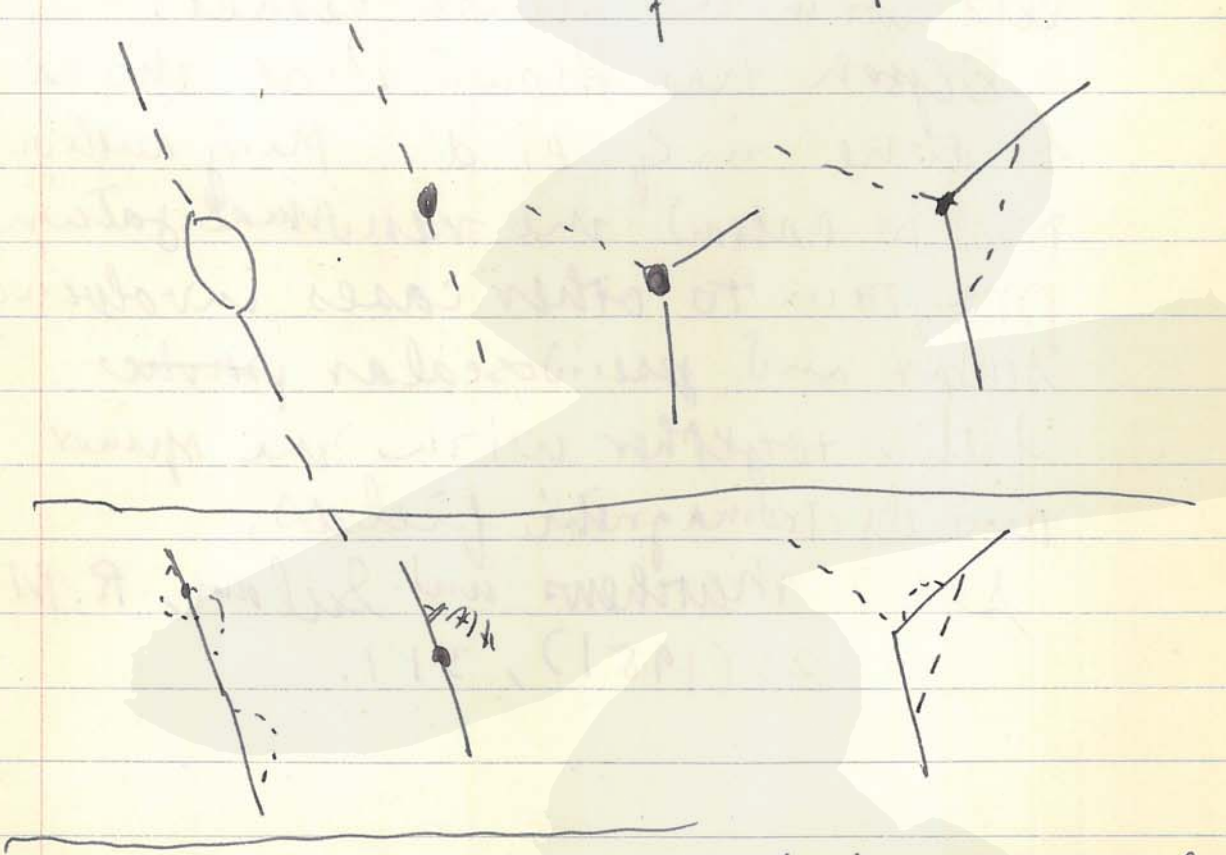
(See Dyson, P.R. 1949).

In other words, ~~all~~ the expansion of S into power series of the coupling constant, all these graphs appears ~~together with the~~ corresponding side by side with the graphs with counter Terms! *



* other part of the graph being identical.

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Actually, the situation is a little more complicated, because to a line or a vertex all kinds of graphs containing divergences are to be added, ~~which is so~~. Thus, the renormalizations are to be such that the counter term just compensate for the totality of all these terms and we are left with the finite result.

Dyson has shown that this can be done in q. e. d. Many authors tried to extend the renormalization program to other cases involving scalar and pseudoscalar ~~particles~~ fields together with the spinor and electromagnetic fields.

See Matthews and Salam, R. M. P. 23 (1951), 311.

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Renormalizable and Unrenormalizable Theories

Sakata, Umezawa and Kamefuchi,
P. T. P. 7 (1452), 377,
Umezawa, P. T. P. 7 (1452), 551.

Renormalizable Cases

	φ (scalar, pseudo-scalar) neutral	φ charged	ψ (spinor) neutral or charged
~~ψ~~	scalar or pseudoscalar coupling $\bar{\psi}\psi\varphi$ or $\bar{\psi}\gamma_5\psi\varphi$		$(\bar{\psi}\bar{\psi}\psi\psi$ unrenormalizable) ††
A_μ		$(\varphi^*\frac{\partial\varphi}{\partial x_\mu} - \varphi\frac{\partial\varphi^*}{\partial x_\mu})A_\mu$ †	$\bar{\psi}\gamma_\mu\psi A_\mu$

† Each vertex gives an addition contribution of +1 to the power of momentum, so that in all it is at most, $+n$, while the contributions from internal lines are now

$2n - 4$ etc.

from Δ_F and

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$2n - E_{ph}$

or the sum is

$$3n - E_{sc} - E_{ph}$$

while the number of internal momentum variables reduces ~~to~~ as before to

$$2n + 4 - 2E_{sc} - 2E_{ph}$$

so that the integration converges if

$$3n - E_{sc} - E_{ph} > 2n + 4 - 2E_{sc} - 2E_{ph} + n$$

or

$$E_{sc} + E_{ph} > 4.$$

Thus there is only a finite number of primitive divergence.!!!

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VI. Non-local Interaction

N.L.(1)

Non-local Interaction

Let us discuss briefly, in connection with the appearance of divergences in current field theories.

See

Bloch, Danish Academy 1952

Kristensen and Møller, ibid. 1952

We take, according to Møller, the case of a nucleon spinor field and scalar, vector, pseudoscalar, neutral meson field interacting with each other. We assume that the field equations can be derived from the variation principle;

$$\delta \int\cdots\int L(x)(dx)^4 = 0$$

$$\delta \bar{L} = 0$$

where

$$\bar{L} = \int\cdots\int \{ L_N(x) + L_M(x) \}(dx)^4$$

$$+ \int\cdots\int L_{int}(x', x'' x''')(dx')(dx'')(dx''')^4$$

with

$$L_N = -\{ \tfrac{1}{2}(\bar{\psi}\gamma_\mu \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma_\mu \psi) + M\bar{\psi}\psi \}$$

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$$h_M = -\frac{1}{2}\{\partial_\mu u \cdot \partial_\mu u + m^2 u^2\}$$

$$h_{int} = -\sum_{\zeta'\zeta''\alpha} \bar{\Psi}_{\zeta'}(x') \Phi^{\alpha}_{\zeta'\zeta''}(x'x''x''') \Psi_{\zeta'}(x')$$

$$(\hbar = c = 1.) \qquad \times u_\alpha(x'') \Psi_{\zeta''}(x''')$$

Let us confine our attention to the case

$$\Phi^{\alpha}_{\zeta'\zeta''} = \Lambda^{\alpha}_{\zeta'\zeta''} F(x'x''x''')$$

In that case, one may write briefly

$$h_{int} = -\bar{\Psi}(x') \Lambda u(x'') \psi(x''') \cdot F(x', x'', x''')$$

local interaction:

$$F = \delta(x'-x'')\,\delta(x'-x''')$$

~~Pseudo~~

Scalar meson:

$$\Lambda F = (g_1 - i\frac{g_2}{\mu}\gamma_\mu \partial_\mu'') F(x', x'', x''')$$

Pseudoscalar meson:

$$\Lambda F = (i g_1 \gamma_5 - i\frac{g_2}{\mu}\gamma_5 \gamma_\mu \partial_\mu'') F.$$

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The equations of motion which are obtained from the variational principle are

$$(\gamma_\mu \partial_{\mu'} + M)\psi(x') = -\int \Phi(x', x'', x''') u(x'') \psi(x''') dx'' dx'''.$$

$$(\Box'' - \nu^2) u(x'') = \int \bar{\psi}(x') \Phi(x', x'', x''') \psi(x''') dx' dx'''$$

$$\bar{\psi}(x'') \overleftarrow{\partial_{\mu''}} \gamma_\mu - M \bar{\psi}(x'') = +\int \bar{\psi}(x') u(x'') \times \bar{\Phi}(x', x'', x''') dx' dx''.$$

In the local theories, the equations of motion guarantee the continuity equation

$$\partial_\mu (i \bar{\psi}(x) \gamma_\mu \psi(x)) = 0$$

for the current density $i\bar{\psi}(x)\gamma_\mu\psi(x)$ of the nucleon. In our case, however, we have

$$\partial_\mu (i \bar{\psi} \gamma_\mu \psi) = -i \{ \int \bar{\psi}(x) \Phi(x, x'', x''') u(x'') \times \psi(x''') dx'' dx''' - \int \bar{\psi}(x') \bar{\Phi}(x', x'', x) u(x'') \psi(x) dx' dx'' \}$$

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Integrating over the whole space of x, -time world we have

$$\iiint (i\bar{\psi}\gamma_\mu\psi)\, d\sigma_\mu = 0$$

surface integral

~~Thus, if the field extends only over the finite space region:~~ As the boundary surfaces, one may chose the surfaces at $t = t_2$ and $t = -\infty$ and we obtain the integral conservation law for the number of nucleons:

$$\int \psi^\dagger(x)\psi(x)(d^{(3)}x)\Big|_{t=t_2} = \int \psi^\dagger(x)\psi(x)(dx)^3\Big|_{t=t_1}$$

provided near the field quantities at large distances from the origin in any space-like direction is sufficiently small. More, precisely this give the conservation of the difference between the total number of nucleons and antinucleons.

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Thus, one may legitimately call

$$N = \int \psi^+ \psi \, (dx)^3$$

a constant of collision, although it is not a constant of motion, because it changes with time.

One can verify similarly the conservation laws of total energy-momentum:

$$t_{\mu\nu}^{(0)} = \frac{1}{2}\{(\bar{\psi}\gamma_\nu \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma_\nu \psi)$$

$$- \delta_{\mu\nu}(\bar{\psi}\gamma_\lambda \partial_\lambda \psi - \partial_\lambda \bar{\psi}\cdot \gamma_\lambda \psi)\} - \delta_{\mu\nu} M \bar{\psi}\psi$$

$$+ \partial_\nu u \, \partial_\mu u - \frac{1}{2}\delta_{\mu\nu}(\partial_\lambda u \partial_\lambda u + m^2 u^2)$$

$$\left(= \sum_\alpha \frac{\partial \psi_\alpha}{\partial x_\mu} \frac{\partial L}{\partial \frac{\partial \psi_\alpha}{\partial x_\nu}} \frac{\partial \psi_\alpha}{\partial x_\mu} - \sum_\alpha \frac{\partial \bar{\psi}_\alpha}{\partial x_\mu} \frac{\partial L}{\partial \frac{\partial \bar{\psi}_\alpha}{\partial x_\nu}} \right.$$

$$\left. - \sum_\alpha \frac{\partial u_\alpha}{\partial x_\mu} \frac{\partial L}{\partial \frac{\partial u_\alpha}{\partial x_\nu}} + L\delta_{\mu\nu} \right.$$

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$$\partial_\nu t^{(01)}_{\mu\nu}(x) = \int \partial_\mu \bar{\psi}(x)\, \Xi(x, x'', x''')\, u(x'')$$
$$\times \psi(x''')\, dx''\, dx''' + \int \bar{\psi}(x')\, \Xi(x', x, x''')$$
$$\times \partial_\mu u(x) \cdot \psi(x''')\, dx'\, dx'''$$
$$+ \int \bar{\psi}(x')\, \Xi(x', x'', x)\, u(x'')\, \partial_\mu \psi(x)$$
$$\times dx'\, dx''\,.$$

By integrating over the whole space-time of x, we obtain

$$\int \cdots \int \partial_\nu t^{(01)}_{\mu\nu}(x)\, (dx)^4 = ,$$

because the right hand side is

$$= -\int \Big(\frac{\partial \Xi(x'x''x''')}{\partial x'} \bar{\psi}(x')\, u(x'')\, \psi(x''')$$
$$+ \frac{\partial \Xi(x'x''x''')}{\partial x''} \quad ,$$
$$+ \frac{\partial \Xi(x'x''x''')}{\partial x'''} \quad \cdots \Big) \cdots$$
$$+ \int \text{surface integral}$$

which is not zero.

But, if we take

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N. L ④

$$L^{(2)}(x) = -\int \Phi(x'x''x''')\,\bar\psi(x')\,u(x)\,\psi(x'')$$

$$= -\int \frac{\partial \Phi}{\partial x}\,\bar\psi(x')\,u(x)\,\psi(x'')\,dx'dx''$$

$$\partial_\mu L^{(2)}(x) = -\int \partial'_\mu \bar\psi(x')\,\Phi(x,x',x''')$$

$$\times\,\bar\psi(x')\,\frac{\partial u}{\partial x_\mu}\,\psi(x''')\,dx'dx'''$$

$$= \int \frac{\partial \Phi}{\partial x}\,\bar\psi(x')\,u(x)\,\psi(x''')\,dx'dx'''$$

term

The latter is the same as

$$+\int \left(\frac{\partial \Phi}{\partial x'} + \frac{\partial \Phi}{\partial x'''}\right)\bar\psi(x')\,u(x)\,\psi(x'')\,dx'dx'''$$

$$-\int \Phi(x,x,x''')$$

because provided that

$$\frac{\partial \Phi}{\partial x'} + \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial x'''} = 0$$

or

Φ does not depend on

$$X = \frac{1}{3}(x' + x'' + x''').$$

but only on other variables

$$\xi = x' - x'', \qquad \eta = x'' - x'''$$

$$(\xi = x''' - x')$$

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This is time, if ~~necessary~~ in order that Φ is invariant with respect to the inhomogeneous Lorentz transformation. In other words, if Φ itself is form invariant, or

$$\int \bar{\psi}(\bar{x}')\,\bar{\Phi}(\bar{x}',\bar{x}'',\bar{x}''')\,\psi(\bar{x}'')\,\psi(\bar{x}''')\,d\bar{x}'\,d\bar{x}''\,d\bar{x}'''$$

has the same form as

$$\int \bar{\psi}(x')\,\Phi(x',x'',x''')\,\psi(x'')\,\psi(x''')\,dx'\,dx''\,dx'''$$

where

$$\bar{x}' = x' + a \qquad \bar{x}'' = x'' + a$$
$$\bar{x}''' = x''' + a$$

and $\psi(\bar{x}') = \psi(x'+a)$

etc.

~~Then~~

$$\bar{\Phi}(\bar{x}',\bar{x}'',\bar{x}''') = \Phi(\bar{x}'+a, \bar{x}''+a, \bar{x}'''+a)$$

~~Hence~~

$$\Phi(x'+a, x''+a, x'''+a)$$

Now ~~by~~ the other hand, $\bar{\Phi}(\bar{x}',\bar{x}'',\bar{x}''')$ is ~~obtained from~~ $\bar{\Phi}(x',x'',x''')$.

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N. L (5)

$\bar{\Psi}(x_1', x_1'', x_1''') = \bar{\Psi}(x' + a, x'' + a, x''' + a)$

if we compare the same Hint in two coordinates.

Hence, we have

$$\int \partial_\nu t^{(2)}_{\mu\nu} \, dx = 0$$

for

$$t^{(2)}_{\mu\nu} = t^{(0)}_{\mu\nu}(x) + \delta_{\mu\nu} L^{(2)}(x).$$

Thus we following Hermitian quantities

$$G_\mu = -i \int t^{(2)}_{\mu 4}(x) (dx)^3$$

$$= G^{(0)}_\mu - i \delta_{\mu\nu} \int L^{(2)}(x) (dx)^3$$

which are constants of collision.

This reduces to the ordinary e.m. tensor in the limit of local interaction, because

$$L^{(2)} \to \underbrace{\bar{\psi}(x) u(x) \psi(x)}_{\text{Hint.}} .$$

$$= L_{\text{int. lor.}}$$

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From what we have discussed, we notice that, if we once introduce a non-local interaction, the whole structure of the field theory changes completely. Above all, we can only talk about the equations of motion and constants of collision, ~~for field variables~~ but ~~we have nothing of the kind~~ the Schrödinger function and the Schrödinger equation can not be found easily its places in our theory.

So the only conceivable way of developing the theory is to deal directly with the field equation also, thereby construct S-matrix according to Yang-Feldmen-Källen's method.

$$\psi(x) = \psi^{(in)}(x) + \int S_M^{(ret)}(x-x') \overline{P}(x', x'', x''') u(x'') \psi(x''') \, dx' \, dx'' \, dx'''$$

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N. L. (6)

$$u(x) = u^{(in)}(x) - \int \bar{\psi}(x') \underline{\mathcal{F}}(x, x', x'') S_F^{(ret)}(x - x'') \psi(x'') dx' dx'' dx'''$$

where $\psi^{(in)}(x)$, $u^{(in)}(x)$ satisfy the field field equations and coincide with the actual fields $\psi(x)$, $u(x)$ at $t = -\infty$. This ~~true~~ is true, if

$$\lim_{x_0 \to -t} \int S_M^{(ret)}(x - x') \underline{\mathcal{F}}(x', x'', x''') u(x'') \psi(x''') dx' dx'' dx''' = 0$$

$$\lim_{x_0 \to -t} \int S^{(ret)}(x - x'') \bar{\psi}(x') \underline{\mathcal{F}}(x', x'', x''') \times \psi(x''') dx' dx'' dx''' = 0$$

hold. This is not so obvious as in the case of local interaction, because $\underline{\mathcal{F}}(x', x'', x''')$ extends over x', x'' which are later than x'' and hence x. But let us suppose that $\underline{\mathcal{F}}$ is so chosen that the integrals above

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~~be~~ vanish, if the interaction (or more precisely the coupling const.) was switch in a remote past.

Similarly, one has

$$\psi(x) = \psi^{(in)}(x) + \int S_M^{(adv)}(x-x')\,\Phi(x',x'',x''') \times u(x'')\psi(x''')\,dx'dx''dx'''$$

$$u(x) = u^{(in)}(x) - \int \bar{\psi}(x')\,\Phi(x',x'',x''') \Delta_m^{(adv)}(x-x'')\,\psi(x''')\,dx'dx''dx'''$$

$$\psi^{(out)} = \lim_{\sigma\to\infty} \psi(x,\sigma)$$

$$u^{(out)} = \lim_{\sigma\to\infty} u(x,\sigma)$$

$$\psi^{(out)}(x) = \psi^{(in)}(x) - \int S_M(x-x')\,\Phi(x',x'',x''') \times u(x'')\psi(x''')\,dx'dx''dx'''$$

$$u^{(out)}(x) = u^{(in)}(x) + \int \bar{\psi}(x')\,\Phi(x',x'',x''') \times \Delta_m(x-x'')\,\psi(x'')\,dx'dx''dx'''$$

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N. L (7)

Quantization of fields:

$$[\psi_\zeta^{(in)}(x), \bar{\psi}_{\zeta'}^{(in)}(x')]_+ = \left(\begin{matrix} * & \\ & * \end{matrix}\right)_{\zeta\zeta'} S_M(x-x')$$

$$[u^{(in)}(x), u^{(in)}(x')] = i\Delta_m(x-x')$$

then, the out-fields satisfy the same com. rel., so that

$$\left.\begin{aligned} \psi^{(out)} &= S^{-1}\psi^{(in)}S \\ u^{(out)} &= S^{-1}u^{(in)}S \\ SS^* &= S^*S = 1 \end{aligned}\right\}$$

In order to find the S-matrix, we ~~carry out~~ make use of the method of successive approximation

$$\psi^{(out)}(x) = \psi^{(in)}(x) - \int S_M(x-x')\,\Phi(x',x'';x''')$$
$$\times u^{(in)}(x'')\,\psi^{(in)}(x''')\,dx'\,dx''\,dx'''$$

$$u^{(out)} = u^{(in)}(x) + \int \bar{\psi}^{(in)}(x')\,\Phi(x',x'',x''')$$
$$\times \Delta_m(x-x'')\,\psi^{(in)}(x''')\,dx'\,dx''\,dx'''$$

However, the first order terms vanish, because of the conservation of total

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energy and momentum. ($S_M(x-x')$, x' appears as if it is the coord. of a particle with rest mass M, so that the integration with respect to $X = \frac{1}{3}(x'+x''+x''')$ vanishes, provided that Φ is indep. of X. Similarly, $\Delta_m(x-x'')$, x'' appears as if it is the coord. of a particle with mass m.)

The second order term can be calculated by taking the first order term of ψ, u:

$$\psi(x) = \psi^{(in)}(x) + \int \bar{S}_M(x-x')\Phi(1,2,3)\, u^{(in)}(x'')\psi^{(in)}(x''')\,dx'dx''dx'''.$$

became

$$\Delta_m^{(ret)}(x) = \bar{\Delta}_m(x) - \frac{1}{2}\Delta_m(x)$$

$$S_M^{(ret)}(x) = \Phi\bar{S}_M(x) - \frac{1}{2}S_M(x)$$

$$u(x) = u^{(in)}(x) - \int \bar{\psi}^{(in)}(x')\Phi(x',x'',x''')\, \bar{\Delta}_m(x-x'')\psi^{(in)}(x'')\,dx'dx''dx'''.$$

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$$\psi^{(out)}(x) = \psi^{(in)}(x) - \int S_M(x-x') \bar{\Phi}(x', x'', x''')$$
$$u^{(in)}(x'') \bar{S}(x''' - x'') \Phi(x'' x'' x'')$$
$$u^{(in)}(x^{v}) \psi^{(in)}(x^{vi}) d(x' \cdots x^{vi})$$
$$+ \int S_M(x-x') \bar{\Phi}(x', x'', x''')$$
$$\times \{ \bar{\psi}^{in}(x^{iv}) \Phi(x^{iv} x^{v} x^{vi}) \bar{\Delta}_m(x'' - x^{v}) \psi^{(in)}(x^{vi}) \}$$
$$\times \psi^{in}(x''') d(x' \cdots x^{vi})$$
$$u^{(out)}(x) = u^{(in)}(x) + \int \bar{\psi}^{(in)}(x') \bar{\Phi}(x', x'', x''')$$
$$\times \Delta_m(x - x'') \bar{S}_M(x'' - x^{iv}) \bar{\Phi} u^{(in)}(x^{v})$$
$$\psi^{in}(x^{vi}) d(1 \cdots 6)$$
$$+ \int \bar{\psi}^{in}(x^{iv}) \bar{\Phi}(x^{iv}, x^{v}, x^{vi}) u^{in}(x^{v})$$
$$\times S_{M}(x'' - x') \bar{\Phi}(x', x'', x''') \Delta_m(x - x'')$$
$$\psi^{in}(x''') d(1, \cdots 6)$$

On the other hand

$$\psi^{(out)} = S \psi^{(in)} S$$

with

$$S = 1 + i\eta + \cdots$$

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where η starts from the second order in q.

Then, we have, ~~up to the second order~~ up to the second order,

$$\psi^{(out)} = \psi^{(in)} - i[\eta, \psi^{(in)}]$$

$$u^{(out)} = u^{(in)} - i[\eta, u^{(in)}]$$

where

$$\eta = -\int \bar{\psi}^{(in)}(x') \Phi(x', x'', x''') u^{(in)}(x'') \times \overrightarrow{S}_\mu(x''' - x'^{\nu}) \Phi(x'^{\nu}, x^{\nu}, x^{\nu\prime}) u^{(in)}(x^{\nu}) \times \psi^{(in)}(x^{\nu\prime}) dx' \cdots dx^{\nu\prime}$$

$$+ \frac{1}{2} \int \bar{\psi}^{(in)}(x') \Phi(x', x'', x''') \{ \bar{\psi}^{(in)}(x'^{\nu}) \Phi(x'^{\nu}, x^{\nu}, x^{\nu\prime}) \bar{\Delta}_m(x''' - x^{\nu}) \psi^{in}(x^{\nu\prime}) \} \times \psi^{(in)}(x''') d(x' \cdots x^{\nu\prime})$$

What we have to do next is to examine the convergence of the various matrix elements of η.

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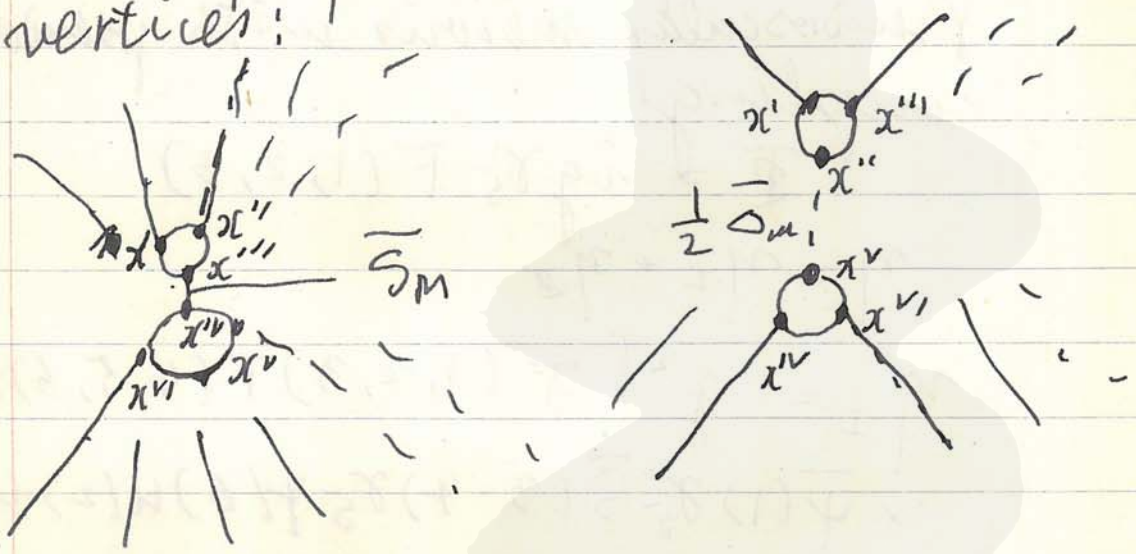
N.L.(9)

By the way, the above method is not at all systematic, but one can see that the above η corresponds to the graph with two "non-local" vertices:



As for more general discussions, see Bleuh. We ~~are only interested in~~ confine our attention here to the problem of convergence in the lowest order.

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Calculation of matrix elements of η.

(i) Self-energy of the meson:

$$\langle \vec{p}'' | \eta | \vec{p}' \rangle_{(1)} = \langle \vec{p}'' | \eta - \langle 0|\eta|0\rangle | \vec{p}' \rangle$$

Pseudoscalar mesons with pseudoscalar coupling:

$$\mathcal{H} = ig\gamma_5 \bar{\tau}(1,2,3)$$

$$\eta = \eta_I + \eta_{II}$$

$$\eta_I = g^2 \int \tau(1,2,3) F(4,5,6) d(1,\cdots 6)$$
$$\times \bar{\psi}(1)\gamma_5 \bar{S}(3-4)\gamma_5 \psi(6) u(2) u(5)$$

$$\eta_{II} = -\tfrac{1}{2} g^2 \int \tau(1,2,3)\tau(4,5,6) d(1\cdots 6)$$
$$\times \bar{\psi}(1)\gamma_5 [\bar{\psi}(4)\gamma_5 \psi(6)] \psi(3) \bar{\Delta}(2-5)$$

where η_{II} does <u>not</u> contribute to the self-energy of the meson because it has the same value for the one-meson state and the vacuum state.

$$\langle \vec{p}'' | \eta | \vec{p}' \rangle_{(1)} = g^2 \int \tau(1,2,3)\tau(4,5,6)$$
$$d(1,\cdots 6) \langle 0|\bar{\psi}(1)\gamma_5 \bar{S}(3-4)\gamma_5 \psi(6)|0\rangle$$
$$\langle \vec{p}'' | u(2) u(5) | \vec{p}' \rangle_{(1)}$$

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N.L (10)

$$\langle 0|\bar{\psi}(1)\gamma_5\bar{S}(3-4)\gamma_5\psi(6)|0\rangle$$

$$=\sum_{\xi_1,\xi_6}(\gamma_5\bar{S}(3-4)\gamma_5)_{\xi_1,\xi_6}\langle 0|\bar{\psi}_{\xi_1}(1)\psi_{\xi_6}(6)|0\rangle$$

$$=\sum_{\xi_1,\xi_6}(\gamma_5\bar{S}(3-4)\gamma_5)_{\xi_1,\xi_6}(-S^{(+)}_{\xi_6\xi_1}(6-1))$$

where

$$S_{\beta\alpha}(x'-x)=-i(S^{(-)}_{\beta\alpha}(x'-x)-S^{(+)}_{\beta\alpha}(x'-x))$$

$$S_{\beta\alpha}(x'-x)=\left(\gamma_\mu\frac{\partial}{\partial x'_\mu}-M\right)_{\beta\alpha}\Delta_M(x'-x)$$

$$\Delta_M(x'-x)=-i(\Delta^{(-)}_M(x'-x)-\Delta^{(+)}_M(x'-x))$$

$$\Delta^{(\mp)}_M(x-y)=\frac{1}{(2\pi)^3}\int\!\!\int\!\!\int\frac{e^{i(\mathbf{p}\cdot\mathbf{r}\mp p_0x_0)}}{2p_0}(dp)^3$$

$$=\frac{1}{2}(\Delta^{(1)}_M(x)\pm i\Delta(x))$$

$$\langle 0|\bar{\psi}(1)\gamma_5\bar{S}(3-4)\gamma_5\psi(6)|0\rangle$$

$$=\mathrm{Tr}\left(\gamma_5(\gamma_\mu\partial^{(3)}_\mu-M)\gamma_5(\gamma_\mu\partial^{(6)}_\mu-M)\right)\times\bar{\Delta}_M(3-4)\Delta^{(+)}_M(6-1)$$

$$\bar{\Delta}_M(x)=\frac{1}{(2\pi)^4}\int\!\!\int\!\!\int(dk)^3\frac{e^{ikx}}{k^2+M^2}$$

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$$= -4\left(\partial_\mu^{(3)}\partial_\mu^{(6)} - M^2\right)\bar{\Delta}_{\mu(3-4)}\Delta_M^{(+)}(6-1)$$

$$\langle \vec{p}'' | u(2)u(5) | \vec{p}' \rangle_{(1)} = \frac{1}{(2\pi)^3}\frac{1}{\sqrt{2p_0'' \cdot 2p_0'}} \times \left\{ e^{i(p'x^{(5)} - p''x^{(2)})} + e^{i(p'x^{(2)} - p''x^{(5)})} \right\}$$

$\because$

$$u = u^{(-)} + u^{(+)}$$

$$u^{(-)} = \frac{1}{\sqrt{(2\pi)^3}} \int \frac{1}{\sqrt{2p_0}} b(\vec{p}) e^{ip_\mu x_\mu} d\vec{p}$$

$$u^{(+)} = \frac{1}{\sqrt{(2\pi)^3}} \int_{p_0>0} \frac{1}{\sqrt{2p_0}} b^*(\vec{p}) e^{+ip_\mu x_\mu} d\vec{p}$$

$$p_4 = ip_0 \qquad x_4 = ix_0$$

$$[b(\vec{p}'')\, b^*(\vec{p}')] = \delta(\vec{p}'' - \vec{p}')$$

$$|\vec{p}\rangle = b^*(\vec{p})|0\rangle$$

$$\langle 0 | \int b(p'') b^*(p') \, d\vec{p}'' d\vec{p}' | 0 \rangle = \int \delta(p_\nu'' - p_\nu') \delta(p_\nu''' - p_\nu) \, (\cdots) d\vec{p}'' d\vec{p}''' \times (\cdots) d\vec{p}' d\vec{p}'''$$

$$\langle 0 | \int b(p'') b(p''') b^*(p''') b^*(p') | 0 \rangle = \int \delta(p_\nu'' - p_\nu''') \delta(p_\nu''' - p_\nu') \, (\cdots) d\vec{p}'' d\vec{p}''' + \int \delta(p_\nu'' - p_\nu') \delta(p_\nu''' - p_\nu''') \, (\cdots)$$

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As for the ~~factor~~ form functions, we consider the Fourier transform;

$$\tilde{F}(x', x'', x''') = (2\pi)^{-8} \int\!\!\int \tilde{F}(l', l'', l''') \exp i\{l'x' + l''x'' + l'''x'''\}\, dl'\, dl''\, dl'''$$

From the requirement that $F(x', x'', x''')$ or Φ itself is invariant with respect to the displacement of the coordinate origin, we have the restriction

$$\tilde{F}(l', l'', l''') = G(l', l''')\, \delta(l' + l'' + l''')$$

From the requirement that

$$\int \mathcal{L}_{int}\, dx'\, dx''\, dx'''$$

is hermitian, we obtain the relation

$$\tilde{F}(x', x'', x''') = \tilde{F}^*(x''', x'', x')$$

$$\therefore \quad \mathcal{L}_{int} = -\bar{\psi}(x')\Lambda u(x'')\psi(x''')\, \tilde{F}(x', x'', x''')$$

$$\mathcal{L}^*_{int} = -\psi^*(x''')\Lambda^* \gamma_4^{u(x'')}\psi(x')\, \tilde{F}^*(x', x'', x''')$$

$$= -\psi^*(x''')\gamma_4 \Lambda u(x'')\psi(x')\, \tilde{F}^*(x', x'', x''')$$

($\Lambda^*\gamma_4 = \gamma_4\Lambda$ must hold in order to have a hermitian local interaction)

or $\quad \bar{\psi}(x''')\Lambda^* u(x'')\psi(x') = \bar{\psi}(x')\Lambda u(x'')\psi(x''')$

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or $\int \sigma_J(l', l''') \exp i\{l'x' + l'''x''' - (l' + l''')x''\}\, dl'\, dl'''$

$= \int \sigma_J^*(l', l''') \exp[-i\{l'x''' + l'''x' - (l' + l''')x''\}]\, dl'\, dl'''$

$\sigma_J(l', l''') = \sigma_J^*(-l''', -l')$

In order that $F(x', x'', x''')$ is Lorentz invariant, σ_J must be an invariant function of l', l''' or of

$L = l' + l'''$, $\quad l = \dfrac{l' - l'''}{2}$

$F(x', x'', x''') = (2\pi)^{-8} \int \sigma_J(L, l)$

$\exp i\{L\left(\dfrac{x' + x'''}{2} - x''\right) + l(x' - x''')\}$

$\times\, dL\, dl$

$= \int$ fn of $R = \dfrac{x' + x'''}{2} - x''$

and $r = x' - x'''$.

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N.L (12)

$$\langle \vec{p}'' | \eta | \vec{p}' \rangle_{(1)} = \frac{g^2}{(2\pi)^3 \sqrt{2p_0' 2p_0''}} \frac{(2\pi)^{26}}{(2\pi)^{14}} \frac{1}{(2\pi)^{16}}$$

$$\int_{(1)} \int d[1\cdots 6]\, d(l^{(1)} l^{(3)} l^{(4)} l^{(6)})\, dK'\, dK$$

$$g(l^{(1)}, l^{(3)})\, g(l^{(4)}, l^{(6)})\, \frac{K'K + M^2}{K'^2 + M^2}\, \delta(K^2 + M^2)$$

$$\times \frac{1 - \varepsilon(K)}{2} \exp i \{ l^{(1)} x^{(1)} + l^{(3)} x^{(3)} - (l^{(1)} + l^{(3)}) x^{(2)}$$

$$+ l^{(4)} x^{(4)} + l^{(6)} x^{(6)} - (l^{(4)} + l^{(6)}) x^{(5)} \}$$

$$\times \exp i \{ K'(x^{(3)} - x^{(4)}) + K(x^{(6)} - x^{(1)})$$

$$+ p' x^{(5)} - p'' x^{(2)}$$

$$+ \text{2nd term}$$

$$= \frac{2g^2}{(2\pi)^2} \frac{\delta(p' - p'')}{p_0'} \int dK\, |g(K, -K + p')|^2$$

$$\times \frac{(p' + K)K + M^2}{(p' + K)^2 + M^2}\, \delta(K^2 + M^2)\, \frac{1 - \varepsilon(K)}{2}$$

$x^{(1)}$:	$\delta(l^{(1)} - K)$	$x^{(2)}$:	$\delta(l^{(1)} + l^{(3)} + p'')$
$x^{(3)}$:	$\delta(l^{(3)} + K')$	$x^{(4)}$:	$\delta(l^{(4)} - K')$
$x^{(5)}$:	$\delta(l^{(4)} + l^{(6)} - p')$	$x^{(6)}$:	$\delta(l^{(6)} + K)$

$+ K' \# K$
$= p'$

$* \varepsilon(K) = \pm 1$ for $K_0 \gtrless 0$

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+ 2nd Term ($p' \to -p''$, $p'' \to -p'$

If in the 2nd term we change the variable of integration K into $-K$, the only change in the result is the change in the sign of $\varepsilon(K)$. Hence, we obtain

$$\langle \vec{p}'' | \eta | \vec{p}' \rangle_{(1)} = \frac{2g^2}{(2\pi)^2} \frac{\delta(p'-p'')}{p_0'} \times \int dK \, |G(K,-K-p')|^2 \frac{(p'+k)K+M^2}{(p'+k)^2+M^2} \times \delta(K^2+M^2)$$

If we introduced in the Lagrangian lead

$$\delta L^{int} = \frac{1}{2} \delta m^2 \cdot u^2,$$

the corresponding S-matrix element in the first order must have been

$$-(2\pi)^4 \frac{\delta(p'-p'')}{2 p_0'} \cdot \frac{\delta m^2}{(2\pi)^3} = -\pi \frac{\delta(p'-p'')}{p_0'} \delta m^2.$$

$\therefore$ $(\Box'' - m^2) u(x'') = (-) \delta m^2 \cdot u(x'')$

$$u^{(out)}(x) = u^{(in)}(x) + \delta m^2 \int \Delta_n(x-x') u(x') \, dx'$$

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N. L (13)

First approximation:

$$u^{(out)} = u^{(in)}(x) + \delta m \int \Delta_m(x-x')\, u^{(in)}(x')\, dx'$$

$$= u^{(in)}(x) \underbrace{\pm \int}_{-i\cdot\delta m^2} [u^{(in)}(x), u^{(in)}(x')]\, u^{(in)}(x')\, dx'$$

$$= u^{(in)}(x) \pm i[\eta, u^{(in)}(x)]$$

$$\eta = \left(-\frac{\delta m^2}{2}\right) \int [u^{(in)}(x')]^2 \varepsilon(x-x')\, dx'$$

Thus, the self-energy term is equivalent to

$$\delta m^2 = -\frac{2g^2}{\pi}(2\pi)^{-2} \int dK\, |G(K, -K-P')|^2 \times \frac{(P'+K)K + M^2}{(P'+K)^2 + M^2}\, \delta(K^2+M^2)$$

In the limit $G = 1$, δm^2 is quadratically divergent as it should be. In order to have a convergence, G must be small for large values of K, so that G ~~must be a fn~~ cannot be a fn of $k = k' + k''$ alone, because $k = -P'$ is fixed in our case.

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Self-energy of the Nucleon:

$$\langle p''\sigma''|\eta|p'\sigma'\rangle_{(1)} = -\delta(p''-p') \times 2\pi\{j_\mu \delta A_\mu + I\,\delta M\}$$

where:

$$j_\mu(\vec{p}';\sigma'',\sigma') = i\bar{v}(\vec{p}'',\sigma'')\gamma_\mu v(\vec{p}',\sigma')$$

$$I(\vec{p}';\sigma'',\sigma') = \bar{v}(\vec{p}'',\sigma'')v(\vec{p}',\sigma')$$

with

$$\psi(x) = (2\pi)^{-\frac{3}{2}}\sum_\sigma \int_{p_0>0}(dp)^3\{b^{(p)}(\vec{p},\sigma)v^{(p)}(\vec{p},\sigma)e^{ipx} + b^{(a)*}(\vec{p},\sigma)\bar{v}^{(a)}(\vec{p},\sigma)e^{-ipx}\}$$

$$\bar{\psi}(x) = (2\pi)^{-\frac{3}{2}}\sum_\sigma \int_{p_0>0}(dp)^3\{b^{(p)*}(\vec{p},\sigma)\bar{v}^{(p)}(\vec{p},\sigma) \times e^{-ipx} + b^{(a)}(\vec{p},\sigma)\bar{v}^{(a)}(\vec{p},\sigma)e^{+ipx}\}$$

$$[b^{(p)}(\vec{p}',\sigma'), b^{(p)*}(\vec{p}'',\sigma'')]_+ = \delta_{\sigma'\sigma''}\delta(\vec{p}'-\vec{p}'')$$

etc,

δM and δA_μ are scalar and four-vector respectively and are given by

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$$\delta A_\mu = \frac{1}{(2\pi)^3} g^2 \Big\{ \int dk \Big\{ |G(P', k-P')|^2 \times \frac{(P'-k)_\mu}{(P'-k)^2+M^2} \delta(k^2+m^2) \frac{1+\varepsilon(k)}{2} + \int dK |G(K,-P')|^2 \times \frac{K_\mu}{(K-P')^2+m^2} \delta(K^2+M^2) \frac{1-\varepsilon(k)}{2} \Big\}$$

$$\delta M = \frac{1}{(2\pi)^3} g^2 \Big\{ \int dk |G(P', k-P')|^2 \times \frac{M}{(P'-k)^2+M^2} \delta(k^2+m^2) \frac{1+\varepsilon(k)}{2}$$
$$+ \int dK |G(K,-P')|^2 \frac{M}{(K-P')^2+m^2} \delta(K^2+M^2) \frac{1-\varepsilon(K)}{2} \Big\}$$

Such terms would appear, if we have introduced

$$\delta L_{int} = i \bar{\psi} \gamma_\mu \psi \delta A_\mu + \bar{\psi} \psi \delta M$$

δM can be regarded as the change in the nucleon mass due to the interaction with the meson field, while δA_μ represents a constant self-potential.

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Convergence of Self-energies:

If we assume G to be real, we have

$$G(l'', l''') = G(-l'', -l')$$

and it can be a fn of

$$l^2, \quad L^2, \quad (lL)^2$$

alone. We have seen already that it cannot be a function of L alone.

Let us assume further that G is a function of

$$\pi^2 = l^2 - \frac{(lL)^2}{L^2}$$

alone,

In the case of the meson self-energy

$$l' = K, \quad l''' = -K - P'$$

$$l = K + \frac{P'}{2} \qquad L = -P'$$

$$\pi^2 = \left(K + \frac{P'}{2}\right)^2 - \frac{\left|\left(K + \frac{P'}{2}\right) P'\right|^2}{P'^2}$$

In particular, for a meson at rest

$$\pi^2 = \vec{K}^2 - K_0^2 - K_0 P' + \frac{m^2}{4}$$

$$+ \frac{\left(\left(K_0 + \frac{m}{2}\right) \cdot m\right)^2}{m^2} = \vec{K}^2 - K_0^2 - K_0 m$$

$$- \frac{m^2}{4} + K_0^2 + K_0 m + \frac{m^2}{4} = \vec{K}^2$$

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$$\frac{1}{(-m^2-2mk_0)}\cdot\frac{1}{(-m^2+2mk_0)} = \frac{-4mk_0}{4m^2(k_0^2-\frac{m^2}{4})}$$

Thus, we have

$$\delta m^2 = -\frac{4}{\pi}\left(\frac{g^2}{4\pi}\right)\int G(k^2)k^2dk \times \frac{-k_0^2}{m^2-2mk_0}\cdot\frac{1}{\sqrt{k^2+M^2}}$$

$$= -\frac{4}{\pi}\left(\frac{g^2}{4\pi}\right)\int \frac{G(k^2)k^2dk\sqrt{k^2+M^2}}{(k^2+M^2-\frac{m^2}{4})}$$

Thus, if $G(k^2) = 1$ for $\pi^2 < \frac{1}{\lambda^2}$ i.e. $k \leq \frac{1}{\lambda}$

$= 0$ for $\pi^2 k^2 > \frac{1}{\lambda^2}$

and if $\frac{1}{\lambda} \ll M$,

$$\delta m^2 = -\frac{4}{\pi}\left(\frac{g^2}{4\pi}\right)\int_0^{\frac{1}{\lambda}} \frac{k^2dk}{M}$$

$$= -\frac{4}{3\pi}\left(\frac{g^2}{4\pi}\right)\frac{1}{M}\left(\frac{1}{\lambda}\right)^3 \qquad \frac{1}{\lambda} \approx \mu$$

or $$\frac{\delta m}{m} = \frac{\delta m^2}{2m^2} = -\frac{2}{3\pi}\left(\frac{g^2}{4\pi}\right)\frac{m}{M}\left(\frac{1}{\lambda m}\right)^3 \left(\frac{M}{m}\right)^3$$

For the nucleon self energy.

$l' = p$, $l'' = k - p$

$l = -\frac{k}{2} + p'$ $\quad L = k$

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for the first term. (This is equivalent to ~~change~~ ~~incorrectly~~ exchange L and l' and to change ~~P', k~~ K, P' into $P', -k$)

$$\pi^2 = l^2 - \frac{(lL)}{L^2} = (p' - \frac{k}{2})^2 - \frac{[(p' - \frac{k}{2})k]^2}{k^2}$$

For a nucleon at rest.

$$\pi^2 = p'^2 - p'k + \frac{k^2}{4} - \frac{(p'k)^2}{k^2} + p'k - \frac{k^2}{4}$$
$$= p'^2 - \frac{(p'k)^2}{k^2} = -M^2 + (Mk_0/m^2)^2 \frac{k^2}{} = -M^2 + M^2 + \frac{M^2}{m^2}\vec{k}^2$$

For the second term

$$l' = K, \quad l''' = -P'$$
$$l = \frac{K}{2} + \frac{P'}{2} \qquad L = K - P'$$

$$\pi^2 = \frac{1}{4}(K + P')^2 - \frac{1}{4}\frac{(K^2 - P'^2)^2}{(K - P')^2}$$
$$= \frac{1}{4}(K^2 + 2KP' + P'^2)$$
$$= \frac{1}{2}(KP' - M^2)$$

For a nucleon at rest,

$$\pi^2 = -\frac{1}{2}(\sqrt{\vec{K}^2 + M^2} + M)M < -M^2$$

In both case, the integral will be convergent, if we ~~take~~ assume

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N.L. (16)

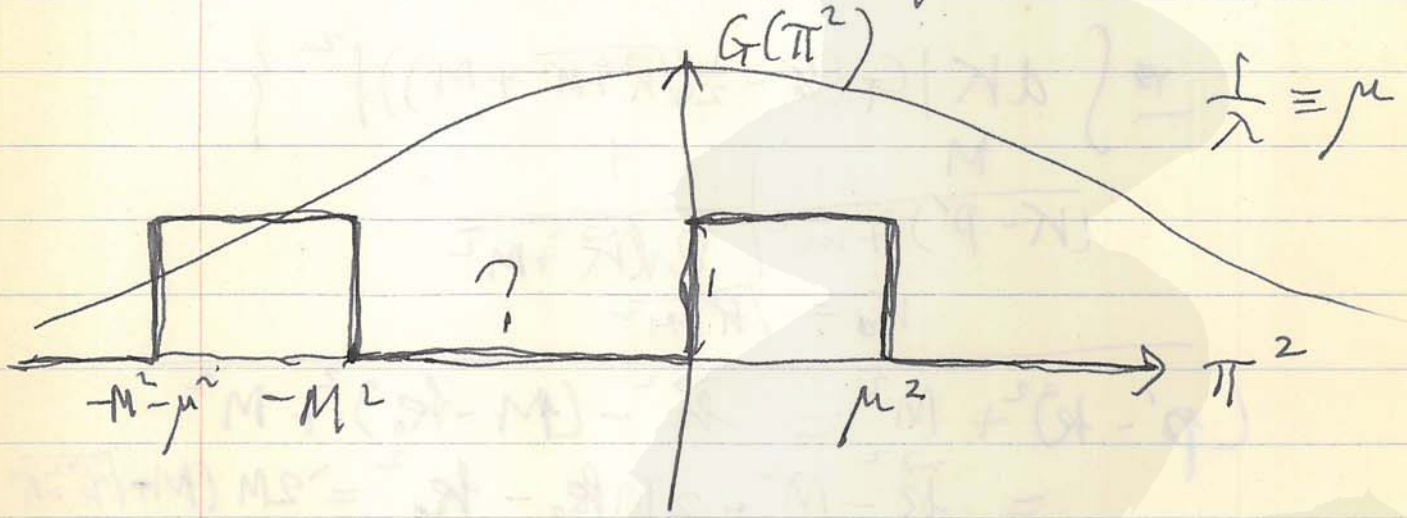
$G(\pi^2) = 1$ for $-M^2 - \frac{1}{\lambda^2} \leq \pi^2 \leq -M^2$

0 for $\pi^2 \leq -M^2 - \frac{2}{\lambda^2}$

together with

$G(\pi^2) = 1$ for $\pi^2 < \frac{1}{\lambda^2}$

$= 0$ for $\pi^2 > \frac{1}{\lambda^2}$



$\frac{1}{\lambda} \equiv \mu$

Of course, one may replace the above discontinuous $G(\pi^2)$ by a contin. fn which decreases sufficiently rapidly with large values of π^2. For instance

$$G(\pi^2) = \exp -(\lambda^2 \pi^2)^2$$

The integrals for δA_μ and δM computed by Møller and Kristensen, ~~In the approximation~~

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For instance

$$\delta M = \frac{1}{(2\pi)^3} g^2 \Big\{ \int d\vec{k} \left| G\left(\frac{M^2}{m^2}\vec{k}^2\right)\right|^2 \frac{M}{(p'-k)^2+M^2}\Big|_{k_0=\sqrt{\vec{k}^2+m^2}} \frac{1}{2\sqrt{\vec{k}^2+m^2}}$$

$$= \int d\vec{K} \left| G\left(\tfrac{M}{m}\vec{K} - \tfrac{1}{2}\left(\sqrt{\vec{K}^2+M^2}+M\right)\right)\right|^2 \Big\} \frac{M}{(K-P')^2+m^2}\Big|_{K_0=-\sqrt{\vec{K}^2+M^2}} \frac{1}{2\sqrt{\vec{K}^2+M^2}}$$

$$(p'-k)^2+M^2 = \vec{k}^2 - (M-k_0)^2 + M^2 = \vec{k}^2 - M^2 + 2Mk_0 - k_0^2 = 2M\left(M+\sqrt{\vec{k}^2+m^2}\right)$$

$$(K-P')^2+m^2 = \vec{K}^2 - K_0^2 + 2MK_0 - M^2 + m^2 = 2MK_0 + m^2 = -2M\sqrt{\vec{K}^2+M^2} + m^2$$

$$\frac{1}{4}\Big[\int_0^\infty \frac{d\vec{k}\ |G(\frac{M^2}{m^2}\vec{k}^2)|^2}{\left(M+\sqrt{\vec{k}^2+m^2}\right)\sqrt{\vec{k}^2+m^2}} + \int_0^\infty \frac{d\vec{K}\ |G(-\frac{1}{2}(\sqrt{\vec{K}^2+M^2}+M))|^2}{\left(\sqrt{\vec{K}^2+M^2}-\frac{m^2}{M}\right)\sqrt{\vec{K}^2+M^2}}\Big]$$

First term: $\vec{K} = \frac{M}{m}\vec{k}$

$$\int \frac{(\frac{m}{M})^3 d\vec{K}\ |G(\vec{K}^2)|^2}{(\frac{m}{M})^2\left(\frac{M^2}{m}-\sqrt{\vec{K}^2+M^2}\right)\sqrt{\vec{K}^2+M^2}}$$

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If $\frac{1}{\lambda} \ll M$, we have approx.

$$-\frac{g^2}{(2\pi)^3}\cdot\frac{4\pi}{4}\cdot\frac{q^3}{3}\left(\frac{1}{\lambda}\right)^3\cdot\frac{1}{M^2}\times\frac{1}{2}$$

$$\frac{\delta M}{M} = -\left(\frac{g^2}{4\pi}\right)\frac{q^3}{3\pi}\frac{1}{M^3}\left(\frac{1}{\lambda}\right)^3 \ll 1.$$

$$\simeq -\left(\frac{g^2}{4\pi}\right)\frac{2}{3\pi}\left[\frac{M}{M}\right]^3$$

$$-\frac{1}{2}\left(\sqrt{K^2+M^2}+M\right) \simeq k \qquad -\sqrt{K^2+M^2} = +2k+M$$

$$-\frac{1}{2}\frac{K\,dK}{M} = dk \qquad K^2+M^2 = M^2+2kM$$

$$K^2 = 2kM.$$

$$K^2\,dK = -2kM\cdot\frac{M\,dk}{\sqrt{K}} = -\sqrt{2kM}\cdot M\,dk$$

$$= 2^{\frac{1}{2}}k^{\frac{1}{2}}M^{\frac{3}{2}}\,dk$$

$$\delta\vec{A} = 0$$

$$\frac{\delta A_0}{M} = \frac{\delta M}{M}$$

Muller concluded from the above results that the change in the masses of the meson and the nucleon

$$\frac{\delta m}{m} \sim 10^{-2}\,\alpha^{-3}$$

$$\frac{\delta M}{M} \sim 10^{-3}\,\alpha^{-3} \qquad \alpha = \left(\frac{m}{M}\right)$$

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are small compared with 1.
However, ~~Since~~ we assumed $\frac{g^2}{4\pi} \sim 10^{-1}$
and $\alpha \sim 1$, This is not true for
the pseudoscalar coupling. Instead,
we have to assume that

$$\frac{g^2}{4\pi} \sim 10 \qquad \alpha^{-1} = \frac{M}{m} > 1$$
$$\sim 3$$

Then

$$\frac{\delta m}{m} \gtrsim 1$$

although

$$\frac{\delta M}{M} \lesssim 1$$

may still hold, provided that m is smaller than M appreciably.

Thus, at least, in the case of the meson self-energy, ~~it is dubious whether~~ we cannot rely much on the lower order in the perturbation calculations.

More generally, one expect that in the case pseudoscalar coupling, we have to take into account, at least,

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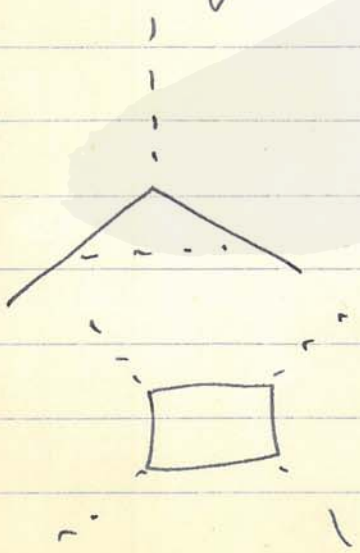
N. L. (18)

there are other the fourth order terms, because the probability of creation and annihilation of a pair of nucleons together with ~~the~~ a meson is larger than that of transition of a nucleon with the simultaneous emission and ~~abb~~ absorption of a meson, provided that the ~~nucleon~~ transition is between low energy states.

Furthermore, there are questions relating to other types of primitive divergence such as the vacuum ~~vector~~ polarization type and the meson-scatter ~~of the meson~~ type.



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Physical Interpretation

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(Elementary Derivation of S-matrix)

Connection between S-matrix Elements and Collision Cross-sections. (S.1)

We realized that the S-matrix played not only an important rôle in the local field theories, but also it furnished us a convenient means of generalizing the field theory. The importance of S-matrix was emphasized by Heisenberg already in 1943. (Z.S.f. Phys. 120 (1943), 513; 673) (1944) According to his argument, the future theory may differ ~~from~~ fundamentally from the present one in that the universal length would come into theory as the third constant in addition to the first two universal constants. (Z.S.f. Phys. 110 (1938), 251) Then, one may expect that the microscopic behavior of physical quantities would be quite different from ~~what we know~~ in the present theory.

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It may well happen that the Schrödinger function ψ Schrödinger equation would be replaced by something else or ~~not even~~ it is even possible that we cannot find any substitute for them in the future theory.
However, Heisenberg thought that the quantity like S-matrix may well survive such a ~~visi~~ vicissitude and may have a well-defined meaning in the future theory, because it is related to the asymptotic behavior of the ~~colli~~ interacting particle at a large average distance before and after collision, ~~but~~ In other words, even if the theory be radically different from the present one as to the behavior of ~~partic~~ interacting particles at very short distances from each other, still one may

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3.2

~~for~~ be able to find a way of determining the S-matrix in future theory. He thought that Hamiltonian operator itself may be too complicated to deal with directly, or it may even be possible that there is no Hamiltonian formalism. Instead, in future theory, the Hermitian ~~S~~ operator η, which is related to S ($S = e^{i\eta}$) may have simple forms which could be ~~chosen~~ so chosen that the correct results would be obtained for various possible collision processes. I ~~do not know whether this~~ doubt very much what conjecture is possible at all in future to guess the form of η or S independ of some underlying fundamental field theory, ~~but~~ However, irrespective of whether Heisenberg is correct (or not) in any specific point,

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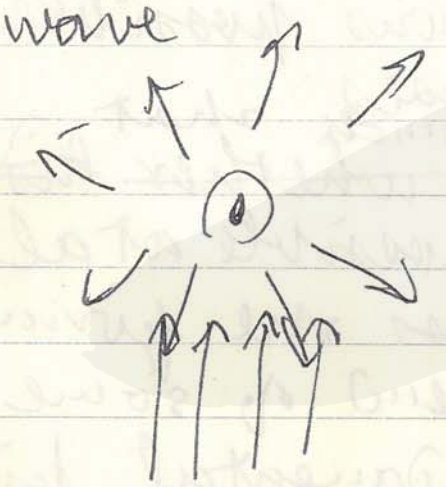
~~It~~ it is interesting to investigate the general properties of S-matrix from a ~~general~~ more ordinary field theory.

C. Moller, Danish Academy. I. II, 1945, 1946

Let us take the case of scattering of two different particles by each other. Then, in the ordinary space-time picture, the relative motion of two particles is to be described by a superposition of a plane incident wave and an spherical diverging wave



as symptotically

Namely, if the incident wave propagates in z-direction.

$$\psi(r,\theta,\varphi) \cong e^{ikz} + \frac{e^{ikr}}{r} \times f(\theta,\varphi)$$

asymptotically, the differential cross-section is given by

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$$d\sigma = |f(\theta,\varphi)|^2 \sin\theta\, d\theta\, d\varphi$$

Now, if we go over to the momentum space, ~~including the time factor $e^{-iK_0 t_0}$~~

$$\psi(x,y,z) \cong e^{iKz} + \frac{e^{iKr}}{r}$$

$$\psi(x,y,z) = \int \Psi(K_r)\, e^{iKr}\, dK_r$$

~~For a stationary state with the energy E~~

Let us take the whole Schrödinger function, for a general stationary state

$$\Psi(x_1', y_1', z_1'; x_2'', y_2'', z_2'')$$

and take the Fourier integral

$$\Psi(x_1, y_1, z_1, x_2, y_2, z_2) = \iint \chi(K_1, K_2)\, e^{i(K_1 r_1 + K_2 r_2)}\, (dK_1)(dK_2)$$

If the total energy of the system is E and the Hamiltonian is

$$H = H_0 + V,$$

we have

$$(E - H_0)\Psi = V\Psi$$

or, in momentum space,

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$$(E - W')\chi(\mathbf{k}_1', \mathbf{k}_2') = \int (\mathbf{k}_1'\mathbf{k}_2'|V|\mathbf{k}_1''\mathbf{k}_2'')\chi(\mathbf{k}_1''\mathbf{k}_2'')\chi(d\mathbf{k}_1'')\chi(d\mathbf{k}_2'')$$

where

$$(\mathbf{k}_1'\mathbf{k}_2'|V|\mathbf{k}_1''\mathbf{k}_2'') = \frac{1}{(2\pi)^3}\int e^{-i(\mathbf{k}_1'\mathbf{r}_1 + \mathbf{k}_2'\mathbf{r}_2)} \times V e^{i(\mathbf{k}_1''\mathbf{r}_1 + \mathbf{k}_2''\mathbf{r}_2)} d\mathbf{r}_1 d\mathbf{r}_2$$

In short,

$$(E - W')\chi(\mathbf{k}') = \int (\mathbf{k}'|V|\mathbf{k}'')\chi(\mathbf{k}'') \times d\mathbf{k}''$$

We are interested in the solution $\chi(\mathbf{k}')$, which reduces to a incident plane wave with the wave vectors

$$V\Psi = \int V \cdot \chi(\mathbf{k}_1''\mathbf{k}_2'') e^{i(\mathbf{k}_1''\mathbf{r}_1 + \mathbf{k}_2''\mathbf{r}_2)} \times (d\mathbf{k}_1'')(d\mathbf{k}_2'')$$

$$\frac{1}{(2\pi)^3}\int e^{-i(\mathbf{k}_1'\mathbf{r}_1 + \mathbf{k}_2'\mathbf{r}_2)} V \Psi_{(\mathbf{k})} d\mathbf{r}_1 d\mathbf{r}_2$$

$$= \int (\mathbf{k}_1'\mathbf{k}_2'|V|\mathbf{k}_1''\mathbf{k}_2'')\chi(\mathbf{k}_1''\mathbf{k}_2'')(d\mathbf{k}_1'')(\quad)$$

where

$$(\quad|V|\quad) = \frac{1}{(2\pi)^3}\int e^{-i(\mathbf{k}_1'\mathbf{r}_1 + \mathbf{k}_2'\mathbf{r}_2)} V e^{i(\mathbf{k}''\mathbf{r}_1 + \mathbf{k}_2''\mathbf{r}_2)} d\mathbf{r}_1 d\mathbf{r}_2$$

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(5.4)

$\Psi_\Phi^{(0)}, \Psi_2^{(0)}$, if there were no interaction.
So one may write

$$\chi(\Psi') = (\Psi' | \chi | \Psi^{(0)}),$$

which depends on both Ψ' and $\Psi^{(0)}$ and can be decomposed into two parts;

$$(\Psi' | \chi | \Psi^{(0)}) = \delta(\Psi' - \Psi^{(0)}) + (\Psi' | T | \Psi^{(0)}),$$

the latter corresponding to the scattered wave.
Since $E = W^{(0)}$, we may write

$$\chi W - W\chi = V\chi$$

where χ is regarded as a matrix which we call wave matrix.
One can then write

$$\chi = 1 + T,$$

so that we have

$$TW - WT = V(1+T)$$

or

$$(W^0 - W')(\Psi' | T | \Psi^{(0)}) = -\frac{1}{2\pi i}(\Psi' | V | \Psi^{(0)})$$

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with

$$U = -2\pi i V(1+T)$$

The general solution is of the form

$$(k'|T|k^{(0)}) = -\frac{1}{2\pi i}(k'|U|k^{(0)}) \times \left[\frac{1}{W^{(0)}-W'} + \lambda\,\delta(W^{(0)}-W')\right]$$

where λ is and arbitrary constant.*
In order that T gives the outgoing wave alone,

$$\lambda = -i\pi$$

or

$$(k'|T|k^{(0)}) = \delta_+(W'-W^{(0)})(k'|U|k^{(0)})$$

where

$$\delta_\pm(W'-W_0) \equiv \frac{\pm 1}{2\pi i(W'-W^{(0)})} + \frac{1}{2}\delta(W'-W^{(0)}_{0})$$

Proof: Dirac, Q. M., p. 195 ~p. 197
Let us take the polar coordinate in relative momentum space and write

$$(k'|T|k^{(0)}) \equiv f(k', W, \chi)\,\delta(k'-k^{(0)})$$

* If the integration with respect to W' is necessary, we take

$$P\int \frac{f(W')}{W^{(0)}-W'}\,dW' \equiv \lim_{\varepsilon\to 0}\left[\int_{W^{(0)}-\varepsilon} + \int_{W^{(0)}+\varepsilon}\right]$$

~~* may depend on variable other than total~~ energy

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$$-\frac{1}{2\pi i}(k'|U|k^{(0)}) \equiv u(k', w, \chi)\,\delta(k'-k^{(0)})$$

then we have

$$(W^{(0)} - W')\,f(k', w, \chi) = u(k', w, \chi),$$

and also we obtain by a Fourier transformation the Schrödinger eq. in coord. space

$$\psi(r, \theta, \varphi) = \iiint e^{ik'r(\cos w \cos\theta + \sin w \sin\theta \times \cos(\chi-\varphi))} \times f(k', w, \chi)\, k'^2\, dk'\, \sin w\, dw\, d\chi$$

In particular, if we take $\theta = 0$.

$$\psi(r, 0, \varphi) = \int_0^\infty p'^2 dp' \int_0^{2\pi} d\chi \int_{-1}^{+1} e^{ip'rx} f(p', w, \chi) \times dx$$

$$\int_{-1}^{+1} e^{ip'rx} f(p', w, \chi)\, dx$$

$$= \left\{ \frac{e^{iprx} f(p', w, \chi)}{ip'r} \right\}_{x=-1}^{x=+1} - \int_{-1}^{+1} dx \frac{e^{ip'rx}}{ip'r} \frac{\partial f}{\partial x}$$

$$= \left[\frac{e^{iprx} f(p', w, \chi)}{ip'r} \right]_{x=-1}^{x=+1} - \left[\frac{e^{ip'rx}}{(ip'r)^2} \frac{\partial f}{\partial x} \right]_{x=-1}^{x=+1} + \int_{-1}^{+1} dx \frac{e^{ip'rx}}{(ip'r)^2} \frac{\partial^2 f}{\partial x^2}.$$

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We are interested in the asymptotic form of ψ for large r, we may ignore the second and third terms:

$$\psi(r\,\theta\,\varphi) \cong -\frac{i}{r}\int_0^{\infty} p'dp'\int_0^{2\pi} d\chi \times\left[e^{ip'r} f(p',0,\chi) - e^{-ip'r} f(p',\pi,\chi)\right]$$

$$\cong -\frac{2\pi i}{r}\int_0^{\infty} p'dp'\left[e^{ip'r} f(p',0,\chi) - e^{-ip'r} f(p',\pi,\chi)\right]$$

Now we substitute the general solution for $f(p',\theta,\chi)$, which is

$$f(p',\theta,\chi) = u(p',\theta,\chi) \times\left[\frac{1}{W^{(0)} - W'} + \lambda\,\delta(W^{(0)} - W')\right]$$

$\lambda(\theta,\chi)$

the first term becomes

$$-\frac{2\pi i}{r}\int_0^{\infty} p'dp' e^{ip'r}\cdot u(p',\theta,\chi) \times\left(\frac{1}{W^{(0)} - W'} + \lambda\,\delta(W^{(0)} - W')\right]$$

$\lambda(0,\chi)$

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(5.6)

"Now there is a relation

$$(P_0')^2 = (W' - W_g')^2 = (\cancel{p - p'}) p'^2 + m^2$$

where W_g' is the energy of the ~~reaction~~ of the centre of mass and m is the reduced mass.

$$\cancel{2W'dW' - 2W_g'dW' = 2p'dp'}$$

or

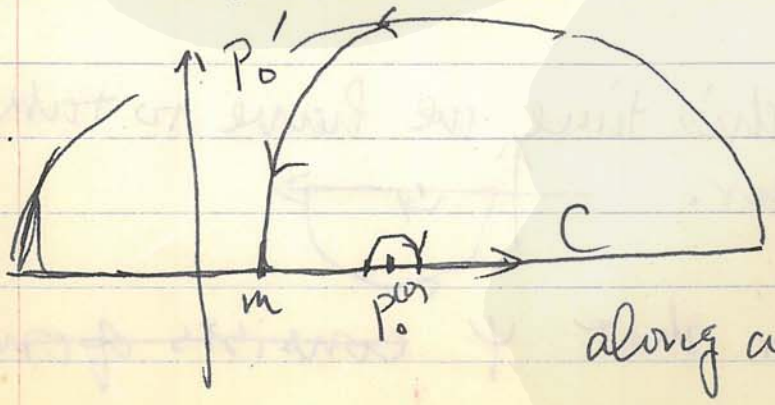
$$2P_0'dP_0' = 2p'dp'$$

$$W^{(0)} - W' \approx P_0^{(0)} - P_0'$$

where

$$P_0^{(0)} \simeq W^{(0)} - W_g' = W^{(0)} - W_g^{(0)}$$

$$-\frac{2\pi i}{r}\left[\int_{\text{res}}^{\infty} \frac{P_0' dP_0' e^{ip'r}}{P_0^{(0)} - P_0'} u(P', 0, \chi) + \cancel{\chi(0,\chi)}\lambda e^{ip^{(0)}r} P_0^{(0)} u(P^{(0)}, 0, \chi)\right]$$



For a large value of r, the contribution to ~~from~~ integration along a contour C in the

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complex $p^{(0)}$-plane, of the ~~part~~ exterior on the upper semi-plane, is small and is proportional to $1/r$.

$$\left(\because \sim \text{const.} \int_0^{\infty} e^{\sim \chi r} dx = \frac{\text{const.}}{r} \right)$$

The contributions from the small semicircle around $p_0^{(0)}$ is

$$+\pi i \, e^{i p_{0}^{(0)} r} \, p_0^{(0)} \, u(p_{,}^{(0)} 0, \chi)$$

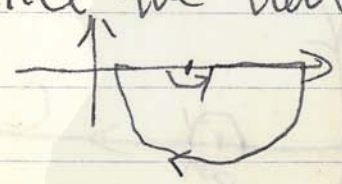
Since the result of integration must be zero, the first term must be

$$-\frac{2\pi i}{r} \{ -\pi i + \lambda \} \, e^{i p_{0}^{(0)} r} \cdot p_0^{(0)} \, u(p_{,}^{(0)} 0, \chi)$$

Similarly, the second term is

$$\frac{2\pi i}{r} [\pi i + \lambda] \, e^{-i p_{0}^{(0)} r} \, p_0^{(0)} \, u(p^{(0)}, \pi, \chi)$$

because this time we have to take the contour.



In order that ψ ~~consists of only~~

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S. W.

contains only diverging (scattered) wave, it is necessary that

$$\lambda = -\pi i$$

Q. E. D.

Now we have

$$\psi(r,\theta,\varphi) = -\frac{2\pi i}{r}[2\pi i]\cdot e^{iP_0 r} P_0^{(0)} u(P_0^{(0)},\theta,\chi)$$

Since the choice of the ~~axes in~~ direction of the axes in polar coord. was completely arbitrary, we have in general

$$\psi(r,\theta,\varphi) = -\frac{2\pi i}{r}[2\pi i]\, e^{iP_0 r} P_0^{(0)} u(P_0^{(0)},\theta,\varphi)$$

To this the incident ~~wave~~ plane wave

$$e^{ip^{(0)}x}$$

is to be superposed.

Thus, the differential cross-section becomes

$$d\sigma = 4\pi^2 \cdot |-2\pi i\, u(P_0^{(0)},\theta,\varphi)|^2 \sin\theta\, d\theta d\varphi$$

$$= 4\pi^2 |(R'|U'|P_{0\parallel}^{(0)})|^2 \sin\theta\, d\Omega$$

where U' is ~~the~~ related to U by

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$$(p'|U|p^{(0)}) = \delta(\quad - \quad)(p'|U'|p^{(0)})$$

Now we introduce

$$(p'|R|p^{(0)}) = \delta(W'-W^{(0)})(p'|U|p^{(0)})$$
$$= \delta(K_\mu'-K_\mu^{(0)})(p'|U'|p^{(0)})$$

~~By definition~~ Since V is hermitian and

$$U = -2\pi i V(1+T)$$
$$= -2\pi i V\Psi$$
$$U^* = 2\pi i \Psi^* V ,$$

we hence we have

$$\Psi^* U + U^* \Psi = 0$$

or $$U + U^* + T^* U + U^* T = 0$$

or

$$(p'|U+U^*|p^{(0)})$$
$$+ \int (p'|U^*|p'')\,dp''\,(p''|U|p^{(0)})$$
$$\times [\delta_+(W'-W'') + \delta_+(W''-W^{(0)})] = 0$$

since

$$(p'|T^*|p^{(0)}) = \delta_+(W'-W^{(0)})(p'|U^*|p^{(0)})$$

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(5,8)

But

$$\int \delta(w'-w^{(0)})[\delta_+(w'-w'') + \delta_+(w''-w^{(0)})]\,..\,dw'$$

$$\simeq \int \delta(w'-w^{(0)})[\delta_+(w^{(0)}-w'') + \delta_+(w''-w^{(0)})]\,..\,dw'$$

$$= \int \delta(w'-w^{(0)})\,\delta(w''-w^{(0)})\,..\,dw'$$

or we have

$$R + R^* + R^*R \simeq 0$$

Thus if we define

$$S = 1 + R$$

$$S^*S = 1.$$

In order to prove

$$SS^* = 1.$$

one has ~~only~~ ~~contracter~~ to replace the diverging wave by a converging wave, (cf. Møller)*

In this way, we obtained a unitary S-matrix, ~~Heisenberg~~ which ~~the~~ can be written as

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$$S = e^{i\eta}$$

where η is a hermitican operator.
Heisenberg thought that in
future theory, instead of the
Hamiltonian, η itself ~~must be~~
may have simple form and one
can start from η.
* Moller has shown also that S-matrix
is ~~not~~ Lorentz invariant.

However, we are not much interested
in Heisenberg theory as such, last.
The above argument is very instructive
in that we could ~~make sure that~~
~~the cross-section~~ find the precise
relation between S-matrix and
the cross-section. The factor $4\pi^2$
will remain unchanged in any theory
because any change in it will destroy
the unitarity of S.

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(S.9)

In this connection, it is interesting that the δ_+-function $(W=W'-W'')$

$$\delta_+(W) = \frac{1}{2\pi i W} + \frac{1}{2}\delta(W)$$

has an intimate ~~connection~~ relation with the chronological factor $\theta(x)$ $x=x-x'$ which appeared in the S-matrix of the field theory. Namely, let the Fourier transform of a function $f(t)$ be $g(W)$:

$$f(t) = \int_{-\infty}^{+\infty} e^{iWt} g(W)\, dW$$

If we take

$$g(W) = \delta(W),$$

then $\quad f(t) = 1.$

If we take

$$g(W) = \frac{1}{W}$$

$$f(t) = P\int_{-\infty}^{+\infty} \frac{e^{iWt}\, dW}{W}$$

$t>0$:

$$P\int_{-\infty}^{+\infty} \frac{e^{iWt}}{W} dW - \pi i = 0$$

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~~$f(t) = 2\pi i$~~ for $t > 0$

$t < 0$

$f(t) = -\pi i$ for $t < 0$

Thus

$$g(w) = \delta_+(w)$$
$$f(t) = \theta(t)$$

where

$\theta(t) = +1$ for $t > 0$
$\quad\;\; 0$ for $t < 0$

This shows that the appearance of $\delta_+(w' - w^{(0)})$ in the relation between

$(\Phi' | T | \Phi^{(0)}) = \delta_+(w' - w^{(0)})(\Phi' | U | \Phi^{(0)})$

and from

$$-\frac{1}{2\pi i} U = V + V T.$$

we obtain the relation between U, V,

$$-\frac{1}{2\pi i} U = V + V \cdot \delta_+ \cdot U$$

corresponds to

~~By~~ leads to the ~~expression can be transformed into~~ the relativistic form:

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(5.10)

$$R(\tau,\tau_0) = \int_{\tau_0''}^{\tau} H(\tau'')\,d\tau'' + \iint_{\tau_0''}^{\tau} H'(\tau'')\,\theta(\tau,\tau'')\,R(\tau'',\tau_0)\,d\tau''\,d\tau''$$

$$R(t\omega,-\omega) = \int_{-\omega}^{t\omega} H'(\tau'')\,d\tau''$$

$$+ \int_{-\omega}^{t\omega}\int H'(\tau'')\,\theta(\tau,\tau'')\,R(\tau'',-\omega)\,d\tau''\,d\tau''$$

etc.

where

$$\frac{1}{2\pi i}\,\delta(\kappa_\mu' - \kappa_\mu^{(0)})\,(\mu|V|\mu^{(0)})$$

$$= \frac{1}{2\pi i}\int dr\,dt\; V(r)\,e^{i(\kappa'-\kappa_0^{(0)})r - i(\kappa_0'-\kappa_0^{(0)})t}$$

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Problem:

The interaction between the nucleon field $\psi(x)$ and the neutral scalar field $\varphi(x)$ is given by the lagrangian density

$$g \sum_a \bar{\psi}_a(x) \psi_a(x) \varphi(x).$$

Compute the second order term in S-matrix element

$$(p_1', p_2' | S | p_1, p_2)$$

for the scattering of two nucleons with the momenta p_1', p_2' and p_1'', p_2'' before and after collision.

Show that

$$(2\pi)^2 \left| (p_1''' p_2'''' | U | p_1' p_2'') \right|^2$$

$$= (2\pi) \left| \sum_I \frac{\bar{H}'_{FI} \bar{H}'_{I0}}{E_I - E_0} \right|^2$$

where $\bar{H}'$ is the Hamiltonian for the interaction between the nucleons and the meson field in the conventional theory and $0, I, F$ denote the initial, intermediate and final states respectively. U is the submatrix of,

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S as defined by

$$(p'''_n p''''_n \underset{S}{W} | p'_n, p''_n) = (p'''_n p''''_n | \cup | p'_n, p''_n)$$
$$\times \delta^{(4)}(p'''_n + p''''_n - p'_n - p''_n)$$
$$\times \delta(W'''_n + W''''_n - W'_n - W''_n)$$

where $W'_n, W''_n, W'''_n, W''''_n$ are the energies of the free nucleons with the momenta $p'_n, p''_n, p'''_n, p''''_n$ respectively. We adopt the unit $\hbar = c = 1$.

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N-N. Scattering Cross-section (1)

$\hbar = c = 1$:

$$d\sigma = \frac{2\pi}{v} \left| \sum_I \frac{H_{FI} H_{IO}}{E_I - E_O} \right|^2 \rho_E \, d\Omega$$

Unit cube

Initial wave function

$$\frac{1}{\sqrt{2}} \{ u_1(1) e^{i p_1 r_1} u_2(2) e^{i p_2 r_2} - u_2(1) e^{i p_2 r_1} u_1(2) e^{i p_1 r_2} \} \Xi_0$$

Final wave fn

$$\frac{1}{\sqrt{2}} \{ u_1'(1) e^{i p_1' r_1} u_2'(2) e^{i p_2' r_2} - u_2'(1) e^{i p_2' r_1} u_1'(2) e^{i p_1' r_2} \} \Xi_0$$

Interaction

$$H' = g \sum_k \frac{1}{\sqrt{2 k_0}} \left(a_k^* (e^{-i k r_1} + e^{-i k r_2}) + a_k (e^{i k r_1} + e^{i k r_2}) \right]$$

$$k_0 = \sqrt{k^2 + \pi^2}$$

Intermediate states:

(i) $\frac{1}{\sqrt{2}} \{ u_1'(1) e^{i p_1' r_1} u_2(2) e^{i p_2 r_2} - u_2(1) e^{i p_2 r_1} u_1'(2) e^{i p_1' r_1} \}$

with $p_1' + k = p_1$

(ii) $\frac{1}{\sqrt{2}} \{ u_1(1) e^{i p_1 r_1} \cdot u_2'(2) e^{i p_2' r_2} - u_2'(1) e^{i p_2' r_1} \cdot u_1(2) e^{i p_2 r_2} \}$

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$$- (\bar{u}_1'(2)u_2(2))(\bar{u}_1(1)u_1(1))$$
$$- (\bar{u}_1'(1)u_2(1))(\bar{u}_2(2)u_1(2))$$

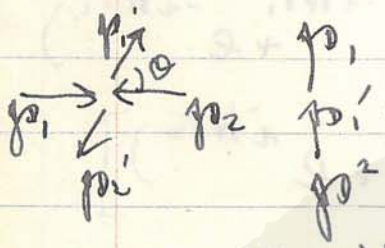
with $p_2' \neq p_2$

(i) $$H_{I0} = \frac{g^{\prime\prime}\cdot \bar{u}_1'(1)u_1(1) + \bar{u}_1'(2)u_1(2)}{2\sqrt{2\sqrt{(p_2-p_1')^2+\pi^2}}}$$

$$H_{FI} = \frac{g^{\prime\prime}\ \bar{u}_2'(1)u_2(1) + \bar{u}_2'(2)u_2(2))}{2\sqrt{2\sqrt{(p_2-p_2')^2+\pi^2}}}$$

$$E_I - E_0 = \sqrt{p_1'^2+M^2} + \sqrt{(p_1-p_1')^2+\pi^2} - \sqrt{p_1^2+M^2}$$

$$\sum_{(i)} \frac{H_{FI}H_{I0}}{E_I-E_0} = \frac{g^2(\bar{u}_2'(1)u_2(1))(\bar{u}_1'(1)u_1(1))}{2\cdot\sqrt{(p_1-p_1')^2+\pi^2}\sqrt{(p_2''-p_2)^2+\pi^2}} \times \left\{\sqrt{p_1'^2+M^2} + \sqrt{(p_1-p_1')^2+\pi^2} - \sqrt{p^2+M^2}\right\}$$



C. M. S, Conservation of Energy-Mom.

$p_1 = -p_2 = p$. $\quad (p_1-p_1')^2 = 2p^2(1-\cos\theta)$

$p_1' = -p_2' = p'$ $\quad (p_2-p_2')^2 = $,,

$p^2 = p'^2$

(ii) $$\sum_{(iii)} \sim \sum_{(i)} \frac{(\bar{u}_2'(1)u_2(1))(\bar{u}_1'(1)u_1(1))}{(p-p')^2+\pi^2}$$

e.l. e.l. -e.l. -e.l.

$$\sum_{(ii)} \frac{M_{FI}M_{I0}}{E_I-E_0} = g^2 \frac{(\bar{u}_2'(1)u_2(1))(\bar{u}_1'(1)u_1(1))}{2\sqrt{(p-p')^2+\pi^2}\cdot\sqrt{(p_1-p_1')^2+\pi^2}}$$

$$\sum_{I} = \frac{(\bar{u}_2'(1)u_2(1))(\bar{u}_1'(1)u_1(1))}{(p-p')^2+\pi^2}$$

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(2)

$$\left|\sum_I\right|^2 = g^4 \frac{\left|(\bar{u}_2(1)u_2(1))(\bar{u}_1'(1)u_1(1))\right|^2}{\{(p-p')^2+\kappa^2\}^2}$$

$$\frac{M}{2}v = p, \qquad v = \frac{2}{M}p$$

$$\rho_E d\Omega = pE\,d\Omega \frac{1}{(2\pi)^3}$$
$$= p\sqrt{p^2+\left(\frac{M}{2}\right)^2}\, d\Omega/(2\pi)^3$$

$$d\sigma = \frac{g^4}{2p}\frac{M}{2}\frac{p\sqrt{p^2+\left(\frac{M}{2}\right)^2}\,d\Omega}{(2\pi)^2\cdot\{(p-p')^2+\kappa^2\}^2}\left[\overbrace{\bar{u}_2'u_2}^{\text{e.e.}}\overbrace{\bar{u}_1u_1'}^{\text{e.e.}}\right]^2$$

$$\left.\begin{aligned} p_x &= p_y = 0 \quad p_z = p \\ p_x &= p\sin\theta \\ p_y &= 0 \quad p_z = p\cos\theta \end{aligned}\right\}$$

$$(p-p')^2 = p^2\sin^2\theta + p^2(1-\cos\theta)^2$$
$$= 2p^2(1-\cos\theta)$$

$$(p-p')^2+\kappa^2 = 2p^2(1-\cos\theta)+\kappa^2$$

$\theta = 2\textcircled{H}$:
$$\cos\theta = \cos^2\textcircled{H} - \sin^2\textcircled{H}$$
$$1-\cos\theta = 1-\cos^2\textcircled{H}+\sin^2\textcircled{H}$$
$$= 2\sin^2\textcircled{H}$$
$$= 4p^2\sin^2\textcircled{H}+\kappa^2$$

$$d\sigma = \frac{2M p\sqrt{p^2+\left(\frac{M}{2}\right)^2}\,d\Omega}{(2\pi)^2\{4p^2\sin^2\textcircled{H}+\kappa^2\}^2}\Big|\qquad\Big|^2$$

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ordinary unit:

$$q^2 \to 4\pi q^2/\hbar c$$

$$p \to \frac{1}{\hbar c} p$$

$$d\sigma = \frac{(4\pi)^2}{(4\pi)^2} \cdot \frac{2q^4 M \sqrt{\quad}}{(\quad)^2} d\Omega$$

$(p_1, E_1), (p_2, E_2)$

(p_1', E_1') (p_2', E_2')

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(3)

(i) $\int \frac{1}{\sqrt{2}}\{\bar{u}_1'(1)e^{-ip_1'r_1} - \bar{u}_2(2)e^{-ip_2r_2} - \bar{u}_2(1)e^{-ip_2r_1}\bar{u}_1'(2)e^{ip_1'r_2}\}$

$\times g\frac{1}{\sqrt{2k_0}}(e^{-ikr_1}+e^{-ikr_2})$

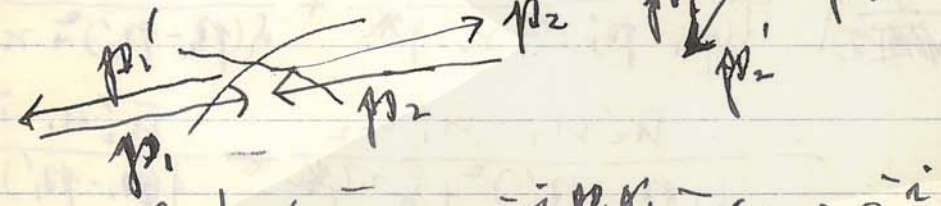
$\times \frac{1}{\sqrt{2}}\{u_1(1)e^{ip_1r_1}u_2(2)e^{ip_2r_2} - u_2(1)e^{ip_2r_1}u_1(2)e^{ip_1r_2}\}dr_1dr_2$

$= \frac{1}{2}\left\{\frac{g}{\sqrt{2}\cdot k[(p_1-p_1')^2+\pi^2]^{1/4}}\{\bar{u}_1'(1)u_1(1)\bar{u}_2(2)u_2(2) + \bar{u}_2'(2)u_2(2)\}\right.$ — $\frac{g}{\sqrt{2}}\}$

$\bar{u}_1'(1)e^{-ip_1'r_1}\bar{u}_2'(2)e^{-ip_2'r_2}$

$g\frac{1}{\sqrt{2k_0}}(e^{-ikr_1}+e^{-ikr_2})$

$u_1(1)e^{ip_1r_1}u_2(2)e^{-ip_2r_2}$



(ii) $\int \frac{1}{\sqrt{2}}\{\bar{u}_1(1)e^{-ip_1r_1}\bar{u}_2'(2)e^{-ip_2'r_2} - \bar{u}_2'(1)e^{-ip_2'r_1}\bar{u}_1(2)e^{-ip_2r_2}\}$

$\times g\frac{1}{\sqrt{2k_0}}(e^{-ikr_1}+e^{-ikr_2})$

$\times \frac{1}{\sqrt{2}}\{u_1(1)e^{ip_1r_1}u_2(2)e^{ip_2r_2} - u_2(1)e^{ip_2r_1}u_1(2)e^{ip_1r_2}\}dr_1dr_2$

$= \frac{1}{2}\left(\frac{g}{\sqrt{2}\{(p_2-p_2')^2+\pi^2\}^{1/4}}\{\bar{u}_2'u_2 + \bar{u}_2'u_2\}\right.$

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$$\bar{u}_1'(1)e^{-ip_1'r_1}\bar{u}_2'(2)e^{-ip_2'r_2} - \bar{u}_2'(1)e^{-ip_2'r_1}\bar{u}_1'(2)e^{-ip_1'r_2}$$

(iii) $\int \frac{1}{\sqrt{2}}\{\bar{u}_2'(1)e^{-ip_2'r_1}\bar{u}_2(2)e^{-ip_2r_2} - \bar{u}_2(1)e^{-ip_2r_1}\bar{u}_2'(2)e^{-ip_2'r_2}\}$

$$\times \frac{g}{\sqrt{2}k_0}(e^{-ikr_1}+e^{-ikr_2})$$

$$\times\{u_1(1)e^{ip_1r_1}u_2(2)e^{ip_2r_2} - u_2(1)e^{ip_2r_1}u_1(2)e^{ip_1r_2}\}dr_1dr_2$$

$$= \frac{1}{2}\frac{g(\bar{u}_2'u_1+\bar{u}_2'u_1)}{\sqrt{2}\{(p_1-p_2')^2+\varkappa^2\}^{1/4}}$$

(iv) $\frac{1}{2}\frac{g(\bar{u}_1'u_2+\bar{u}_1'u_2)}{\sqrt{2}\{(p_2-p_1')^2+\varkappa^2\}^{1/4}}$

$$\frac{g}{2}\left\{\frac{\bar{u}_2'u_2\,\bar{u}_1'u_1}{\{(p_1-p_1')^2+\varkappa^2\}^{1/2}} + \frac{\bar{u}_2'u_2\,\bar{u}_1'u_1}{\{(p_2-p_2')^2+\varkappa^2\}^{1/2}} - \frac{\bar{u}_2'u_1\cdot\bar{u}_1'u_2}{\{(p_1-p_2')^2+\varkappa^2\}^{1/2}} - \frac{\bar{u}_2'u_1\cdot\bar{u}_1'u_2}{\{(p_2-p_1')^2+\varkappa^2\}^{1/2}}\right\}$$

$$= g\left\{\frac{\bar{u}_2u_2\cdot\bar{u}_1'u_1}{\{(p_1-p_1')^2+\varkappa^2\}^{1/2}} + \frac{\bar{u}_1'u_1\cdot\bar{u}_1'u_2}{\{(p_2-p_2')^2+\varkappa^2\}^{1/2}}\right\}$$

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$$d\sigma = g^4 \frac{M}{2(2\pi)^2} \frac{\sqrt{p^2+\left(\frac{M}{2}\right)^2}\, d\Omega}{\{(p\cdot p')^2 + \mu^2\}^2}$$

$$\times \left| \frac{\bar{u}_2 u_2 \cdot \bar{u}_1' u_1}{(p-p')^2+\mu^2} - \frac{\bar{u}_2' u_1 \cdot \bar{u}_1' u_2}{(p+p')^2+\mu^2} \right|^2$$

$W = 2E$:

$$\frac{p^2 dp}{dW} = \frac{pE\,dE}{dW} = \frac{1}{2} pE$$

v

$\rightarrow p \quad \leftarrow -p$

$$p' = \frac{p+\beta E}{\sqrt{1-\beta^2}}$$

$$\beta = \frac{p}{E} \quad \left(= \frac{2p}{\sqrt{1-\left(\frac{p}{E}\right)^2}}\right)$$

$$p' = \frac{2pE}{M}$$

$$E' = \frac{E+\beta p}{\sqrt{1-\beta^2}} = \frac{E^2+p^2}{\sqrt{E^2-p^2}}$$

$$= \frac{E^2+p^2}{M}$$

$$\frac{p'}{E'} = \frac{2pE}{E^2+p^2}$$

$$\frac{1}{2} pE \frac{E^2+p^2}{2pE} = \frac{E^2+p^2}{4}$$

$$= \frac{p^2+M^2+p^2}{4} = \frac{p^2+\frac{M^2}{2}}{2}$$

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Møller: I. p. 19

$$dQ = \frac{dN}{\rho_1^\circ \rho_2^\circ |u_1^\circ - u_2^\circ|}$$

$$u_1^\circ = -u_2^\circ = \frac{p}{E}$$

$$\frac{\frac{1}{2} p E}{2 p/E} = \frac{E^2}{4} = \frac{p^2 + M^2}{4}$$

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H.1

Hint: unit cube

$$\varphi(x)=\sum_k \sqrt{\frac{1}{2k_0}}\left(a_k e^{i(kx-k_0x_0)}+a_k^* e^{-i(kx-k_0x_0)}\right)$$

$$\psi_\alpha(x)=\sum_p \left(b^{(j)}_p e^{i(px-p_0x_0)}(u^{(j)}_p)_\alpha + c^{*(j)}_p e^{-i(px-p_0x_0)}(v^{(j)}_p)_\alpha\right)$$

$$\bar\psi_\alpha(x)=\sum_p \left(b^{(j)*}_p(\bar u^{(j)}_p)_\alpha e^{-i(px-p_0x_0)} + c^{(j)}_p(\bar v^{(j)}_p)_\alpha e^{i(px-p_0x_0)}\right)$$

$$(p''',p^{IV}|S_2|p',p'')=-g^2(\cdot\cdot)\int \bar\psi_\alpha(x)\psi_\alpha(x)\varphi(x)$$

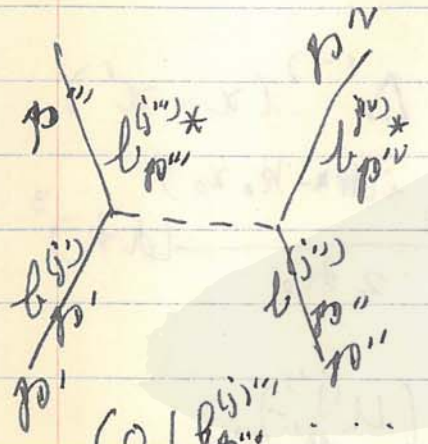
$$\times\theta(x,x')\bar\psi_\beta(x')\psi_\beta(x')\varphi(x')\,dx\,dx'(\cdot\cdot)$$

$$=-g^2\int(0|\varphi(x)\varphi(x')|0)$$

$$\times(0|b^{(j''')}_{p'''}b^{(j^{IV})}_{p^{IV}}\bar\psi_\alpha(x)\psi_\alpha(x)\theta(x,x')$$

$$\times\bar\psi_\beta(x')\psi_\beta(x')b^{(j')*}_{p'}b^{(j'')*}_{p''}|0)$$

$$dx\,dx'$$



$$(0|b^{(j)'''}_{p'''}\cdots|0)\approx$$

$$b^{(j')}_{p'}b^{(j'')}_{p''}\Big\{b^{(j')*}_{p'}b^{(j'')}_{p''}\,b^{(j''')*}_{p'''}b^{(j^{IV})}_{p^{IV}}\;+\;b^{(j')*}_{p'}b^{(j^{IV})}_{p^{IV}}\,b^{(j''')*}_{p'''}b^{(j'')}_{p''}\;+\;b^{(j''')*}_{p'''}b^{(j'')}_{p''}\,b^{(j')*}_{p'}b^{(j^{IV})}_{p^{IV}}\;+\;b^{(j''')*}_{p'''}b^{(j^{IV})}_{p^{IV}}\,b^{(j')*}_{p'}b^{(j'')}_{p''}\Big\}\cdot u^{(j')}_{p'}e^{i(p'x'-p_0x'_0)}$$

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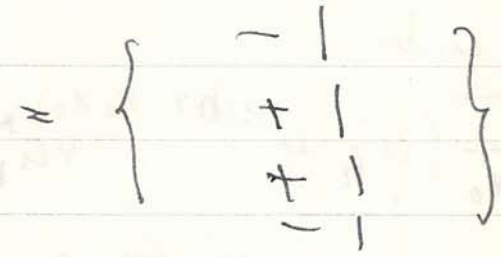

$$\left\{ \begin{array}{l} (\bar{u}^{(j'')}_{p''})_\alpha e^{-i(p''_0 x - p''_0 x_0)} (u^{(j')}_{p'})_\alpha e^{i(p'' x - p''_0 x_0)} \\ \times (\bar{u}^{(j''')}_{p'''})_\rho e^{-i(p''' x' - p'''_0 x'_0)} (u^{(j)}_{p})_\rho e^{i(p' x' - p'_0 x'_0)} \\ \int e^{ikx} dx \end{array} \right.$$

$$(0|\varphi(x)\varphi(x')|0) = \Delta^{(-)}(x-x')$$

$$\Delta^{(-)}(x-x') = \frac{1}{(2\pi)^3} \iiint \frac{e^{i(kx - k_0 x_0)}}{2k_0} (dk)^3$$

$$(\bar{u}^{(j'')}_{p''})_\alpha (u^{(j''')}_{p'''})_\alpha (\bar{u}^{(j''')}_{p'''})_\beta (u^{(j')}_{p'})_\beta$$
$$\exp i\{(p'' - p''')x - (p''_0 - p'''_0)x_0$$
$$+ (p' - p''')x' - (p'_0 - p'''_0)x_0\} \; dx\,dx'$$
$$(2\pi)^4 \delta(p' + p'' - p''' - p'') \delta(p'_0 + \cdots)$$

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Hw 2

$$\int_{-\infty}^{+\infty} \exp i \frac{p'' - p''' - p' + p''' \overbrace{+2k}}{2} (x - x') \quad (d(x-x'))^3$$

$$\int_0^{\infty} \exp -i \frac{p_0'' - p_0''' - p_0' + p_0'''}{2} (x_0 - x_0') \quad d(x_0 - x_0')$$

$$\overbrace{+2k_0}$$

$$= \delta(p'' - p''' - p' + p''' + k_0) \left(\frac{1}{2\pi i (p_0'' - p_0''' - p_0' + p_0''' + 2k_0)} \right.$$

$$\left. + \frac{1}{2} \delta(p_0'' - p_0''' - p_0' + p_0''' + 2k_0) \right)$$

$$\frac{1}{(2\pi)^3} \int \frac{d}{2k_0} \delta(p'' - p''' - p' + p''' + k_0)$$

$$\times \left(\frac{1}{2\pi i (p_0'' - p_0''' - p_0' + p_0''' + k_0)} \right.$$

$$\left. + \frac{1}{2} \delta(p_0'' - p_0''' - p_0' + p_0''' + k_0) \right)$$

$p' + p'' = p''' + p'''$

$p''' + p'' + k = p''' + p' - k$

$= p' + p'' - p''' + p' - k$

$= 2p' + p'' - p''' - k$

$(p', p'')(p'' + k, p'')$

$(p''', p'' - k)$ $p' + p'' - p$

$p''' + k + p'' = p' + p''' - k$

$= p' + p' + p'' - p''' - k$

$p''' + k = p' + p''' - k$

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$$2p = p' + p'' - p'' - p'''$$

$$p' + p'' = p''' + p''$$

$$2p = 2p' - 2p''$$

$$p' = p'' + p$$

$$\int e^{iwt}\, dw = 2\pi i$$

$$t\int_{-\infty}^{t\infty} \frac{1-\cos x}{x^2}\, dx = \pi t$$

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$\Psi(r_1, r_2)$ = $\int (r_1 r_2 | p_1' p_2')$

$\times dp_1' dp_2' \, (p_1' p_2' | \Psi | p_1^{(0)} p_2^{(0)})$

asymptotic form for $r \to \infty$

$\Psi(r_1, r_2)_{r\to\infty} = (2\pi)^{-3} e^{iK^{(0)}X} e^{ik^{(0)}r}$

$- (2\pi)^{-3} e^{iK^{(0)}X} (2\pi i) \frac{e^{ikr}}{r} (p_1' p_2' | U_{k^{(0)} W^{(0)}} | p_1^{(0)} p_2^{(0)})$

$\times \left(\frac{k_2'}{\partial W'/\partial k_2'} \right)$ $\quad r = r_2 - r_1$

$(2\pi)^2 \frac{k_2'^2}{(\partial W'/\partial k_2')^2} |(p_1' p_2' | U_{k^{(0)} W^{(0)}} | p_1^{(0)} p_2^{(0)})|^2$

$\times |(u_2' - u_1') e_2'| \cdot d\Omega$

$(u_2' - u_1') e_2' = \frac{\partial W_1'}{\partial k_2'} + \frac{\partial W_2'}{\partial k_2'} = \frac{\partial W'}{\partial k_2'}$

$\left(= \frac{k_2'}{W_2'} - \frac{k_1'}{W_1'} \right)$

$k_2' = |k_2'| \cdot e_2'$

$k_1' = K^{(0)} - |k_2'| e_2'$ $\Big\}$

$dQ_{scat.} = (2\pi)^2 \cdot \frac{k_2'^2 |C \quad 1 \quad 1 \quad)|^2}{\sqrt{|u_1^{(0)} - u_2^{(0)}|^2 - |u_1^{(0)} \times u_2^{(0)}|^2}} \cdot d\Omega$

$|(u_2' - u_1') e_2'|$

C.M.S. $k' = \frac{1}{2}(k_2' - k_1')$ $\quad u_1^{(0)} - u_2^{(0)} = w^{(0)}$

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Møller, Dan. Acad. Proc. 23 (1945), Nr. 1,

Stationary Solution of Schrödinger equation in momentum space.

$$\overline{H}\Psi = E\Psi.$$

$$\overline{H} = \overline{H}_0 + \overline{H}'$$

$\Psi(\mathfrak{p}_1', \mathfrak{p}_2')$ ~~$\neq$~~ $\qquad W' = \sqrt{\mathfrak{p}_1'^2+\pi^2} + \sqrt{\mathfrak{p}_2'^2+\pi^2}$

$$(E - W')\,\Psi(\mathfrak{p}_1', \mathfrak{p}_2') = \int (\mathfrak{p}_1'\mathfrak{p}_2'|\overline{H}'|\mathfrak{p}_1''\mathfrak{p}_2'')\, d\mathfrak{p}_1''\, d\mathfrak{p}_2'' \; \Psi(\mathfrak{p}_1'', \mathfrak{p}_2'')$$

$$(\mathfrak{p}'|\Psi|\mathfrak{p}^{(0)}) = \Psi_{\mathfrak{p}^{(0)}}(\mathfrak{p}_1', \mathfrak{p}_2') = \delta(\mathfrak{p}_1' - \mathfrak{p}_1^{(0)})\,\delta(\mathfrak{p}_2' - \mathfrak{p}_2^{(0)}) + (\mathfrak{p}_1'\mathfrak{p}_2'|T|\mathfrak{p}_1^{(0)}\mathfrak{p}_2^{(0)})$$

$$T = \frac{1}{2\pi i}(\mathfrak{p}'|U|\mathfrak{p}^{(0)})$$

$$(W^{(0)} - W')(\mathfrak{p}'|\Psi|\mathfrak{p}^{(0)}) = \int (\mathfrak{p}_1'\mathfrak{p}_2'|\overline{H}'|\mathfrak{p}_1''\mathfrak{p}_2'')\, d\mathfrak{p}_1''\, d\mathfrak{p}_2'' \times (\mathfrak{p}_1''\mathfrak{p}_2''|\Psi|\mathfrak{p}_1^{(0)}, \mathfrak{p}_2^{(0)})$$

$$(\Psi W - W'\Psi = \overline{H}'\cdot\Psi)$$

in the partial (configuration) space.

$$(W^{(0)} - W')(\mathfrak{p}'|T|\mathfrak{p}^{(0)}) = (\mathfrak{p}'|\cdot|U|\mathfrak{p}^{(0)})$$

$-\frac{1}{2\pi i}$

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where

$$U = (-2\pi i)(\bar{H}' + \bar{H}' T)$$

$$(k'|T|k^{(0)}) = (k'|U|k^{(0)})\,\delta_+(W' - W^{(0)})$$

$$\delta_\pm(x) = \frac{\pm 1}{2\pi i x} + \frac{1}{2}\delta(x)$$

Conservation of total momentum:

$$(k'|U|k^{(0)}) = \delta(K' - K^{(0)})(k'|U_{K^{(0)}}|k^{(0)})$$

$$U + U^\dagger + T^\dagger U + U^\dagger T = 0$$

$$(k'|R|k^{(0)}) = \delta(W' - W^{(0)})(k'|U|k^{(0)})$$

$S = 1 + R$: unitary

Perturbation theory:

$$U = -2\pi i V$$

$S = e^{i\eta}$: $(k'|\eta|k^{(0)}) = -2\pi\delta(W' - W^{(0)})$
$\times (k'|V|k^{(0)})$

$$(k'|\Psi\Psi^\dagger|k'') = \sum_r \Psi_r(k')\Psi_r^\dagger(k'') = (k'|1|k'')$$

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June 19, 1953

S-Matrix in Momentum Space M.I.

$$S = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar c}\right)^n \int\cdots\int H'(x') \theta(x', x'') H''(x'') \times \theta(x'', x''') H'''(x''') \cdots H^{(n)}(x^{(n)}) \times (dx')^4 \cdots (dx^{(n)})^4$$

$$\simeq \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar c}\right)^n \frac{1}{n!} \int\cdots\int P(H'(x'), H''(x''), \cdots H'(x^{(n)}))(dx')^4 \cdots (dx^{(n)})^4$$

$$\psi(x) = \int \left(\frac{1}{(2\pi)^3}\sqrt{\frac{1}{2k_0}}\right) \{a_k e^{i(kx - k_0 x_0)} + a_k^* e^{-i(kx - k_0 x_0)}\} \zeta(dk)$$

$$\psi(x) \simeq b_p e^{i(px}$$

$$\varphi(x) \simeq \int v_{\mu}(k_\mu) \delta(k_\mu k_\mu + \varkappa^2) e^{i k_\mu x_\mu} \times (dk_\mu)^4$$

$$\left.\begin{aligned} \psi_\alpha(x) &\simeq \int u_\alpha(p_\mu) \delta(p_\mu p_\mu + M^2) e^{i p_\mu x_\mu} (dp_\mu)^4 \\ \overline{\psi}_\alpha(x) &\simeq \int \bar{u}_\alpha(p_\mu) \delta(p_\mu p_\mu + M^2) e^{i p_\mu x_\mu} (dp_\mu)^4 \end{aligned}\right\}$$

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$$\theta(x', x'') =$$

$$\theta(t) = P\int_{-\infty}^{+\infty} \frac{1}{\pi i W} e^{iWt}\, dW$$

$$= 1 \quad \text{for } t > 0$$

$$= 0$$

$$\theta(t) = P\left\{\int_{-\infty}^{+\infty} \left\{\frac{e^{iWt}}{2\pi i W} + \frac{1}{2}\delta(W)\right\} \times e^{iWt}\right\} dW\}$$

$$\simeq 1 \quad \text{for } t > 0$$

$$= 0 \quad \text{for } t < 0$$

$W_0 = -k_0$

~~$\theta(x', x'') = \frac{1}{2\pi i}$~~

$$\theta(x_0) = P\left\{\int_{-\infty}^{+\infty} \left\{\frac{e^{-ik_0 x_0}}{-2\pi i k_0} + \frac{1}{2}\delta(k_0) e^{-ik_0 x_0}\right\} \times dk_0\right\}$$

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$n_\mu = n_1, n_2, n_3, n_4 = i n_0$

$n_\mu n_\mu = -1$

$n_\mu x_\mu = n_1 x_1 + n_2 x_2 + n_3 x_3 - n_0 x_0$

~~$\theta(-n_\mu x_\mu) =$~~

$x_0' = -n_\mu x_\mu$ }

$n_4' = -i n_\mu x_\mu$ }

$\theta(-n_\mu x_\mu) = P\{\int \left(\frac{1}{-2\pi i k_0'} + \frac{1}{2}\delta(k_0')\right)$

$\times \exp\{-i (n_\mu k_\mu)(n_\mu x_\mu)\} d k_0'$

$\int H'(x') \theta(x', x'') H'(x'') dx' dx''$

$= g^2 \cdot \int v(k_\mu) \delta(k_\mu k_\mu + \varkappa^2) \bar{u}_\alpha(p_\mu'') \delta(p_\mu'' p_\mu'' + M^2)$

$u_\alpha(p_\mu'') \delta(p_\mu'' p_\mu'' + M^2)$

$\times \theta(x', x'') v(k_\mu') \delta(k_\mu' k_\mu' + \varkappa^2) \bar{u}_\beta(p_\mu'')$

$\delta(p_\mu'' p_\mu'' + M^2) u_\beta(p_\mu''') \delta(p_\mu''' p_\mu''' + M^2)$

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$$\exp\left\{ i\left[(k_\mu + p_\mu + p'_\mu)x' + (k'_\mu + p''_\mu + p'''_\mu)x'' \right] \right\}$$

$$dk_\mu\, dp\, dp'\, dk'\, dp''\, dp'''\, dx'\, dx''$$

$$\Rightarrow \quad X = \frac{1}{2}(x' + x'') \qquad r = x' - x''$$

$$(2\pi)^4 \cdot \delta(k_\mu + p_\mu + p'_\mu + k'_\mu + p''_\mu + p'''_\mu)$$

$$\int \theta(x', x'')\, e^{+i \cdot \frac{k_\mu + p_\mu + p'_\mu - k'_\mu - p''_\mu - p'''_\mu}{2}}$$

$$x(x'_\mu - x''_\mu)\, dr$$

$$= \int \theta(r)\, e^{-i l_\mu r_\mu}\, dr$$

$$l_\mu = \frac{-1}{2}(k_\mu + p_\mu + p'_\mu - k'_\mu - p''_\mu - p'''_\mu)$$

~~l_μ : time-like~~

$l_\mu r_\mu = l_1 x_1 + l_2 x_2 + l_3 x_3 - l_0 x_0$

~~If, further, $l_0 \to 0$.~~

$$\int \theta(r_1, r_2, r_3, r_0)\, e^{-i l_\mu r_\mu} (dr)^4$$

$$= \int \theta(r_1, r_2, r_3) \left(-\frac{1}{i l_0} + \pi \delta(l_0) \right)$$

$$(2\pi)^3 \cdot \delta(l_1)\delta(l_2)\delta(l_3) \times$$

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M.3

$$\int e^{+ik_\mu r'_\mu} (dk)^4 \int \theta(r_1, r_2, r_3, r_0)\, e^{-il_\mu r_\mu} (dr)^4$$

$$= (2\pi)^4 \int \theta(r_1, r_2, r_3, r_0)\, \delta(k_\mu - k'_\mu) \times (dr)^4$$

$$= (2\pi)^4\, \theta(r'_1, r'_2, r'_3, r'_0)$$

$$= P\int e^{+il_\mu r'_\mu} (dl)^4 \frac{\delta(l_1)\delta(l_2)\delta(l_3)}{} \left(\frac{-1}{il_0} + \pi\delta(l_0)\right)$$

$$= (2\pi)^3 \cdot P\int e^{-il_0 r'_0} \left(\frac{-1}{il_0} + \delta(l_0)\right) dl_0$$

or $\theta(r'_1, r'_2, r'_3, r'_0)$

$$= P\int e^{-il_0 r'_0} \left(\frac{-1}{2\pi i l_0} + \frac{1}{2}\delta(l_0)\right) dl_0$$

$= 1$ for $r'_0 > 0$.

$= 0$ for $r'_0 < 0$.

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$$(2\pi)^4 \delta(\vec{l}) \left(\frac{1}{2\pi i l_0} + \frac{1}{2} \delta(l_0) \right)$$

Old variables: New variables:

k_μ
p_μ
p'_μ
k'_μ
p''_μ
p'''_μ

$L_\mu = k_\mu + p_\mu + p'_\mu + k'_\mu + p''_\mu + p'''_\mu$

$l_\mu = \frac{1}{2}(k_\mu + p_\mu + p'_\mu - k'_\mu - p''_\mu - p'''_\mu)$

k'_μ
p'_μ
k_μ
p''_μ

$p'_\mu = \frac{1}{2} L_\mu - l_\mu - k'_\mu - p_\mu$

$p'''_\mu = \frac{1}{2} L_\mu + l_\mu - k'_\mu - p'_\mu$

Jacobian

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & +\frac{1}{2} & +\frac{1}{2} & +\frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

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$M>4$

$$= -\frac{1}{2}\begin{vmatrix} -2 & -2 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = -\frac{1}{2}\begin{vmatrix} -2 & -2 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= -\frac{1}{2}\begin{vmatrix} -2 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -\frac{1}{2}\{x^2 = +1$$

$$P\int\cdots\int v(k_\mu)\delta(k_\mu k_\mu + \varkappa^2)\,\bar{u}_\alpha(p_\mu)\delta(p_\mu p_\mu + M^2)$$
$$u_\alpha(p'_\mu)\delta(p'_\mu p'_\mu + M^2)$$
$$(2\pi)^4\{\delta^3(\vec{l})\left(\frac{-1}{2\pi i l_0} + \frac{1}{2}\delta(l_0)\right\} v(k')\delta(k'k' + \varkappa^2)$$
$$\bar{u}_\beta(p'')\delta(p''p'' + M^2)\,u_\beta(p''')\delta(p'''p''' + M^2)$$
$$\exp i\{(2\pi)^4\delta^4(L)(dL)^4(dl)^4(dk)^4$$
$$(dp)^4(dk')^4(dp''')^4.$$

$$L_\mu = k_\mu + p_\mu + p'_\mu + k'_\mu + p''_\mu + p'''_\mu$$
$$l_\mu = -\frac{1}{2}(k_\mu + p_\mu + p'_\mu - k'_\mu + p''_\mu - p'''_\mu)$$
$$p'_\mu = \frac{1}{2}L_\mu - l_\mu - k_\mu - p_\mu$$
$$p'''_\mu = \frac{1}{2}L_\mu + l_\mu - k'_\mu - p''_\mu$$

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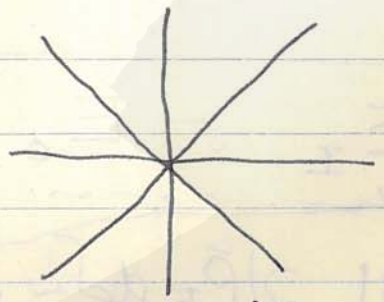
l_μ: space-like

$L_\mu = 0$.

$p_\mu' = -l_\mu - k_\mu - p_\mu$

$p_\mu'' = l_\mu - k_\mu' - p_\mu''$

$$\int \delta^3(\vec{l})\left(-\frac{1}{2\pi i l_0} + \frac{1}{2}\delta(l_0)\right)(d\vec{l})^3(dl_0)$$



$l_0 dl_0 = m dm$.

$$\frac{dm}{l_\mu l_\mu + m^2} + \frac{1}{2}\delta^4(l_\mu)$$

$$(2\pi)^{-\frac{3}{2}\times 3\times 2} = (2\pi)^{-9} \times (2\pi)^{8}$$
$$= (2\pi)^{-1}.$$

$$(2\pi)^{-\frac{3}{2}\times 3\times 2 - 1} \times (2\pi)^{8} = (2\pi)^{-2}$$

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$$\int f(l)\,\frac{\delta^3(\vec{l})\,(dl)^4}{l_0}$$

$$\frac{}{E_I - E_0}$$

$$(2\pi)^2 \times \left/ (2\pi)^{\frac{3}{2}\times 3} \right)^2 = (2\pi)^{-9} \quad (2\pi)^{-1}$$

$$\left((\cdot\cdot) \int H'(x)\,(d\vec{x})^3 \Big/ \cdot \right) (\cdot\cdot) \int H'(x')\,(d\vec{x})^3 \,|\cdot\cdot)$$

$$(2\pi)^8 \cdot \frac{}{E_I - E_0}$$

$$(2\pi)^2 \cdot \frac{1}{i} \left(\quad |\hat{S}| \right)^2 \rho_E \, d\sigma\Omega.$$

$$2\pi \rho_E \, |\overline{H'}|^2$$

$$2\pi \rho_E \sum_{n'} \frac{\overline{H}_{n n'} \, \overline{H}_{n' n}}{E_{n'} - E_{n'_0}}$$

$$\left\{ (2\pi)^8 \cdot (2\pi)^{-9} \cdot (2\pi)^{-1} \right\}^2 \quad (2\pi)^{-2}$$

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Problem:

~~Prove~~ Show that, in the case of the scattering of a nucleon by a nucleon, the differential cross-section ~~is~~ as given by

$$\frac{(2\pi)^2}{v}\left|\langle p_1', p_2' | S | p_1, p_2\rangle\right|^2 \rho_E \, d\Omega$$

reduces to the familiar form

~~$\sum \langle | \bar{H}' | \rangle \langle | \bar{H}'' | p_1, p_2 \rangle$~~

$\bar{H}'$

the interaction between

Suppose that the nucleon field ψ ~~interacts~~ and with the neutral scalar meson field φ according is given by the ~~Lagrangian~~ Lagrangian density

$$\mathcal{H} = g \sum_\alpha \bar{\psi}_\alpha^{(x)} \psi_\alpha^{(x)} \varphi(x) .$$

Compute the ~~S-matrix~~ second order term in S-matrix corresponding to the scattering of two nucleons, ~~Discuss the reason~~ ~~Show that~~ element $\langle p_1', p_2' | S | p_1, p_2 \rangle = \delta(p_1' p_2' | p_1, p_2)\, \delta(p_1' + p_2' - p_1 - p_2)$ the absolute square of the S-matrix element agrees with the ~~expression~~ where p_1, p_2 and p_1', p_2' are momenta of collision, two nucleons b. and a.

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$$\frac{1}{(2\pi)^2}\left|\sum_I \frac{\bar{H}'_{FI}\,\bar{H}'_{I0}}{E_I - E_0}\right|^2$$

which is ~~obtained by application~~ familiar in

Shows that

$$(2\pi)^{\frac{3}{2}}|(p_1', p_2'|B|p_1, p_2)|^2$$

agrees with the familiar expression

$$\left|\sum_I \frac{H'_{FI}\,H'_{I0}}{E_I - E_0}\right|^2$$

in the conventional perturbation theory, where 0, I, F ~~designate~~ denote initial, intermediate and final states respectively and H' is the Hamiltonian for the interaction between the nucleons and the meson field in the conventional theory.